Study Notes for AI (CS 440/520)
Lecture 13: Temporal State Estimation

Corresponding Book Chapters: 15.1-15.2

Note: These notes provide only a short summary and some highlights of the material covered in the corresponding lecture based on notes collected from students. Make sure you check the corresponding chapters. Please report any mistakes in or any other issues with the notes to the instructor.

1 Temporal State Estimation

The focus of temporal state estimation is to solve problems where the state of an agent has to be estimated under uncertainty in sensing and action. To address such dynamic problems we first break them into a sequence of time slices. At each time step we have the following parameters:

X_t: unobservable state variables at time t
E_t: denotes the set of observable evidence variables at time t
(an actual evidence value will be denoted as e_t)

We will focus on a subset of the general case of temporal state estimation problems. Specifically to problems that satisfy the following assumptions:

1. **Stationary Process:** The process with which the variables change over time does not change:

$$P(X_t|Parents(X_t)) \text{ is the same } \forall t$$

2. **Markov Assumption:** The current state depends only on a finite history of previous states. If the current state depends only on the previous state then we have a first-order Markov process:

$$P(X_t|X_{0:t-1}) = P(X_t|X_{t-1}) \text{ transition model}$$

$$P(E_t|X_{0:t}) = P(E_t|X_t) \text{ observation model}$$

In order to be able to solve the problems that satisfy the above assumptions we must have the following parameters available:

- Prior (initial) probability: $P(X_0)$
- The transition model: $P(X_t|X_{t-1})$
- The observation model: $P(E_t|X_t)$

The dynamic Bayesian network that represents temporal state estimation problems is shown in Figure 1. Given this dynamic Bayesian network we can then represent the joint probability distribution of our problem as follows (we just apply the independence and conditional independence properties implied by the dynamic Bayesian network):

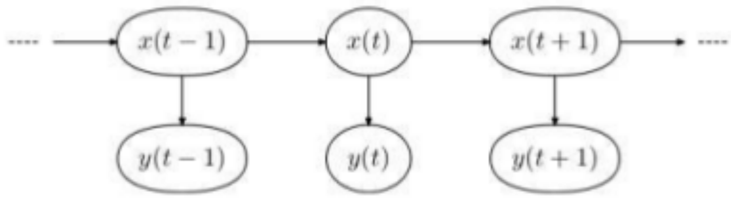

Figure 1: Dynamic Bayesian Network for Temporal State Estimation. In this figure the variable y corresponds to the evidence variables.

$$P(X_0, X_1, \ldots, X_t, E_1, \ldots, E_t) = P(X_0) \prod_{i=1}^{t} P(X_i|X_{i-1}) P(E_i|X_i)$$

Later on we describe the types of problems we can solve with such dynamic Bayesian networks.

1.1 Problems

- **Filtering:** Compute the current belief state given all evidences so far

$$P(X_t|E_{1:t})$$

- **Prediction:** Predict a future state based on all the evidences up to this point in time t

$$P(X_{t+k}|E_{1:t}), \quad k > 0$$

- **Smoothing:** Given all the evidence up to time t, what is the most likely value of a previous state?

$$P(X_k|e_{1:t}), \quad 0 \leq k < t$$

- **Most Likely Explanation:** Given the observations up to this point, what is the sequence of states most likely to generate these observations?

$$argmax_{X_{1:t}}\{P(X_{1:t}|e_{1:t})\}$$

1.2 Derivation of Filtering

$$P(X_t|E_{1:t})$$

The basic idea is that this problem can be solved in an online, iterative fashion: given the results of filtering up to time t, one can easily compute the result for $t+1$ given evidence e_{t+1}:

$$\text{recursive estimation: } P(X_{t+1}|e_{1:t+1}) = f(e_{t+1}, P(X_t|e_{1:t}))$$

This is a two stage process:

1. project the current estimation forward in time from time t to $t+1$ using the transition model
2. update it using the new evidence e_{t+1}

$$
\begin{aligned}
P(X_{t+1}|e_{1:t+1}) &= P(X_{t+1}|e_{1:t}, e_{t+1}) && \text{separate the evidences} \\
&= \alpha P(e_{t+1}|X_{t+1}, e_{1:t}) P(X_{t+1}|e_{1:t}) && \text{Bayes rule} \\
&= \alpha \underbrace{P(e_{t+1}|X_{t+1})}_{\text{Observation model}} \underbrace{P(X_{t+1}|e_{1:t})}_{\text{Predictive Formula}} && \text{Markov assumption}
\end{aligned}
$$

$$
\begin{aligned}
P(X_{t+1}|e_{1:t}) &= \textstyle\sum_{X_t} P(X_{t+1}|X_t, e_{1:t}) P(X_t|e_{1:t}) \\
&= \textstyle\sum_{X_t} \underbrace{P(X_{t+1}|X_t)}_{\text{transition model}} \underbrace{P(X_t|e_{1:t})}_{\text{prior belief}}
\end{aligned}
$$

We can apply the following computations:

In order to be able to answer the filtering problem $P(X_{t+1}|e_{1:t+1})$, we actually have to compute the predictive probability $P(X_{t+1}|e_{1:t})$. We can compute this probability by applying conditioning over the previous states X_t:

Overall, the filtering problem can be solved by the following equation:

$$
P(X_{t+1}|e_{1:t+1}) = \alpha \underbrace{P(e_{t+1}|X_{t+1})}_{\text{observation model}} \sum_{X_t} \underbrace{P(X_{t+1}|X_t)}_{\text{transition model}} \underbrace{P(X_t|e_{1:t})}_{\text{prior belief}}
$$

Filtering has constant time and space requirements per update.

1.3 Rain-Umbrella Example

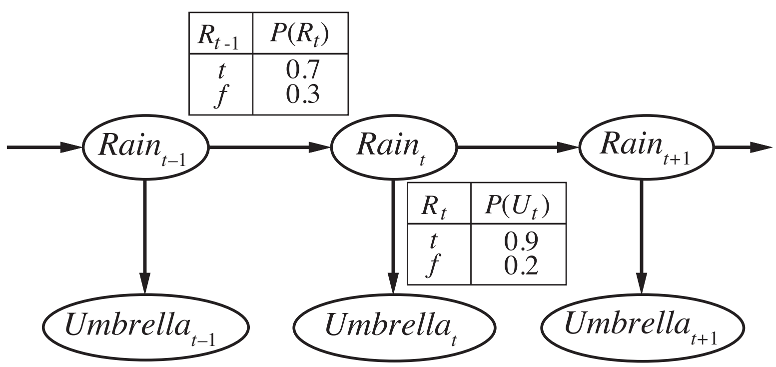

Figure 2: The dynamic Bayesian network for the rain-umbrella example.

Let's try an example. Consider a person who is never exiting a building and is not able to directly observe the weather. This person is able, however, to observe other people entering the building and notices whether they hold or not an umbrella each day. Assume that the prior probability for rain and not rain is uniform initially and that the transition and observation models are given by Figure 2. What is the probability of rain on the second day if both the first and second day the people entering the building were holding an umbrella?

The prior probability is uniform, which means that:

$$P(R_0) =< 0.5, 0.5 >$$

where R_i is the event that it rains on day i. After the first day, when people enter with an umbrella:

$$P(R_1) = \sum_{r_0} P(R_1|r_0)P(r_0) =< 0.7, 0.3 > \times 0.5 + < 0.3, 0.7 > \times 0.5 = < 0.5, 0.5 >$$

Then:

$$\begin{aligned} P(R_1|u_1) \quad &= \alpha P(u_1|R_1)P(R_1) \\ &= \alpha < 0.9, 0.2 > \times < 0.5, 0.5 > \\ &= \alpha < 0.45, 0.1 > \\ &\approx < 0.818, 0.182 > \end{aligned}$$

The transition to the second day will result in the predictive probability:

$$\begin{aligned} P(R_2|u_1) \quad &= \textstyle\sum_{r_1} P(R_2|r_1)P(r_1|u_1) \\ &=< 0.7, 0.3 > \times < 0.818 > + < 0.3, 0.7 > \times < 0.182 > \\ &=< 0.627, 0.373 > \end{aligned}$$

After observing that the people enter the building with umbrella for a second day:

$$\begin{aligned} P(R_2|u_1, u_2) \quad &= \alpha P(u_2|R_2)P(R_2|u_1) \\ &= \alpha < 0.9, 0.2 > \times < 0.627, 0.373 > \\ &= \alpha < 0.565, 0.075 > \\ &\approx < 0.883, 0.117 > \end{aligned}$$

Consequently, the probability is approximately 88.3% that it is raining on the second day.

1.4 Derivation of Prediction

$$P(X_{t+k}|E_{1:t}), \quad k > 0$$

We have already seen as part of the solution to the filtering problem that we can solve one-step prediction problems just by applying conditioning (conditioning rule: $P(x) = \alpha \sum_y P(x|y)P(y)$):

$$P(X_{t+1}|e_{1:t}) = \sum_{X_t} \underbrace{P(X_{t+1}|X_t)}_{\text{transition model}} \underbrace{P(X_t|e_{1:t})}_{\text{prior belief}}$$

The above rule generalizes as follows for k-step prediction:

$$P(X_{t+k+1}|e_{1:t}) = \sum_{X_{t+k}} \underbrace{P(X_{t+k+1}|X_{t+k})}_{\text{transition model}} \underbrace{P(X_{t+k}|e_{1:t})}_{\text{prior belief}}$$

As the value of k increases, the above probability converges to a stationary distribution, which is actually the prior distribution. For the rain-umbrella example, the stationary probability would be $< 0.5, 0.5 >$.