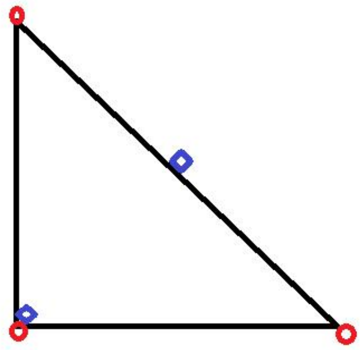
1. Let us consider three data points (0,0) , (1,0) , (0,1) and the initial cluster centers are at the points (0,0) and the mid point of hypotenuse (0.5, 0.5) as shown in fig



 = data point ,  = cluster centre . Now (0,0) will belong to cluster 1 and (1,0) and (0,1) will belong to cluster 2 and k means will stop at this point . This is local solution and its not a global solution . The global solution will have points (0,0) and (0,1) belong to cluster1 with cluster centre at (0,0.5) , and point (1,0) will belong to cluster 2 with cluster centre at (1,0) . This is an example of global convergence failure .

2 . The Objective function for k means can be written as

$$\mathrm{F}(\mathrm{Z}_{1:\mathrm{N}}\,,\mathrm{M}_{1:\mathrm{k}}) \;=\; \sum_{\mathrm{n}=1}^{\mathrm{N}} \mathrm{Dist}(\mathrm{X}_\mathrm{n}\,,\mathrm{M}_{\mathrm{Z}_\mathrm{n}})$$

Where $\mathrm{Dist}(\mathrm{X}_\mathrm{n}\,,\mathrm{M}_{\mathrm{Z}_\mathrm{n}})$ = distance between a point and its cluster centre , $\mathrm{M}_{\mathrm{Z}_\mathrm{n}}$ = cluster centre of data point n .

$\mathrm{Dist}(\mathrm{X}_\mathrm{n}\,,\mathrm{M}_{\mathrm{Z}_\mathrm{n}})$ can be written as the following .

$$\mathrm{Dist}(\mathrm{X}_\mathrm{n}\,,\mathrm{M}_{\mathrm{Z}_\mathrm{n}}) \;=\; Z_n^T\,.\,Dist_n$$

Where Z = [0,0,0,0,.........0, 1 , 0 ,0 0,0..........] , Z contains all zeros and a 1 is present at the Z_n^{th} position .

$Dist_n \;=\; [\,Dist(X_n\,,M_1)\,,\ Dist(X_n\,,M_2)\,,\ldots\ldots\ldots\ldots,\,Dist(X_n\,,M_k)\,]$

Now we can write our equation as

$$\mathrm{F}(\mathrm{Z}_{1:\mathrm{N}}\,,\mathrm{M}_{1:\mathrm{k}}) \;=\; \sum_{\mathrm{n}=1}^{\mathrm{N}} Z_n^T\,.\,Dist_n$$

We have our equation in the form f(x , y) = x.y , if we have a function having the product of independent variables as value of the function than that function is a non-convex function .

Here we have $\mathrm{F}(\mathrm{Z}_{1:\mathrm{N}}\,,\mathrm{M}_{1:\mathrm{k}})$ as a product of two independent variables Z_n^T , $Dist_n$ hence the objective function $\mathrm{F}(\mathrm{Z}_{1:\mathrm{N}}\,,\mathrm{M}_{1:\mathrm{k}})$ is a non-convex function .
We Know that a non-convex function has many local optima but it doesn't lead to a global optima
Hence all non-convex functions converge to local optima . Since our objective function $\mathrm{F}(\mathrm{Z}_{1:\mathrm{N}}\,,\mathrm{M}_{1:\mathrm{k}})$ is a non-convex function it will also converge to a local optima .

Hence K-Means Converge to a Local Optima .