Inner Product Spaces

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1 Introduction

1.1 A bilinear Form

let E be an R-vs and $\theta : E \times E \to R$, we say, θ is a bilinear form if:
$\forall (u, v, w) \in E^2$ and $\forall a \in R$,

- $\theta(u + v, w) = \theta(u, v) + \theta(u, w)$
- $\theta(au, v) = a\theta(u, v)$

Proposition 1
let E be an R-vs of finite dimension 'n'. We have B = (e_1,e_2, ... , e_n) the basis of E.
$\theta : E \times E \to R$ a bilinear form.
Then $\forall (x, y) \in E^2, \theta(x, y) = X^t \cdot M \cdot Y$
Where M is the matrix of the bilinear form defined as:
$M = \theta(e_i, e_j)$
And X and Y are the coordinates of 'x' and 'y'.

2 Inner product

A space (E, θ) is said to be an inner product space iff

- The bilinear form is '*symmetric*'
- The form is positive definite.

2.1 Symmetry

A bilinear form is symmetric iff
$\forall (x, y) \in E^2$
$\theta(x, y) = \theta(y, x)$

2.2 Positive definite

A bilinear form is said to be positive definite iff
$\forall x \in E, \theta(x, x) \geq 0$, and
$\forall x \in E, \theta(x, x) = 0 \Longrightarrow x = 0$

3 Theorems

3.1 Cauchy-Schwartz and Minkowski Theorems

Cauchy-Schwartz: Let E be an R-vs and $\theta : E \to R$ a positive definite and symmetric bilinear form, then

$\forall(x,y) \in E^2 \mid \theta(x,y) \mid \leq \sqrt{\theta(x,x)} \times \sqrt{\theta(y,y)}$

Minkowski's: Let (E, θ) be an inner product space on R, then:
$\forall(x,y) \in E^2$, $\sqrt{\theta(x+y, x+y)} \leq \sqrt{\theta(x,x)} + \sqrt{\theta(y,y)}$

4 Orthogonality

We call $N : E \rightarrow R$ a norm $\forall(x,y) \in E^2$ and $\forall\lambda \in R$ we have:

- $N(x) \geq 0$
- $N(\lambda \cdot x) = \lambda \cdot N(x)$
- $N(x) = 0 \iff x = 0$
- $N(x+y) \leq N(x) + N(y)$

and we say that 'N' is a norm. In geometry, this is what we call a triangular inequality.
Proposition 2 The norm for any vector $x \in E$ is $\sqrt{\theta(x,x)}$.

4.1 Pythagorean Theorem

for (E, θ) an inner product space, and $\forall(x,y) \in E^2$ such that $< x, y >= 0$, we have:
$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

4.2 The orthogonal of a subspace

For (E, θ) an inner product space, $A^\perp$ is defined as:
$\{\forall x \in E, \forall y \in A, < x, y >= 0\}$ where $A \subset E$

Some important remarks are:

- $A \subset B \Rightarrow B^\perp \subset A^\perp$
- $A^\perp = span(A)^\perp$
- $A \cap A^\perp \subset \{0\}$

5 Orthonormal sets

Definition
Let $(E, <,>)$ be an inner product space on R and $X = \{x_1, x_2, ..x_n\} \subset E$. And $B = \{e_1, e_2, ..e_n\}$ a basis of E. we say that X is an orthonormal set in E if:
$< x_i, x_j >= \delta_{ij}$ and
$< e_i, e_j >\geq \delta_{ij}$
$\delta_{ij} = 1$ if $i = j$
$\delta_{ij} = 0$ if $i \neq j$

Proposition 7: Any orthogonal set / family of vectors of non-zero vectors from an inner product space is linearly independent.

6 Gram-Schmidt Theorem

Let $(E, <,>)$ be a eucledian space and $B = \{e_1, e_2, ..., e_n\}$ a basis of E. Then there exists an orthogonal basis $O = \{f_1, f_2, ...f_n\}$ from E. such that $\forall k \in [1, n]$, $f_k \in span(B)$.

7 Theorem of orthogonal supplementary

Let $(E, <,>)$ be a eucledian space and 'F' a sub-vector space of E, then:
$E = F \oplus F^{\perp}$
Corollary: $F^{\perp\perp} = F$, only in a eucledian space, i.e, Finite dimension.

8 Orthogonal Projections

Definition
Let $(E, <,>)$ be a eucledian space and F a sub-vector space of E. We call orthogonal projector on 'F' and we denote P_F the projector on F parallel to $F^{\perp}$ i.e., $P_F \in L(E)$ such that:
$P_F^2 = P_F$, and $Im(F) = F$ and $ker(F) = F^{\perp}$
Proposition: $\forall x \in E$

$$P_F(x) = \sum_{i=1}^{n} < x, e_i > \cdot e_i$$

9 Distance to a subspace

Proposition: Let F be a sub-vector space of a eucledian space $(E, <, >)$. Then, the map:
$y \Rightarrow \|x - y\| \ \forall x \in E$ reaches it's minimum value at $P_F(x)$.
We call distance of x to F and denote $d(x, F)$ the real number: $\|x - P_F(x)\|$.

10 Extra remarks

- If F is of finite dimension, then $E = F \oplus F^{\perp}$
- if E is finite, $F^{\perp\perp} = F$, but if it's not, then $F \subset F^{\perp\perp}$