Phonons as Goldstone Bosons

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Outline

Introduction

- Phonons are Goldstone bosons generated by spontaneously breaking translation symmetry.
- Phonons self-interact; sound wave necessarily nonlinear.
- We'll first set up an effective Lagrangian upto second order in the fields. This runs into difficulties. To cure the difficulties, we'll use a covariant formulation. Then we analyze third order terms in the fields, use the symmetries, and reduce the problem to the covariant one.

Effective Lagrangian

- We'll model a solid using the displacement vectors $\xi_a(t, \vec{x})$ as fields.
- We demand space reflection, time reversal, and, for simplicity, rotational symmetry for the solid.
- To second order in the fields, the effective Lagrangian takes the form:

$$\mathcal{L}_2 = \frac{1}{2}\rho_0 \dot{\xi}_a \dot{\xi}_a - \frac{\mu}{2}\partial_a \xi_b \partial_a \xi_b - \frac{1}{6}\left(\mu + 3K\right)\partial_a \xi_a \partial_b \xi_b + l_0 \xi_a \xi_a. \quad (1)$$

  Even number of time derivatives, even number of space derivatives, up to two ξ's.
- We'll now show that $l_0 = 0$; phonons are massless.

Energy-Momentum Tensor

- To first order, the momentum density takes the form

$$\theta^{0a}(t, \vec{x}) = \rho_0 \dot{\xi}_a(t, \vec{x}). \tag{2}$$

- Rotational invariance constrains the stress tensor to take the form:

$$\theta^{rs} = -\mu\left(\partial_r \xi_s + \partial_s \xi_r\right) + \left(\frac{2\mu}{3} - K\right)\delta^{rs}\partial_a \xi_a. \tag{3}$$

The constants μ and K are the *torsion* and *compression modules*.

Equations of Motion

- Momentum conservation gives

$$\rho_0 \ddot{\xi}_a - \mu \nabla^2 \xi_a - \left(K + \frac{\mu}{3}\right) \partial_a (\partial \cdot \xi) = 0. \tag{4}$$

- Energy conservation fixes the form of θ^{00} to be

$$\theta^{00} = -\rho_0 (\partial \cdot \xi) . \tag{5}$$

- Euler-Lagrange equations of

$$\mathcal{L}_2 = \frac{1}{2}\rho_0 \dot{\xi}^2 - \frac{1}{2}\mu \partial_a \xi_b \partial_a \xi_b - \frac{1}{6}(\mu + 3K)\, \partial_a \xi_a \partial_b \xi_b \tag{6}$$

reproduce Eq. (4).

Is there a problem?

- By definition, θ^{00} is the energy density, and must agree with

$$\begin{aligned}\mathcal{H}_2 &= \frac{\partial \mathcal{L}_2}{\partial \dot{\xi}_a}\dot{\xi}_a - \mathcal{L}_2 \\ &= \frac{1}{2}\rho_0 \dot{\xi}^2 + \frac{1}{2}\mu \partial_a \xi_b \partial_a \xi_b + \frac{1}{6}(\mu + 3K)\partial_a \xi_a \partial_b \xi_b.\end{aligned} \tag{7}$$

  Clearly there is a mismatch with Eq. (5).
- Components of the energy-momentum tensor, using Noether's theorem, fail to be symmetric.
- If we start with second-order in the energy-momentum tensor, then the conservation equation requires a Lagrangian of third-order.

Covariant Formulation

- A foolproof way to get a symmetric energy-momentum tensor is to start with a Lorentz invariant Lagrangian and consider it in curved space.
- The energy-momentum tensor then becomes

$$\theta^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{8}$$

- Instead of the displacement vectors, we'll use

$$z_a(t,\vec{x}) = x_a(t) - \xi_a(t,\vec{x}) \tag{9}$$

as our fields. Any point on the worldline of the solid satisfies $z_a(x) = \text{const}$.

- We'll build a field theory using the scalars:

$$H_{ab} = g^{\mu\nu} \partial_\mu z_a \partial_\nu z_b. \tag{10}$$

Any Lagrangian that is a function of the above fields will be translation invariant and have a symmetric energy-momentum tensor.

- On flat space,

$$H_{ab} = \delta_{ab} - \bar{H}_{ab} = \delta_{ab} - \partial_a \xi_b - \partial_b \xi_a - \partial_c \xi_a \partial_c \xi_b + \frac{1}{c^2} \dot{\xi}_a \dot{\xi}_b. \tag{11}$$

- And the Lagrangian that takes into account all terms upto third-order in ξ is:

$$\mathcal{L}_{eff} = \sqrt{-g} \sqrt{\det(H)} \{ -\rho_0 c^2 - \frac{1}{8} \left(K - \frac{2\mu}{3} \right) \mathrm{Tr}(\bar{H})^2 - \frac{\mu}{4} \mathrm{Tr}(\bar{H}^2) + L_1 \mathrm{Tr}(\bar{H})^3 + L_2 \mathrm{Tr}(\bar{H}) \mathrm{Tr}(\bar{H}^2) + L_3 \mathrm{Tr}(\bar{H}^3) \}. \tag{12}$$

- The co-efficients are matched to second-order to give $\mathcal{L}_2$ from before.
- We also need to expand the determinant in the front using:

$$\sqrt{\det(H)} = \sqrt{\det(1-\bar{H})} = \exp\left[\frac{1}{2}\,\mathrm{Tr}\left(\log(1-\bar{H})\right)\right]. \tag{13}$$

- To third-order, we get three new coupling constants L_1, L_2, and L_3. Had we restricted ourselves to only order H^2, we'd still get terms cubic and quartic in ξ.
- To get a symmetric energy-momentum tensor, we are forced to include higher-order terms in ξ.

Third-Order in ξ

- Last time, we kept upto second-order terms in the Lagrangian for a solid. Now, we'll include terms upto third-order and write:

$$\begin{aligned}\mathcal{L}_3 = {} & l_1\dot{\xi}_a\dot{\xi}_a\partial_b\xi_b + l_2\dot{\xi}_a\dot{\xi}_b\partial_a\xi_b + l_3\partial_a\xi_a\partial_b\xi_b\partial_c\xi_c \\ & + l_4\partial_a\xi_a\partial_b\xi_c\partial_c\xi_b + l_5\partial_a\xi_a\partial_b\xi_c\partial_b\xi_c + l_6\partial_a\xi_b\partial_a\xi_c\partial_b\xi_c \\ & + l_7\xi_a\xi_a\partial_b\xi_b + l_8\xi_a\dot{\xi}_a\partial_b\dot{\xi}_b + l_9\xi_a\partial_b\xi_b\partial_c\partial_c\xi_a\end{aligned} \tag{14}$$

- To preserve space and time reflection symmetries, we kept odd space derivatives and even time derivatives.
- The momentum density will now be quadratic in ξ:

$$\begin{aligned}\theta_2^{0a} = {} & p_1\dot{\xi}_a\partial_b\xi_b + p_2\dot{\xi}_b\partial_a\xi_b + p_3\dot{\xi}_b\partial_b\xi_a \\ & + p_4\xi_a\partial_b\dot{\xi}_b + p_5\xi_b\partial_a\dot{\xi}_b + p_6\xi_b\partial_b\dot{\xi}_a.\end{aligned} \tag{15}$$

- We can use the ambiguity

$$\theta^{\mu\nu} \to \theta^{\mu\nu} + \partial_\lambda \chi^{\lambda\mu\nu}, \tag{16}$$

where $\chi^{\lambda\mu\nu} = -\chi^{\mu\lambda\nu}$, to eliminate p_4, p_5, and p_6 and get a translation invariant momentum density.

- Since θ^{0a} is translation invariant, so is θ^{00}—which takes the form

$$\theta^{00} = e_1 \dot{\xi}_a \dot{\xi}_a + e_2 \partial_a \xi_b \partial_a \xi_b + e_3 \partial_a \xi_b \partial_b \xi_a + e_4 \partial_a \xi_a \partial_b \xi_b. \tag{17}$$

- Comparing the energy conservation equation with the equation of motion in Eq. (4), we read off

$$e_1 = -\frac{\rho_0 p_2}{2\mu}, \quad e_2 = -\frac{p_2}{2}, \quad e_3 = -\frac{p_3}{2}, \quad e_4 = -\frac{p_1}{2}. \tag{18}$$

Further, for consistency, we must have

$$\mu(p_1 + p_3) = p_2 \left(K + \frac{\mu}{3} \right). \tag{19}$$

Stress tensor

- The stress-tensor has 15 coupling constants:

$$
\begin{aligned}
\theta_2^{ab} = {} & s_1 \dot{\xi}_a \dot{\xi}_b + s_2 \partial_c \xi_a \partial_c \xi_b + s_3 (\partial_a \xi_c \partial_c \xi_b + \partial_b \xi_c \partial_c \xi_a) + s_4 \partial_a \xi_c \partial_b \xi_c \\
& + s_5 (\partial_a \xi_b + \partial_b \xi_a) \partial_c \xi_c + s_6 (\xi_a \partial_c \partial_c \xi_b + \xi_b \partial_c \partial_c \xi_a) + s_7 \xi_c \partial_a \partial_b \xi_c \\
& + s_8 \xi_c (\partial_a \partial_c \xi_b + \partial_b \partial_c \xi_a) + s_9 (\xi_a \partial_b \partial_c \xi_c + \xi_b \partial_a \partial_c \xi_c) + s_{10} \delta_{ab} \dot{\xi}_c \dot{\xi}_c \\
& + s_{11} \delta_{ab} \partial_c \xi_d \partial_c \xi_d + s_{12} \delta_{ab} \partial_c \xi_d \partial_d \xi_c + s_{13} \delta_{ab} \partial_c \xi_c \partial_d \xi_d \\
& + s_{14} \delta_{ab} \xi_c \partial_d \partial_d \xi_c + s_{15} \xi_c \partial_c \partial_d \xi_d .
\end{aligned}
\tag{20}
$$

- From momentum conservation and the Euler-Lagrange equations that follow from $\mathcal{L}_3$, it turns out we can eliminate 13 of the above constants in terms of l's and p's.

Current Algebra

- If we quantize the theory, then

$$P^a = \int d^3x\, \theta^{0a}(x), \quad H = \int d^3x\, \theta^{00}(x), \tag{21}$$

  being the generator of space and time translations, satisfy

$$[P^a, \theta^{\mu\nu}] = i\hbar\partial_a\theta^{\mu\nu}, \quad [H, \theta^{\mu\nu}] = -i\hbar\partial_0\theta^{\mu\nu}. \tag{22}$$

- At the classical level, this is reflected in the Poisson Brackets

$$\{P^a, \theta^{\mu\nu}\} = -\partial_a\theta^{\mu\nu}, \quad \{H, \theta^{\mu\nu}\} = \partial_0\theta^{\mu\nu}. \tag{23}$$

- To evaluate these Poisson brackets, we use

$$\{\xi_a, \xi_b\} = \{\pi^a, \pi^b\} = 0,\ \{\pi^a(x), \xi_b(y)\} = \delta^a_b\delta^3(x-y). \tag{24}$$

- The canonical momentum for the full Lagrangian $\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_3$ is

$$\pi^a = \rho_0 \dot{\xi}_a + 2 l_1 \dot{\xi}_a \partial_b \xi_b + l_2 \dot{\xi}_b \left(\partial_a \xi_b + \partial_b \xi_a\right) + l_8 \xi_a \partial_b \dot{\xi}_b - l_8 \dot{\xi}_b \partial_b \xi_a - l_8 \xi_b \partial_a \dot{\xi}_b . \tag{25}$$

- Using all the above constraints there are only three independent coupling constants left.
- The symmetry violating terms l_7, l_8, and l_9 vanish.
- The resulting effective Lagrangian is the same as what we got from the covariant formulation with three independent coupling constants L_1, L_2, and L_3. Matching co-efficients, we can determine $(l_1, \dots, l_6)$ in terms of L.

Conclusion

- At second-order, the mass term vanishes due to energy-momentum conservation. This led us to interpret phonons as Goldstone bosons.
- The conservation laws and current algebra gave us a covariant and translation invariant Lagrangian.
- Nonlinear terms can't be ignored. Phonons interact.

References

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