1 i)
```
n = input('n=');
a = ones(n, n) + n * eye(n, n);
A = a;
b = sum(a, 2)
I = eye(n);
tic
for k = 1:n-1
    l(k+1:n, k) = a(k+1:n, k)/a(k,k)';
    for j = k:n
        for i = k+1:n
            a(i,j) = a(i,j) - a(i,k) * a(k,j);
        end
    end
end
toc
U = a; L = I;
```

ii)

kij		kji	
500	4.77	500	4.86
1000	49.26	1000	38.87
2000	413.92	2000	317.08

The program is cubic.

iii) This has to do with how an array is stored in the memory, row wise or columnwise. In case of Matlab the array is stored in a columnwise manner because of which kji is a better method

4 i) f

2i)

```
function [L, U, P] = lup (A)
  a = A
  n = size (A, 1);
  l = zeroes (n)
  perm = eye(n)
  idx = [ ];
  for k = 1 : n - 1
      loc = a(1:n, k);
      if (~isempty(idx)
          loc (idx) = 0;
      end

  [~, I] = max ( abs(loc));
  idx = [idx; I];
  for j = k:n
     for i = 1:n
        if j = = k && isempty ( find (idx==1, 1 ))
            I (i, k)= a(i, k) / a (I, k);
     end
     if isempty ( find (idx = = 1, 1))
         a (i,j) = a(i, j) - I (i, k) * a(1,j);
     end
   end
end
```

```
for itr2 = 1:n
    if isempty(find(idx == itr2, 1))
        idx = [idx; itr2];
    end
end
L = I(idx, :) + eye(n);
U = a(idx, :);
P = perm(idx, :);
end.
```

2ii) Procedure described below.

- Factorize $A = P_0 L U$
- Let $Ux = z$, solve $Lz = Pb$ by forward substitution.
- Then solve, $Ux = z$ by backward substitution

For large matrices the condition number, $cond(A) = ||A^{-1}|| \, ||A||$ becomes large. Hence, the solution becomes progressively inaccurate.

3. $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $p = [1 \ 2 \ 3]$

$K = 1$, $\max\{|0|, |-1|, |-2|\} = 2$,
interchange row 3 with 1, $p = [3, 2, 1]$

$$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Multiply the first equation by $\frac{1}{2}$ and subtract from the second to obtain:

$$\begin{bmatrix} -2 & 0 & 1 \\ (\frac{1}{2}) & 2 & -\frac{1}{2} \\ (0) & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

$K = 2$ $\max\{2, 1\} = 2$ so no interchange and
$p = [3, 2, 1]$

Multiply the second equation with $\frac{1}{2}$ and subtract from the third equation

$$\begin{bmatrix} -2 & 0 & 1 \\ \left(\frac{1}{2}\right) & 2 & -\frac{1}{2} \\ (0) & \left(\frac{1}{2}\right) & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{3}{4} \end{bmatrix}$$

So,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

To derive P we start with the unit matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and the permutation vector $p = [3, 2, 1]$

The first row becomes the third and the third becomes the first

$$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Test using Matlab.

```
>> A = [ 0  1 -1
        -1  2  0
        -2  0  1 ]

>> [L U P] = lu(A)

L =  1.0    0    0
     0.5   1.0   0
     0     0.5  1.0

U = -2.0    0    1.0
     0     2.0  -0.5
     0      0   -0.75

P =  0  0  1
     0  1  0
     1  0  0

>> P * L * U
ans =  0  1  -1
      -1  2   0
      -2  0   1
```

There are two ways of solving $Ax = B$

Perform the elimination steps for the right hand side vector b at the same time you derive PLU. Then perform Backward substitution. For the example above we have

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{3}{4} \end{bmatrix}$$

$$-\frac{3}{4} x_3 = -\frac{3}{4} \Rightarrow x_3 = 1$$

$$2x_2 - \frac{1}{2}x_3 = \frac{3}{2} \Rightarrow 2x_2 = \frac{3}{2} + \frac{1}{2} * 1 \Rightarrow x_2 = 1$$

$$-2x_1 + 0x_2 + x_3 = -1 \Rightarrow -2x_1 = -1 - x_3$$
$$\Rightarrow -1 - 1 = -2 = -2x_1 \Rightarrow x_1 = 1$$

We determine PLU without performing elimination steps o b and then perform the final solves : $PLUx = b$.

To solve this equation we split it into linear systems by making the following substitutions $PLz = b$, $Ux = z \Rightarrow P^T PLz = P^T b$ and since we know that $P^T P = I$ we get $Lz = P^T b$, $Ux = z$. We now have two systems to solve one using forward substitution and the other backward substitution

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -1/2 \\ 0 & 0 & -3/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\ \frac{1}{2}&1&0\\ 0&\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}0&0&1\\ 0&1&0\\ 1&0&0\end{bmatrix}\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$$

$= \begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$, We now perfom forward substitution : $z_1 = -1$

$\frac{1}{2} z_1 + z_2 = 1 \Rightarrow z_2 = 1 + \frac{1}{2} = \frac{3}{2}$,

$\frac{1}{2} z_2 + z_3 = 0 \Rightarrow z_3 = -\frac{3}{4}$

iv) $\det(A) = \det(PLU) = \det(P)\det(L)\det(U)$

$= -1 * 1 * \left(-2 * 2 * \left(\frac{-3}{4}\right)\right)$

$= -3$

v) $p = [1, 2, 3] \rightarrow$ find the maxiomum and interchange $\rightarrow p = [3, 2, 1]$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$) \det(U)$

$-\frac{3}{4}))$

m

$p = [3, 2, 1]$

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & +\frac{4}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Testing with Matlab

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

>> inv (A)

$$ans = \begin{matrix} -0.6667 & 0.3333 & -0.6667 \\ -0.3333 & 0.6667 & -0.3333 \\ -1.3333 & 0.6667 & -0.3333 \end{matrix}$$

vi) The better method is $PLUx = b$ because we do not have to take out the inverse

```
4i) function [L U a] = kij(a)
     m = size (a);          // size of a
     n = m(1);              // number of columns
     for   k = 1:n          // kij loop
       for i = k+1:n
         a(i,k) = a(i,k)/a(k,k);
         for j = k+1:n
             a(i,j) = a(i,j) - a(i,k)*a(k,j);
         end
       end
     end
     L =  tril (a, -1)       //extract lower
     U  = a - L             // triangular matrix
     L  = L + eye(N)        // add identity
     end                       matrix
```

4 (ii)
```
function [L U a] = kji(a)
m = size(a);
n = m(1);
for k = 1:n
    a(k+1:n, k) = a(k+1:n, k)/a(k,k)
    for j = k:n
        for i = k+1:n
            a(i,j) = a(i,j) - a(i,k)*a(k,j)
        end
    end
end
L = tril(a, -1);
U = a - L;
L = L + eye(n);
end
```

5.i) $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $A - D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

$D^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$M = D^{-1}(A - D) = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

$\|M\|_1 = \max\left\{ \frac{1}{2}+1, \frac{1}{2}+1, \frac{1}{2}+1 \right\}$

$= \frac{3}{2} > 1$

$$\begin{matrix} 1 & 1 \\ 0 & -2 \\ 1 & 0 \end{matrix}\Bigg]$$

$$\|M\|_\infty = \max\left\{\frac{1}{2}+\frac{1}{2},\ \frac{1}{2}+1,\ 1+1\right\}$$

$$= 2 > 1$$

Hence the method does not converge.

checked using MATLAB the method did not converge. :)

5 ii) $A = \begin{bmatrix} -6 & 1 & 1 \\ 1 & 4 & -2 \\ 1 & 1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$

$D = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $A - D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}$

$D^{-1} = \begin{bmatrix} -1/6 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

$M = D^{-1}(A - D) = \begin{bmatrix} 0 & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$

$$\|M\|_1 = \max\left\{ \frac{1}{4} + \frac{1}{3}, \frac{1}{6} + \frac{1}{3}, \frac{1}{6} + \frac{1}{2} \right\}$$

$$= \frac{1}{6} + \frac{1}{2} = \frac{1+3}{6} = \frac{4}{6} = \frac{2}{3} > 1$$

$$\|M\|_\infty = \max\left\{ \frac{1}{6} + \frac{1}{6}, \frac{1}{4} + \frac{1}{2}, \frac{1}{3} + \frac{1}{3} \right\}$$

$$= \frac{2}{3} > 1$$

Hence the method converges

Checked using MATLAB the method did converge.