Analysis of the Probability of Successful Packet Delivery

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In this analysis, we find an expression for the probability of successful packet delivery considering RSU scheduling, packet collision and retransmission. The expression for probability can be split into 2 parts - namely probability of successful channel access, and then followed by the probability that the RSU is on and can receive a packet.

1 Original Model

First, we define $P(l, n, w, k)$ $(1 \leq l \leq w$ and $1 \leq k \leq n)$ as the probability that n vehicles select backoffs from a contention window of w slots, $(l-1)$ empty slots pass before the first transmission attempt, and k vehicles transmit in the l^{th} slot.

$$P(l,n,w,k) = \left(1 - \frac{l-1}{w}\right)^n . \binom{n}{k} . \left(\frac{1}{w-l+1}\right)^k . \left(1 - \frac{1}{w-l+1}\right)^{n-k} \tag{1}$$

Here, the first term in the probability of no transmission in the first $l-1$ slots, the second term is the number of ways we can choose k vehicles out of n vehicles, the third term in the probability of k vehicles choosing uniformly at random the l^{th} slot to transmit out of the remaining $w-l+1$ slots, while the final term is the probability that the remaining $n-k$ vehicles choose some other slot for transmission. Next, we define $X(t, w, n)$ as the mean number of successful transmissions during an interval, given that there are t slots left in this interval, at most w contention slots left at the vehicles' back-off counter and n vehicles have not attempted to transmit yet. If we take the transmission time (given in terms of number of time slots) for successful transmission and collision as s and c respectively,

$$X(t,w,n) = \sum_{l=1}^{min(w,t)} \Big\{ P(l,n,w,1)[1 + X(t-l+1-s, w-l, n-1)] + \sum_{k=2}^{n} P(l,n,w,k) X(t-l+1-c, w-l, n-k) \Big\} \tag{2}$$

The above equation computes the expectation of the number of successful transmissions, with $P(l, n, w, 1$ being the probability that only one vehicle transmits in the l^{th} slot, a success, and $P(l, n, w, k)$ being the probability that more than k $(k > 1)$ vehicles transmit in the l^{th} time slot, a failure due to collision. In the former case, the number of successful transmissions is the mean number of successful transmissions of the remaining $n-1$ vehicles in the leftover $t-l+1-s$ time slots (since l slots have elapsed and s slots are used up during the successful transmission, preventing any other vehicle from transmitting) with at most $w-l$ contention slots left at the vehicles' backoff counter. The latter case deals with a failure due to collision, in which case the number of successful transmissions is equal to the number of successes in the remaining $t-l+1-c$ time slots (c is used instead of s because of the collision) by the remaining $n-1$ vehicles. Thus, $X(T, W, N)$ gives us the number of successful transmissions of N vehicles in T time slots, with a maximum contention window size of W.

2 Incorporating Hidden Terminals

The earlier analysis was done under the assumption that no hidden terminals exist. If we were to incorporate hidden terminals as well into our model, $X(t,w,n)$ is rewritten as

$$X(t,w,n) = \sum_{l=1}^{\min(w,t)} \Bigg\{ P(l,n,w,1)[1 + X(t-l+1-s, w-l, n-1)] \prod_r (1 - P_s(l,w)) \\ + P(l,n,w,1) X(t-l+1-c, w-l, n-1) \sum_r P_s(l,w) \\ + \sum_{k=2}^{n} P(l,n,w,k) X(t-l+1-c, w-l, n-k) \Bigg\} \quad (3)$$

where $P_s(l,w)$ is the probability of there being an ongoing transmission at one of the r neighbouring RSUs during time slot l, out of a contention window of size w (Assumption: Max contention window size for all RSUs is W, time slots are all in sync so time slot l will happen at the same time everywhere, leading to similar slot updates everywhere). The original equation has been rewritten to allow a successful transmission (ie., $P(l,n,w,1)$) only if there is no node currently transmitting at a neighbouring RSU (Hence the product of probabilities of no transmission at all neighbouring nodes). If there is a concurrent transmission happening, then the transmission for the node under consideration fails, and is accounted as a collision, giving rise to the similar $X(t-l+1-c, w-l, n-1)$ as in the $P(l,n,w,k)$ case (This happens if there is a simultaneous transmission at even one neighbouring RSU, and hence the summation over r). As $P_s(l,w)$ is the probability of a transmission happening (at a neighbouring RSU) **during** the l^{th} slot from a contention window of w slots,

$$P_s(l,w) = \sum_{i=0}^{s} \sum_{j=0}^{N} P_t(l-i, j, w) \quad (4)$$

(where $P_t(l,n,w)$ is the probability of a successful transmission starting **in** the l^{th} slot when there are n vehicles, selecting backoffs from a contention window of w slots), since a successful transmission lasts for the following s time slots. Now, $P_t(l,n,w)$ is given by:

$$\begin{aligned} P_t(l,n,w) &= \binom{n}{1} \frac{1}{w-l+1} \left(1 - \frac{1}{w-l+1}\right)^{n-1} P\,(\text{no successful transmission in the previous } s \text{ slots}) \\ &= \frac{n}{w-l+1} \left(1 - \frac{1}{w-l+1}\right)^{n-1} (1 - P\,(\text{successful transmission in the previous } s \text{ slots})) \\ &= \frac{n}{w-l+1} \left(1 - \frac{1}{w-l+1}\right)^{n-1} \left(1 - \sum_{i=1}^{s} P\,(\text{successful transmission in slot } l-i)\right) \\ &= \frac{n}{w-l+1} \left(1 - \frac{1}{w-l+1}\right)^{n-1} \left(1 - \sum_{i=1}^{s} P_t(l-i, n+1, w+i)\right) \end{aligned} \quad (5)$$

The above equation arises as a result of choosing one of n vehicles, and choosing the l^{th} slot uniformly at random from the remaining slots. Thus,

$$\begin{aligned} P_s(l,w) &= \sum_{i=0}^{s} \sum_{j=0}^{N} P_t(l-i, j, w) \\ &= \sum_{i=0}^{s} \sum_{j=0}^{N} \frac{j}{w-l+i+1} \left(1 - \frac{1}{w-l+i+1}\right)^{j-1} \left(1 - \sum_{k=1}^{s} P_t(l-i-k, j+1, w+i)\right) \end{aligned} \quad (6)$$

(Another alternative, simpler non-recursive formulation I came up with is (I don't know if the two are equivalent, or if either is correct) assuming n is the no. of vehicles that have already transmitted, instead of it being the no. of vehicles left to transmit, as above. Then,

$$P_t(l,n,w) = \begin{cases} \left(\frac{l-1-s}{w}\right)^n \left(1-\frac{s}{w-l+s}\right)^{N-n} \binom{N-n}{1}\frac{1}{w-l+1}\left(1-\frac{1}{w-l+1}\right)^{N-n-1} & l > s \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Here, we say that the n vehicles should have transmitted in the first $l-1-s$ slots, the following s slots shouldn't have any transmission by the remaining $N-n$ vehicles, then we choose 1 out of $N-n$ vehicles and make it transmit in the l^{th} slot, and finally require the remaining $N-n-1$ vehicles to not transmit in the l^{th} slot. If $l <= s$, then whatever be the value of n, it is impossible for the transmission to have ended by the l^{th} slot, in which case a successful transmission cannot start in the l^{th} slot. Hence, the value is zero. Now,

$$\begin{aligned} P_s(l,w) &= \sum_{i=0}^{s}\sum_{j=0}^{N} P_t(l-i,j,w) \\ \implies P_s(l,w) &= \begin{cases} \sum_{i=0}^{s}\sum_{j=0}^{N}\left(\frac{l-1-s}{w}\right)^j \left(1-\frac{s}{w-l+s}\right)^{N-j} \binom{N-j}{1}\frac{1}{w-l+1}\left(1-\frac{1}{w-l+1}\right)^{N-j-1} & l > s \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{8}$$

Here we don't end up with a recursive formulation, and it is also $O(sN)$ as opposed to $O(s^2N)$ for the previous formulation.)
Now, if we just want to model the success probability of a packet without retransmissions, we can directly compute

$$P_{succ} = \frac{X(T,W,N)}{N}$$

3 Including Retransmissions

Now, we take into account re transmissions due to collision. Here, we compute the probability of successful packet delivery by considering K transmission intervals. Assume at the start of the first analysis, there are N vehicles attempting to transmit N packets (and contention window of W). We now track the N packets that were attempted for transmission in the first slot, across K interval, each with T slots.
Define x_k as the number of packets of the initially considered N packets that get delivered successfully in the k_{th} time interval, let's denote $X(T,W,N)$ as X for simplicity of notation.

$$x_1 = X$$

To find x_2, we assume that N vehicles again transmit in the $2nd$ interval with contention window W, which is a reasonable assumption. Now, x_1 packets got delivered successfully in the $1st$ time interval, so $N-x_1$ packets will be retransmitted in this time interval. So, the number of packets of the initially considered N packets that get delivered successfully in the 2_{nd} time interval is given by:

$$x_2 = \left(\frac{N-x_1}{N}\right)X$$

Generalizing this, we get

$$x_k = \frac{\left(N - \sum\limits_{i=1}^{k-1} x_i\right)}{N}X$$

We see that $x_{i+1} = (N - x_1 - x_2 - \cdots - x_i)\,X/N$. Replacing x_i with $(N - x_1 - x_2 - \cdots - x_{i-1})\,X/N$, we get

$$
\begin{aligned}
x_{i+1} &= \left(\frac{N^2 - Nx_1 - \cdots - Nx_{i-1} - (N - x_1 - x_2 - \cdots - x_{i-1})\, X}{N}\right)\frac{X}{N} \\
&= \frac{(N-X)\, X}{N^2}\,(N - x_1 - x_2 - \cdots - x_{i-1}) \\
&= \frac{(N-X)\, X}{N^2}\left(x_i \frac{N}{X}\right) = \left(\frac{N-X}{N}\right) x_i
\end{aligned}
$$

Since $x_1 = X$, we obtain

$$
x_k = \left(\frac{N-X}{N}\right)^{k-1} X \tag{9}
$$

Now, assuming the maximum number of attempts for any packet is K, we can find out the number of packets (of the initially considered N packets) delivered in successfully X' as:

$$
X' = \sum_{k=1}^{K} x_k = \sum_{k=1}^{K} \frac{\left(N - \sum\limits_{i=1}^{k-1} x_i\right)}{N} X = \sum_{k=1}^{K} \left(\frac{N-X}{N}\right)^{k-1} X
$$

We see that $X' = X\left[1 + \left(\frac{N-X}{N}\right) + \left(\frac{N-X}{N}\right)^2 + \cdots + \left(\frac{N-X}{N}\right)^{K-1}\right]$, which is the sum of geometric progression, given by $S_n = \frac{a(1-r^n)}{1-r}$. Hence,

$$
X' = \frac{X\left(1 - \left(\frac{N-X}{N}\right)^K\right)}{1 - \left(\frac{N-X}{N}\right)} = \frac{N^K - (N-X)^K}{N^{K-1}}
$$

The probability of successful packet delivery can be given as:

$$
P_{succ} = \frac{X'}{N} = \frac{N^K - (N-X)^K}{N^K} = 1 - \left(\frac{N-X}{N}\right)^K \tag{10}
$$

It is interesting to note that we could have arrived at the same result if we were to track a single packet instead, over K retransmissions. In this case, in each retransmission, only X of the N packets are successfully transmitted, giving us a failure probability of $(N-X)/N$ for each retransmission attempt. Thus, for a single packet to fail after K retransmissions, it must fail on each retransmission, giving a total failure probability of

$$
P_{fail} = \left(\frac{N-X}{N}\right)^K
$$

$$
\implies P_{succ} = 1 - P_{fail} = 1 - \left(\frac{N-X}{N}\right)^K
$$

We could also have computed x_k as the expected number of packets to succeed on the k^{th} try, after failing for the first $k-1$ tries. Thus,

$$
x_k = \left(\frac{N-X}{N}\right)^{k-1}\left(\frac{X}{N}\right)\cdot N = \left(\frac{N-X}{N}\right)^{k-1} X
$$

To take into account RSU scheduling, we assume that the vehicles arrive as an exponential distribution. So, it's reasonable to assume that the probability of an RSU being OFF is also follows the same distribution. So, we define:

$$
P(\text{RSU} = \text{OFF}) = \lambda e^{-\lambda N} \tag{11}
$$

where there are N vehicles in the system. Since $P\,(\text{RSU} = \text{OFF} | N = 0) = \lambda$, and we know that the RSU is shut off when there are no vehicles, we get $\lambda = 1$. Thus, $P(\text{RSU} = \text{ON}) = 1 - e^{-N}$. Since the two events, successful packet transmission and RSU being on/off are independent, we can simply multiply the two probabilities to obtain the final probability of successful packet delivery. Thus,

$$P_{succ-new} = P_{succ} \times P(\mathrm{RSU} = \mathrm{ON})$$

$$\implies P_{succ-new}\left(T, W, N, K\right) = \left(1 - \left(\frac{N - X\left(T, W, N\right)}{N}\right)^{K}\right)\left(1 - e^{-N}\right)$$