Improper Integrals

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1 Definition

When a function is not defind for either or both of the bounds in an integral, it is known as an improper integral.
$\int_a^b f(t)dt$
$f : [a, b[\to R$
f continous on $[a, b[$

E.g:
$\int_1^{+\infty} e^{-t}dt$

2 Convergence and Divergence

let $f : [a, b[\to R$ and f be continous on $[a, b[$.
we say that the integral is convergent if

$$\lim_{x \to b} \int_a^x f(t)dt$$

is a finite value.
else, it is divergent.

2.1 Proposition 1

let f and g be two functions on $[a, b[$ such that, both integrals $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ converge.
Then $\forall \alpha \in R$, $\int_a^b (\alpha f(t) + g(t))$ converges and linearity can be used to split the two integrals.

2.2 Riemann Functions

Theorem:

- $\int_0^1 dt/t^\alpha$ converges $\Leftrightarrow \alpha < 1$
- $\int_1^{+\infty} dt/t^\alpha$ converges $\Leftrightarrow \alpha > 1$
- $\int_0^{+\infty} dt/t^\alpha$ is always divergent.

2.3 Positive funtions improper integrals

let $f : [a, b[\to \mathrm{R}_+$ and $g : [a, b[\to \mathrm{R}_+$, continous on $[a, b[$.
if $f(t) < g(t)$, then

- if $\int_a^b g(t)dt$ is conv $\Rightarrow \int_a^b f(t)dt$ is conv.
- if $\int_a^b f(t)dt$ is divergent $\Rightarrow \int_a^b g(t)dt$ is divergent.
- if $f(t) = o(g(t))$ then if $\int_a^b g(t)dt$ is conv. $\Rightarrow \int_a^b f(t)dt$ is conv.
- if $f(t) \sim g(t)$ then $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ have the same nature

3 Cheat sheet

3.1 Different Integration methods

3.2 Integration by parts

When the integral is in the form of $f(t) \cdot g(t)$ we can do the integration by splitting them into the following:
$\int_a^b u \cdot v' = u \cdot v + \int_a^b v \cdot u'$

4 Abel's Theorem

let f and g be two continous functions over $[a, b[$ such that:
1) f is decreasing and $\lim_{t\to+\infty} f(t) = 0$
2) $\exists k \in R$, $\forall x \in [a, b[$, $\int_a^x \mid g(t) \mid dt \le k$
then the integral $\int_a^x f(t) \cdot g(t)dt$ is convergent.

4.1 Integration by substitution

When we have an integral $\int_a^b f(x)dx$ then we try to find a suitable u where:
$u = f(x)$ and replace the integral as follows:
$\int_{f(a)}^{f(b)} udu$

5 Some important integrals

- **LEJEUNE-DIRICHLET's** integral represented as $\int_0^{+\infty} \frac{sin(x)}{x} dx$
- **LAPLACE - GAUSS's** integral represented as $\int_0^{+\infty} e^{-t^2} dt$ which is equal to $\frac{\sqrt{\pi}}{2}$

6 Last Remarks

General redefinition of a function:
$\forall f : t \to f(t)$ defined on $]a, b]$
if $\lim_{x \to a} f(x) = k \ \forall k \in R$ then:
$f(x)$ is continous on $[a, b]$