The Entropy of Hawking Radiation

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Introduction

- Black Hole Information Paradox – deepest mystery of the universe.
- Black holes radiate[1]:

$$T_H = \frac{\hbar \kappa}{2\pi}, \tag{1}$$

  where κ is the surface gravity.
- Unitary evolution leading to information loss?[2]

[1] Hawking, S.W., 1975. Particle creation by black holes. Communications in mathematical physics, 43(3), pp.199-220.

[2] Hawking, S.W., 1976. Breakdown of predictability in gravitational collapse. Physical Review D, 14(10), p.2460.

The Paradox (Version 1)

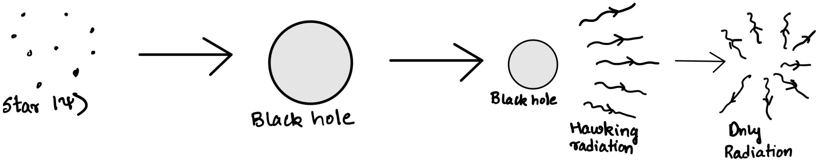


Figure: A star in a pure state collapses to a black hole which evaporates and leaves thermal radiation behind.

- If the fundamental theory is unitary, the above process is impossible.
- Where did we go wrong? To answer this, we consider

Entropy in Quantum Systems

- Given a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the *von Neumann entropy* is
$$S_A = -\operatorname{Tr}\rho_A \log \rho_A = S_B = -\operatorname{Tr}\rho_B \log \rho_B, \tag{2}$$
where $\rho_A = \operatorname{Tr}_B |\psi\rangle \langle\psi|$.
- Alternatively, the $n \to 1$ limit of the Rényi entropies:
$$S^{(n)}(\rho) = \frac{1}{1-n} \log \operatorname{Tr} \rho^n. \tag{3}$$
- In field theories, there is the *replica trick* to compute $\operatorname{Tr}(\rho^n)$.

Replica Trick

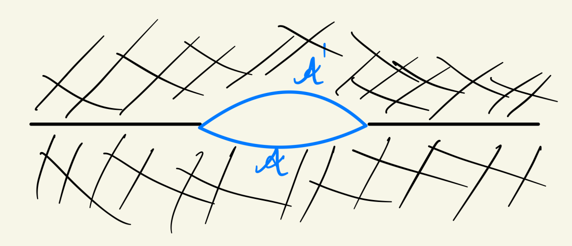


Figure: The density matrix ρ_A in a $(1+1)$-dimensional QFT.

- States are prepared by a path integral with fixed Cauchy data.
- To get ρ_A for a region A on a Cauchy slice, take two copies, glue A^c, and path integrate.

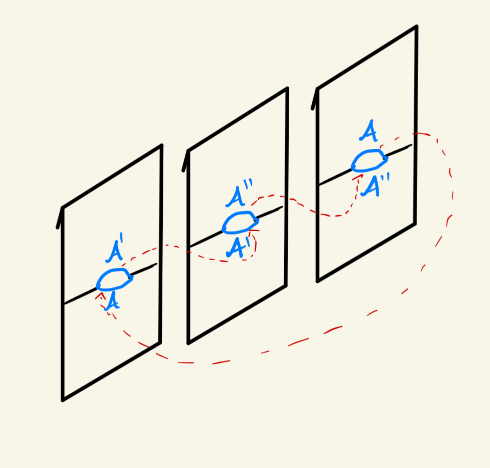
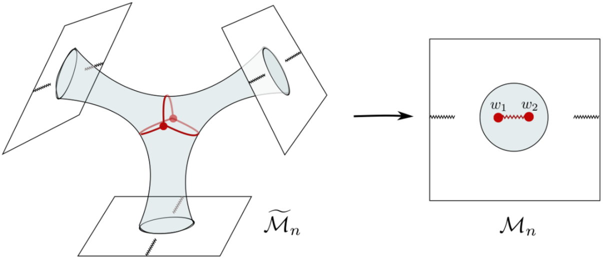


Figure: The computation for $\operatorname{Tr} \rho^3$.

- To get $\operatorname{Tr} \rho_A^n$, we take n copies of the sheet and glue the A's cyclically.

Holographic Entanglement Entropy



- Let $\tilde{\mathcal{M}}_n$ be the ramified cyclic cover. We assume it has a symmetry under Z_n[3]. Let $\mathcal{M}_n = \tilde{\mathcal{M}}_n / Z_n$ be the quotient space.
- Insert cosmic branes with tension

$$4G_N T_n = 1 - \frac{1}{n} \tag{4}$$

to reproduce conical singularities.

[3] Lewkowycz, A. and Maldacena, J., 2013. Generalized gravitational entropy. Journal of High Energy Physics, 2013(8), pp.1-29.

- The gravitational action is then

$$\frac{1}{n} I_{\mathsf{grav}}[\tilde{\mathcal{M}}_n] = I_{\mathsf{grav}}[\mathcal{M}_n] + T_n \int_{\Sigma_{d-2}} \sqrt{g}. \tag{5}$$

- The position of the cosmic branes are fixed by Einstein's equations.
- The entanglement entropy on $\mathcal{M}_n$ with the cosmic branes is

$$S = \frac{A_{\mathsf{minimal}}}{4G_N}. \tag{6}$$

Quantum Extremal Surface

- In the presence of matter, the gravitational entropy is

$$S_{\text{gen}} = \frac{A(X)}{4G_N} + S_{\text{matter}}(B), \tag{7}$$

where B is the region between minimal surface X and the boundary.

- The QES proposal[4] states that

$$S = \min_X \left\{ \text{ext}_X \left[\frac{A(X)}{4G_N} + S_{\text{matter}}(\Sigma_X) \right] \right\}, \tag{8}$$

where Σ_X is the region between X and the boundary.

- We'll use this to compute entropy of the black hole and Hawking radiation.

[4] Engelhardt, N. and Wall, A.C., 2015. Quantum extremal surfaces: holographic entanglement entropy beyond the classical regime. Journal of High Energy Physics, 2015(1), pp.1 -27.

Paradox (Version 2)

- The *fine-grained entropy* of a state ρ is the von Neumann entropy:

$$S(\rho) = -\operatorname{Tr} \rho \log \rho. \tag{9}$$

- Say $\{O_i\}$ are some simple observables we're interested in.
- The *coarse-grained entropy* of a state ρ is

$$S_{cg}(\rho) = \max_{\tilde{\rho}} S(\tilde{\rho}) \tag{10}$$

where the $\tilde{\rho}$ are such that

$$\operatorname{Tr} \tilde{\rho} O_i = \operatorname{Tr} \rho O_i. \tag{11}$$

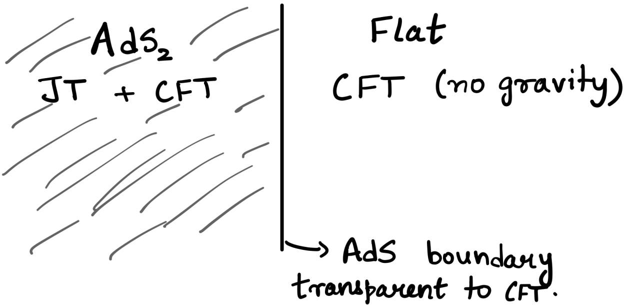
- The Bekenstein entropy

$$S = \frac{A}{4G_N} \tag{12}$$

  is to be thought of as the coarse-grained entropy of the black hole[5].
- We run into a paradox if the entropy of radiation exceeds the Bekenstein entropy.

[5] Almheiri, A., Hartman, T., Maldacena, J., Shaghoulian, E. and Tajdini, A., 2021. The entropy of Hawking radiation. Reviews of Modern Physics, 93(3), p.035002.

JT Gravity + CFT



- The model we'll consider is JT Gravity coupled to a CFT. In addition, the same CFT in flat space is glued to the AdS boundary.

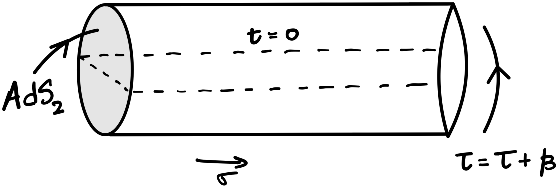


Figure: Thermofield double prepared as a path integral.

- The matter CFT is free to leak out.
- The action for this theory is

$$S = \frac{S_0}{4\pi}\int_{\Sigma_2} R + \int_{\Sigma_2} \frac{\phi}{4\pi}\,(R+2) + \log Z_{\mathsf{CFT}[g]} + \mathsf{GHY}. \quad (13)$$

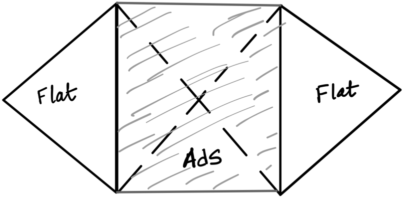
- We'll consider a simple thermofield double state in this theory.

- On the manifold $\tilde{\mathcal{M}}_n$, the gravitational part of the action becomes (ignoring the Gibbons-Hawking term)

$$-\frac{1}{n} I_{\mathsf{grav}} = \frac{S_0}{4\pi} \int R + \int \frac{\phi}{4\pi}(R+2) - \left(1 - \frac{1}{n}\right) \sum_i \left[S_0 + \phi(w_i)\right], \tag{14}$$

where w_i are the positions of the cosmic branes.

The Geometry of the black hole



- The JT gravity theory above describes an AdS_2 black hole glued to flat space.
- In complex co-ordinates, the metric is

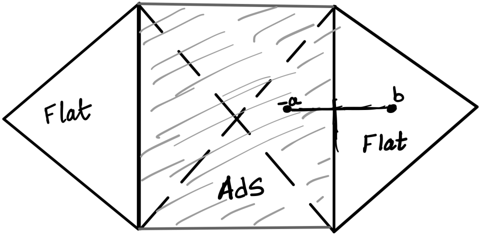
$$ds^2_{\text{in}} = \frac{4\pi^2}{\beta^2}\frac{dy d\bar{y}}{\sinh^2 \frac{\pi}{\beta}(y+\bar{y})} \tag{15}$$

inside, and

$$ds^2_{\text{out}} = \frac{1}{\epsilon^2} dy d\bar{y} \tag{16}$$

outside.

Single Interval



- In this setup we consider a single interval $B = [0, b]$ in the flat region joined to the AdS boundary.
- In $\mathcal{M}_n$, we introduce a cosmic brane at $-a$ as shown above.
- The generalized entropy is

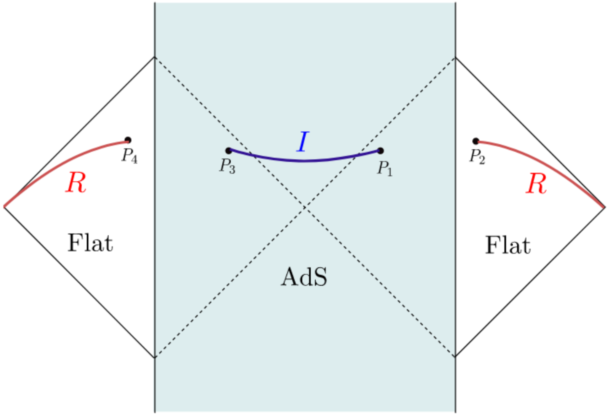
$$S_{\text{gen}} = S_0 + \phi(-a) + S_{\text{CFT}}([-a, b]). \tag{17}$$

- We know the black hole metric. We can solve for the dilaton from the action.
- The entanglement entropy of an interval in a CFT is known[6].
- Extremizing S_{gen} as a function of a, we get

$$\sinh\left(\frac{2\pi a}{\beta}\right) = \frac{12\pi\epsilon\phi}{\beta c}\frac{\sinh\left(\frac{\pi}{\beta}(b+a)\right)}{\sinh\left(\frac{\pi}{\beta}(a-b)\right)}. \tag{18}$$

[6] Calabrese, P. and Cardy, J., 2009. Entanglement entropy and conformal field theory. Journal of physics A: Mathematical and Theoretical, 42(50), p.504005.

Two Intervals



- Collect the Hawking radiation in R shown above.

- From the QES proposal, let's choose X so that we have the island I contribution. At late times, this dominates the Hawking saddle.
- Say, $P_1 = (-a, t_a)$, $P_2 = (b, t_b)$, then $P_3 = (-a, -t_a + i\pi)$, and $P_4 = (b, -t_b + i\pi)$.
- The generalized entropy is

$$S_{\text{gen}} = 2S_0 + 2\phi(-a) + S_{\text{CFT}}([P_4, P_3] \cup [P_1, P_2]). \tag{19}$$

- The entropy of two intervals is non-universal. It depends on the CFT under consideration.
- At late times, the island contribution dominates, and the QES proposal gives

$$S_{\text{radiation}} = \min\{S_{\text{gen}}^{\text{island}}, S_{\text{gen}}^{\text{no island}}\}, \tag{20}$$

and reproduces the expected Page curve.