Errors and more practice problems

We start with an interval $[a, b]$, a function $f(x)$ and n+1 points in that interval $\{(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))\}$ and find a polynomial $P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$. We want to find the error $|f(t) - P_n(t)|$ for all points in the given interval. As we increase n the distance between two consecutive points becomes smaller. For example equidistant points are defined as $x_i = a + h * (i - 1),\ i = 1,2, \dots, n,\ h = \frac{b-a}{n}$ and as n goes to ∞, h the distance between two consecutive points goes to 0. We will show that the error satisfies the following inequality:

$$\max_{a \le x \le b} |f(x) - p_n(x)| \le \max_{a \le x \le b} \frac{|\Omega_n(x)|}{(n+1)!} \max_{a \le x \le b} |f^{(n+1)}(x)|$$

$$\Omega_n(x) = (x - x_0)(x - x_1) \dots (x - x_n).$$

The proof is simple. We add a point $(t, f(t))$ in our $n + 1$ interpolation points. We now have $n + 2$ points and we can find an interpolation polynomial of degree$n + 1$. This polynomial has the form

$$P_{n+1}(x) = P_n(x) + f[x_0, x_1, \dots, x_n, t](x - x_0)(x - x_1) \dots (x - x_n)$$

From the definition of interpolation we know that

$$P_{n+1}(t) = f(t) = P_n(t) + f[x_0, x_1, \dots, x_n, t](t - x_0)(t - x_1) \dots (t - x_n)$$

Which implies that

$$f(t) - P_n(t) = f[x_0, x_1, \dots, x_n, t]\Omega_n(t).$$

This is one form of the error. In other words the Newton's finite differences must become smaller as we add more points in the interval. Another form is to show that

$$f[x_0, x_1, \dots, x_n, t] = \frac{f^{n+1}(\mu)}{(n+1)!}$$

For some μ in the interval $[a, b]$. For simplicity of the proof let's assume equidistant points. Proving for $n = 0$. We have $f[x_0, t] = \frac{f(t) - f(x_0)}{t - x_0} = f'(\mu)$ from Taylor's theorem. Let's prove it for $n = 1$. We have $f[x_0, x_1, t] = f[x_0, t, x_1] = \frac{f[t, x_1] - f[x_0, t]}{x_1 - x_0}$, we have $f(x) = f(t) + (x - t)f'(t) + (x - t)^2 \frac{f''(\mu)}{2!}$, setting $x = x_0$ we get

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$f(x_0) = f(t) + (x_0 - t)f'(t) + (x_0 - t)^2 \frac{f''(\mu)}{2!}$ Which can be rewritten as $\frac{f(x_0)-f(t)}{x_0-t} = f'(t) + (x_0 - t)\frac{f''(\mu)}{2!}$

Implying that $f[x_0, t] = f'(t) + (x_0 - t)\frac{f''(\mu)}{2!}$. Similarly we can show that $f[x_1, t] = f'(t) + (x_1 - t)\frac{f''(\mu)}{2!}$

We have $f[x_1, t] - f[x_0, t] = \frac{(x_1-x_0)f''(\mu)}{2!}$ Implying $\frac{f[x_1,t]-f[x_0,t]}{x_1-x_0} = \frac{f''(\mu)}{2!} = f[x_0, t, x_1] = f[x_0, x_1, t]$

The proof is similar for higher n. We give several examples below.

1. Given the functions below what is the linear and quadratic interpolations polynomials and the error for equidistant points?
   i. $f(x) = e^x$, [0,1]
   ii. $f(x) = x^2$, [0,1]
   iii. $f(x0 = x^3$ [0,1]

2. The composite Trapezoidal rule is given by

   $\int_a^b f(x)dx = T(h) + E(h)$ where $T(h) = \frac{h}{2}(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$
   and $E(h) = -\frac{(b-a)h^2 f''(\xi)}{12}$.

   i. How small h should be chosen so that the error $|E(h)| < \frac{1}{2}\, 10^{-2}$ for the 3 functions given above, and $a = 0, b = 1$.
   ii. Compute the actual results and compare them with your error bounds. How good are the error bounds in predicting the actual results?

3. One wants to solve the equation *x+ln(x)=0*, whose root p≈0.5, by iteration, and one chooses among the following iteration formulas:

   I1: $x_{n+1} = -\ln(x_n)$ *I2:* $x_{n+1} = \exp(-x_n)$ *I3:* *xn+1* $= (x_n + \exp(-x_n))/2$

   (a) Which formula can be used?
   (b) Which formula *should* be used?
   (c) Give an even better formula.

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4. The function $f(x)=x^2+x-2$ has two roots in the intervals [0,3] and [-3,0].
   (a) Perform 3 steps of the bisection method for the root in [0, 3]. How many steps will you need so that the error in the n^{th} iteration of the bisection method is less than 0.0001?
   (b) Perform 2 steps of Newton's method for both roots staring with $x_0=2$ and $x_0=-2$. How many steps will you need to get to 0.0001 error in Newton's method.
   (c) What are the roots?
   (d) A fixed point iteration can be derived by re-writing $x^2+x-2=0$ into its equivalent form $x=2-x^2=g(x)$. Will this iteration converge to any of the two roots? Explain.
   (e) Another fixed point iteration can be derived by re-writing the equation as $x=sqrt(2-x)$. Will this iteration converge to any of the roots?

5. Given the data points (0,2), (1,1), find the following:
   (a) The straight line interpolating these data
   (b) The function $f(x)=a+be^x$ interpolating these data.
   (c) The function $f(x)=a/(b+x)$ interpolating these data.

6. Construct a divided difference table and then determine the interpolation polynomial for the following data:

i.

x	1	$\frac{3}{2}$	0	2
$f(x)$	3	$\frac{13}{4}$	3	$\frac{5}{3}$

ii.

x	0	2	3	4
$f(x)$	1	11	28	63

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7.

(a) In the following finite difference table is it possible to find the values of the boxes with ? marks in them?

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0	**0**	**2**		
			?	
1	**1**	?		**4/6**
			?	
2	**2**	**2/3**		

(b) Given the Lagrange Polynomials $L_i(x)$ for a polynomial of *degree <=3.* Prove that

$$L_0(x) + L_1(x) + L_2(x) + L_3(x) = 1.$$

HINT: There is a hard way and an easy way to prove this. Select the right function for interpolation and use the Lagrange interpolation formula.

8. From the Interpolation polynomial error

$$f[x_0, x_1, \dots, x_i] = \frac{f^{n+1}(\mu)}{(n+1)!}$$

We can see that the difference is zero if the data come from a polynomial of degree n and for all $i > n$. Verify that this is the case for the following data:

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x	1	-2	0	3	-1	7
$F(x)$	-2	-56	-2	4	-16	376

9. Another form of the error for Trapezoidal rule can be given by The General Euler MacLaurin formula is defined by ,

$$\int_a^b f(x)dx = h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a) + f(b)) - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} h^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) + R$$

Where the Bernoulli numbers are given by

$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}, \dots$

and

$h = \frac{b-a}{n}, \quad x_i = a + ih, \ i = 0, \dots, n.$

a. Use the Bernoulli Generating equation to verify few of the numbers above:
$\frac{t}{e^t - 1} = \sum_{j=1}^{\infty} B_j \frac{t^j}{j!}$ where $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

b. Compute $\sum_{i=1}^{n} i^2$ and for $\sum_{i=1}^{n} i^4$ using the above formula for $a = 0, b = n, \ f(x_i) = f(i)$.

10. The mean value theorem for integrals from Calculus sates that $\int_a^b w(x)g(x)dx = w(\mu)\int_a^b g(x)dx$ provided that $g(x)$ does not change sign in the interval of integration. Use this theorem and the interpolation error to show that :

$$\int_0^1 f(x)dx - \frac{1}{2}[f(0) + f(1)] = -\frac{1}{12}f''(\sigma)$$