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Decision Analytics

Lecture 20-21: Simplex algorithm

Linear program

- A linear program in **standard form** is seeking to maximise a linear objective function

$$f = c_0 + c^T x$$

- Subject to the linear constraints

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

- We will also assume that all $b_i \geq 0$, which can be easily achieved by multiplying the i-th row with -1 where necessary

Tableau notation

- To describe the linear program we will collect all the linear equations (including the objective function) in a single matrix as follows

$$
\begin{array}{c|ccc|c}
-f & x_1 & \cdots & x_n & \\
\end{array}
$$

$$
\left(\begin{array}{c|ccc|c}
1 & c_1 & \cdots & c_n & -c_0 \\
\hline
0 & a_{11} & \cdots & a_{1n} & b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_{m1} & \cdots & a_{mn} & b_m
\end{array}\right)
$$

Tableau notation

- For example, the linear program

$$f = 3 + 2x_1 - 5x_2 + x_3$$

- Subject to

$$\begin{aligned} 5x_1 - 2x_2 - x_3 &= 4 \\ -3x_1 + 2x_2 - 4x_3 &= 5 \end{aligned}$$

- Will be denoted as

$$\begin{array}{ccccc} \boldsymbol{-f} & \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} & \\ \end{array}$$
$$\left(\begin{array}{c|cccc|c} 1 & 2 & -5 & 1 & -3 \\ \hline 0 & 5 & -2 & -1 & 4 \\ 0 & -3 & 2 & -4 & 5 \end{array}\right)$$

Linear equation systems

- Two linear equation systems

$$
\begin{aligned}
Ax &= b \\
\bar{A}x &= \bar{b}
\end{aligned}
$$

- are considered equivalent, if they have the same solution, i.e. if

$$\{x: Ax = b\} = \{x: \bar{A}x = \bar{b}\}$$

Pivoting

- Let's assume the element of A in row r and column s is $a_{rs} \neq 0$

- Multiplying the row r of the linear equation system with the factor $\frac{1}{a_{rs}}$ will create an equivalent linear equation system with the element $\overline{a_{rs}} = 1$

- Then, for all $i \neq r$, subtracting the product of a_{is} and the row r from the i-th row will create an equivalent linear equation system with the element $a_{is} = 0$

- These two operations are the basis for the Gaussian algorithm for matrix inversion

Pivoting

- For example

$$\begin{pmatrix} 3 & 2 & 6 \\ 4 & 5 & 2 \\ 4 & 3 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$

- Then pivoting on a_{22} first transforms this into

$$\begin{pmatrix} 3 & 2 & 6 \\ 4/5 & \mathbf{1} & 2/5 \\ 4 & 3 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 2/5 \\ 5 \end{pmatrix}$$

- Followed by

$$\begin{pmatrix} 3 - 8/5 & 0 & 6 - 4/5 \\ 4/5 & \mathbf{1} & 2/5 \\ 4 - 12/5 & 0 & 1 - 6/5 \end{pmatrix} x = \begin{pmatrix} 4 - 4/5 \\ 2/5 \\ 5 - 6/5 \end{pmatrix}$$

Pivoting

- In summary, using the matrix

$$P = \begin{pmatrix} 1 & & & -\frac{a_{1s}}{a_{rs}} & & & \\ & \ddots & & \vdots & & 0 & \\ & & 1 & -\frac{a_{r-1,s}}{a_{rs}} & & & \\ & 0 & & \frac{1}{a_{rs}} & & 0 & \\ & & & -\frac{a_{r+1,s}}{a_{rs}} & 1 & & \\ & 0 & & \vdots & & \ddots & \\ & & & -\frac{a_{ms}}{a_{rs}} & & & 1 \end{pmatrix}$$

- We transform the equation system $Ax = b$ into the equivalent equation system $PAx = Pb$
- The column s of the resulting $\bar{A} = PA$ has $\overline{a_{rs}} = 1$ and $\overline{a_{is}} = 0$ for $i \neq s$

Basis

- Let's assume for now that there are m variables

$$B = \{x_{i_1}, \dots, x_{i_m}\}$$

- For which all cost coefficients are zero, i.e. $c_{i_1} = 0, \dots, c_{i_m} = 0$
- and the $m \times m$ submatrix of A corresponding to these variables is the identity matrix

- The set B is called a **basis**
- The variables $x_i \in B$ are called **basic variables**
- The variables $x_i \notin B$ are called **non-basic variables**

Basis

- For example the following linear program (in tableau notation)

$$\begin{array}{c|cccccc|c} -f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \end{array}$$

$$\left(\begin{array}{c|cccccc|c} 1 & 20 & 16 & 12 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -10 \\ \hline 0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 4 \\ 0 & 2 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 10 \\ 0 & 2 & 2 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 16 \end{array}\right)$$

- Has the basis $B = \{x_4, x_5, x_6\}$ and the non-basic variables $N = \{x_1, x_2, x_3\}$

Basic feasible solution

- A linear program in this form has two main advantages

  - Given values for the non-basic variables we can easily compute the values for the basic variables

  - Assigning zero to all non-basic variables is a **basic feasible solution** of the linear programming problem (because we assume $b \geq 0$)

- The simplex algorithm is generating a sequence of such basic feasible solutions that converges to the optimal solution

Basic feasible solution

- For example the following linear program (in tableau notation)

$$
\begin{array}{cccccccc}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\end{array}
$$

$$
\left(\begin{array}{c|cccccc|c}
1 & 20 & 16 & 12 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -10 \\
\hline
0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 4 \\
0 & 2 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 10 \\
0 & 2 & 2 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 16
\end{array}\right)
$$

- We can easily compute the basic variables from the non-basic variables

$$
\begin{aligned}
x_4 &= 4 - x_1 \\
x_5 &= 10 - 2x_1 - x_2 - x_3 \\
x_6 &= 16 - 2x_1 - 2x_2 - x_3 \\
f &= 10 + 20x_1 + 16x_2 + 12x_3
\end{aligned}
$$

Basic feasible solution

- For example the following linear program (in tableau notation)

$$
\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\end{array}
$$

$$
\left(\begin{array}{c|cccccc|c}
1 & 20 & 16 & 12 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -10 \\
\hline
0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 4 \\
0 & 2 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 10 \\
0 & 2 & 2 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 16
\end{array}\right)
$$

- The first basic feasible solution now is $x_1 = 0, x_2 = 0, x_3 = 0$

$$
\begin{aligned}
x_4 &= 4 - x_1 = 4 \\
x_5 &= 10 - 2x_1 - x_2 - x_3 = 10 \\
x_6 &= 16 - 2x_1 - 2x_2 - x_3 = 16 \\
f &= 10 + 20x_1 + 16x_2 + 12x_3 = 10
\end{aligned}
$$

Basic feasible solution

- Let's analyse this basic feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$
$$x_4 = 4 - x_1 = 4$$
$$x_5 = 10 - 2x_1 - x_2 - x_3 = 10$$
$$x_6 = 16 - 2x_1 - 2x_2 - x_3 = 16$$
$$f = 10 + 20x_1 + 16x_2 + 12x_3 = 10$$

- The coefficient corresponding to x_1 is larger than zero $c_1 > 0$
- Therefore, if we increase only x_1 the objective function increases
- This does not affect the other non-basic variables x_2 and x_3
- However, we need to ensure that the basic variables remain positive, i.e.

$$x_4 = 4 - x_1 \geq 0$$
$$x_5 = 10 - 2x_1 \geq 0$$
$$x_6 = 16 - 2x_1 \geq 0$$

- Obviously, the best we can do is set $x_1' = 4$ which is a feasible solution and increases the objective to

$$f' = 90$$

- Also note, that $x_4' = 0$ now

The simplex method

- Apparently setting $x_4 = x_2 = x_3 = 0$ is better than $x_1 = x_2 = x_3 = 0$

- We therefore want to transform the linear programming problem so that the basis transforms from $B = \{x_4, x_5, x_6\}$ to $B' = \{x_1, x_5, x_6\}$

- The variable x_4 is said to be leaving the basis, while the variable x_1 is entering the basis

- This transformation can be achieved by pivoting the linear equations at the element connecting x_1 and x_4, i.e. at a_{11}

The simplex method

- Pivoting the linear equation at a_{11} transforms

$$
\begin{array}{c}
\begin{array}{cccccccc} -f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{array} \\
\left(\begin{array}{c|cccccc|c}
1 & 20 & 16 & 12 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -10 \\
\hline
0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 4 \\
0 & 2 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 10 \\
0 & 2 & 2 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 16
\end{array}\right)
\end{array}
$$

- Into

$$
\begin{array}{c}
\begin{array}{cccccccc} -f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{array} \\
\left(\begin{array}{c|cccccc|c}
1 & \mathbf{0} & 16 & 12 & -20 & \mathbf{0} & \mathbf{0} & -90 \\
\hline
0 & 1 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & 4 \\
0 & \mathbf{0} & 1 & 1 & -2 & \mathbf{1} & \mathbf{0} & 2 \\
0 & \mathbf{0} & 2 & 1 & -2 & \mathbf{0} & \mathbf{1} & 8
\end{array}\right)
\end{array}
$$

- Note that also the objective function is transformed by this operation

The simplex method

- This is an equivalent linear program, for with the basic feasible solution $x_2 = x_3 = x_4 = 0$ evaluates the objective function to $f = 90$

$$
\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\end{array}
$$

$$
\left(\begin{array}{c|cccccc|c}
1 & \mathbf{0} & 16 & 12 & -20 & \mathbf{0} & \mathbf{0} & -90 \\
\hline
0 & \mathbf{1} & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & 4 \\
0 & \mathbf{0} & 1 & 1 & -2 & \mathbf{1} & \mathbf{0} & 2 \\
0 & \mathbf{0} & 2 & 1 & -2 & \mathbf{0} & \mathbf{1} & 8
\end{array}\right)
$$

- Again, we observe that the coefficient $c_2 = 16$ is larger than zero, therefore increasing x_2 should increase the cost function
- In this case $x_1 = 4, x_5 = 2 - x_2 \geq 0, x_6 = 8 - 2x_2 \geq 0$, therefore x_5 is the tightest of these inequalities and we pivot at a_{22}

The simplex method

- Pivoting the linear program at a_{22}

$$\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
1 & \mathbf{0} & 16 & 12 & -20 & \mathbf{0} & \mathbf{0} & -90 \\
\hline
0 & \mathbf{1} & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & 1 & -2 & \mathbf{1} & \mathbf{0} & 2 \\
0 & \mathbf{0} & 2 & 1 & -2 & \mathbf{0} & \mathbf{1} & 8
\end{array}$$

- Yields

$$\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
1 & \mathbf{0} & \mathbf{0} & -4 & 12 & -16 & \mathbf{0} & -122 \\
\hline
0 & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & 1 & -2 & 1 & \mathbf{0} & 2 \\
0 & \mathbf{0} & \mathbf{0} & -1 & 2 & -2 & \mathbf{1} & 4
\end{array}$$

The simplex method

- The objective function has increased again to $f = 122$ for the basic feasible solution $x_3 = x_4 = x_5 = 0$

$$
\begin{array}{c|ccccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & \mathbf{0} & \mathbf{0} & -4 & 12 & -16 & \mathbf{0} & -122 \\
\hline
0 & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & 1 & -2 & 1 & \mathbf{0} & 2 \\
0 & \mathbf{0} & \mathbf{0} & -1 & 2 & -2 & \mathbf{1} & 4
\end{array}
$$

- Again, the coefficient $c_4 > 0$, therefore we can improve by having x_4 enter the basis, which means one of the following will leave the basis

$$x_1 = 4 - x_4 \geq 0, x_2 = 2 + 2x_4 \geq 0, x_6 = 4 - 2x_4 \geq 0$$

- The tightest constraint can be determined from the matrix using the **min-ratio-test**

$$\min_{i:a_{is}>0} \frac{b_i}{a_{is}} = \min\left\{\frac{4}{1}, \frac{4}{2}\right\}$$

- This means x_6 will leave the basis, so we need to pivot again at a_{34}

The simplex method

- Pivoting at a_{34}

$$\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & \mathbf{0} & \mathbf{0} & -4 & 12 & -16 & \mathbf{0} & -122 \\
\hline
0 & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & 1 & -2 & 1 & \mathbf{0} & 2 \\
0 & \mathbf{0} & \mathbf{0} & -1 & \mathbf{2} & -2 & \mathbf{1} & 4
\end{array}$$

- Yields

$$\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & -4 & -6 & -146 \\
\hline
0 & \mathbf{1} & \mathbf{0} & 1/2 & \mathbf{0} & 1 & -1/2 & 2 \\
0 & \mathbf{0} & \mathbf{1} & 0 & \mathbf{0} & -1 & 1 & 6 \\
0 & \mathbf{0} & \mathbf{0} & -1/2 & \mathbf{1} & -1 & 1/2 & 2
\end{array}$$

The simplex method

- Now $c_3 > 0$, so one final pivot at a_{13} transforms

$$
\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & -4 & -6 & -146 \\
\hline
0 & \mathbf{1} & \mathbf{0} & \mathbf{1/2} & \mathbf{0} & 1 & -1/2 & 2 \\
0 & \mathbf{0} & \mathbf{1} & 0 & \mathbf{0} & -1 & 1 & 6 \\
0 & \mathbf{0} & \mathbf{0} & -1/2 & \mathbf{1} & -1 & 1/2 & 2
\end{array}
$$

- into

$$
\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -8 & -4 & -154 \\
\hline
0 & 2 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 2 & -1 & 4 \\
0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & -1 & 1 & 6 \\
0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 & 4
\end{array}
$$

The simplex method

- This is the final optimal solution

$$
\begin{array}{c|cccccc|c}
-f & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & -4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -8 & -4 & -154 \\
\hline
0 & 2 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 2 & -1 & 4 \\
0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & -1 & 1 & 6 \\
0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 & 4
\end{array}
$$

- Because the objective function now reads

$$f = 154 - 4x_1 - 8x_5 - 4x_6$$

- Clearly there is no positive $x_1 \geq 0, \mathrm{x}_5 \geq 0, \mathrm{x}_6 \geq 0$ so that the value increases, therefore the solution is can now be taken directly from the tableau (remember, pivoting does not change the solution of a linear system):

$$x_1 = 0, x_5 = 0, x_6 = 0$$
$$f = 154$$
$$x_2 = 6, x_3 = 4, x_4 = 4$$

The simplex method

- In summary, the simplex method works as follows:
- Starting from an initial tableau

$$\begin{array}{c} \begin{array}{ccccc} -f & x_1 & \cdots & x_n & \end{array} \\ \left(\begin{array}{c|ccc|c} 1 & c_1 & \cdots & c_n & -c_0 \\ \hline 0 & a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{m1} & \cdots & a_{mn} & b_m \end{array}\right) \end{array}$$

- That can be partitioned into a basis $B = \{x_{j_1}, \dots, x_{j_m}\}$ so that all $c_{j_i} = 0$ and $a_{k j_i} = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases}$
- We repeat until all $c_j < 0$:
  - Pick a $c_s > 0$ and find r such that $\frac{b_r}{a_{rs}} = \min_{i: a_{is} > 0} \frac{b_i}{a_{is}}$ and pivot the tableau on a_{rs}
  - Replace x_{j_r} by x_s in B

The simplex method

- In each iteration where the simplex algorithm pivots on a_{rs} the value of the objective function is increased by $\frac{b_r c_s}{a_{rs}}$
- Therefore, as long as all the b_r are always strictly positive the algorithm converges
- If in any iteration one of the b_r is zero, this creates ambiguity in the min-ratio test
- A linear program is called **degenerate** if this happens and the simplex algorithm can in this case cycle through intermediate states without converging
- This can be broken by resolving the ambiguities according to some ordering of variables (Bland's anti-cycling pivoting rule)

The simplex method

- So are we done yet?

- The simplex method relies on a rather special structure of the linear program, with the existence of a basis of the size equal to the number of constraints

- Sometimes this is the case, for instance if we convert the constraints
$$Ax \leq b$$
- into standard form using slack variables (and if all $b > 0$ already)

- However, in general we need to transform the linear program into an equivalent linear program that has this specific form

Pre-processing

- In case the linear program does not have the required form we add m additional variables $x_1^a, \dots, x_m^a$ and consider the following linear program
- Maximise

$$w = -\sum_{i=1}^{m} x_i^a$$

- Subject to

$$Ax + Ix_a = b$$
$$x \geq 0, x^a \geq 0$$

- In case the original linear program has a feasible solution $Ax = b$, the maximum should be $w = 0$, with all $x_a = 0$

Pre-processing

- The constraint can be re-written as

$$x_a = b - Ax$$

- And substituted into the objective function (with $e = (1 \quad \cdots \quad 1)^T$)

$$w = -e^T b + e^T A x$$

- Subject to

$$Ax + Ix_a = b$$
$$x \geq 0, x^a \geq 0$$

- This linear program has the required structure and can be solved using the simplex algorithm with the initial basis being $B = \{x_1^a, \dots, x_m^a\}$

Pre-processing

- After solving this problem there are three potential outcomes
  1. The objective function of this problem has been reduced below zero, i.e. $w < 0$. In this case the original linear program does not have any feasible solution
  2. The objective function of this problem has been reduced to zero, i.e. $w = 0$, and the basis at the end of the simplex algorithm contains only original variables, i.e. $\nexists x_i^a \in B$. In this case we can simply delete the columns corresponding to the artificial variables x_a and replace the objective function with the original objective function expressed in terms of the variables in B.
  3. The objective function has been reduced to zero, i.e. $w = 0$, but there are still artificial variables in the basis after the execution of the simplex procedure, i.e. $\exists x_i^a \in B$. We will see how to remove these artificial variables from the basis in the following slides.

Pre-processing

- Let's look at the following example

$$\begin{array}{c|cccc|c} -f & x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left(\begin{array}{c|cccc|c} 1 & 20 & 16 & 12 & 5 & 0 \\ \hline 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 \end{array}\right)$$

- It is not in the required form, so we substitute the cost function and not that after this substitution the variable x_1 can already be considered for a basis. Hence, we only need two additional variables and get

$$\begin{array}{c|cccc:cc|c} -w & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\ \end{array}$$

$$\left(\begin{array}{c|cccc:cc|c} 1 & \mathbf{0} & 2 & 2 & 5 & \mathbf{0} & \mathbf{0} & 4 \\ \hline 0 & \mathbf{1} & 0 & 1 & 2 & \mathbf{0} & \mathbf{0} & 4 \\ 0 & \mathbf{0} & 1 & 2 & 3 & \mathbf{1} & \mathbf{0} & 2 \\ 0 & \mathbf{0} & 1 & 0 & 2 & \mathbf{0} & \mathbf{1} & 2 \end{array}\right)$$

Pre-processing

- Executing the simplex method, i.e. pivoting on a_{22}, we obtain

$$
\begin{array}{c}
\begin{array}{cccccccc}
-w & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a &
\end{array}
\\
\left(\begin{array}{c|cccc:cc|c}
1 & \mathbf{0} & \mathbf{0} & -2 & -1 & -2 & \mathbf{0} & 0 \\
\hline
0 & \mathbf{1} & \mathbf{0} & 1 & 2 & 0 & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & 2 & 3 & 1 & \mathbf{0} & 2 \\
0 & \mathbf{0} & \mathbf{0} & -2 & -1 & -1 & \mathbf{1} & 0
\end{array}\right)
\end{array}
$$

- The objective function $w = 0$, therefore the original LP has a feasible solution
- At this point x_2^a is still a basic variable and has to leave the basis
- Note that the corresponding equation is always degenerate, i.e. the right-hand side is $b_3 = 0$

Pre-processing

- The fact that $b_3 = 0$ enables us to pivot on any non-zero $a_{3i} \neq 0$ without affecting the objective function value. Choosing a_{34}

$$
\begin{array}{c|cccc:cc|c}
-w & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
1 & \mathbf{0} & \mathbf{0} & -2 & -1 & -2 & \mathbf{0} & 0 \\
\hline
0 & \mathbf{1} & \mathbf{0} & 1 & 2 & 0 & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & 2 & 3 & 1 & \mathbf{0} & 2 \\
0 & \mathbf{0} & \mathbf{0} & -2 & \mathbf{-1} & -1 & \mathbf{1} & 0
\end{array}
$$

- transforms into

$$
\begin{array}{c|cccc:cc|c}
-w & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
1 & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & -1 & -1 & 0 \\
\hline
0 & \mathbf{1} & \mathbf{0} & -3 & \mathbf{0} & -2 & 2 & 4 \\
0 & \mathbf{0} & \mathbf{1} & -4 & \mathbf{0} & -2 & 3 & 2 \\
0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{1} & 1 & -1 & 0
\end{array}
$$

Pre-processing

- Now the basis $B = \{x_1, x_2, x_4\}$ only contains original variables

$$
\begin{array}{c|cccc:cc|c}
-\boldsymbol{w} & \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} & \boldsymbol{x_4} & \boldsymbol{x_1^a} & \boldsymbol{x_2^a} & \\
1 & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & -1 & -1 & 0 \\
\hline
0 & \mathbf{1} & \mathbf{0} & -3 & \mathbf{0} & -2 & 2 & 4 \\
0 & \mathbf{0} & \mathbf{1} & -4 & \mathbf{0} & -2 & 3 & 2 \\
0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{1} & 1 & -1 & 0
\end{array}
$$

- The basic feasible solution is $x_3 = x_1^a = x_2^a = 0$, therefore

$$x_1 = 4, x_2 = 2, x_4 = 0$$

- Substituting this into the original objective function we get

$$f = 20x_1 + 16x_2 + 12x_3 + 5x_4 = 112$$

Pre-processing

- Putting the original objective function

$$f = 20x_1 + 16x_2 + 12x_3 + 5x_4 = 112$$

- back into the tableau and deleting the columns for the two additional variables finally yields

$$\begin{array}{c|cccc|c}
-\boldsymbol{f} & \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} & \boldsymbol{x_4} & \\
\end{array}$$

$$\left(\begin{array}{c|cccc|c}
1 & \mathbf{0} & \mathbf{0} & 12 & \mathbf{0} & -112 \\
\hline
0 & \mathbf{1} & \mathbf{0} & -3 & \mathbf{0} & 4 \\
0 & \mathbf{0} & \mathbf{1} & -4 & \mathbf{0} & 2 \\
0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{1} & 0
\end{array}\right)$$

- This equivalent linear program now complies with the requirements of the simplex algorithm and can be used as input to phase 2 of the procedure

Matrix representation

- In every iteration the tableau looks like this (assuming some re-ordering of variables to group together the basis variables x_B)

$$\begin{array}{c|cc|c} \boldsymbol{-f} & \boldsymbol{x_B} & \boldsymbol{x_A} & \\ \end{array}$$

$$\left(\begin{array}{c|cc|c} 1 & 0 & \bar{c}_N^T & -\bar{c}_0 \\ \hline 0 & I & \bar{A}_N & \bar{b} \end{array}\right)$$

- Every pivoting operation can be expressed as the multiplication with an invertible matrix, therefore for some matrix P

$$\bar{A}_N = PA_N$$
$$\bar{b} = Pb$$

- And also

$$I = \bar{A}_B = PA_B$$

- Which implies

$$P^{-1} = A_B$$

Matrix representation

- Using $P = A_B^{-1}$ we can express the current linear program in each iteration in terms of the original linear program for each basis B

$$\bar{A}_N = A_B^{-1} A_N$$
$$\bar{b} = A_B^{-1} b$$

- The basic feasible solution implies

$$x_B = \bar{b} - \bar{A}_N x_N$$

- Also the objective functions are equivalent

$$c_0 + c_B^T x_B + c_N^T x_N = \bar{c}_0 + \bar{c}_N^T x_N$$

- If we assume that the original cost function offset is $c_0 = 0$ (this does not change the solution of the linear program) then

$$c_B^T \bar{b} - c_B^T \bar{A}_N x_N + c_N^T x_N = \bar{c}_0 + \bar{c}_N^T x_N$$

- This implies

$$\bar{c}_0 = c_B^T \bar{b} = c_B^T A_B^{-1} b$$
$$\bar{c}_N^T = c_N^T - c_B^T \bar{A}_N = c_N^T - c_B^T A_B^{-1} A_N$$

Matrix representation

- Putting it all together we have

$$\begin{aligned} \bar{A}_N &= A_B^{-1} A_N \\ \bar{b} &= A_B^{-1} b \\ \bar{c}_0 &= c_B^T A_B^{-1} b \\ \bar{c}_N^T &= c_N^T - c_B^T A_B^{-1} A_N \end{aligned}$$

- Using the substitution

$$y^T = c_B^T A_B^{-1}$$

- this can be summarised in the tableau as follow

$$\begin{array}{cccc} -f & x_B & x_A & \\ \multicolumn{4}{c}{\left(\begin{array}{c|cc|c} 1 & 0 & c_N^T - y^T A_N & -y^T b \\ \hline 0 & I & A_B^{-1} A_N & A_B^{-1} b \end{array}\right)} \end{array}$$

Summary

- The simplex algorithm comprises two stages
  - First the input linear program in standard form is transformed into an equivalent linear program that makes the basis explicit
  - This explicit basis allows to directly calculate a (non-optimal) basic feasible solution for the linear program
  - Then this linear program is transformed through a sequence of pivoting operations into an equivalent linear program where the basic feasible solution is the optimum

- The geometric intuition of the simplex method is that it moves along the edges of the simplex until it arrives at the optimal corner vertex

- The worst-case runtime is determined by the number of selections of basis vectors and has therefore $\binom{n}{m}$ iterations, however in most practical applications convergence is reached much faster

Thank you for your attention!