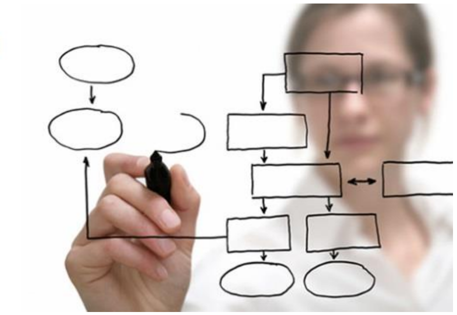
CORK INSTITUTE OF TECHNOLOGY
INSTITIÚID TEICNEOLAÍOCHTA CHORCAÍ

Computing@CIT

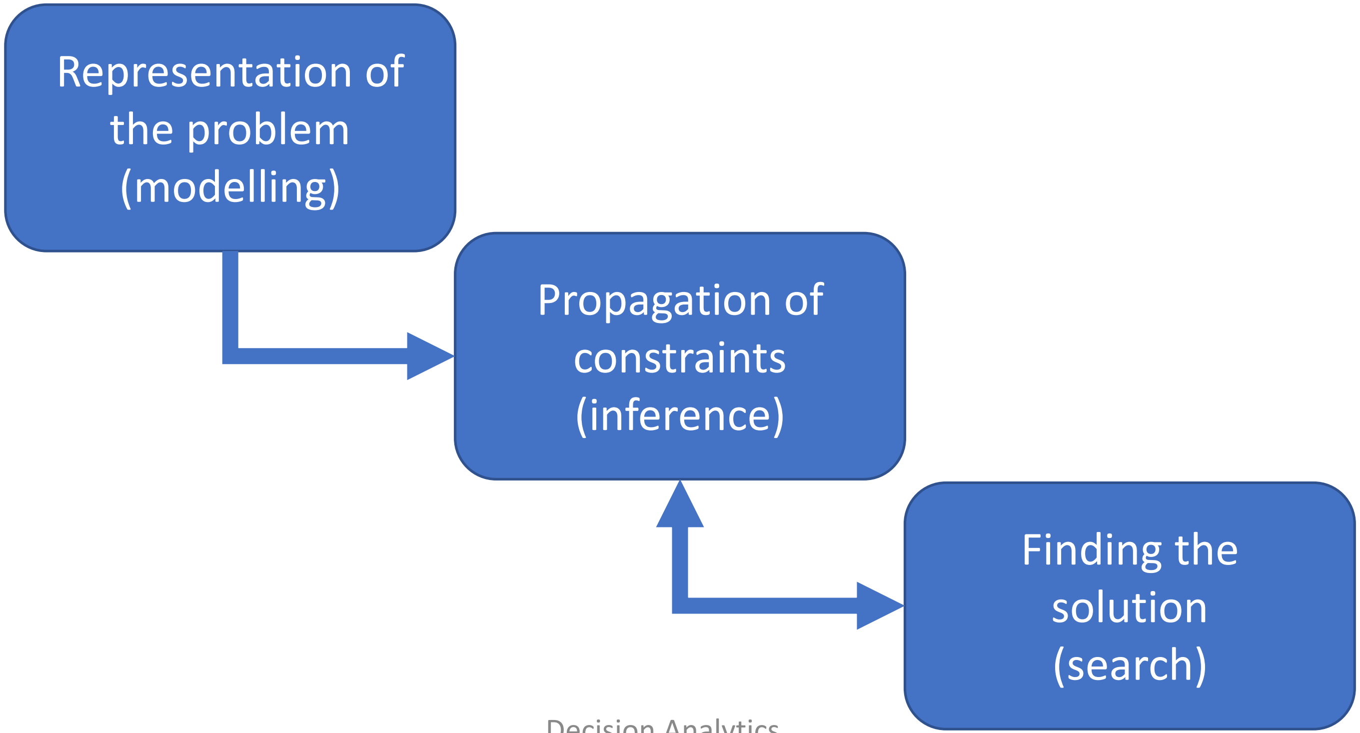


Decision Analytics

Lecture 16: Backtracking search

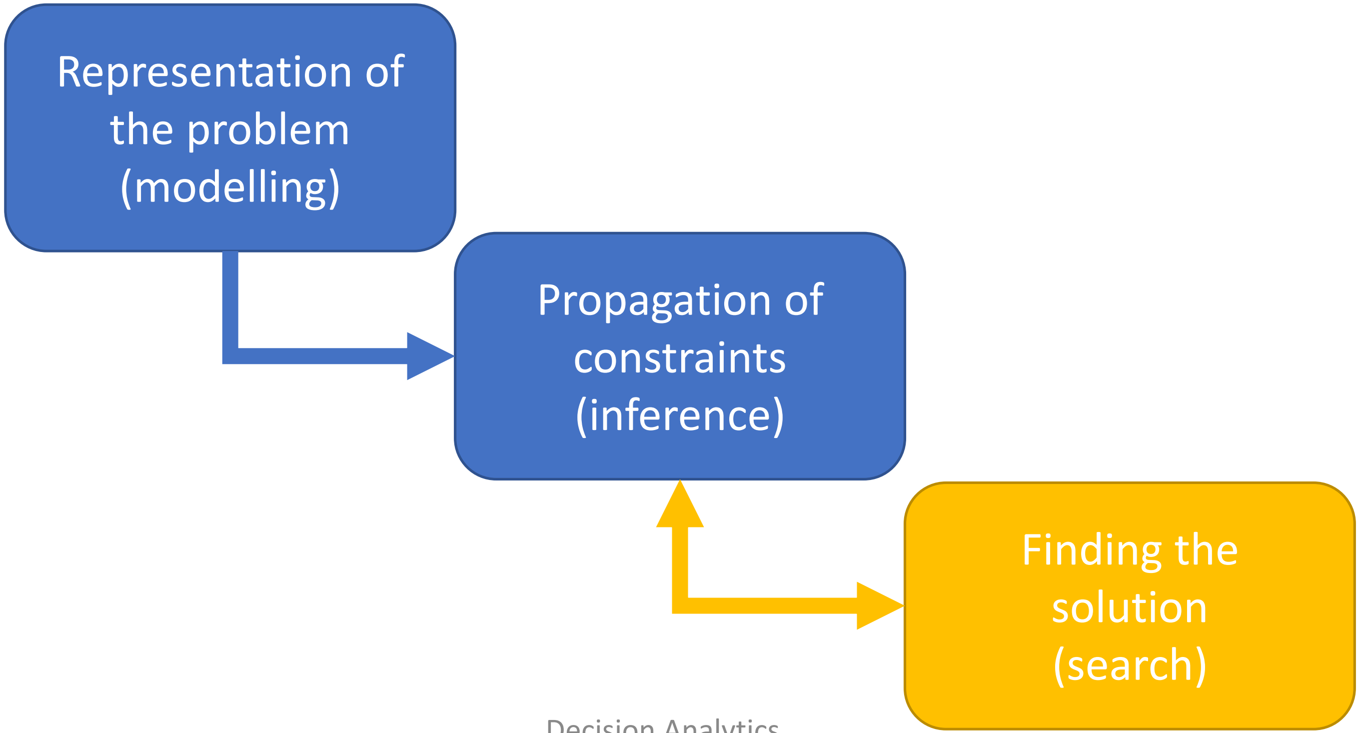
Constraint Programming

- Constraint Programming (CP) is a paradigm for solving combinatorial constraint satisfaction and constrained optimisation problems using a combination of modelling, propagation, and search



Constraint Programming

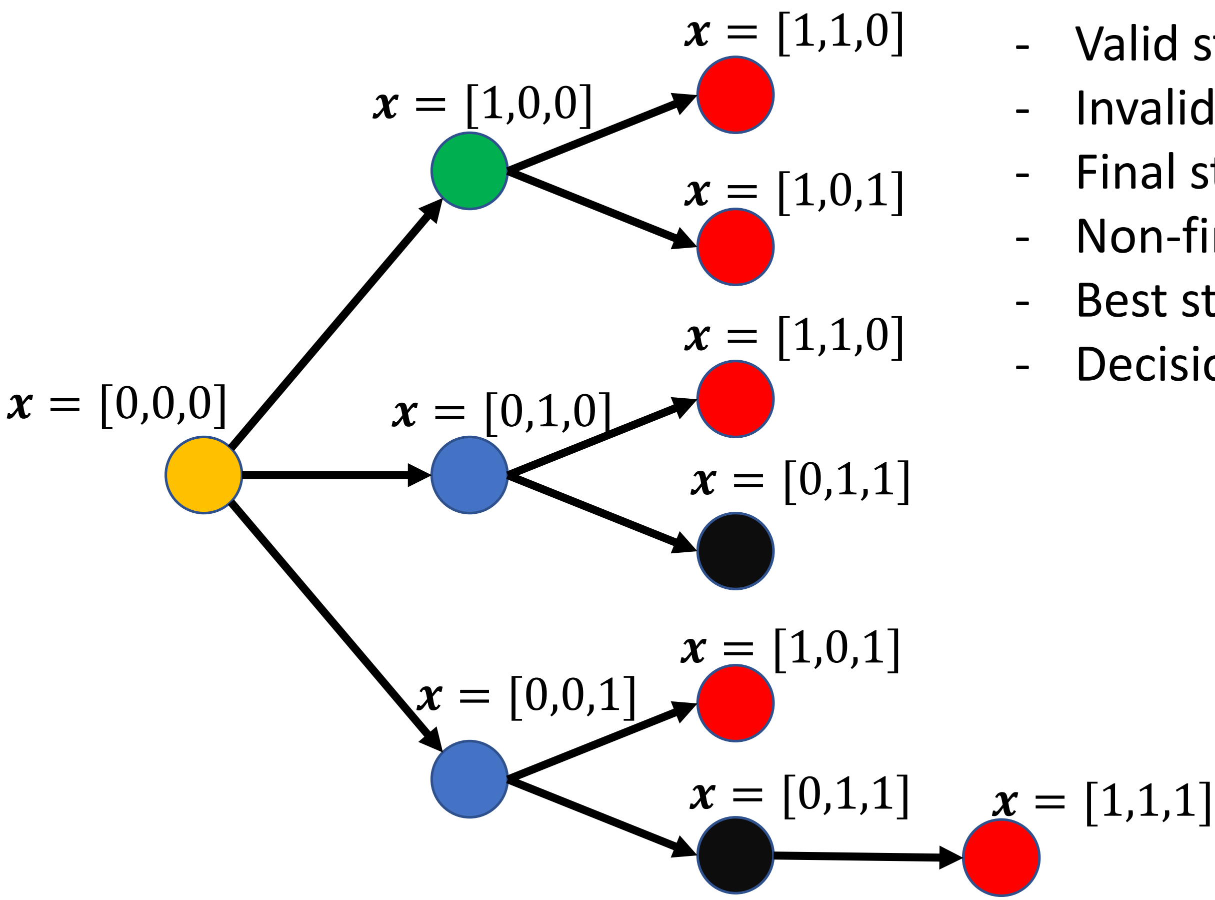
- Constraint Programming (CP) is a paradigm for solving combinatorial constraint satisfaction and constrained optimisation problems using a combination of modelling, propagation, and search
- This lecture is about **backtracking search**, which very closely linked with constraint propagation



Constraint propagation

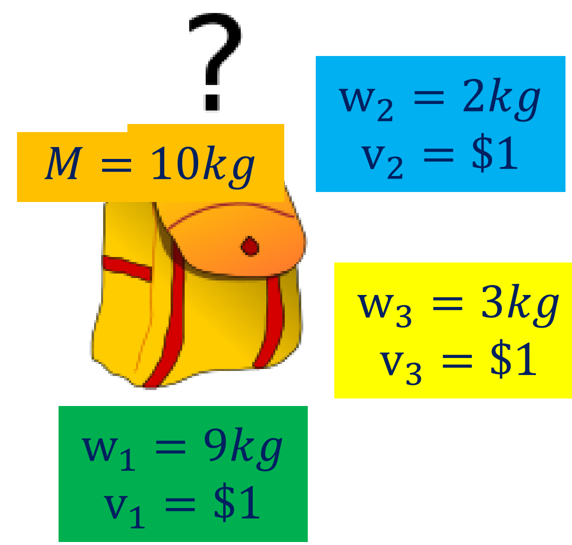
- The outcome of constraint propagation is a tightened network $N = (X, D, C)$
- Unless all domains have become instantiated in this process, i.e. $|D(x_i)| = 1$ for all variables x_i, we need to find the solution by trying out all remaining instantiations
- The number of remaining valid instantiations is $\prod_i |D(x_i)|$, which in general is exponential in the number of variables
- As we have seen, bringing the number of remaining instantiations down (for instance by enforcing higher order consistencies) is also exponential in runtime and/or space requirements
- Therefore, we are looking for a compromise in terms of constraint propagation performance and search performance
- As this problem is NP-hard, there is no known “rule” how this can be achieved

The search tree for the knapsack problem



We explored the following concepts:

- Initial state
- Valid state
- Invalid state
- Final state
- Non-final state
- Best state
- Decision making



Branching constraints

- How can we formalise this consistent with the concepts from constraint propagation?

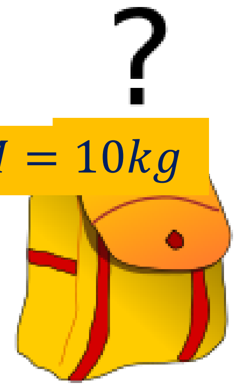
- In every node of the search tree we need to make a **decision** what variable we want to explore further and what value we want to assign during this exploration
- More generally, we need to decide on a **branching constraint** b that we want to explore by looking at the network $N' = (X, D, C \cup \{b\})$ and see if it can be tightened to a fully instantiated network
- Note, that a generic constraint does not need to be limited to just 1 variable, nor does it need to be assigning only 1 value to that variable

- This decision making process is called a **branching strategy**

Branching strategy

- Every node in the tree is defined by the sequence of branching constraints $p = \{b_1, \dots, b_j\}$, which is the path of length j from the root to the node
- We call b_i the branching constraint **posted** at level i
- A node $p = \{b_1, \dots, b_j\}$ is extended by adding the branches
$$p \cup \left\{b_{j+1}^1\right\}, \dots, p \cup \left\{b_{j+1}^k\right\}$$
- As we are usually trying to find a solution in each of the branches in sequence, we try to apply an **ordering heuristic** to make sure that the most promising branch is explored first

Example: Knapsack problem



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Example: Knapsack problem

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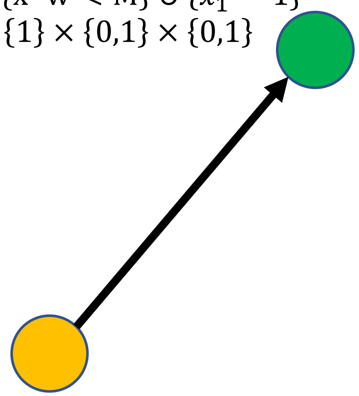

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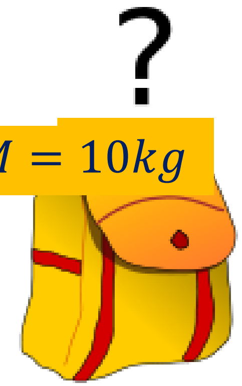

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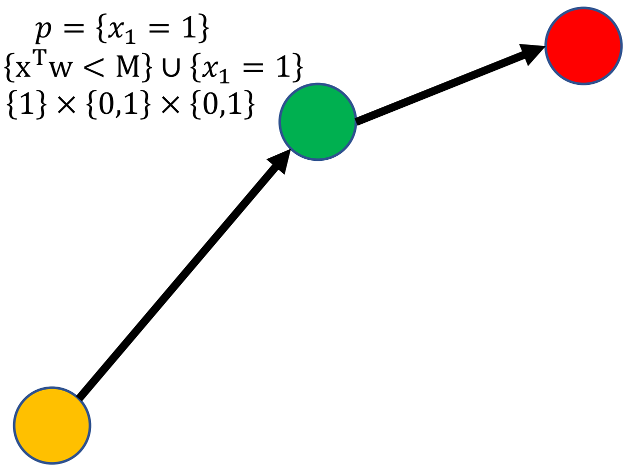

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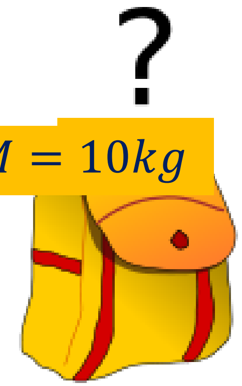

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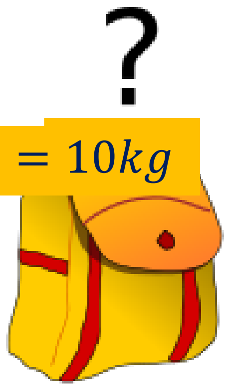

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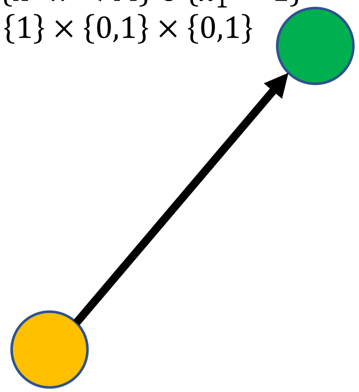

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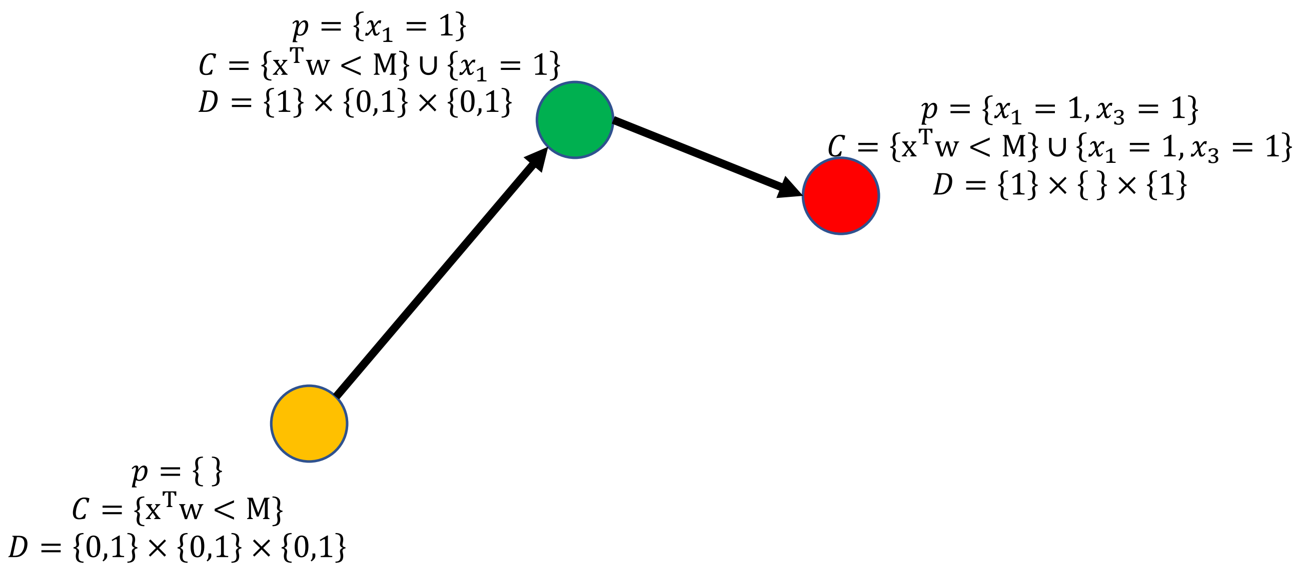

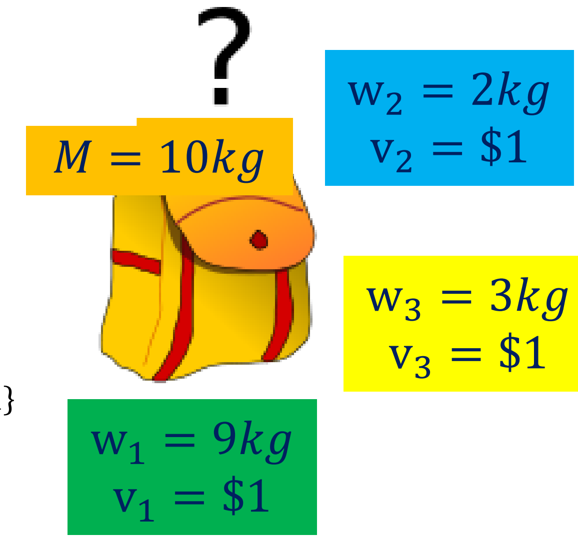

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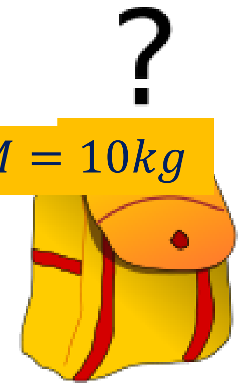

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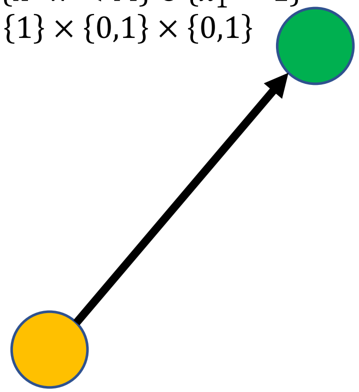

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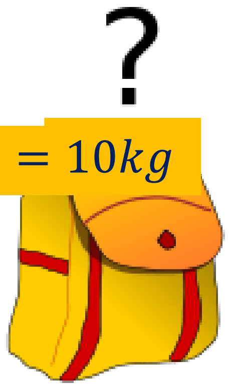

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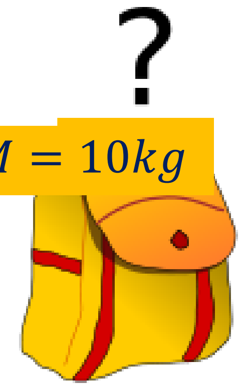

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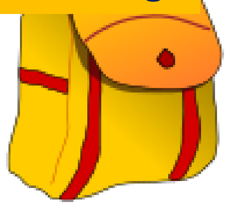

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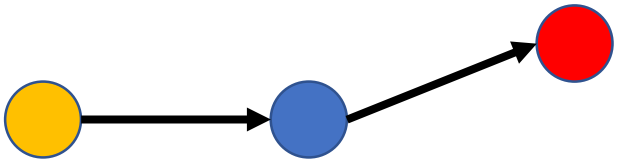

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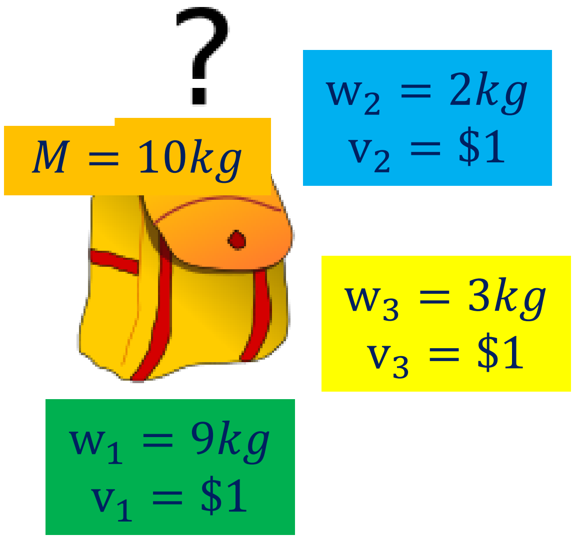

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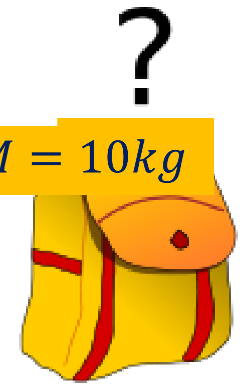

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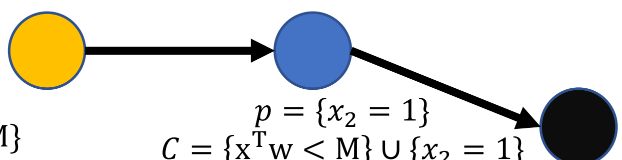

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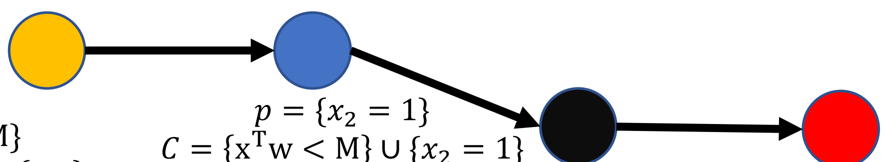

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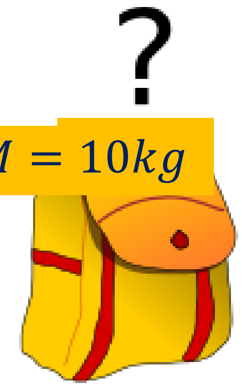

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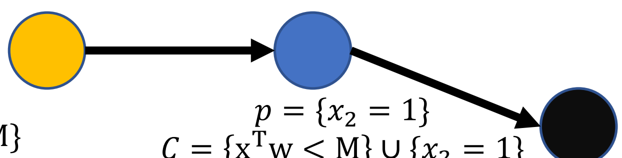

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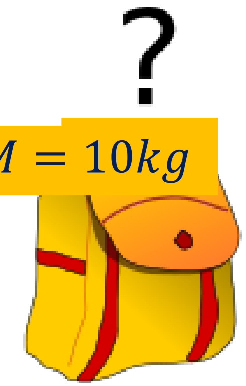

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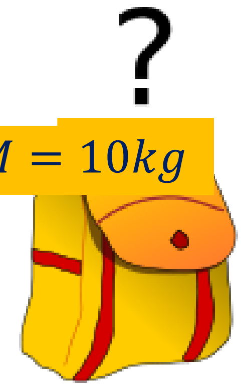

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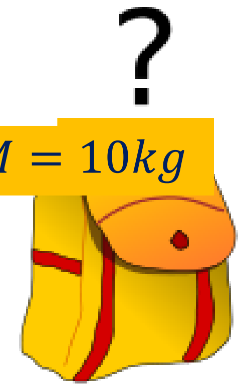

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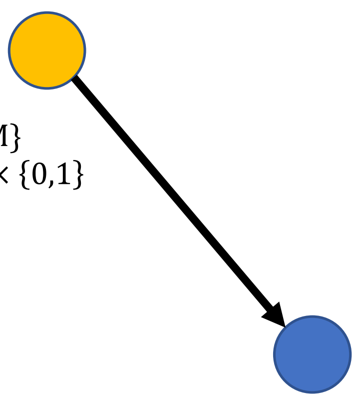

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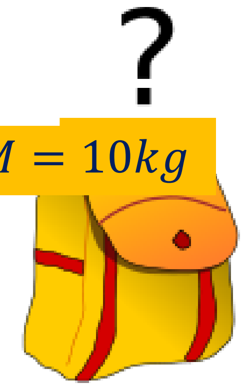

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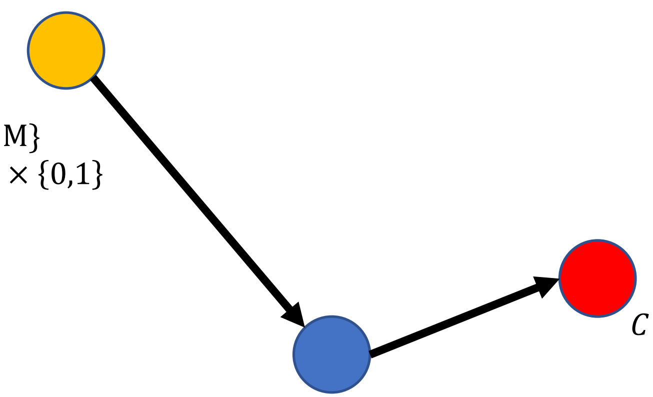

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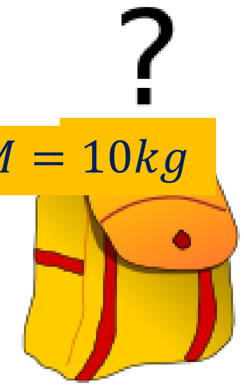

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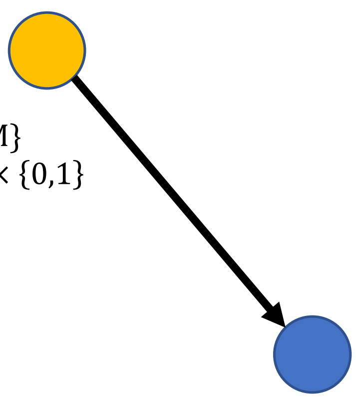

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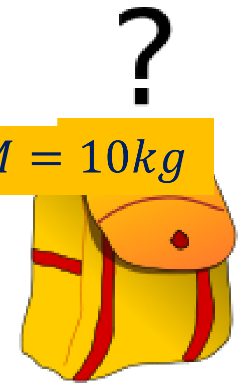

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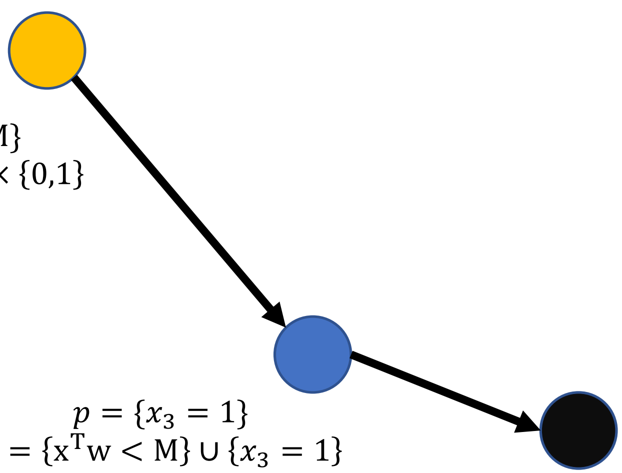
$\mathrm{w}_1 = 9kg$
$\mathrm{v}_1 = \$1$

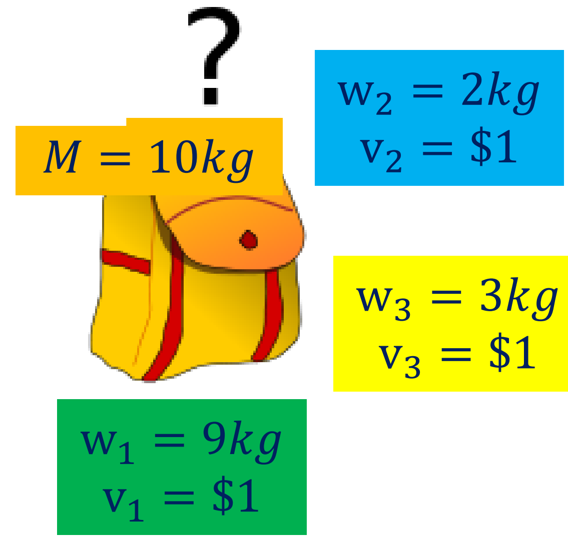


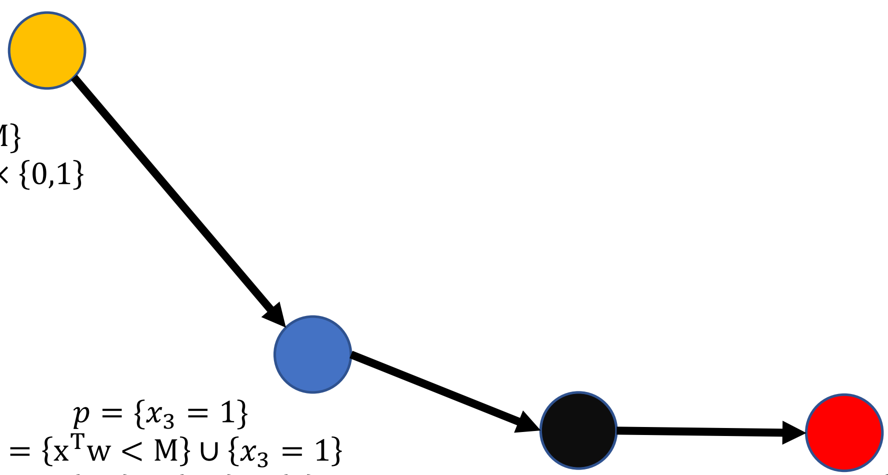
$p = \{\,\}$
$C = \{\mathrm{x^T w < M}\}$
$D = \{0,1\} \times \{0,1\} \times \{0,1\}$

$p = \{x_3 = 1\}$
$C = \{\mathrm{x^T w < M}\} \cup \{x_3 = 1\}$
$D = \{0,1\} \times \{0,1\} \times \{1\}$

$p = \{x_3 = 1, x_2 = 1\}$
$C = \{\mathrm{x^T w < M}\} \cup \{x_3 = 1, x_2 = 1\}$
$D = \{0,1\} \times \{1\} \times \{1\}$

Example: Knapsack problem





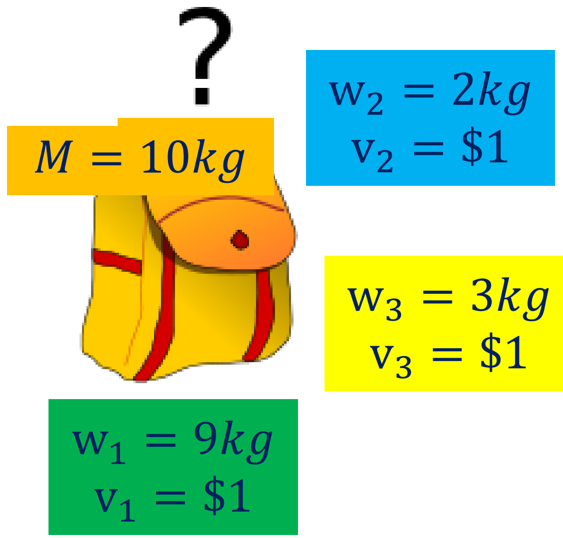
$p = \{\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\}$
$D = \{0,1\} \times \{0,1\} \times \{0,1\}$

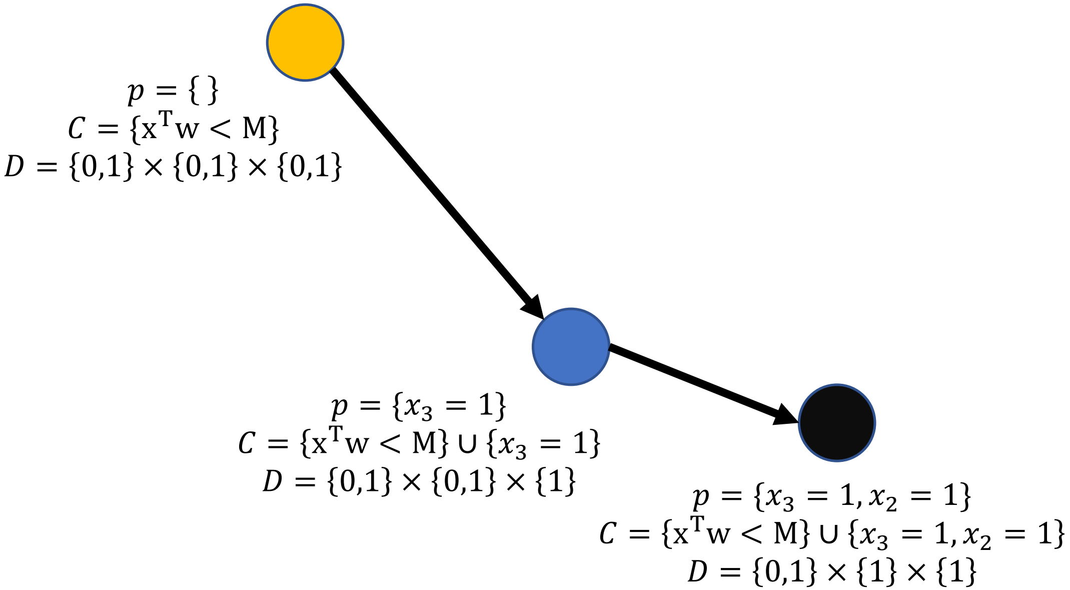
$p = \{x_3 = 1\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\} \cup \{x_3 = 1\}$
$D = \{0,1\} \times \{0,1\} \times \{1\}$

$p = \{x_3 = 1, x_2 = 1\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\} \cup \{x_3 = 1, x_2 = 1\}$
$D = \{0,1\} \times \{1\} \times \{1\}$

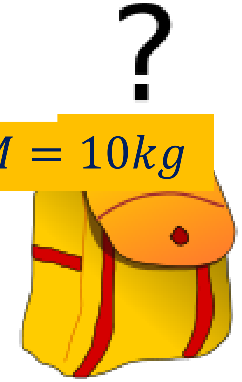
$p = \{x_3 = 1, x_2 = 1, x_1 = 1\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\} \cup \{x_3 = 1, x_2 = 1, x_1 = 1\}$
$D = \{\} \times \{\} \times \{\}$

Example: Knapsack problem





Example: Knapsack problem

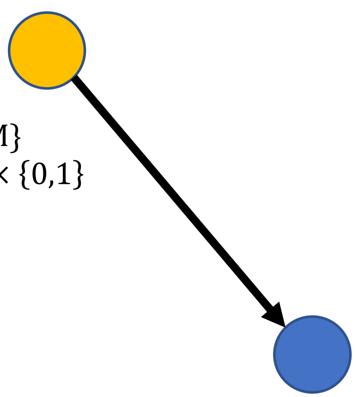


$M = 10kg$

$\mathrm{w}_2 = 2kg$
$\mathrm{v}_2 = \$1$

$\mathrm{w}_3 = 3kg$
$\mathrm{v}_3 = \$1$

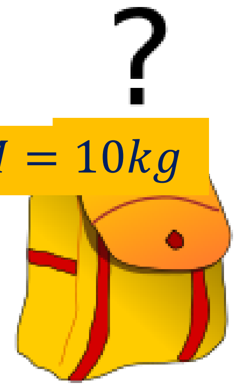
$\mathrm{w}_1 = 9kg$
$\mathrm{v}_1 = \$1$



$p = \{\,\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\}$
$D = \{0,1\} \times \{0,1\} \times \{0,1\}$

$p = \{x_3 = 1\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\} \cup \{x_3 = 1\}$
$D = \{0,1\} \times \{0,1\} \times \{1\}$

Example: Knapsack problem



$M = 10kg$

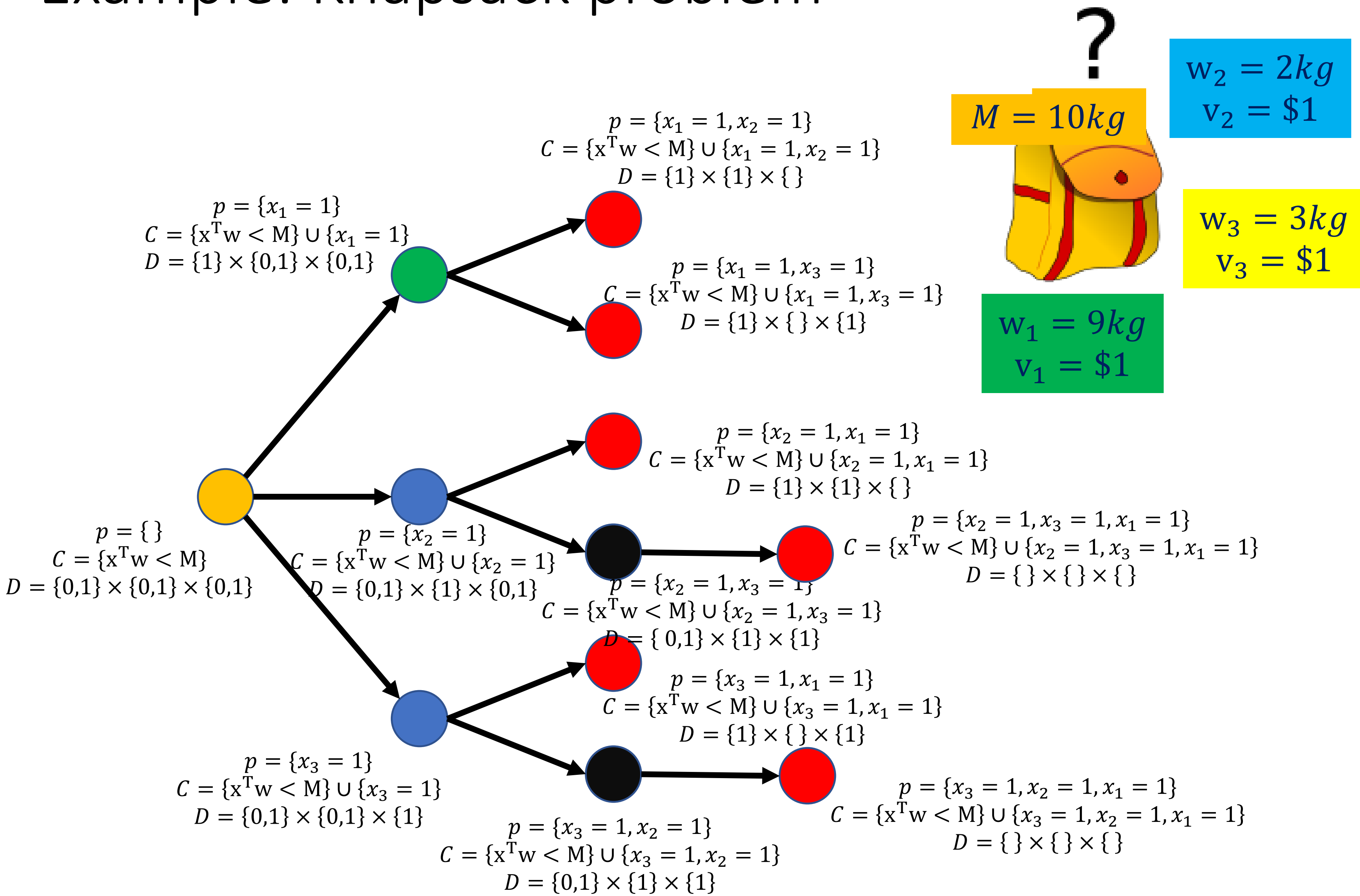
$\mathrm{w}_2 = 2kg$
$\mathrm{v}_2 = \$1$

$\mathrm{w}_3 = 3kg$
$\mathrm{v}_3 = \$1$

$\mathrm{w}_1 = 9kg$
$\mathrm{v}_1 = \$1$

$p = \{\,\}$
$C = \{\mathrm{x}^{\mathrm{T}}\mathrm{w} < \mathrm{M}\}$
$D = \{0,1\} \times \{0,1\} \times \{0,1\}$

Example: Knapsack problem



Branching strategy

- In practice most branching strategies post unary constraints only, i.e. a single variable $x_i \in X$ is chosen to be explored further
- The second decision is then how to restrict the domain $D(x_i)$ of the variable
- Popular decision strategies are for this are
  - **Enumeration**: create branches for each value in the domain $D(x_i) = \{v_1, \dots, v_k\}$
    $$b_{j+1}^1 = \{x_i = v_1\}, \dots, b_{j+1}^k = \{x_i = v_k\}$$
  - **Binary choice points**: branch into the assignment of a variable to a value v vs. the assignment to every other value
    $$b_{j+1}^1 = \{x_i = v\} \qquad b_{j+1}^2 = \{x_i \neq v\}$$
  - **Domain splitting**: branch into two sub-branches covering a portion of the domain each
    $$b_{j+1}^1 = \{x_i \leq v\} \qquad b_{j+1}^2 = \{x_i > v\}$$
- Obviously, all these three strategies are equivalent for SAT problems

Variable ordering heuristics

- The first branching decision for posting unary constraints is which variable to choose
- A common choice is to look at the size of the domain $|D(x_i)|$ and select the variable with the
  - Smallest remaining domain size, which is the one that probably has the highest chance of being reduced to either 0 or 1
  - Largest remaining domain size, which is the one that needs to be broken down first in order to track down the solution
  - The lowest lower/highest upper bound, which are the ones that could prevent consistencies from propagating further
- As with all heuristics, none is provably superior to the other and it depends on the application

Branching strategy

- The OR Tools provide some limited control mechanism over the branching strategy heuristics for selected variables

```
model.AddDecisionStrategy(variables,
                          cp_model.CHOOSE_FIRST,
                          cp_model.SELECT_MIN_VALUE)
```

```
CHOOSE_FIRST
CHOOSE_LOWEST_MIN
CHOOSE_HIGHEST_MAX
CHOOSE_MIN_DOMAIN_SIZE
CHOOSE_MAX_DOMAIN_SIZE
```

```
SELECT_MIN_VALUE
SELECT_MAX_VALUE
SELECT_LOWER_HALF
SELECT_UPPER_HALF
```

Thank you for your attention!