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HIGH-FREQUENCY VIBRATION ISOLATION

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Concepts applicable to high-frequency vibration isolation are reviewed with the primary intent of providing insight into the important phenomena. Generalized expressions for transmissibility and isolation effectiveness are presented, which are applicable to isolation of one point of a linear system and which take into account isolator mass effects as well as vibration source and receiver characteristics. The high-frequency mobilities of simple structures are reviewed, and standing waves in resilient elements are discussed in relation to isolation. The isolation performance of two-stage mounts, consisting of a mass between two resilient elements, is analyzed. Optimization of such mount design is discussed, and experimental results illustrating their performance are presented.

1. INTRODUCTION

In many segments of industry the trend in the past few years has been towards more complex equipments and machines, which are lighter and more compact than their predecessors and which operate at greater speeds and higher power ratings. To the vibration engineer this trend has meant more problems associated with high-frequency vibrations: i.e., more excitation available at these high frequencies and more components likely to be affected adversely by them. As a consequence it has become increasingly important to provide vibration isolation systems that will retain their effectiveness at high frequencies, where classical isolators often fail to perform satisfactorily.

Classical vibration isolation theory deals with isolation of rigid masses supported on massless resilient members (springs). At high frequencies, however, the items to be isolated do not behave as rigid masses, and effects of the distributed mass of the resilient members may come into play. This paper is concerned (a) with extensions of the classical vibration isolation theory that take the aforementioned high-frequency phenomena into account, (b) with providing some insight into how these phenomena affect the isolation characteristics of given installations, and (c) with providing guidelines for the design of isolation systems with improved high-frequency characteristics.

Much information related to the high-frequency isolation problem is available in the technical literature. A considerable fraction of this published work, however, deals with detailed analyses of special cases or highly idealized systems and with experiments aimed at demonstrating the importance of the various phenomena or at providing some data on practical systems. It appears that no general summary is available which shows how the various available analyses fit into the matrix provided by a generalized isolation theory, and which points out the effects of the generally important parameters. The intent of this paper is to provide just such a summary.

The first of the following sections summarizes the classical isolation theory in somewhat generalized terms and points out its implications. This theory is further generalized in the second section. Non-rigid isolated structures and wave effects in isolators are discussed in the third section, followed in the fourth section by analysis of two-stage mounting systems which are particularly suitable for high-frequency isolation.

Since the aim of this paper is the provision of an overall view and some intuitive insight, the discussion here is limited essentially to isolation of motion of a single point in a single direction (corresponding to the classical single-degree-of-freedom spring-and-point-mass system). However, the concepts presented here may readily be generalized for more complex situations. Also in keeping with the aforementioned aim, all of the analyses here are presented in terms of mobilities rather than impedances. Mobility diagrams are advantageous in that they appear very much like diagrams of the corresponding mechanical system (1); in addition, they permit one to obtain an intuitive appreciation of the action of a system of mobilities simply by visualizing high mobilities as soft and low mobilities as stiff springs.

2. THE VIBRATION ISOLATION PROBLEM

As indicated in Figure 1, the general vibration isolation problem involves a source of vibrations, a "receiver" (i.e., an item to be protected from vibrations), and an isolator inserted in the path between the source and receiver. The isolation problem consists of providing an isolator that, for a given source and receiver, reduces the vibrations of the receiver to acceptable levels.

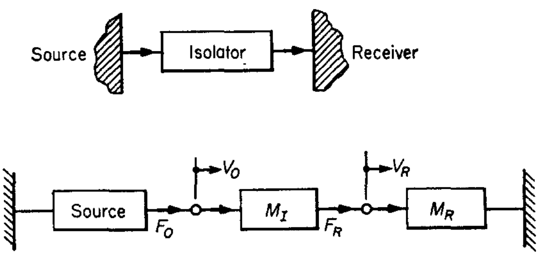


Figure 1. Schematic and mobility diagram of source, isolator and receiver.

2.1. MEASURES OF ISOLATOR PERFORMANCE

The most commonly used measure of performance of an isolator in a given situation is the transmissibility T. Transmissibility may be defined as the ratio of the velocity amplitude at the receiver to the velocity amplitude on the source side of the isolator. In terms of the mobility diagram of Figure 1,

$$T = |V_R/V_O|, \tag{1}$$

where V denotes a complex velocity amplitude (or "phasor") from which one may obtain the velocity as a function of time in the usual manner, as $v(t) = Re[V e^{i\omega t}]$. Clearly, a good isolator results in a low receiver velocity V_R for a given excitation and thus has a *low* transmissibility.

The classical definition of transmissibility is based on a system consisting of a spring-mounted mass (Figure 2a), driven by an oscillatory motion of the support, which itself is unaffected by the system response. The force transmissibility T_f, also often used in relation to masses mounted on isolators, is defined as the ratio $|F_e/F_g|$ of force amplitudes, where F_e corresponds to an oscillating force (produced by a source external to the mass-isolator system) acting on the mass, and F_g corresponds to the resulting force that the isolator exerts on a rigid "ground" (see Figure 2(b)). For systems like those of Figure 2 one may show that $T = T_f$. For systems with more complex sources the T_f ratio is of little value, since it involves the artifice of a rigidly fixed termination. Hence, the transmissibility definition of equation (1) is used throughout the remainder of this paper.

The transmissibility, it must be noted, is not a characteristic of an isolator itself, but depends on the entire system in which the isolator is used. The dependence of isolation on the entire system is perhaps somewhat more clearly evident from another measure of isolation performance, namely the "isolation effectiveness" E. This effectiveness is defined as the ratio of the receiver amplitude obtained when the receiver is connected

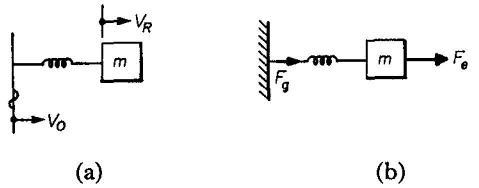


Figure 2. Classical mass-on-spring isolation. (a) Velocity excitation at support; (b) force excitation at mass.

directly to the source to that amplitude obtained when the isolator is inserted between the source and the receiver (2). Thus,

$$E = \left|\frac{V_{RO}}{V_R}\right| = \left|\frac{F_{RO}}{F_R}\right| \tag{2}$$

where the added subscript O indicates velocities and forces obtained if the isolator is replaced by a rigid (zero mobility) and massless connection between source and receiver, and where the absence of such a subscript refers to conditions with the actual isolator in place. The two ratios above are equal, since $V_R = M_R F_R$ and since the receiver mobility M_R does not depend on what is connected to the receiver. The isolation effectiveness thus indicates how much good an isolator does in a given situation, with $E = 1$ corresponding to no improvement over a rigid connection and greater effectiveness values corresponding to better isolation. (It should be noted that also other labels, such as "response ratio" or "insertion ratio", have been used in the literature (3, 4) either for the ratio that has been designated here as E or for its reciprocal.)

If an isolator between a source and a receiver (Figure 1) is replaced by a massless rigid connector, then the velocities and forces on its two ends are equal, or $V_{OO} = V_{RO}$ and $F_{OO} = F_{RO}$. (Here, as before, the added subscript O designates quantities obtained with the rigid connector.) Thus, one may also rewrite equation (2) as $E = |V_{OO}/V_R| = |F_{OO}/F_R|$. By comparing this expression with equation (2) one finds that $E = 1/T$ only for the special case where $V_O = V_{OO}$, $F_O = F_{OO}$; that is, the *effectiveness is equal to the reciprocal of the transmissibility only if the source output is independent of the load* on which the source acts. This limitation on the relation between effectiveness and transmissibility appears not to have been stated explicitly before this, probably since most of the prior literature has dealt (at least tacitly) with either constant force or constant velocity sources.

2.2. SOURCE CHARACTERIZATION

The previous paragraph provides an indication that the characteristics of the vibration source may affect isolator performance and thus influence the choice of an optimum isolator. One may illustrate this concept rather simply and dramatically by considering the isolator of Figure 1 to be massless. It then transmits all the force that is applied to it, but not all the velocity; one finds

$$\frac{V_R}{F_O} = M_R, \quad \frac{V_R}{V_O} = \frac{M_R}{M_I+M_R}, \tag{3}$$

where M_R denotes the mobility of the receiver and M_I that of the isolator.

For a force-source (one that generates F_O, regardless of V_O) one obtains $F_O = F_{OO}$ and $E = 1$. The isolator is seen to be completely ineffective—a not too surprizing result, since a massless isolator by definition does not attenuate the force acting on the receiver. On the other hand, for a velocity source one obtains $V_O = V_{OO}$ and $E = |1 + (M_I/M_R)|$. For this type of source, the isolator does serve a useful function.

Force-source-like behavior may, for example, be observed at mounting points at which rigid frames of unbalanced rotating machines are attached to other structures; the unbalanced centrifugal forces here may be expected to be little affected by the magnitudes of the mounting point velocities. Velocity-source-like behavior may be produced, for example, by the action of a piston in a reciprocating machine operating at constant rotational speed. Most realistic sources depart from these ideal behaviors, of course. If the aforementioned machine frames are massive, for example, they will reduce the force observed at the mounting points, and this force reduction will increase with increasing mounting point velocity amplitude. Further, as a perhaps more realistic example, one may expect that the force (or velocity) obtained from a source consisting of a panel excited by sound or by boundary layer turbulence will be strongly affected by loading of the source: e.g., by attaching resilient connections between the panel and other structural members.

It is useful to consider a general linear source, whose force output F_O is related to its velocity output V_O as

$$F_O = F_S - V_O/M_S = (V_S - V_O)/M_S. \tag{4}$$

Here M_S denotes the source mobility, F_S the output of a force-source, and V_S that of a velocity source. Two equivalent mobility representations (1) of these sources are sketched in Figure 3. The quantities F_S, V_S and M_S corresponding to a given source may readily be identified:

$$\begin{aligned} F_S &= F_O\Big|_{V_O = 0} = \text{“blocked force”} \equiv F_{bl}; \\ V_S &= V_O\Big|_{F_O = 0} = \text{“free velocity”} \equiv V_{fr}; \\ M_S &= V_S/F_S = V_{fr}/F_{bl}. \end{aligned} \tag{5}$$

The general linear source expression (4) may also readily be specialized to represent the two previously discussed ideal cases. For infinite M_S the output force F_O is equal to the blocked force F_{bl}, regardless of V_O; and for $M_S = 0$ the output velocity V_O is equal to the free velocity V_{fr} regardless of F_O. Thus, a force-source corresponds to $M_S = \infty$, a velocity source to $M_S = 0$.

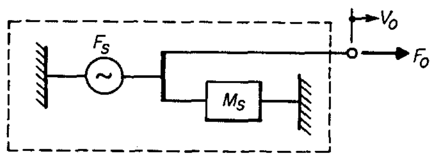
If one considers the source of Figure 1 to be a general linear one with mobility M_S, and if one limits oneself to *massless isolators*, one finds that one may express the isolator effectiveness as

$$E = \left|1 + \frac{M_I}{M_S+M_R}\right|. \tag{6}$$

This result is equivalent to those obtained by Muster and Plunkett (4) and by Sykes (2) using a somewhat less general approach.

Equation (6) indicates that if an isolator is to be effective, its mobility M_I must exceed the sum M_S+M_R considerably. Thus, a criterion for good isolator performance is

(a)



(b)

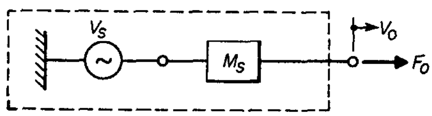


Figure 3. Two equivalent representations of a general linear source. (a) With force source F_s (b) with velocity source V_s.

$|M_I| \gg |M_S+M_R|$. One may again note that for an ideal force-source ($M_S=\infty$) the foregoing inequality cannot be satisfied, and that a massless isolator is ineffective for such a source. On the other hand, for a velocity source ($M_S=0$) one may expect that the inequality $|M_I| \gg |M_R|$ may be satisfied relatively easily with a properly chosen M_I.

2.3. FAILURE OF CLASSICAL MODEL

The classical model consisting of a rigid mass mounted on a massless isolator fails to account for two conditions that are encountered at high frequencies:

1. a realistic mounted item ceases to move as a rigid mass at high enough frequencies; instead, it behaves like a multi-degree-of-freedom system, whose mobility (as measured at the isolator attachment point) passes through successive maxima and minima with changing frequency;
2. a realistic mount is not massless; at sufficiently high frequencies the distributed mass and elasticity of the mount give rise to standing waves in the mount.

As discussed further subsequently, both of these conditions in general serve to reduce the effectiveness of an isolation system.

3. HIGH FREQUENCY BEHAVIOR

3.1. DRIVING POINT MOBILITIES OF MOUNTED ITEMS (RECEIVERS)

If a prototype of the item to be isolated is available, then one may in general determine its driving point mobility (or the reciprocal thereof, the driving point impedance), essentially by exciting the item sinusoidally at the driving point and measuring the steady-state driving force and response velocity simultaneously (5, 6). Unfortunately, the required instrumentation is rather complex, and the measurements usually are beset with experimental difficulties, which tend to become greater and more numerous at higher frequencies (7, 8).

The driving point mobility of a mounted item may also be calculated, at least in concept, using classical techniques based on the wave equations that describe the behavior of its various component structural elements. For such calculations one requires exact descriptions of the dimensions, mass and stiffness distributions, and interconnections between these elements. Because of the close spacing of modes at high frequencies, small errors in the values of the foregoing quantities may cause considerable errors in the mobility at a given frequency, so that this classical approach is here generally of little practical utility.

On the other hand, Skudrzyk (10, 11) has demonstrated that the driving point impedances of elastic structures may be represented by a "background level" plus contributions from a limited number of modes. He has shown that at intermediate frequencies only the mode nearest the excitation frequency contributes significantly, whereas at high frequencies the background level alone is of importance. For ideal structural elements, such as uniform plates or beams in flexure or rods in extensional or torsional motion, it turns out that the impedance background level (and thus the driving point impedance at high frequencies) is equal to the driving point impedance of a similar structural element of infinite extent.

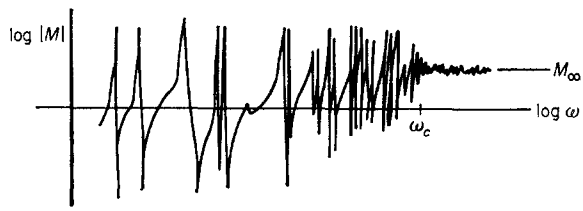


Figure 4. Typical driving point mobility of finite plate.

Figure 4 is a conceptual sketch showing how the driving point mobility of a typical realistic structural plate varies with frequency, based on data presented by Skudrzyk (10). As indicated, at frequencies greater than a limiting frequency ω_c the observed mobility M is essentially equal to M_∞, the mobility of an infinite plate. At lower frequencies the mobility alternates rapidly between very large values (in conjunction with which a given isolator can perform effectively) and very small values (for which an isolator is ineffective).

The driving point impedance of a finite system may be expected to be equal to that of an infinite one at frequencies ω at which the bandwidth associated with a mode is greater than the average frequency interval between modes (10, 11); i.e., whenever

$$\omega\eta > \Delta\omega_n. \tag{7}$$

Here $\Delta\omega_n$ denotes the average frequency interval between the structural modes excited by a point force, and η represents the structural loss factor. (One may also visualize that for large enough $\omega\eta$ the structural wavelengths will be small enough and the attenuation per wavelength will be high enough so that waves reflected from the system boundaries will have no appreciable effect at the driving point (12).)

From Skudrzyk's work and the well-established relations between frequency, wavelength, and the properties of elastic systems (13), one may determine that the mobilities (or impedances) of infinite systems are observed at the driving points of similar finite systems if the following inequalities are satisfied:

(1) for plates vibrating flexurally,

$$\omega\eta > \frac{4c_l h}{\sqrt{3}A} \quad \text{or} \quad \frac{A}{\lambda_b^2}\eta > \frac{2}{\pi^2}; \tag{8a}$$

(2) for beams vibrating flexurally,

$$\omega\eta^2 > (4\pi)^2\frac{rc_l}{L^2} \quad \text{or} \quad \frac{L}{\lambda_b}\eta > 2; \tag{8b}$$

(3) for rods vibrating longitudinally,

$$\omega\eta > \omega_{1l} \quad \text{or} \quad \frac{L}{\lambda_l}\eta > \tfrac{1}{2}; \tag{8c}$$

(4) for rods vibrating torsionally,

$$\omega\eta > \omega_{1t} \quad \text{or} \quad \frac{L}{\lambda_t}\eta > \tfrac{1}{2}. \tag{8d}$$

In the above relations h denotes plate thickness, A plate surface area, r beam cross-sectional radius of gyration, and L beam or rod length. The symbol λ represents wavelength, with subscript b referring to flexural (bending), l to longitudinal and t to torsional oscillations. The frequencies ω_{1l} and ω_{1t} are the fundamental resonance frequencies for longitudinal and torsional vibrations, respectively, and are given by

$$\omega_{1l} = \pi c_l/L, \quad \omega_{1t} = \pi c_t/L \tag{9}$$

with c_l and c_t denoting the longitudinal and torsional wave velocities (13), respectively.

With an estimated loss factor $\eta \approx 10^{-3}$ for typical homogeneous metal structures one may calculate that infinite system driving point mobilities are obtained, for example,

(1) above 7×10^3 Hz† for a 1/10 in. thick plate measuring 20 by 50 in.,
(2) above $2{\cdot}5\times10^8$ Hz for flexural vibrations
(3) above 10^6 Hz for longitudinal vibrations
(4) above 7×10^5 Hz for torsional vibrations
} for a 100 in. long rod with 1 in. diam.

For structures with somewhat greater effective damping, say $\eta \approx 10^{-2}$ as might be produced for example by structural joints [which are particularly effective at higher frequencies (14)] transition to infinite system behavior may be expected to occur a decade (or two, for beam flexure) lower than the above indicated values.

It is instructive to compare the driving point mobilities of some finite elastic systems at high frequencies [i.e., the infinite system mobilities M_∞ (13, 15)] with the mobilities M_{mo} of the same systems considered as rigid masses. For a plate in flexure one finds

$$\left|\frac{M_\infty}{M_{mo}}\right| = \left|\frac{i\omega\rho Ah}{8\sqrt{\rho hD}}\right| = \frac{\pi^2}{2}\frac{A}{\lambda_b^2}, \tag{10a}$$

for a beam driven laterally at a point far from an end

$$\left|\frac{M_\infty}{M_{mo}}\right| = \left|\frac{i\omega\rho SL}{2(1+i)\rho S\sqrt{\omega r c_l}}\right| = \frac{\pi}{\sqrt{2}}\frac{L}{\lambda_b}, \tag{10b}$$

for a semi-infinite rod driven longitudinally

$$\left|\frac{M_\infty}{M_{mo}}\right| = \left|\frac{i\omega\rho SL}{\rho S c_l}\right| = 2\pi\frac{L}{\lambda_l}, \tag{10c}$$

and for a semi-infinite rod driven torsionally

$$\left|\frac{M_\infty}{M_{mo}}\right| = \left|\frac{i\omega\rho J L}{\rho J c_t}\right| = 2\pi\frac{L}{\lambda_t}. \tag{10d}$$

In the above expressions ρ denotes the material density, D the plate flexural rigidity, S the beam cross-section area, and J the polar moment of inertia of that area. All of these relations indicate very clearly that $|M_\infty| \gg |M_{mo}|$ at high frequencies, where the structural dimensions are many wavelengths long. Thus, all these structures are very much more

† The symbol Hz is used in this paper to represent c/s, in accordance with a proposed new international standard.

mobile at high frequencies than they would be if they would remain rigid, and a much more mobile isolator is needed to provide a given effectiveness for these nonrigid than for similar rigid structures.

3.2. WAVE EFFECTS IN ISOLATORS

At frequencies well below their first standing-wave resonances isolators behave very nearly like ideal springs, so that their low frequency mobilities may be obtained directly from calculated or measured spring rates. (The effects of isolator damping are, of course, unimportant at low frequencies, except at system resonances.)

The frequencies at which standing wave resonances occur in an isolator depend on the distributions of mass and elasticity in it, whereas its behavior at and near these resonances depends largely on the damping. The standing wave resonances, as well as the isolator effectivenesses or related quantities, have been calculated for some isolators whose resilient elements have geometries and deformation behaviors that lend themselves to simple mathematical description: e.g., for rods or pads in compression or shear (16–19), and for leaf and helical springs (16, 19). For complex geometries and deformations, however, one generally must resort to experimental means for determining isolation at high frequencies; measurement procedures and results have been reported by several authors (4, 16, 18).

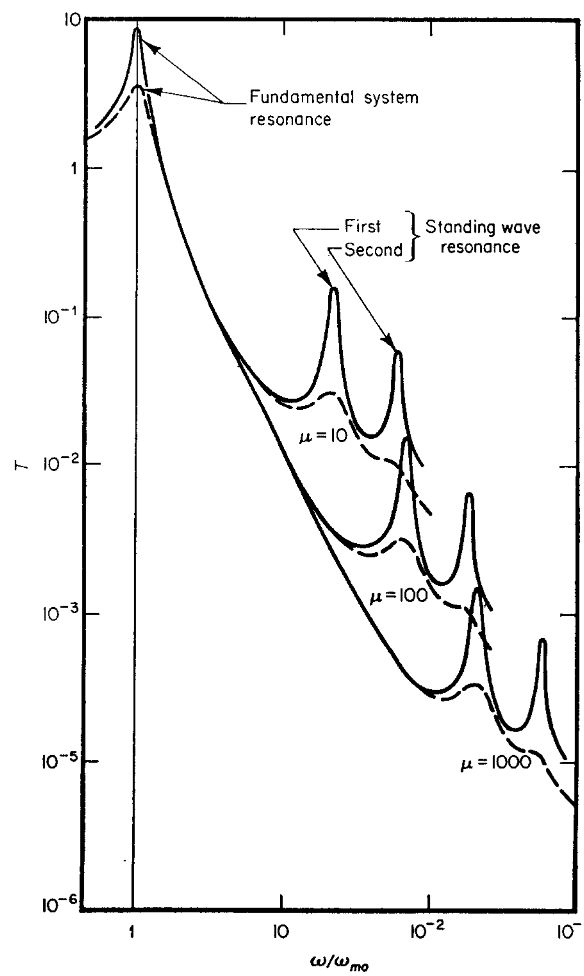


Figure 5. Effect of isolated isolator mass ratio and of damping on high-frequency transmissibility. ——, $\eta = 0$; – – –, $\eta = 0{\cdot}6$.

Figure 5 illustrates how transmissibility for a rigid mounted mass is affected by isolator damping and by the ratio μ of the mounted mass to the isolator mass (19). The peaks in the transmissibility curves are associated with resonances; the peak at $\omega = \omega_{mo}$ corresponds to the fundamental resonance of the mounted mass on an essentially massless spring, the higher frequency resonances (of which only two are shown for each condition for the sake of clarity) correspond to standing waves in the spring. The frequency ratios at which standing waves occur are proportional to $\sqrt{\mu}$, whereas the transmissibility peak associated with a given standing wave resonance is inversely proportional to the mass ratio μ (for constant loss factor η).

The curves of Figure 5 pertain to springs whose elastic properties and loss factors do not change with frequency. They thus differ from the often-discussed case of viscously damped springs, since a constant viscous damping coefficient c corresponds to a frequency dependent loss factor $\eta = \omega c/k$, where k denotes the elastic spring constant. However, the damping of practical isolator materials generally is better represented by a constant loss factor than by a constant damping coefficient, particularly at high frequencies (22). Figure 5 shows that, as one may expect, damping has no appreciable effect, except near resonances, and the transmissibility at all resonances is decreased by increasing damping.

Although the frequency ratio ω_1/ω_{mo} at which the first standing wave resonance occurs is proportional to $\sqrt{\mu}$ for a given isolator configuration and material, different ratios apply for isolators of different materials and configurations. For a set of geometrically similar isolators one finds that ω_1 is proportional to c_l/L_o, where c_l denotes the longitudinal wave velocity in the isolator material and L_0 stands for a characteristic dimension of the isolator. Thus, $\omega_1 \sim c_l\sqrt{\mu}/L_o$ and, for example, the first standing wave resonance in a small and light metallic isolator may be expected to occur at a much higher frequency than in a similar heavier (smaller μ), larger (greater L_o), elastomeric (lower c_l) isolator of equal static stiffness.

3.3. EFFECTIVENESS AND TRANSMISSIBILITY CONSIDERING ISOLATOR MASS EFFECTS

For cases where isolator mass cannot be ignored (e.g., at frequencies where wave effects occur in isolators), the effectiveness expression (6) and the transmissibility relation obtained by substitution of equation (13) into (1) do not apply, since equations (3) and (6) are based on the assumption of massless isolators. If one analyzes the system of Figure 1 without introducing the assumption of negligible isolator mass, but instead characterizes the isolator as a general linear "four-pole" system, then one may obtain, as Molloy essentially has shown (20), that the effectiveness is given by

$$E = \left|\frac{\alpha}{M_S+M_R}\right| \left|1+\frac{M_S}{M_{1b}}+\frac{M_R}{M_{2b}}\left(1+\frac{M_S}{M_{1f}}\right)\right| \tag{11}$$

where

$$\frac{1}{\alpha^2} = \frac{1}{M_{2b}}\left(\frac{1}{M_{1b}}-\frac{1}{M_{1f}}\right). \tag{12}$$

Here M_{1b} denotes the mobility of the isolator measured on the source side, with the receiver side blocked (i.e., with $V_R = 0$), M_{1f} denotes the same, but with the receiver side free ($V_R = 0$), and M_{2b} denotes the mobility measured on the receiver side of the isolator with the source side blocked. One may readily verify that equation (11) reduces to equation (6) for a massless isolator, for which $M_{1b} = M_{2b} = M_I$, $M_{1f} = \infty$.

One may also determine that the transmissibility T, defined as in equation (1), here obeys

$$\frac{1}{T} = \left|\alpha\left(\frac{1}{M_R}+\frac{1}{M_{2b}}\right)\right| \tag{13}$$

where α is given by equation (12). By combining equations (11) and (13) one may find that

$$E.T = \left| \frac{1+\dfrac{M_S}{M_{1b}}+\dfrac{M_R}{M_{2b}}\left(1+\dfrac{M_S}{M_{1f}}\right)}{\left(1+\dfrac{M_S}{M_R}\right)\left(1+\dfrac{M_R}{M_{2b}}\right)} \right|. \tag{14}$$

As equation (12) indicates, the mobility parameter α depends only on properties of the isolator. Thus equation (13) shows that the transmissibility depends only on the receiver and isolator mobilities, whereas according to equation (11) the effectiveness depends on the source mobility as well. Further, equation (14) indicates that E and T are always inversely proportional to each other, but are reciprocally related only in special cases (e.g., for a velocity source, for which $M_S=0$).

4. TWO-STAGE ISOLATORS

Since a massless isolator transmits to a receiver all the force applied to it by a source, as has been previously discussed, such an isolator is clearly useless for force isolation purposes. This fact leads one to consider what sort of isolator might serve to attenuate transmitted forces.

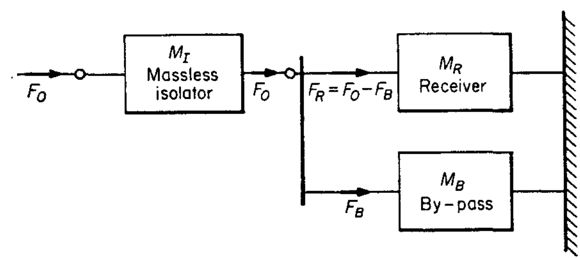


Figure 6. Reduction of receiver force by by-pass mobility.

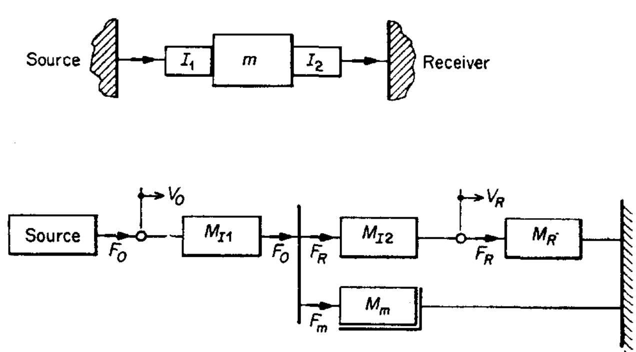


Figure 7. Two-stage isolator.

In dealing with this problem in terms of mobility diagrams one notes that a massless isolator represents essentially a direct connection through which force "flows" from source to receiver. The force arriving at the receiver may be reduced (for a given force generated by the source) only if additional branches for force flow are provided to by-pass the receiver. Such by-pass branches for force flow to the "mechanical ground" are obtained by connecting other mobilities in parallel with the receiver mobility (Figure 6). Since the mobility of a pure (rigid) mass always has one terminal grounded, such a mobility requires no additional grounding provision; hence such a mass is a natural choice for a by-pass element.

The sort of reasoning summarized in the previous paragraph leads one to consider the addition of a mass either directly at the receiver or within the mount itself. Isolators incorporating one or more lumped masses have been called "compound isolators"; isolators incorporating a single mass and two resilient elements (Figure 7) have also been called "two-stage isolators".

4.1. EFFECTIVENESS

The transmissibilities and effectivenesses of two-stage isolators supporting rigid masses have been studied both theoretically and experimentally, notably by Exner (21) and Snowdon (22). By generalizing their results, by applying equations (11, 12, 13), or by a straightforward analysis based on the mobility diagram of Figure 7 one finds that one may express the effectiveness of a two-stage mount consisting of a mass between two massless isolators as

$$E = \left| 1 + \frac{1}{M_S + M_R}(M_{I1} + M_{I2}) + \Delta E \right|, \tag{15}$$

where

$$\Delta E = \frac{(M_{I1} + M_S)(M_{I2} + M_R)}{M_m(M_S + M_R)}. \tag{16}$$

The symbols M_{I1}, M_{I2} represent the mobilities of the two resilient isolator components and $M_m = 1/i\omega m$ that of the rigid mass m, as indicated in Figure 7.

The ΔE term represents the contribution to the effectiveness that is made by virtue of the included mass. If the mass is infinitely mobile (corresponding to zero mass and/or zero frequency) then the two-stage mount of Figure 7 reduces to the single-stage mount of Figure 1 with the total isolator mobility $M_I = M_{I1} + M_{I2}$. For infinite M_m the contribution ΔE vanishes and equation (15) reduces to equation (6), which was derived for a massless mount.

It is of interest to note the symmetry of equations (15) and (16) in the source and receiver terms. Interchanging M_{I1} with M_{I2}, and M_R with M_S, is seen to have no effect on the effectiveness E. Thus, the source and receiver mobilities play similar roles, as Sykes (16) has also pointed out for massless isolators.

4.2. OPTIMIZATION

It is clear from equation (16) that a greater included mass, to which corresponds a lower mobility M_m, results in an increased effectiveness contribution ΔE due to the mass. In general, however, one desires to obtain the greatest effectiveness with a given included mass and a given total isolator mobility $M_I = M_{I1} + M_{I2}$ (which corresponds to a given static deflection for a given supported weight).

By setting

$$M_{I1} = rM_I, \quad M_{I2} = (1-r)M_I, \tag{17}$$

in equation (16) one may determine that ΔE takes on its greatest value $\Delta E_{\max}$ for $r = r_{\text{opt}}$, where†

$$r_{\text{opt}} = \tfrac{1}{2}\left[1 + \frac{M_R - M_S}{M_I}\right]$$

$$\Delta E_{\max} = \frac{(M_I + M_S + M_R)^2}{4M_m(M_S + M_R)}. \tag{18}$$

† Although the various quantities involved here are complex, they are treated here as if they were real. This treatment leads to correct qualitative results and avoids more lengthy discussions and complicated expressions which detract from the development of an intuitive understanding.

From equation (16) one may conclude that $|M_{I1}|$ must be considerably greater than $|M_S|$ if the isolator part near the source is to have any appreciable effect on ΔE. For the isolator part connected to the receiver to be similarly effective one requires $|M_{I2}| \gg |M_R|$. If both of these requirements are satisfied, then $|M_I|$ exceeds $|M_S - M_R|$ considerably as well, and equations (18) reduce to

$$r_{\text{opt}} \approx 1/2,$$

$$\Delta E_{\max} \approx \frac{(M_I)^2}{4M_m(M_S+M_R)}. \tag{19}$$

Thus, if the two mount portions are very mobile compared to the source and receiver one may realize a considerable effectiveness increase ΔE; the greatest possible ΔE may be obtained by the use of two resilient elements of equal mobility.

4.3. MASS-LOADING OF RECEIVER OR SOURCE

Attaching a mass directly to the receiver has in some instances been advocated instead of mounting a mass between two isolators. The effectiveness increase ΔE_{mR} associated with such mass-loading of the receiver, as found directly by setting $M_{I1} = M_I$ and $M_{I2} = 0$ in equation (16), is given by

$$\Delta E_{mR} = \frac{(M_I + M_S) M_R}{M_m(M_S+M_R)} \approx \frac{M_I M_R}{M_m(M_S+M_R)}, \tag{20}$$

where the approximate right-hand expression corresponds to good isolation: i.e., to $|M_I| \gg |M_S|$. Comparison of this result with equation (19) shows that inclusion of a given mass between two equal isolators is more advantageous than attaching it directly to the receiver as long as $|M_I| > 4|M_R|$—a condition which one is likely to be able to satisfy in many practical circumstances.

For the case where the mass is attached directly to the source one finds similarly that the effectiveness increase is given by

$$\Delta E_{mS} = \frac{(M_I + M_R) M_S}{M_m(M_S+M_R)} \approx \frac{M_I M_S}{M_m(M_S+M_R)}, \tag{21}$$

where the approximate equality holds for $|M_I| \gg |M_R|$. Again, ΔE_{mS} is clearly less than $\Delta E_{\max}$ for $|M_I| > 4|M_S|$.

The special case where M_S is large (i.e., where the source acts essentially like a force-source) merits some special attention, since for such a source one cannot choose $|M_{I1}|$ and $|M_I|$ to exceed $|M_S|$. For a force-source equation (16) becomes

$$\Delta E\Big|_{M_S = \infty} = \frac{M_{I2} + M_R}{M_m}, \tag{22}$$

which implies that the greatest effectiveness may be obtained by making M_{I2} as large as possible: i.e., by making $M_{I2} = M_I$ and $M_{I1} = 0$, which corresponds to attaching the mass directly to the source.

4.4. SOME EXPERIMENTAL RESULTS

Figures 8 and 9 show data obtained from a series of measurements carried out on an electronic package, where it was desired to protect interior circuit-board-mounted components from high frequencies to which they were sensitive. The package was roughly in the shape of a 3 ft^3 and weighed about 300 lb. All of the measurements summarized here were obtained using excitation provided by a high frequency shaker, driven by $\frac{1}{3}$-octave band filtered white noise, and by measuring responses in the same $\frac{1}{3}$-octave bands.

The topmost curve of Figure 8 indicates the transmissibility that was measured from the mounting point on the reinforcing frame (see sketch) to the circuit board, in absence of any isolator. The second curve from the top represents the transmissibility measured from the input mounting point of a conventional elastomeric isolator to the same circuit board. The remaining three curves represent similarly measured transmissibilities for two-stage

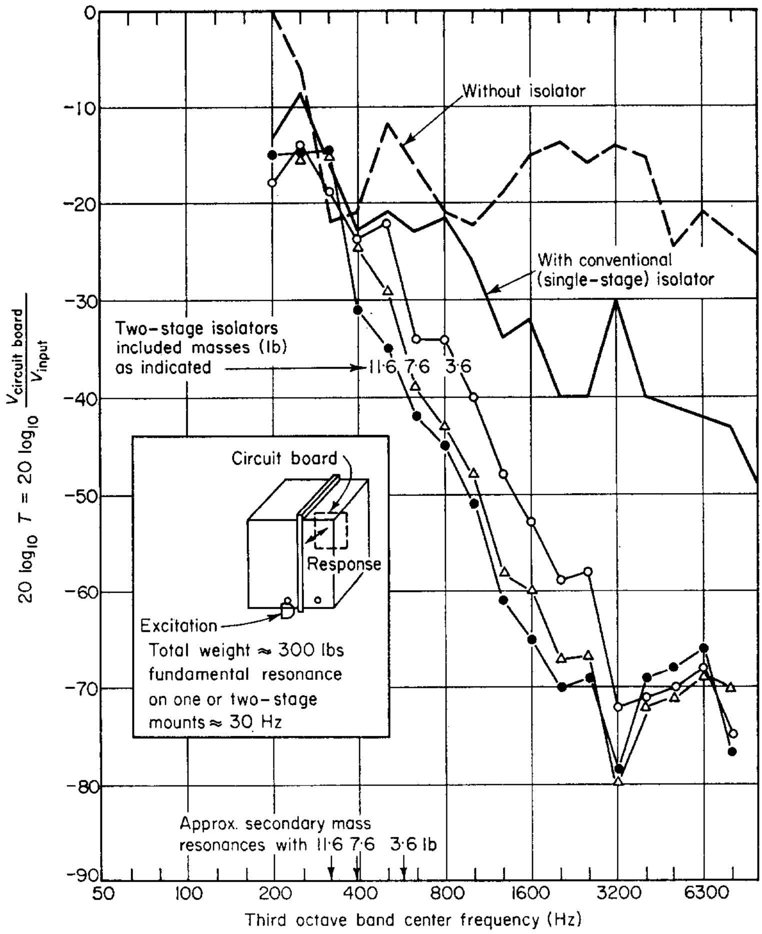


Figure 8. Comparison of performance of single and two-stage isolators on an electronic package.

mounts with three different included masses, but all with the same static stiffness (low frequency mobility) as the conventional isolator.

The fundamental resonance of the entire package of Figure 8 supported on four identical isolators is roughly 30 Hz. The frequencies associated with resonances of the three different masses within the two-stage isolators are indicated at the bottom of the figure. The advantage of the two-stage over the single stage mounts at frequencies above these resonance frequencies is clearly evident.

In the intermediate frequency region, where the two-stage isolators evidently perform as intended, one may also observe the effect of the magnitude of the included mass. The data there agree reasonably well with the theoretically predicted 6 dB decrease in transmissibility per doubling of mass.

The two-stage transmissibility increases evident in Figure 8 above 3200 Hz are probably associated with standing wave resonances in the resilient elements of the isolators. The isolators to which the reported test results pertain contain a damping provision that evidently limited these resonances; earlier versions of these isolators contained no such provision and performed much like those for which results are shown in the figure, except that the transmissibility curves exhibited much higher peaks in the 3200 to 8000 Hz region.

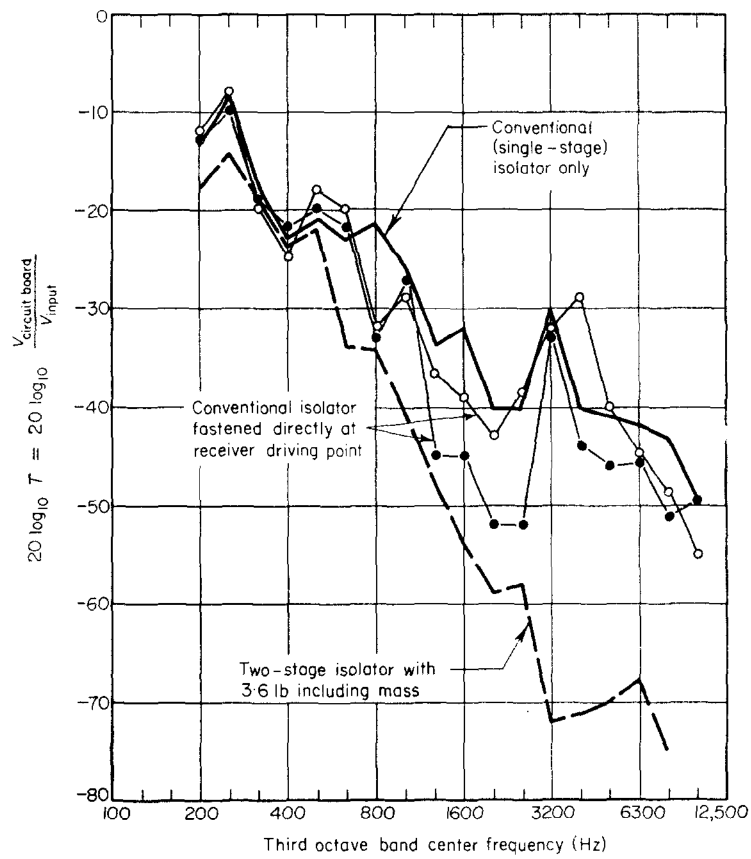


Figure 9. Comparison of effects of mass additions at receiver and in two-stage mount for electronic package of Figure 8.
○, 3·6 lb; ●, 7·6 lb.

Figure 9 indicates the circuit board vibration reductions that were obtained by attaching masses directly at the mounting point on the reinforcing frame, atop the conventional isolator. The transmissibility obtained with the conventional isolator, without added mass, and that obtained with a two-stage isolator incorporating a 3·6 lb mass, are repeated here from Figure 8 for the sake of comparison. It is evident that addition of a 3·6 lb mass directly atop the conventional isolator had very little effect, whereas inclusion of such a mass in a two-stage isolator provided a considerably greater reduction in transmissibility than direct attachment of nearly twice as much mass at the receiver input.

It is important to note that all data presented here pertain to transmissibilities, which do not depend on vibration source characteristics. On the other hand, isolation effectiveness does depend on the behavior of the source. Thus, a given measured reduction in transmissibility does not mean that a similar increase in isolation effectiveness will necessarily be realized in a practical situation where the source behavior is unknown.

5. CONCLUSION

5.1. SUMMARY

The transmissibility and isolation effectiveness concepts pertaining to vibration isolation of one point of a linear system have been reviewed and generalized, both for massless isolators and for isolators in which mass effects are important. In particular, it has been demonstrated that the characteristics of the vibration source influence the effectiveness of an isolator and enter the relation between effectiveness and transmissibility. These two measures of isolation performance have been shown to be inversely proportional to each other for a given system operating at a given frequency, but to be reciprocals of each other essentially only for the case where the excitation is provided by a velocity source.

It has been pointed out that in order to be effective an isolator must in general possess a mobility which exceeds that of the source and of the receiver. The failure of isolation schemes that perform well at low frequencies to provide adequate high frequency isolation has accordingly been ascribed to two main causes:

1. non-rigid behavior of the items to be isolated, which causes their mobilities to be greater than those of similar rigid items;
2. resonant (standing wave) effects in resilient isolators, which essentially decrease isolator mobility.

The high-frequency mobility characteristics of simple structures have been discussed. It has been pointed out that at high enough frequencies the driving point mobilities of finite structural components approach those of similar components of infinite extent, and that the frequency above which a structure exhibits the driving point mobility of an infinite structure decreases with increasing damping. It has been demonstrated that the high-frequency mobilities of elementary structures far exceed the mobilities they would exhibit if they were to remain rigid.

Standing wave effects in isolators have been shown to be responsible for transmissibility peaks at high frequencies. The frequencies corresponding to such peaks have been shown to increase with decreasing isolators size and increasing longitudinal wave velocity in the isolator material. The ratio of the first standing wave resonance to the fundamental resonance of a spring-mounted mass has been shown to vary as the square root of the ratio of the mounted mass to the isolator mass. Damping of the isolator (as described reasonably realistically by a frequency-independent loss factor) has been shown to decrease transmissibility and improve isolator performance at the transmissibility peaks associated with primary system and standing wave resonances, and to have little effect at other frequencies.

The isolation effectiveness of two-stage isolators consisting of a rigid mass between two resilient elements has been analyzed. For cases where effective isolation can be achieved, i.e., where both resilient elements can be made more mobile than the source and receiver, the use of two resilient elements of equal mobility has been shown to be optimum. Some experimental results have been presented which illustrate the utility of two-stage isolation.

5.2. PRACTICAL IMPLICATIONS

In dealing with vibration isolation problems one does well to keep in mind that the isolation effectiveness E obeys

$$E = \left|1 + \frac{M_I}{M_S + M_R}\right|, \tag{6}$$

and that for effective isolation one requires $|M_I| \gg |M_S+M_R|$, where M_I, M_S, M_R represent the isolator, source, and receiver mobilities, respectively. (Although the foregoing relation is strictly applicable only to massless isolators, it is also convenient to visualize the behavior of isolators with mass in terms of it.) To increase isolation effectiveness one thus must increase the isolator mobility and/or decrease the source and receiver mobilities.

The driving point mobilities of structures (which may be sources or receivers) generally fluctuate widely with changes in frequency, alternating rapidly between maxima and minima. Reduced isolation effectiveness may be expected to be associated with mobility maxima. Since increased structural damping reduces these maxima, it thus serves to increase isolation effectiveness. However, if the damping is high enough so that in a frequency range of interest the bandwidth associated with resonance of a mode is greater than the frequency spacing between modes, then the structure will respond essentially like an infinite one, and further increases in damping will not produce measurable further improvements in isolation effectiveness.

Standing wave resonances in resilient isolator elements in effect decrease isolator mobility (as sensed at the isolator terminals). Thus, one does well to use elements whose first standing wave resonance occurs at the highest possible frequency. For this reason it is desirable to choose resilient elements of small physical dimensions, and made of a material with a high longitudinal wave velocity. Since damping reduces standing wave amplitudes, damping is generally beneficial. The detrimental effect of damping at off-resonant frequencies is small for most practical materials. (Viscous damping has a marked detrimental effect at high frequencies, but the damping of actual materials is generally better represented by a frequency-independent loss factor than by a constant viscous damping coefficient, which corresponds to a loss factor which is proportional to frequency.)

For isolation at high frequencies use of a two- (or multi-) stage isolator generally is advantageous. Good performance of such an isolator requires that both resilient elements have mobilities that considerably exceed the source and receiver mobilities; it has been shown that if this relation among the mobilities is satisfied, then one obtains the best isolation by making the mobilities of the two resilient elements equal.

Use of resilient elements with equal mobilities may be generally advocated, particularly for instances where one knows relatively little about the source and receiver mobilities. If one does have full information about these latter mobilities, and if one cannot use resilient elements that are much more mobile than the source and receiver, then one may optimize design of the two-stage isolator by use of equation (18) or of a similar relation which takes better account of the fact that the source and receiver impedances are complex.

The high-frequency isolation effectiveness of well-designed two-stage isolators is very nearly proportional to the included mass. In nearly all cases the addition of a mass between two resilient elements to form a two-stage isolation system is more advantageous than attaching the mass directly to either the source or receiver and using a single resilient element. One notable exception occurs for the case of a force-source: i.e., for a source whose force amplitude output remains constant as its output velocity amplitude changes. For such a source it is best to attach the mass directly to the source.

It is important to note that the superior high-frequency performance of two-stage isolators is predicated upon the included mass remaining "rigid" at the frequencies of interest. This condition is probably not realized in many practical present installations that incorporate two stages of isolation. For example, in some engine installations an engine is resiliently mounted to a frame which in turn is spring-mounted on a substructure. The usual frame designs, however, incorporate extended structural members which exhibit modal behavior at acoustic frequencies; thus, such frames do not act as rigid masses at these frequencies and the advantages of a two-stage mounting system are lost. In many such installations it is likely that better high-frequency isolation, plus perhaps a saving in weight, may be obtained essentially by replacing the frame with compact masses.

Finally, it should be pointed out that the inclusion of a mass to change a one-stage to a two-stage system introduces an additional resonance. This resonance is associated essentially with the added mass as supported by the resilient elements, and must be kept as far as possible below the frequency range where good isolation is desired. (This is accomplished, of course, by choosing soft enough isolators and large enough masses.) This concept appears to be violated in many practical situations: e.g. where a machine is separated from its baseplate by a relatively thin layer of rubber or cork, with the base plate mounted on conventional isolators. Such an arrangement may result in a severe resonance of the baseplate—rubber layer—machine system in the high-frequency range, and thus in much reduced isolation effectiveness.

At high frequencies isolation systems consisting of several stages have theoretical advantages over two-stage systems. However, each added mass contributes an additional resonance and decreases the mobility of the resilient elements between masses (for a given total low-frequency mobility or static deflection). Thus, multi-stage systems introduce many resonances, including some at relatively high frequencies, and therefore tend to lose isolation effectiveness at these frequencies. In addition, of course, multi-stage systems necessarily are heavy and extremely difficult to construct practically for multi-axis isolation.

REFERENCES

1. S. H. Crandall 1958 *Mechanical Impedance Methods*, ed. by R. Plunkett. New York: ASME. Section 2: Impedance and mobility analysis of lumped parameter systems.
2. A. O. Sykes 1956 *Shock and Vibration Instrumentation*. New York: ASME. The evaluation of mounts isolating nonrigid machines from nonrigid foundations.
3. J. C. Snowdon 1961 *J. acoust. Soc. Am.* **33**, 1466. Reduction of the response to vibration of structures possessing finite mechanical impedance.
4. D. Muster and R. Plunkett 1960 *Noise Reduction*, ed. by L. L. Beranek. New York: McGraw-Hill. Chapter 18: Isolation of vibrations.
5. G. M. Coleman 1958 *Mechanical Impedance Methods*, ed. by R. Plunkett. New York: ASME, section 5. Methods for the measurement of mechanical impedance.
6. F. Schloss 1963 *David Taylor Model Basin Report* 1584. Recent advances in the measurement of structural impedance.
7. F. Schloss 1963 *David Taylor Model Basin Report* 1698. Inherent limitations of accelerometers for high-frequency vibration measurements.
8. Anon. *Wilcoxon Research Bulletin No. 4. Bethesda, Maryland.* Inherent limits of measurable impedances.
9. D. V. Wright 1958 *Mechanical Impedance Methods*, ed. by R. Plunkett New York: ASME. Section 3: Impedance analysis of distributed mechanical systems.
10. E. J. Skudrzyk 1958 *J. acoust. Soc. Am.* **30**, 1140. Vibrations of a system with a finite or an infinite number of resonances.
11. E. J. Skudrzyk 1959 *J. acoust. Soc. Am.* **31**, 68. Theory of noise and vibration insulation of a system with many resonances.
12. R. H. Lyon 1963 *J. acoust. Soc. Am.* **35**, 797(A). Rôle and limitations of damping in the control of vibration and noise.
13. E. E. Ungar 1964 *Mechanical Design and Systems Handbook*, ed. by H. A. Rothbart. New York: McGraw-Hill. Section 6: Mechanical vibrations.
14. E. E. Ungar 1964 *Air Force Flight Dynamics Laboratory FDL–TDR*–64–98. Energy dissipation at structural joints; mechanisms and magnitudes.
15. M. A. Heckl 1961 *Bolt Beranek and Newman Inc. Report No.* 774. Compendium of impedance formulas.
16. A. O. Sykes 1960 *Noise Control* **6**, 115. Vibration isolation when machine and foundation are resilient and wave effects occur in the mount.

17. J. C. Snowdon 1965 *J. Sound Vib.* **2**, 175. Rubberlike materials, their internal damping and rôle in vibration isolation.
18. M. Harrison, A. O. Sykes and M. Martin 1952 *J. acoust. Soc. Am.* **24**, 62. Wave effects in isolation mounts.
19. E. E. Ungar 1963 *J. Engng Ind. Series B Trans. ASME* **85**, 243. Wave effects in viscoelastic leaf and compression spring mounts.
20. C. T. Molloy 1958 *Mechanical Impedance Methods*, ed. by R. Plunkett. New York: ASME. Section 4: Four pole parameters in vibration analysis.
21. M. Exner 1952 *Acustica* **2**, 213. Schalldämmung durch Gummi- und Stahlfedern.
22. J. C. Snowdon 1956 *Acustica* **6**, Beiheft 1, 118. The reduction of structure-borne noise.