Improper Integrals

Sriram Vadlamani

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Contents

1 Definition

When a function is not defind for either or both of the bounds in an integral, it is known as an improper integral.
$\int_a^b f(t)dt$
$f : [a, b[\to R$
f continous on $[a, b[$

E.g:
$\int_1^{+\infty} e^{-t}dt$

2 Convergence and Divergence

let $f : [a, b[\to R$ and f be continous on $[a, b[$.
we say that the integral is convergent if

$$\lim_{x \to b} \int_a^x f(t)dt$$

is a finite value.
else, it is divergent.

2.1 Proposition 1

let f and g be two functions on $[a, b[$ such that, both integrals $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ converge.
Then $\forall \alpha \in R$, $\int_a^b (\alpha f(t) + g(t))$ converges and linearity can be used to split the two integrals.

2.2 Riemann Functions

Theorem:

- $\int_0^1 dt/t^\alpha$ converges $\Leftrightarrow \alpha < 1$
- $\int_1^{+\infty} dt/t^\alpha$ converges $\Leftrightarrow \alpha > 1$
- $\int_0^{+\infty} dt/t^\alpha$ is always divergent.

2.3 Positive funtions improper integrals

let $f : [a, b[\rightarrow \mathrm{R}_+$ and $g : [a, b[\rightarrow \mathrm{R}_+$, continous on $[a, b[$.
if $f(t) < g(t)$, then

- if $\int_a^b g(t)dt$ is conv $\Rightarrow \int_a^b f(t)dt$ is conv.
- if $\int_a^b f(t)dt$ is divergent $\Rightarrow \int_a^b g(t)dt$ is divergent.
- if $f(t) = o(g(t))$ then if $\int_a^b g(t)dt$ is conv. $\Rightarrow \int_a^b f(t)dt$ is conv.
- if $f(t) \sim g(t)$ then $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ have the same nature