Midtem-2 practice-with solutions

1. Another form of the error for Trapezoidal rule can be given by The General Euler McLaurin formula is defined by ,

$$\int_a^b f(x)dx = h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a)+f(b)) - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} h^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) + R$$

Where the Bernoulli numbers are given by

$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30},\dots$

and

$h = \frac{b-a}{n},\ \ x_i = a + ih,\ i = 0,\dots,n.$

a. Use the Bernoulli Generating equation to verify few of the numbers above:
$\frac{t}{e^t-1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}$ where $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

b. Compute $\sum_{i=1}^{n} i^2$ and for $\sum_{i=1}^{n} i^4$ using the above formula for $a = 0, b = n,\ f(x_i) = f(i)$.

SOLUTIONS:

a. $\frac{1}{1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} = B_0 + \frac{B_1 t}{1!} + \frac{B_2 t^2}{2!} + \cdots ==> 1 = B_0 + B_1 t + \frac{B_2 t^2}{2} + .. + \frac{B_0 t}{2} + \frac{B_1 t^2}{2} + \frac{B_2 t^3}{4} + \cdots. + \frac{B_0 t^2}{6} + \cdots. = B_0 + \left(B_1 + \frac{B_0}{2}\right)t + \left(\frac{B_2}{2} + \frac{B_1}{2} + \frac{B_0}{6}\right)t^2 + \cdots. ==>$
$B_0 = 1, B_1 + \frac{B_0}{2} = 0, \frac{B_2}{2} + \frac{B_1}{2} + \frac{B_0}{6} = 0, \dots ==> B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \dots$

b. $\int_0^n x^2 dx = \sum_{i=0}^{n} i^2 - \frac{1}{2}(n^2+0) - \frac{1}{2*6}(2n-0) ==> \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \sum_{i=0}^{n} i^2$

c. $\int_0^n x^4 dx = \sum_{i=0}^{n} i^4 - \frac{1}{2}(n^4) - \frac{1}{12}(3n^3) + \frac{1}{30} * \frac{1}{4!}(4*3*2n) = \frac{n^5}{5} ==>$
$\sum_{i=0}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$

SOLUTION: $\int_0^1 f(x)dx - \frac{1}{2}[f(0)+f(1)] = \int_0^1 [f(x) - P_1(x)] = \int_0^1 \frac{f''(\theta)x(x-1)}{2} dx = \frac{1}{2}\int_0^1 f''(\theta)x(x-1)dx = \frac{1}{2}f''(\sigma)\int_0^1 x(x-1)dx = \frac{1}{2}f''(\sigma)[\frac{x^3}{3} - \frac{x^2}{2}]0..1 = \frac{1}{2}f''(\sigma)[\frac{1}{3} - \frac{1}{2}] = -\frac{1}{12}f''(\sigma)$

2. The equation from 1 can be rewritten as follows:

$$I = \int_a^b f(x)dx = T(h) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} h^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)$$
$$= T(h) + A_1 h^2 + A_2 h^4 + A_3 h^6 + \cdots$$

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Where

$T^{(0)}(h) = \ h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a) + f(b)) = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) \ + f\,(x_n)$

is the trapezoidal rule. We will derive Romberg's integration. The basic idea is that if you know the error or a form of the error you can get a better approximation of your integration. Let us assume that we have computed $T^{(0)}(h)$ and $T^{(0)}\left(\frac{h}{2}\right)$ then we have

$$I = T^{(0)}(h) + A_1h^2 + A_2h^4 + A_3h^6 + \cdots$$

$$I = T^{(0)}\left(\frac{h}{2}\right) + A_1\left(\frac{h}{2}\right)^2 + A_2\left(\frac{h}{2}\right)^4 + A_3\left(\frac{h}{2}\right)^6 + \cdots$$

Multiply the second equation by 2^2 we get

$$2^2 I = 2^2 T^{(0)}\left(\frac{h}{2}\right) + A_1h^2 + 2^2A_2\left(\frac{h}{2}\right)^4 + 2^2A_3\left(\frac{h}{2}\right)^6 + \cdots$$

Subtracting the two equations above will eliminate A_1h^2 factor:

$$2^2I - I$$

$$= 2^2T^{(0)}\left(\frac{h}{2}\right) + A_1h^2 + 2^2A_2\left(\frac{h}{2}\right)^4 + 2^2A_3\left(\frac{h}{2}\right)^6 - T^{(0)}(h) - A_1h^2$$

$$- A_2h^4 - A_3h^6 + \cdots = 2^2T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h) + A_2^*h^4 + \cdots.$$

Or equivalently: $= T^{(1)}(h) + A_2^*h^4 + \cdots$. where $T^{(1)}(h) = \frac{4\,T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h)}{4-1}$ so we can say that the error for $O(h^4)$ which is more accurate than the error of trapezoidal rule $O(h^2)$. This new rule is called the corrected trapezoidal rule. The following table shows that trapezoidal rule and corrected Trapezoidal rule for different h:

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$

$T^0(h)$			
$T^0(\frac{h}{2})$	$T^{(1)}(h) = \frac{4\,T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h)}{4-1}$		

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$T^0(\frac{h}{2^2})$	$T^{(1)}\left(\frac{h}{2}\right) = \frac{4\,T^{(0)}\left(\frac{h}{2^2}\right) - T^{(0)}\left(\frac{h}{2}\right)}{4-1}$	$T^{(2)}(h) = \frac{4^2\,T^{(1)}\left(\frac{h}{2}\right) - T^{(1)}(h)}{4^2-1}$	
$T^0(\frac{h}{2^3})$	$T^{(1)}\left(\frac{h}{2^2}\right) = \frac{4\,T^{(0)}\left(\frac{h}{2^3}\right) - T^{(0)}\left(\frac{h}{2^2}\right)}{4-1}$	$T^{(2)}(h) = \frac{4^2\,T^{(1)}\left(\frac{h}{2^2}\right) - T^{(1)}\left(\frac{h}{2}\right)}{4^2-1}$	$T^{(3)}(h) = \frac{4^3\,T^{(2)}\left(\frac{h}{2}\right) - T^{(2)}(h)}{4^3-1}$

i. Show how we derive the $O(h^6)$ formula in the table by using $I = T^{(1)}(h) + A_2^* h^4 + \cdots$ and the same idea we used for deriving the $O(h^4)$ formula.

SOLUTION: $I = T^{(1)}(h) + A_2^* h^4 + \cdots ==> I = T^{(1)}\left(\frac{h}{2}\right) + A_2^*\left(\frac{h}{2}\right)^4 + \cdots ==>$

$4^2 I = 4^2 T^{(1)}\left(\frac{h}{2}\right) + A_2^* h^4 ==> subtract: (4^2-1)I =$

$4^2 T^{(1)}\left(\frac{h}{2}\right) - T^{(1)}(h)\ \ solve\ to\ get\ O(h^6)$

ii. Show how we derive the $O(h^8)$ formula in the table by using $I = T^{(2)}(h) + A_3^* h^6 + \cdots$ and the same idea we used for deriving the $O(h^4)$ formula.

$$I = T^{(2)}(h) + A_2^* h^6 + \cdots ==> I = T^{(2)}\left(\frac{h}{2}\right) + A_2^*\left(\frac{h}{2}\right)^6 + \cdots ==> 4^3 I$$

$$= 4^3 T^{(2)}\left(\frac{h}{2}\right) + A_2^* h^6 ==> subtract: (4^6-1)I$$

$$= 4^3 T^{(2)}\left(\frac{h}{2}\right) - T^{(2)}(h)\ \ solve\ to\ get\ O(h^8)$$

iii. Write a MATLAB program to compute the trapezoidal method $T^{(0)}(h)$. Remember that $n = \frac{b-a}{h}$ the number of subintervals in the trapezoidal rule $T^{(0)}(h) = h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a) + f(b)) = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i)\ + f(x_n)$

and $x_i = a + ih, i = 0{:}n$. Submit your program along with your table results.

```
function integral=trapezoidal(a,b,h,index_f)
n=(b-a)/h;

% Initialize for trapezoidal rule.
```

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```
sumend = (f(a,index_f) +f(b,index_f))/2;
sum = 0;

for i=1:1:n-1
   sum = sum + f(a+i*h,index_f);
end
integral = h*(sumend + sum);


function f_value = f(x,index)
%
% This defines the integrand.

switch index
case 1
    f_value = x;
case 2
    f_value = x^7;

end
```

iv. Use the table to derive approximation for $\int_0^1 x^7 dx = \frac{1}{8}$ starting with $h = 1 - 0 = 1$

The table formulae are also known as Romberg's integration.

$O(h\text{^}2)$	$O(h\text{^}4)$	$O(h\text{^}6)$	$O(h\text{^}8)$
0.500000000000000			
0.253906250000000	0.171875000000000		
0.160339355468750	0.129150390625000	0.126302083333333	
0.134043693542480	0.125278472900390	0.125020345052083	0.125000000000000=1/8

3. Derive the extrapolation formula for $I = T(h) + Ah + Bh^3$.

Solution: $I = T\left(\frac{h}{2}\right) + \frac{Ah}{2} + B\left(\frac{h}{2}\right)^3$, ➔$2I = 2T\left(\frac{h}{2}\right) + Ah + \frac{Bh^3}{4}$

Subtract to get $I = 2T\left(\frac{h}{2}\right) - T(h) + B^*h^3 = T^1(h) + B^*h^3, T^1(h) = 2T\left(\frac{h}{2}\right) - T(h)$ You can continue the Extrapolation formula. $I = T^1(h) + B^*h^3$, $I = T^1\left(\frac{h}{2}\right) + B^*(\frac{h}{2})^3$ ➔ multiply by 2^3 to get $2^3 I = 2^3 T^1\left(\frac{h}{2}\right) + B^*h^3$, ➔ Subtract the two and you get $(2^3 - 1)I = 2^3 T^1\left(\frac{h}{2}\right) - T^1(h)$ ➔ $I = (2^3 T^1\left(\frac{h}{2}\right) - T^1(h))/\,(2^3 - 1)$

4. The method of undetermined coefficients assumes that the error in a quadrature is zero for a given set of functions and then determines the coefficient by solving a

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system of linear equations. Apply this method to the following integrals. Determine A, B ,C and c so that the formula is exact when $f(x)$ is polynomial of the highest possible degree.

a.

$$\int_0^1 f(x)dx \approx A\,f(0) + Bf\left(\frac{1}{2}\right) + C\,f(1)$$

b.

$$\int_0^1 \sqrt{x}\,f(x)dx \approx Af(c) + Bf(1)$$

Determine the error of the above integration formulae.

c. Use the above formulae to estimate the integral for $f(x) = x^3$.

SOLUTION:a. :$f(x) = 1, x, x^2$ $A + B + C = 1,\ \frac{B}{2} + C = \frac{1}{2}, \frac{B}{4} + C = \frac{1}{3}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{12} \end{bmatrix} \rightarrow C = \frac{1}{6}, B = \frac{4}{6}, A = \frac{1}{6}$$

Error:

$\int_0^1 f(x)dx - \frac{1}{6}[f(0) + 4f\left(\frac{1}{2}\right) + f(1)] = Df''' + Ef^{iv} + \cdots +$

Set $f(x) = x^3$

$$\int_0^1 x^3 dx - \frac{1}{6}[0 + 4\left(\frac{1}{2}\right)^3 + 1] = D3! ==> \frac{1}{4} - \frac{1}{6}\left[\frac{3}{2}\right] = 6D ==> 0 = D$$

Next we try $f(x) = x^4$

$$\int_0^1 x^4 dx - \frac{1}{6}[0 + 4\left(\frac{1}{2}\right)^4 + 1] = E4! ==> \frac{1}{5} - \frac{1}{6} * \frac{5}{4} = 24E ==> -\frac{1}{5 * 6 * 4} = 24E ==> E$$

$= 2880$

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B

$\int_0^1 \sqrt{x}\,dx \approx A + B = \frac{2}{3},\ \ Ac + B = \frac{2}{5},$

$Ac^2 + B = \frac{2}{7} ==> A(1-c) = \frac{4}{15},\qquad A(c-c^2) = \frac{4}{35} ==> c = \frac{\frac{4}{35}}{\frac{4}{15}} = \frac{15}{35},\qquad A = \frac{\frac{4}{15}}{\frac{20}{35}} = \frac{35*4}{15*20} = \frac{35}{75}, B = \frac{45}{225}$

Error: $\int_0^1 \sqrt{x}\ x^3 dx - \frac{35}{75} * \left(\frac{15}{35}\right)^3 - \frac{45}{225} = 3!\,D ==> D = \frac{\left[\frac{2}{9} - \frac{35}{75} * \left(\frac{15}{35}\right)^3 - \frac{45}{225}\right]}{3!} = -0.0024$

Error== $-0.0024 f'''(\mu)$

C. $\int_0^1 x^3 dx \approx \frac{1}{6}\left(0 + 4\left(\frac{1}{2}\right)^3 + 1\right) = \frac{1}{4}$ Exact because the error is zero for this function.

$$\int_0^1 \sqrt{x}\ x^3 dx \approx \frac{35}{75} * \left(\frac{15}{35}\right)^3 + \frac{45}{225} = 0.236734693877551$$

$$exact\ error = \left|\frac{2}{9} - 0.236734693877551\right| = 0.014512471655329$$

$error\ from\ formula = 0.0024 * 3! = 0.0144$

Formula AGREE

5. We want to solve Ax=b. Solve the system by using Gaussian elimination with partial pivoting for the following linear systems:

i. $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Find PLU and solve $PLUx = b$

ii. Solution: $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, p = [1\ 2\ 3].$

$K = 1, max\{|0|, |-1|, |-2|\} = 2$, interchange row 3 with 1, $p = [3,2,1]$

$$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Multiply the first equation by $\frac{1}{2}$ and subtract from the second to obtain:

$$\begin{bmatrix} -2 & 0 & 1 \\ (\frac{1}{2}) & 2 & -\frac{1}{2} \\ (0) & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

$K = 2\ max\{2,1\} = 2$ so No interchange and , $p = [3,2,1]$

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iii. What is PLU of the matrix A. Show all steps and then verify your result by using matlab solver $[L\ U\ P] = lu(A)$

iv. What is the complexity of Gaussian elimination when applied to general nxn matrix. Show all steps in the derivation of the complexity.

v. What is the det(A)? which method is the best in finding the determinant(EXPLAIN)

vi. Find the inverse of A by using Gauss-Jordan method—eliminating both above and below the diagonal at the same time using the extended system:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify your result by using $inv(A)$ in Matlab. Then derive the computational complexity of inverting a general nxn matrix using the Gauss-Jordan method.

vii. We want to solve Ax=b by using two different methods- $PLUx = b$ or $x = A^{-1}b$. Which one of these two methods is better? Explain why.

6. The kij form of Gaussian Elimination without partial pivoting is given by :

```
function [L U a] = kij( a )
m=size(a);
n=m(1);
for k=1:n
    for i=k+1:n
        a(i,k)=a(i,k)/a(k,k);
        for j=k+1:n
            a(i,j)=a(i,j)-a(i,k)*a(k,j);
        end
    end
end
L = tril(a,-1);
U=a-L;
L=L+eye(n);
end
```

i. Explain each step of the above program by using the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$. Help tril in matlab will explain what the function does—it extracts the lower triangular part of the matrix.

ii. Modify this program to perform the kji form of LU decomposition.

iii. Write a separate function that takes as input $L\ U$, b and outputs the solution x of Ax=b. Your function should perform two steps $LUx = b ==> Lz = b, Ux = z$, i.e a forward substitution first and a backward substation next.

iv. Test your program with the system in question 1 and present the results.

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7. The Gauss-Jacobi iterative method is defined as follows: $Ax = b ==> [D + (A - D)]x = b ==> Dx = b - (A - D)x ==> x = D^{-1}b - D^{-1}(A - D)x$ so we can derive an iterative method:

$x^{k+1} = D^{-1}b - D^{-1}(A - D)x^k$ Write a matlab program that implements this method using Matlab matrix definitions e.g. $D = tril(A) - tril(A, -1)$; $x(k + 1) = inv(D) * (b - (A - D) * x(k))$, you can use as stopping criterion $norm(x(k + 1) - x(k), \gamma), \gamma = 2, or\ \gamma = 1, or\ \gamma = inf$ for the 3 norms that we learned in class, the 2 norm or the 1 norm or the infinity or maximum norm.

Here is an example on how the iterations in your program should work

```
A =[

     1     1    -1
     1     2    -2
    -2     1     1 ]
 >> D=tril(A)-tril(A,-1)

D =

     1     0     0
     0     2     0
     0     0     1
b =[

     0
    -1
    -1]
>> x0=zeros(3,1)

x0 =

     0
     0
     0
x1=inv(D)*(b-(A-D)*x0)

x1 =

         0
   -0.5000
   -1.0000
```

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Now if you continue iterating like that or use your program you will see that this iteration does not converge for this matrix. The reason being is that the norm of the iteration matrix $D^{-1}(A-D)$ is greater than one.

```
>> norm(inv(D)*(A-D),2)

ans =

   2.3284
>> norm(inv(D)*(A-D),1)

ans =

   2.5000
>> norm(inv(D)*(A-D),inf)

ans =

    3
```

i. Will this method converge for $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ check your program and explain if YES or NO.

ii. Repeat the same question with $A = \begin{bmatrix} -6 & 1 & 1 \\ 1 & 4 & -2 \\ 1 & 1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$

8. Given $P(x) = c_1P_1(x) + c_2P_2(x) + \cdots c_nP_n(x)$ and data $(x_i, y_i), i = 1{:}m,\ m > n$. The Normal equation that determines the best $P(x)$ in the least squares sense: Minimization of the squares error: Error=$\sum_{i=1}^{m}(P(x_i) - y_i)$^2 is given by $A^TA = A^Ty$. For special case of $m = 3, n = 2$, $P_1(x) = 1, P_2(x) = x$,The equations look like

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i. For the data $(0,0), (1,1), (2,0)$ find the best least squares line $P(x) = c_1 + c_2x$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ➔ $\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ➔ solution

$c_1 = \frac{1}{3}, c_2 = 0.\ P(x) = 1/3$.

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ii. Given the data $(0,0),(1,1),(2,4)$ determine the best linear and quadratic least squares approximation.