

Fourier series and transforms.

Sriram Vadlamani

December 31, 2019

Introduction

Fourier series are a way of approximating discontinuous functions into continuous functions. For example the PDE heat equation can be approximated as an 'infinite' sum of sine waves of different frequencies. The key word here is definitely *infinite*. A finite sum is simply not capable of giving a discontinuous function as a finite sum of two continuous functions is still continuous. What are the different frequencies of these sine waves? That will be explored later in the notes. What may simplify or rather generalize the hypothesis is by assuming the input as real numbers and the output as a 2 dimensional complex plane.

Basic Math

$$f : \begin{cases} [0, 1] & \mapsto & c^2 \\ x & \mapsto & e^{ix} \end{cases}$$

The function 'f' maps any real number between zero to one to a complex number e^{ix} . The i gives a rotation to the number.

let's begin by applying every real number to this function that maps it to a specific complex number as below:

$$-1 : f(x) = e^{-2\pi ix} \forall x \in [0, 1]$$

$$2 : f(x) = e^{2 \cdot 2\pi ix} \forall x \in [0, 1]$$

$$3 : f(x) = e^{3 \cdot 2\pi ix}$$

and so on.

$$\forall n \in \mathbb{R}, f(x) = e^{n \cdot 2\pi ix} \forall x \in [0, 1]$$

As we can see above, $e^{2\pi ix}$ rotates the vector in a circular motion as x goes from zero to one.

What if, now we want these vectors to have a specific initial length and angle?

We can do that by multiplying each of these outputs by c_n where c_n is a complex constant. For example if we want a value to have an initial angle of 45 degrees,

c_n would be $e^{i\pi/4}$

$$= \dots + c_0 e^{0 \cdot 2\pi ix} + c_1 e^{2\pi ix} + c_2 e^{2 \cdot 2\pi ix} + \dots$$

Now, we should be able to approximate any function $f(x)$ as an infinite sum of such values and it would look something like this:

$$f(x) = c_0 e^{2\pi ix \cdot 0} + c_1 e^{2\pi ix} + c_2 e^{2 \cdot 2\pi ix} + \dots$$

1 Finding the complex constants

Now, in order to draw out any shape we like, we need to determine these constants c_n . Let's start with the simplest one c_0 . The point c_0 acts like a center of mass for the function that we have and thus would be the average of all the other outputs from this function.

In other words, it would be an integral:

$$c_0 = \int_0^1 f(x) dx$$

That would equal:

$$\int_0^1 (c_1 e^{2\pi i x} + c_2 e^{2 \cdot 2\pi i x} + \dots) dx$$

Hence, we can deduce a general formula by that, $\forall n \in \mathbb{Z}$

$$c_n = c_n = \int_0^1 f(x) \cdot e^{-n2\pi i x} dx$$