LAB – 1: Fourier Series Analysis and Synthesis

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Team Name: Noicifiers

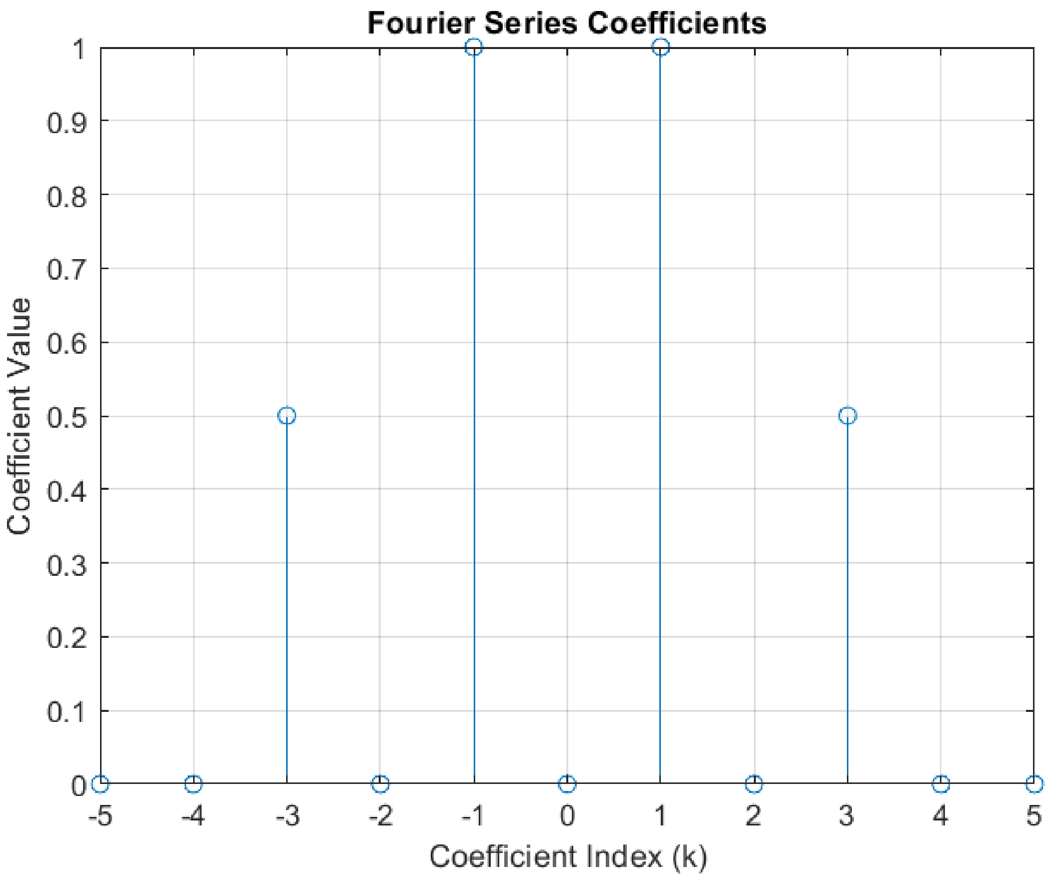
QUESTION – 1.1:

(a):

$x(t) = 2\cos(2\pi t) + \cos(6\pi t)$

$T = 1$

$N = 5$



Analytical verification:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk w_0 t}\, dt$$

$T = 1$

$w_0 = 2\pi/T = 2\pi$

$x(t) = \cos(2\pi t) + 2\cos 6\pi t$

$$= \frac{1}{1}\int_{-1/2}^{1/2} x(t)\, e^{-jk2\pi t}\, dt$$

$$= \int_{-1/2}^{1/2}\left(\frac{e^{j2\pi t} + e^{-j2\pi t}}{2} + 2\times\frac{e^{j6\pi t} + e^{-j6\pi t}}{2}\right) e^{-jk2\pi t}\, dt$$

$$= \int_{-1/2}^{1/2}\left[\frac{e^{j2\pi t(1-k)} + e^{-j2\pi t(1+k)}}{2} + e^{j2\pi t(3-k)} + e^{-j2\pi t(3+k)}\right] dt$$

1) $K = 0$

$$a_0 = \int_{-1/2}^{1/2}\left(\frac{e^{j2\pi t}}{2} + \frac{e^{-j2\pi t}}{2} + e^{j6\pi t} + e^{-j6\pi t}\right) dt$$

$$= \left(\frac{e^{j2\pi t} - e^{-j2\pi t}}{4j\pi} + \frac{e^{j6\pi t} - e^{-j6\pi t}}{6\pi j}\right)\Bigg|_{-1/2}^{1/2}$$

$$= \frac{(e^{j\pi} - e^{-j\pi}) - (e^{-j\pi} - e^{j\pi})}{4j\pi} + \frac{(e^{3j\pi} - e^{-3j\pi}) - (e^{-3j\pi} - e^{3j\pi})}{6\pi j}$$

$\Rightarrow \boxed{a_0 = 0}$

2) $\underline{\underline{k = \pm 1}}$

$$a_{\pm 1} = \int_{-1/2}^{1/2} \left(\frac{1}{2} + \frac{e^{-j4\pi t}}{2} + e^{j4\pi t} + e^{-j8\pi t} \right) dt$$

$$= \frac{1}{2}(1) + \frac{e^{-j2\pi t} - e^{j2\pi t}}{-8\pi j} + \frac{e^{2\pi j} - e^{-2\pi j}}{4\pi j} \pm \frac{e^{-4\pi j} - e^{4\pi j}}{8\pi j}$$

$$= \frac{1}{2} + 1/2$$

$$= 1$$

$\Rightarrow \boxed{a_1 = 1}$ $\boxed{a_{-1} = 1}$

||'y we can solve for $\pm 2, \pm 3, \pm 4, \pm 5$ -

we get

$a_{-5} = 0$ $a_{-4} = 0$ $a_{-3} = 1/2$ $a_{-2} = 0$ $a_{-1} = 1$

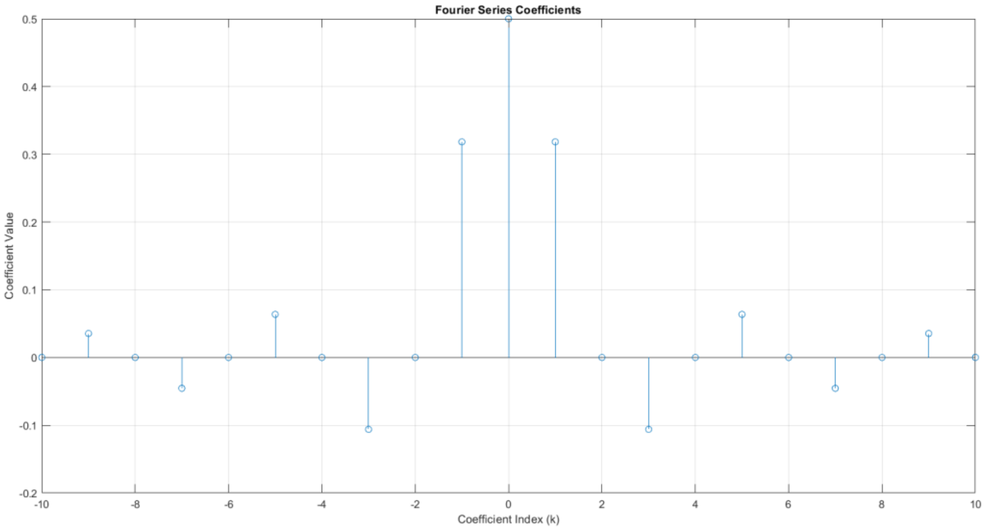
$a_5 = 0$ $a_4 = 0$ $a_3 = 1/2$ $a_2 = 0$ $a_1 = 1$ $a_0 = 0$

(b):

N = 10; T = 1; T_1 = T/4;

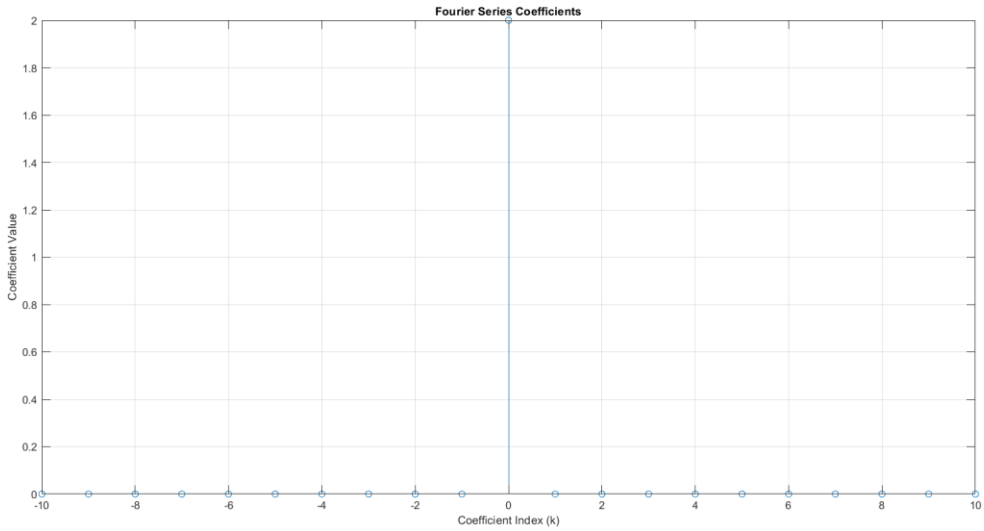
x(t) = 1; if - T_1 <= t <= T_1

0; if T_1 |t|<T/2



(c):

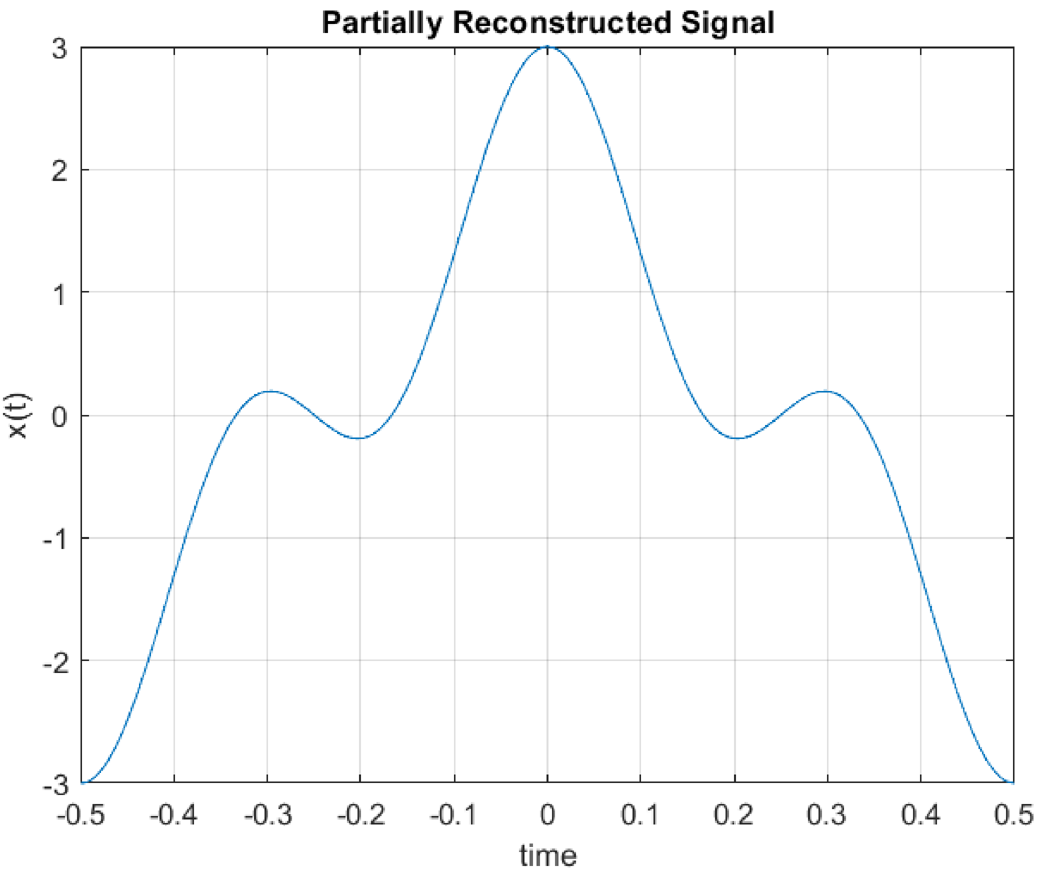
N = 10; T = 5; T_1 = 0.1; x(t) = 2



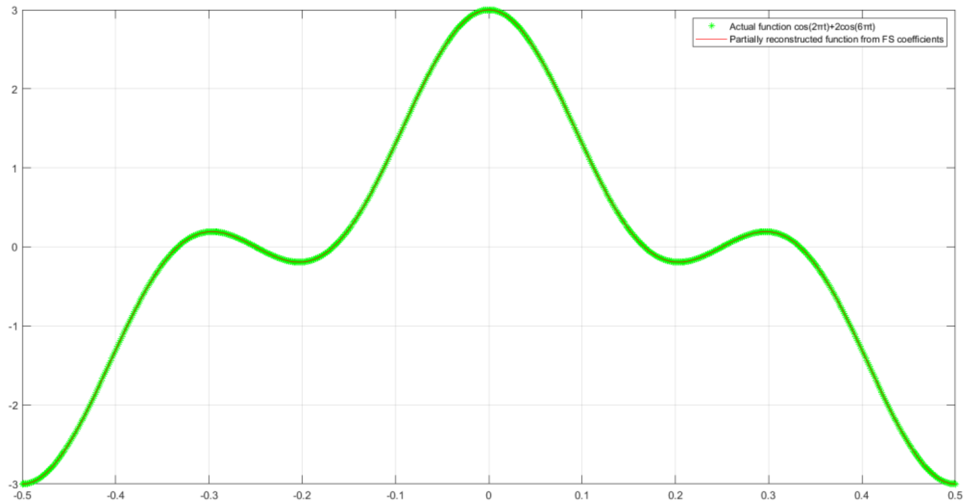
QUESTION – 1.2:

(a):

T = 1; time grid = -0.5:0.01:0.5d



(b):

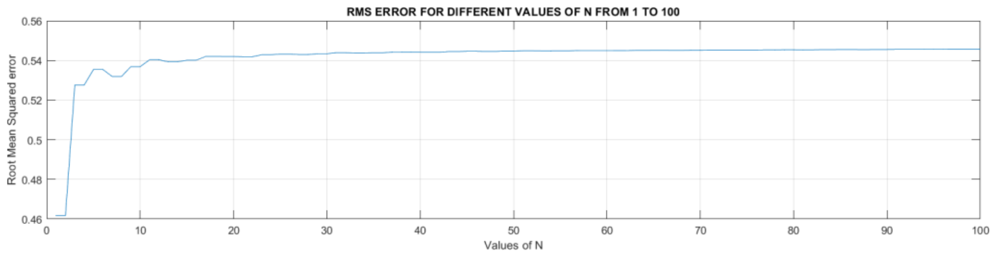


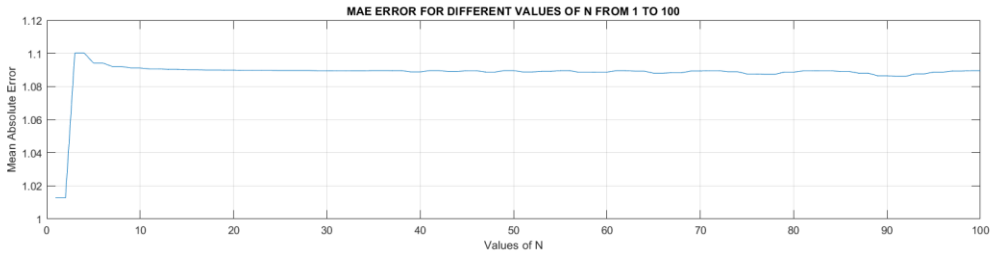
(c):

After running the script, we will get the error as

1) Max Absolute Error = 4.4755e-16
2) Root Mean Square Error = 1.0657e-16

(d):





QUESTION – 1.3:

(a):

$$x(t) = \begin{cases} 1; & -T_1 \le t \le T_1 \\ 0; & -\frac{T}{2} \le t \le -T \text{ and } T_1 \le t \le \frac{T}{2} \end{cases}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_{-T/2}^{-T_1} 0\, dt + \int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt + \int_{T_1}^{T/2} 0\, dt$$

$$= \int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt.$$

1) $k = 0$ $\quad a_0 = \int_{-T_1}^{T_1} dt = 2T_1$

2) $k \neq 0$ $\quad a_k = \int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = \left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right)_{-T_1}^{T_1}$

$$= \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{-jk\omega_0}$$

$$= \frac{\cancel{2j} \sin(k\omega_0 T_1)}{\cancel{j} k \left(\frac{2\pi}{1}\right)}$$

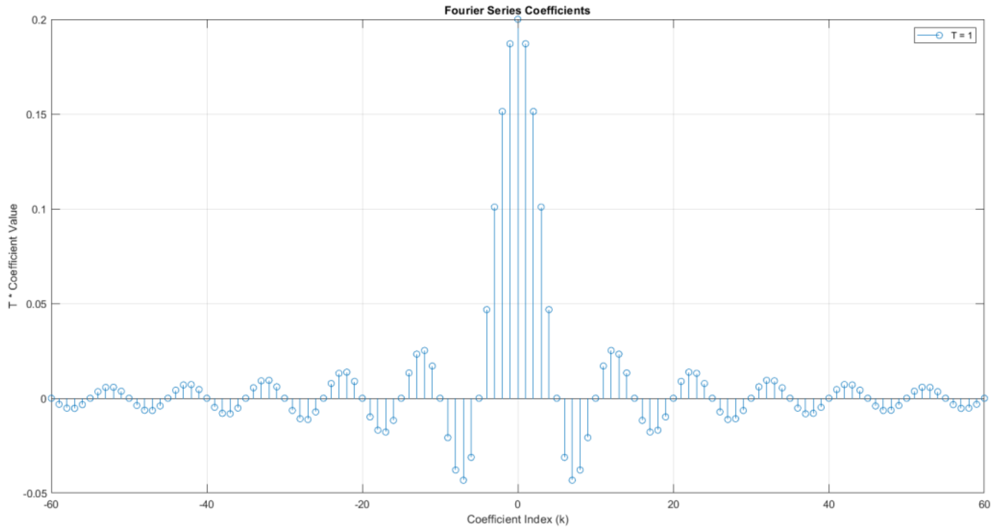
$$= \frac{\sin(2kT_1\pi)}{\pi k}$$

$$\Rightarrow \boxed{a_k = \begin{cases} 2T_1; & k = 0 \\ \dfrac{-\sin(2\pi k T_1)}{\pi k}; & k \neq 0 \end{cases}}$$

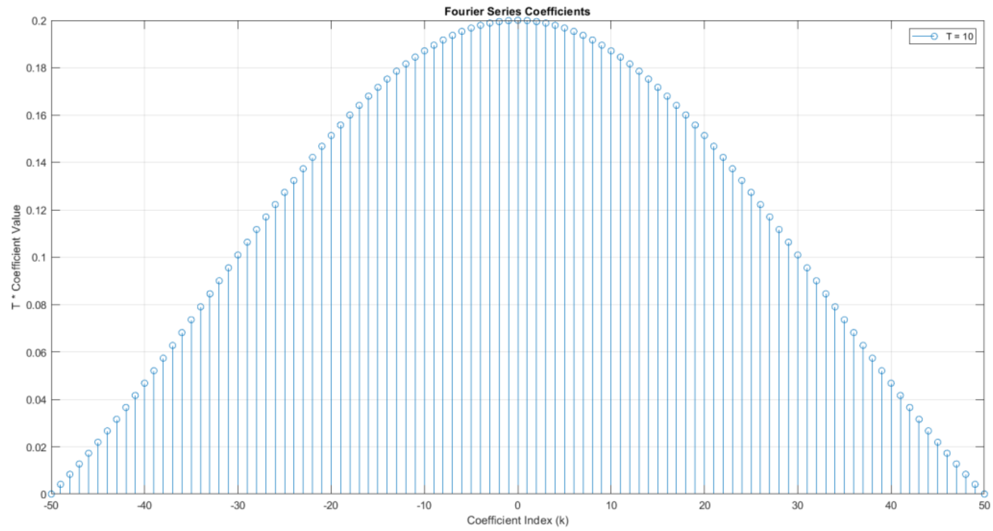
(b):

T1 = 0.1; T=1; N = 10*T

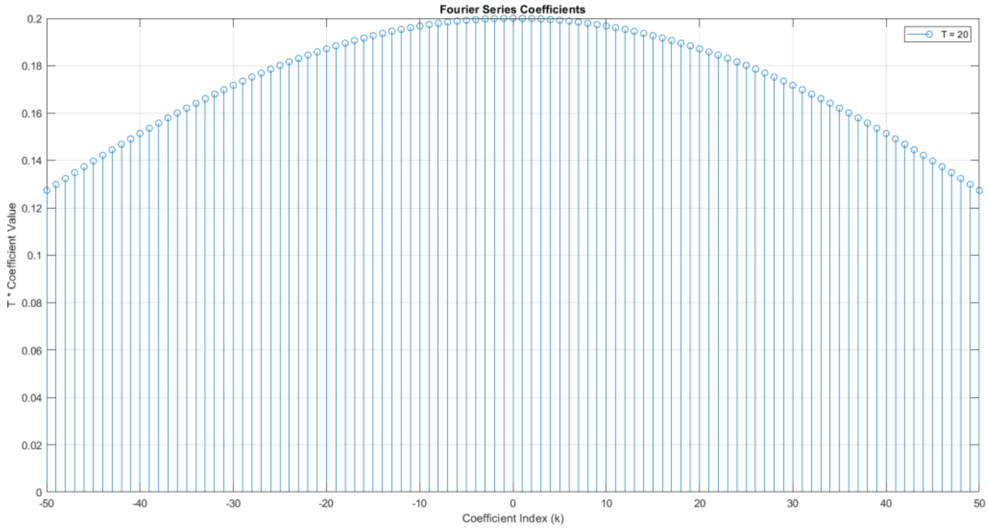
T = 1:



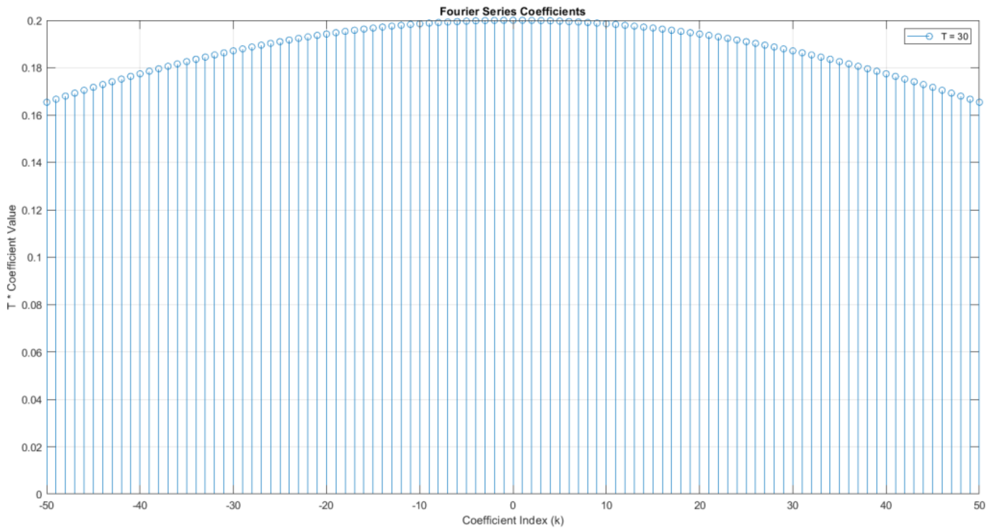
T = 10:



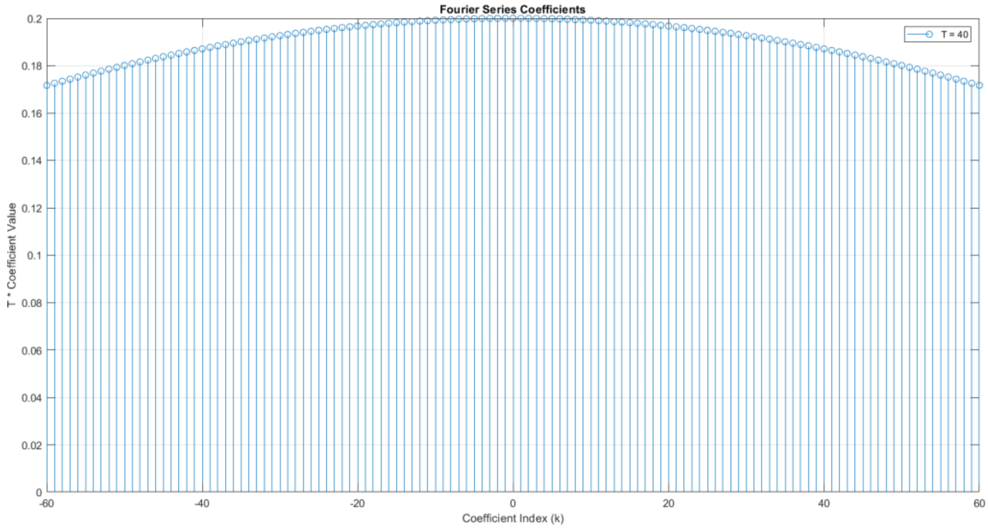
T = 20:


Fourier Series Coefficients
T = 20
T * Coefficient Value
Coefficient Index (k)
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-50
-40
-30
-20
-10
0
10
20
30
40
50

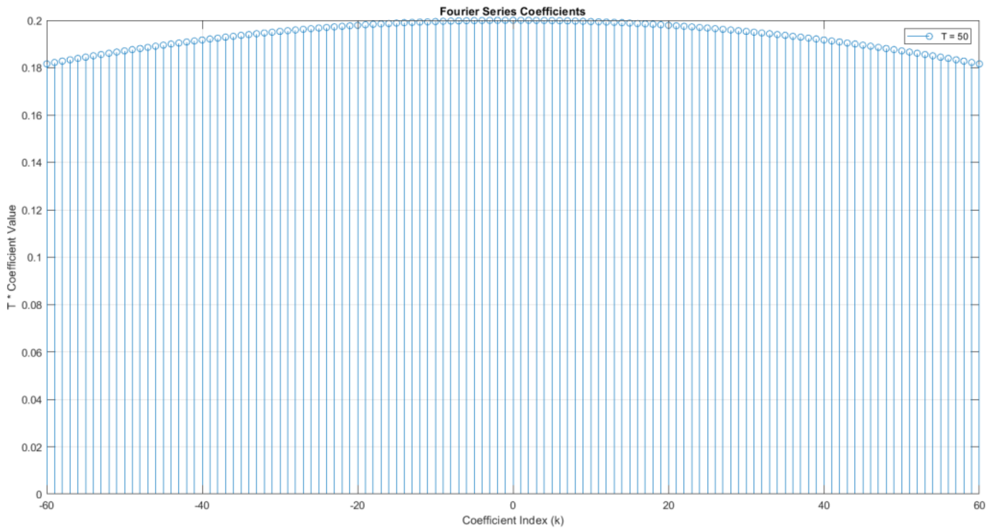
T = 30:


Fourier Series Coefficients
T = 30
T * Coefficient Value
Coefficient Index (k)
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-50
-40
-30
-20
-10
0
10
20
30
40
50

T = 40:


Fourier Series Coefficients
T = 40
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
T * Coefficient Value
-60
-40
-20
0
20
40
60
Coefficient Index (k)

T = 50:


Fourier Series Coefficients
T = 50
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
T * Coefficient Value
-60
-40
-20
0
20
40
60
Coefficient Index (k)

$$x(t) = \begin{cases} 1 ; & abs(t) \le 0.1 \\ 0 ; & \text{otherwise} \end{cases} \quad (\because T \to \infty)$$

$$\Rightarrow a_k = 1/T \int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi t}{T}} dt$$

$$= \frac{1}{T} \int_{0.1}^{0.1} e^{-jk\frac{2\pi t}{T}} dt$$

$$= \frac{e^{-jk\frac{2\pi t}{T}}}{jk2\pi \cancel{T}} \quad ; \text{ if } k \neq 0$$

$$0.2/T \quad ; \text{ if } k = 0$$

$$\Rightarrow a_k = \begin{cases} \frac{0.2}{T} ; & k = 0 \\ \left(\frac{e^{-jk\frac{2\pi t}{T}}}{j2k\pi T} \right)_{-0.1}^{0.1} ; & k \neq 0 \end{cases}$$

$$= \frac{e^{-jk\frac{2\pi}{T}0.1} - e^{+jk\frac{2\pi}{T} \times 0.1}}{jk2\pi \cancel{T}}$$

$$= \frac{\sin\left(k\frac{2\pi}{T} 0.1\right)}{jk2\pi \cancel{T}}$$

$T \to \infty$

$\Rightarrow \frac{1}{T} \to 0$

$\Rightarrow \frac{0.2k\pi}{T} \to 0$

$\Rightarrow \sin\left(\frac{0.2k\pi}{T}\right) \to \frac{0.2k\pi}{T}$

$\therefore \quad a_k = \begin{cases} 0.2/T ; & k=0 \\ \frac{0.2k\cancel{\pi}/T}{\cancel{k\pi}} & k\neq 0 \end{cases}$

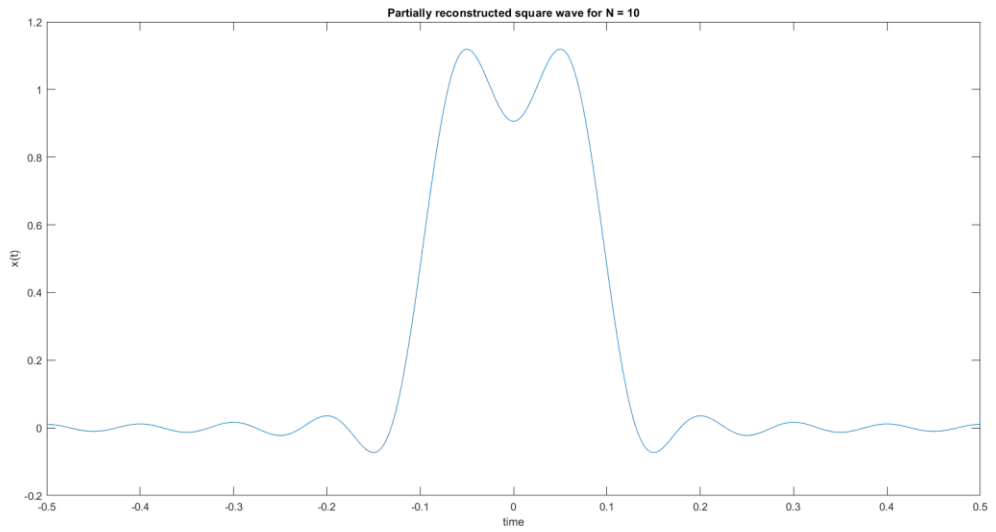
$\Rightarrow \quad T * a_k = \begin{cases} 0.2 ; & k=0 \\ \to 0.2 ; & k \neq 0 \end{cases}$

$\therefore$ as $T \to \infty \quad T * a_k \to 2T$, i.e 0.2

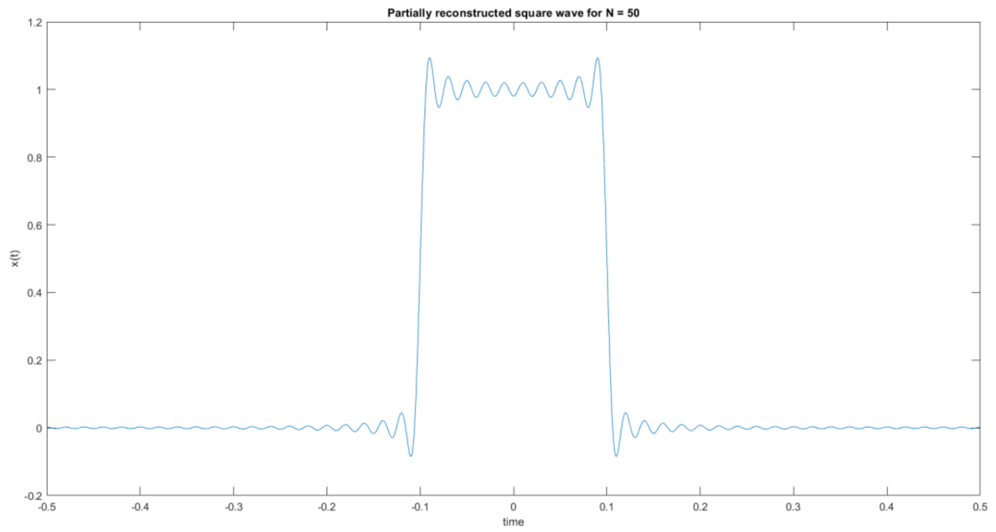
(c):

T1 = 0.1; T = 1; t = -0.5:0.01:0.5; N = [10 50 100 1000];

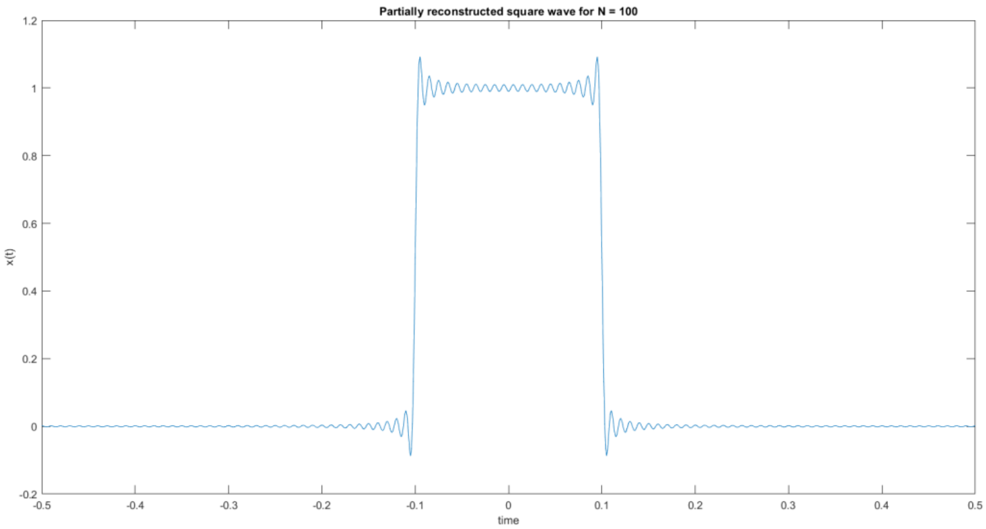
N = 10:



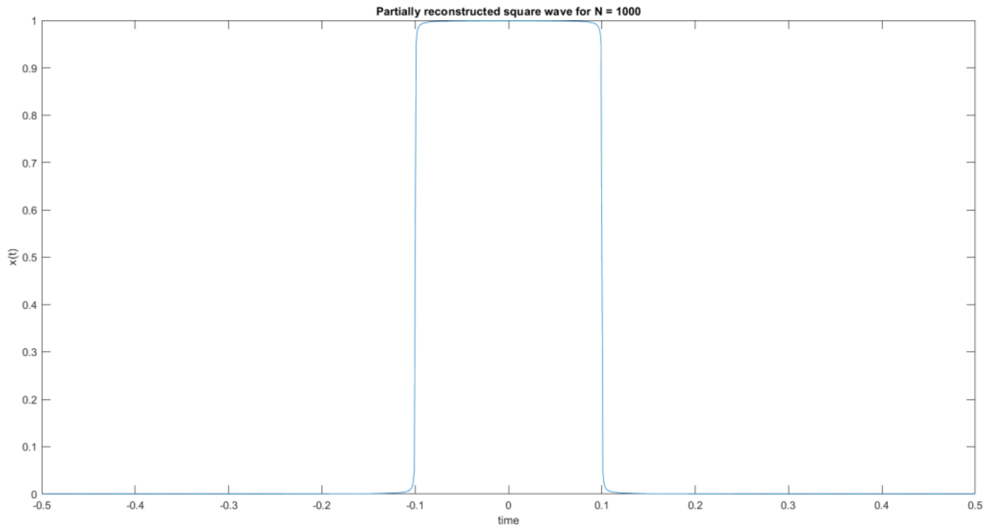
N = 50:



N = 100:



N = 1000:

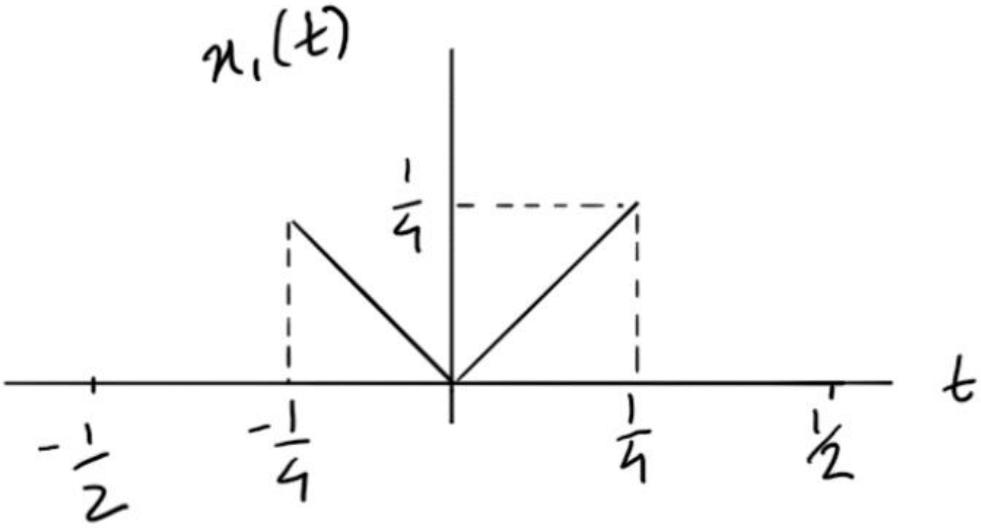


As the number of Fourier coefficients (N) taken to reconstruct a function using its Fourier series increases, the reconstructed signal does indeed approach the original signal in terms of minimizing the mean squared error, particularly within the interval where the function is well-behaved. However, when there are points of discontinuity, such as edges in an image or sudden transitions in a signal, the Fourier series tends to exhibit overshoot or oscillations around these points, even as N becomes very large.

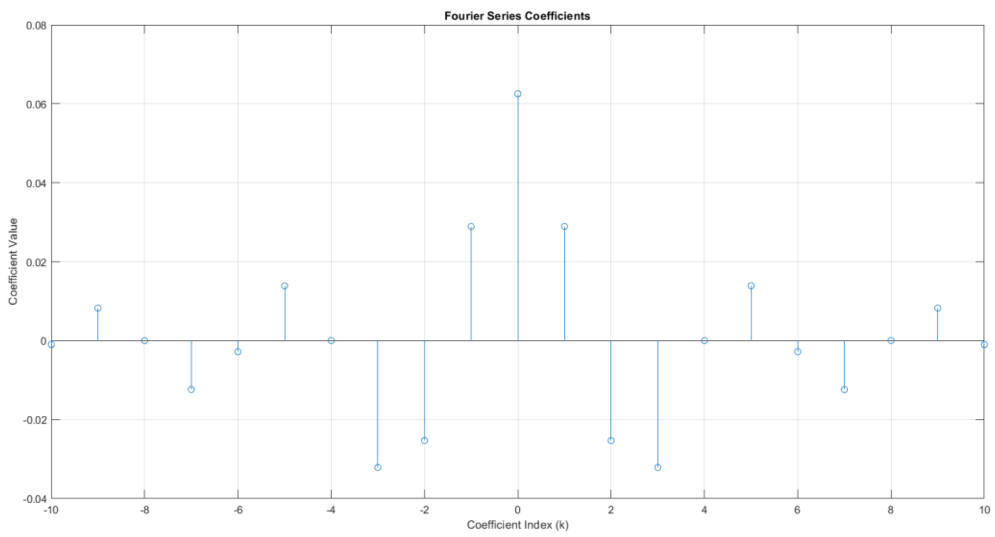
In other words, the overshoot or oscillations, known as the Gibbs phenomenon, do not vanish with increasing N. Instead, the overshoot becomes more pronounced and is typically localized around the discontinuities. This behaviour occurs because the Fourier series converges in a least-squares sense, minimizing the overall error, but it struggles to capture rapid changes or sharp corners. The phenomenon is most prominent when the function being approximated has step-like transitions, such as a square wave.

QUESTION – 1.4:

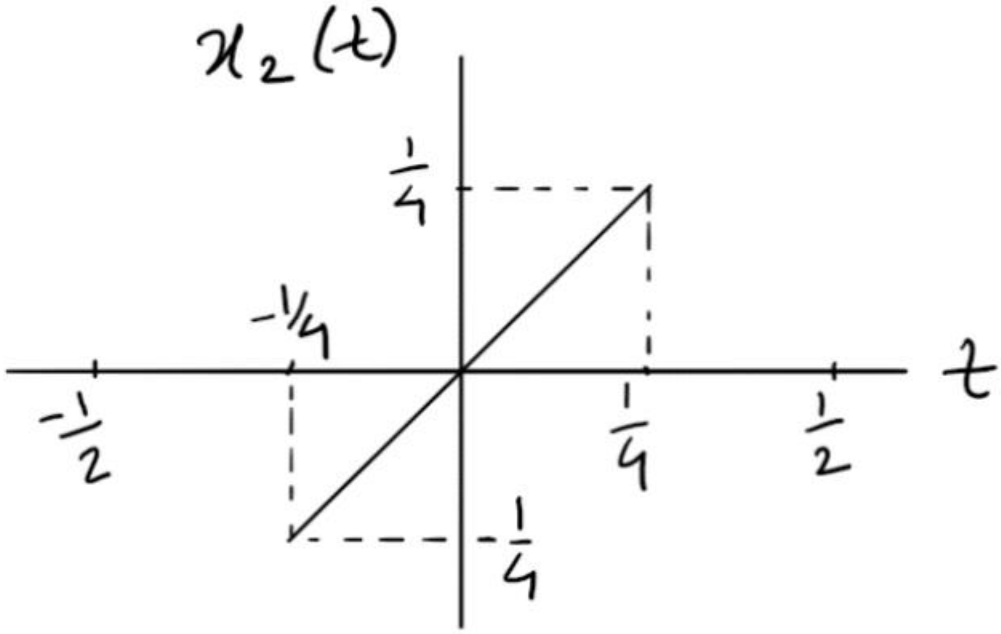
(a):



T = 1;
N = 10;

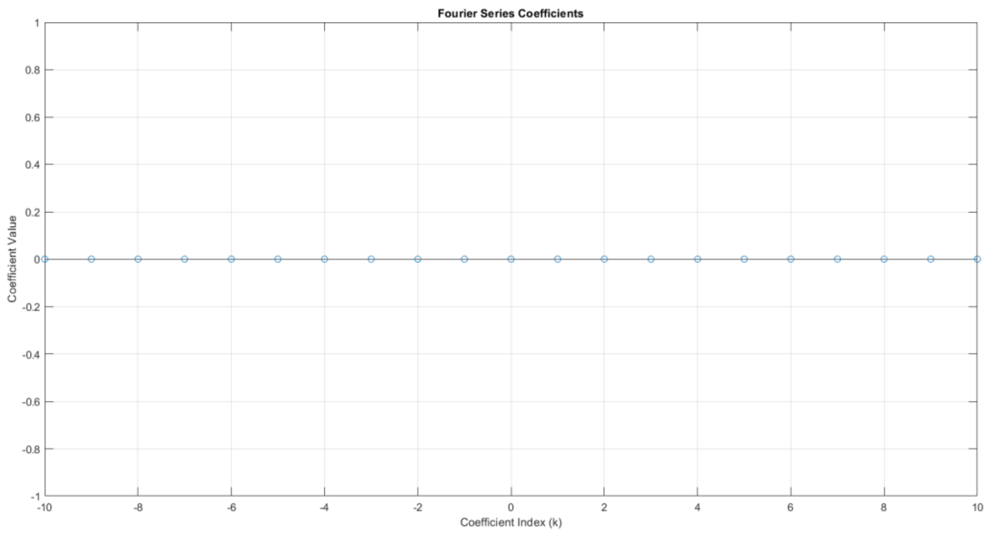


(b):



$x_2(t)$
$\frac{1}{4}$
$-1/4$
t
$-\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{2}$
$-\frac{1}{4}$

T = 1;
N = 10;



Fourier Series Coefficients
Coefficient Value
Coefficient Index (k)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-10
-8
-6
-4
-2
0
2
4
6
8
10

(c):

Analysis of 1.4(a):

$$x(t) = \begin{cases} \text{abs}(t); & -\frac{1}{4} \le t \le 1/4 \\ 0; & \frac{1}{4} \le \text{abs}(t) \le 1/2 \end{cases} \qquad T = 1$$

$$a_k = \int_{-1/2}^{1/2} x(t) e^{-jk\omega_0 t} \, dt$$

$$= -\int_{-1/4}^{0} t e^{-jk2\pi t} \, dt + \int_{0}^{1/4} t e^{-jk2\pi t} \, dt$$

$$k = 0 \longrightarrow -\int_{-1/4}^{0} t\,dt + \int_{0}^{1/4} t\,dt$$

$$= \left(-t^2/2\right)_{-1/4}^{0} + \left(\frac{t^2}{2}\right)_{0}^{1/4}$$

$$= \left(+\frac{1}{32} - 0\right) + \left(1/32 - 0\right)$$

$$= \underline{\underline{\frac{1}{16}}}$$

$$k \neq 0 \longrightarrow -\left(\frac{e^{-jk2\pi t}}{jk2\pi} \times t - \frac{e^{-jk2\pi t}}{(2\pi k)^2}\right)_{-1/4}^{0} +$$

$$\left(\frac{e^{-jk2\pi t}}{jk2\pi} t - \frac{e^{-jk2\pi t}}{(2\pi k)^2}\right)_{0}^{1/4}$$

$$\frac{-1}{4\pi k^2} + \frac{e^{+jk\pi/2}}{jk2\pi} - \frac{e^{-jk\pi/2}}{4k^2\pi^2} +$$

$$a_k = \underbrace{-\frac{e^{-jk\pi/2}}{jk2\pi} + \frac{e^{-jk\pi/2}}{4k^2\pi^2}}$$ — (1)

It has both complex parts and Real parts

But $a_k = a_{-k} \quad \forall\ k \in \mathbb{Z}$

$\Rightarrow$ It has even symmetry

Which can be reflected in the above graph.

$$a_k = \begin{cases} 1/16 \ ; & k = 0 \\ \text{eqn } (1) \ ; & k \neq 0 \end{cases}$$

The given signal has even symmetry.

The other signal analysis is done by my teammate. It has odd symmetry.