Fundalmental matrix F and Essential matrix in stereo

(https://en.wikipedia.org/wiki/Fundamental_matrix_(computer_vision))

(https://en.wikipedia.org/wiki/Essential_matrix)

The fundamental matrix $\boldsymbol{F}$ encapsulates the epipolar intrinsic geometry. It is a 3 × 3 matrix of rank 2, i.e. its determinant is zero. If a point in the 3D space $\boldsymbol{X}$ is imaged as m_1 in the first view, and $m_2$in the second, then the image points satisfy the relation:

$$\boldsymbol{\tilde{m}_2^T * F * \tilde{m}_1 = 0}$$

The essential matrix $\boldsymbol{E}$ can be seen as a precursor to the fundamental matrix $\boldsymbol{F}$. Both matrices establish constraints between matching image points, but the essential matrix can only be used in relation to calibrated cameras since the inner camera parameters must be known in order to achieve pixel normalization. The relationship between normalized pixels and the essential matrix is given by the following equation:

$$\boldsymbol{\hat{m}_2^T * E * \hat{m}_1 = 0}$$

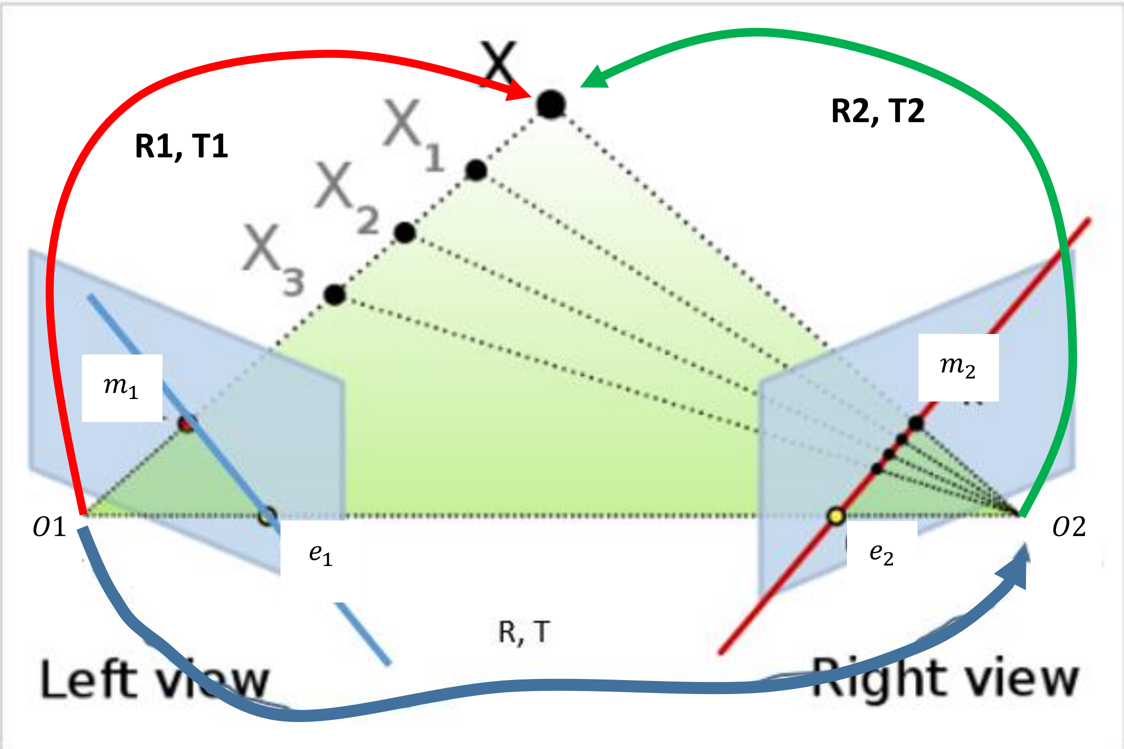
Where $\boldsymbol{\hat{m}_i = A_i^{-1} * \tilde{m}_i}$ **are the normalized pixels, i.e. the values of $\boldsymbol{\hat{m}_i}$ are in mm at f = 1mm.**

Thus, it follows that:

$$\boldsymbol{E = A_2^T * F * A_1}$$

as $\tilde{m}_i = A_i * \hat{m}_i$ and $\tilde{m}_2^T * F * \tilde{m}_1 = 0$

$$\boldsymbol{F = A_2^{-T} * E * A_1^{-1}}$$



Epipolar geometry

The transpose of these equations is also zero. Recall that $(A * B)^T = B^T * A^T$.

$$(\widehat{\boldsymbol{m}}_2^T * \boldsymbol{E} * \widehat{\boldsymbol{m}}_1)^T = \widehat{\boldsymbol{m}}_1^T * \boldsymbol{E}^T * \widehat{\boldsymbol{m}}_2 = \boldsymbol{0}$$

$$(\widetilde{\boldsymbol{m}}_2^T * \boldsymbol{F} * \widetilde{\boldsymbol{m}}_1)^T = \widetilde{\boldsymbol{m}}_1^T * \boldsymbol{F}^T * \widetilde{\boldsymbol{m}}_2 = \boldsymbol{0}$$

Thus, the **essential and fundamental matrix of the inverse transformation is their transpose**.

A point M in the line of projection of m_1 expressed in the first camera frame can be written as ${}^{1}\boldsymbol{M} = \boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1$. If we know the rotation *R* and translation *T* between the frames of the right and left camera, then: ${}^{2}\boldsymbol{M} = \boldsymbol{R}^{\boldsymbol{T}} * ({}^{1}\boldsymbol{M} - \boldsymbol{T})$. The projection of this point into the camera {2} frame:

$$\widetilde{\boldsymbol{m}}_2 = \boldsymbol{A}_2 * {}^{2}\boldsymbol{M};\ \widehat{\boldsymbol{m}}_2 = \boldsymbol{A}_2^{-1} * \boldsymbol{A}_2 * {}^{2}\boldsymbol{M} = {}^{2}\boldsymbol{M}$$

$$\widehat{\boldsymbol{m}}_2 = (\boldsymbol{R}^{\boldsymbol{T}} * {}^{1}\boldsymbol{M} - \boldsymbol{R}^{\boldsymbol{T}} * \boldsymbol{T}) = \boldsymbol{R}^{\boldsymbol{T}} * (\boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1 - \boldsymbol{T}).$$

Thus, $\widehat{\boldsymbol{m}}_2^T = (\boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1 - \boldsymbol{T})^{\boldsymbol{T}} * \boldsymbol{R}$.

It is easy to show that if:

$$\boldsymbol{E} = \boldsymbol{R}^{\boldsymbol{T}} * [\boldsymbol{T}]_x;$$

$$\boldsymbol{F} = \boldsymbol{A}_2^{-\boldsymbol{T}} * \boldsymbol{R}^{\boldsymbol{T}} * [\boldsymbol{T}]_x * \boldsymbol{A}_1^{-1}$$

Where $[\boldsymbol{T}]_x$ corresponds to the skew symmetric matrix of *T*.

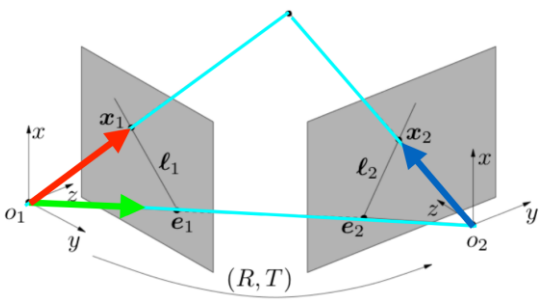
Then $\widehat{\boldsymbol{m}}_2^T * \boldsymbol{E} * \widehat{\boldsymbol{m}}_1 = \boldsymbol{0}$. Effectively:

$$\widehat{\boldsymbol{m}}_2^T * \boldsymbol{E} * \widehat{\boldsymbol{m}}_1 = (\boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1 - \boldsymbol{T})^{\boldsymbol{T}} * \boldsymbol{R} * \boldsymbol{E} * \widehat{\boldsymbol{m}}_1 = (\boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1 - \boldsymbol{T})^{\boldsymbol{T}} * \boldsymbol{R} * (\boldsymbol{R}^{\boldsymbol{T}} * [\boldsymbol{T}]_x) * \widehat{\boldsymbol{m}}_1$$

$$\widehat{\boldsymbol{m}}_2^T * \boldsymbol{E} * \widehat{\boldsymbol{m}}_1 = (\boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1 - \boldsymbol{T})^{\boldsymbol{T}} * [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1 = \boldsymbol{\lambda} * \widehat{\boldsymbol{m}}_1^T * [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1 - \boldsymbol{T}^{\boldsymbol{T}} * [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1 = \boldsymbol{0}.$$

Due to the fact that $\widehat{\boldsymbol{m}}_1^T * [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1$ is the dot product of $\widehat{\boldsymbol{m}}_1$ **and** $(\boldsymbol{T} \times \widehat{\boldsymbol{m}}_1)$, that is zero, as they are perpendicular. The same holds for $\boldsymbol{T}^{\boldsymbol{T}} * [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1$.

Geometric Proof



The vector $\widehat{\boldsymbol{m}}_2$ in frame {1} is equal to $\boldsymbol{R} * \widehat{\boldsymbol{m}}_2$. A vector perpendicular to the epipolar plane formed by the baseline, $\widehat{\boldsymbol{m}}_1$ and $\widehat{\boldsymbol{m}}_2$ can be define as $\boldsymbol{T} \times \widehat{\boldsymbol{m}}_1 = [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1$. Finally, the vector $\widehat{\boldsymbol{m}}_2$ is perpendicular to $[\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1$, i.e. $(\boldsymbol{R} * \widehat{\boldsymbol{m}}_2)^T * [\boldsymbol{T}]_x * \widehat{\boldsymbol{m}}_1 = \widehat{\boldsymbol{m}}_2^T * (\boldsymbol{R}^{\boldsymbol{T}} * [\boldsymbol{T}]_x) * \widehat{\boldsymbol{m}}_1 = \boldsymbol{0}.$

Note: If we look at the vector $\widehat{\boldsymbol{m}}_1$ in frame {2}, as perpendicular to the vector $(\boldsymbol{R}^{\boldsymbol{T}} * \boldsymbol{T}) \times \widehat{\boldsymbol{m}}_2$, we also deduce that $\widehat{\boldsymbol{m}}_2^T * ([\boldsymbol{R}^{\boldsymbol{T}} * \boldsymbol{T}]_x * \boldsymbol{R}^{\boldsymbol{T}}) * \widehat{\boldsymbol{m}}_1 = \boldsymbol{0}$, so the essential matrix can also defined as $\boldsymbol{E} = [\boldsymbol{R}^{\boldsymbol{T}} * \boldsymbol{T}]_x * \boldsymbol{R}^{\boldsymbol{T}}$.

This result is very important as the identification of the essential matrix *E*, allows also identification of the rotation matrix *R*, and the direction of the vector *T*. Effectively, if *E* is the essential matrix, so is any factor $\lambda * E$, as $\widehat{m}_2^T * \lambda * E * \widehat{m}_1 = \lambda * 0 = 0$, and any vector $\boldsymbol{\lambda * [T]_x}$ **is also an essential matrix.**

Finally, the **determinant of the essential matrix is equal to zero**. Effectively, $\mathbf{det}(\boldsymbol{E}) = \mathbf{det}(\boldsymbol{R^T} * [\boldsymbol{T}]_{\boldsymbol{x}}) = \mathbf{det}(\boldsymbol{R^T}) * \mathbf{det}([\boldsymbol{T}]_{\boldsymbol{x}}) = \mathbf{1} * \mathbf{0} = \mathbf{0}$. Additionally, the **determinant of the fundamental matrix is zero**, as $\mathbf{det}(\boldsymbol{F}) = \boldsymbol{det}(\boldsymbol{A_2^{-T}} * \boldsymbol{E} * \boldsymbol{A_1^{-1}}) = \boldsymbol{det}(\boldsymbol{A_2^{-T}}) * \boldsymbol{det}(\boldsymbol{E}) * \boldsymbol{det}(\boldsymbol{A_1^{-1}})$.

Last but not least, the **singular values of the essential matrix are equal to cero and other two equal eigenvalues**. To prove it, we first remember that the singular values of a matrix *A* are the square root of the eigenvalues of $(A^T * A)$. Then:

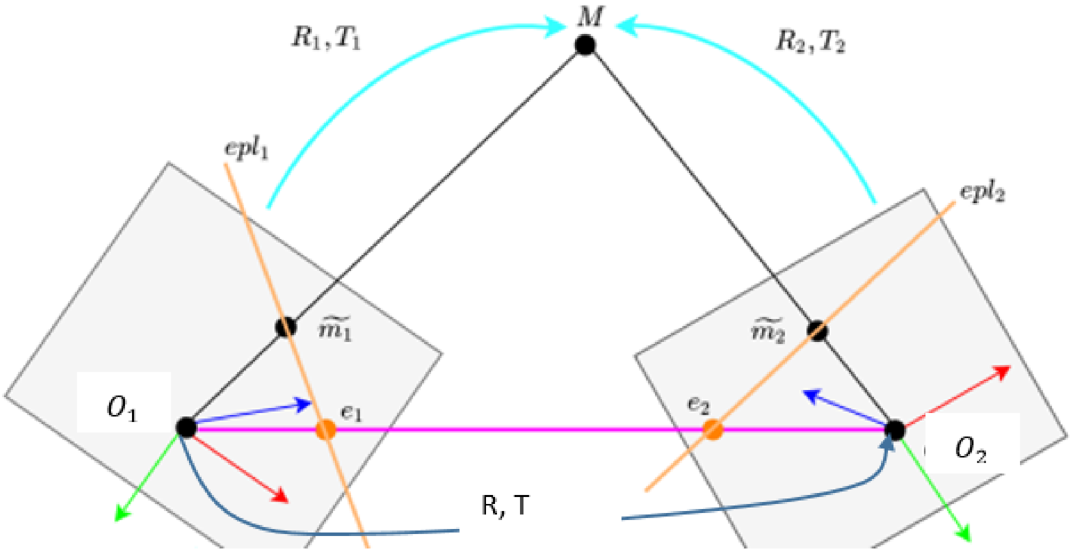
$$\boldsymbol{E^T * E} = \left(\boldsymbol{R^T} * [\boldsymbol{T}]_{\boldsymbol{x}}\right)^T * \left(\boldsymbol{R^T} * [\boldsymbol{T}]_{\boldsymbol{x}}\right) = [\boldsymbol{T}]_{\boldsymbol{x}}^{\boldsymbol{T}} * \boldsymbol{R} * \boldsymbol{R^T} * [\boldsymbol{T}]_{\boldsymbol{x}} = [\boldsymbol{T}]_{\boldsymbol{x}}^{\boldsymbol{T}} * [\boldsymbol{T}]_{\boldsymbol{x}} = -[\boldsymbol{T}]_{\boldsymbol{x}}^{\boldsymbol{2}}$$

We show it in Matlab:

```
>> v = sym('v', [3, 1])
v =
v1
v2
v3

>> skew(v)
ans =

[  0, -v3,  v2]
[ v3,   0, -v1]
[-v2,  v1,   0]

>> skew(v)^2
ans =

[- v2^2 - v3^2,        v1*v2,        v1*v3]
[       v1*v2, - v1^2 - v3^2,        v2*v3]
[       v1*v3,        v2*v3, - v1^2 - v2^2]

>> eig(-skew(v)^2)

ans =
             0
v1^2 + v2^2 + v3^2
v1^2 + v2^2 + v3^2
```

So, the singular values of *E* are 0 and two other positive real values.

<u>**Other properties of the fundamental matrix**</u>



The **epipoles $e1$, $e2$ are the projections of the O_2 and O_1 optical centers in cameras 1 and 2**, as can be seen from the picture. Thus,

$$\tilde{e}_2 = A_2 * [R_2 \quad T_2] * {}^{w}\widetilde{O}_1$$

$$\tilde{e}_1 = A_1 * [R_1 \quad T_1] * {}^{w}\widetilde{O}_2$$

If we know the R, T transformation, then:

$$\tilde{e}_1 = A_1 * [R \quad T] * [0\ 0\ 0\ 1]^T = A_1 * T$$
$$\tilde{e}_2 = A_2 * [R^T \quad -R^T * T] * [0\ 0\ 0\ 1]^T = -A_2 * R^T * T$$

Also for any points in the epipolar lines $\widetilde{m}_2^T * F * \widetilde{m}_1 = 0$ and in particular $\widetilde{m}_2^T * F * \tilde{e}_1 = 0$ **for all points $\widetilde{m}_2$**, as $\tilde{e}_1$ is the epipole common to all epipolar lines. Thus,

$$F * \tilde{e}_1 = 0.$$

Also, from the equation $\widetilde{m}_1^T * F^T * \widetilde{m}_2 = 0$ it follows that $\widetilde{m}_1^T * F^T * \tilde{e}_2 = 0$ **for all points $\widetilde{m}_1$.** Thus,

$$F^T * \tilde{e}_2 = 0$$

The equation of a epipolar line can be written as $a * x + b * y + c = 0$. In matrix form $[x\ y\ 1] * [a\ b\ c]^T = 0$, where $[x\ y\ z]^T$ is a point of the epipolar line in homogeneous coordinates, and $[a\ b\ c]^T$ is the coefficients defining the epipolar line. We call $\widetilde{ep}_1$ **and $\widetilde{ep}_2$ the coefficients defining a epipolar in the first and in the second camera, resulting in $\widetilde{m}_i^T * \widetilde{ep}_i = 0$.**

As $\widetilde{m}_2^T * \widetilde{ep}_2 = 0 = \widetilde{m}_2^T * F * \widetilde{m}_1 = 0$, **for all points** $\widetilde{m}_2$, it follows that:

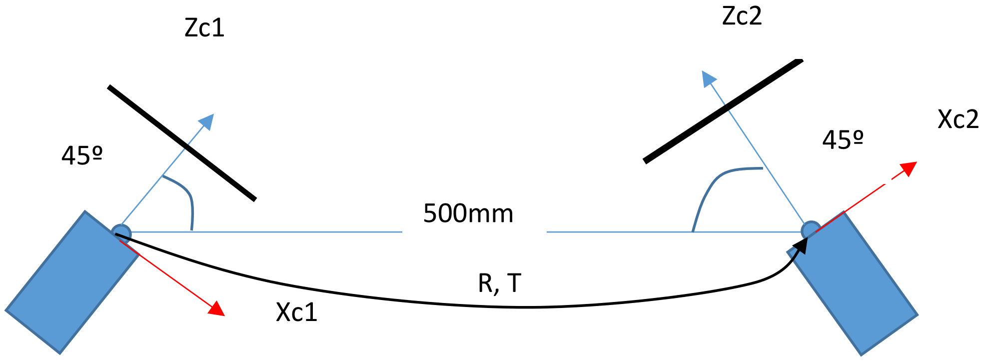
$$\widetilde{ep}_2 = F * \widetilde{m}_1$$

As $\widetilde{m}_1^T * \widetilde{ep}_1 = 0 = \widetilde{m}_1^T * F^T * \widetilde{m}_2 = 0$, **for all points** $\widetilde{m}_1$, it follow that:

$$\widetilde{ep}_1 = F^T * \widetilde{m}_2$$

Exercise

Suppose that we have two cameras, with focal length f1 = 25 mm, f2 = 35mm. Both cameras have a pixel size of 5 microns, and a resolution of 1280 x 1024 pixels. We assume that the lenses don´t have distortion and that they are tilted in the Y axes by 45º as they are shown in the Figure. The optical center is at half the size of the camera, i.e. Cx = 1280/2, Cy = 1024/2.



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a) Calculate the internal matrices of the cameras.

b) Calculate R, T, that relates the second camera to the first one.

c) Calculate the essential and fundamental matrices.

d) Calculate the epipoles.

e) Given a point M in the first camera frame, $\boldsymbol{M} = [\mathbf{10}, \mathbf{20}, \mathbf{350}]^{\boldsymbol{T}}$, calculate the proyections $\widetilde{\boldsymbol{m}}_{\mathbf{1}}$, $\widetilde{\boldsymbol{m}}_{\mathbf{2}}$ in the first and second cameras. Also calculate $\widehat{\boldsymbol{m}}_{\mathbf{1}}$ and $\widehat{\boldsymbol{m}}_{\mathbf{2}}$.

f) Calculate the epipolar lines $\widetilde{\boldsymbol{ep}}_{\mathbf{1}}, \widetilde{\boldsymbol{ep}}_{\mathbf{2}}$ for the points $\widetilde{\boldsymbol{m}}_{\mathbf{1}}$, $\widetilde{\boldsymbol{m}}_{\mathbf{2}}$.

g) Verify all the properties of the fundamental and essential matrix.

Do the programming in Matlab.