EXAM-2

1. Given the matrix A = $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

   a. What is $||A||_1$ and $||A||_\infty = 4$
   b. Find the inverse A^{-1} using Gauss Jordan method.

$$\begin{matrix} 0.7500 & -0.2500 & -0.2500 \\ -0.2500 & 0.7500 & -0.2500 \\ -0.2500 & -0.2500 & 0.7500 \end{matrix}$$

   c. Find the condition number $cond(A,'1')$ and $ond(A,'\infty')$ =5?
   The $ond(A,'n') = ||A||_n \, ||A^{-1}||_n$. Where n is a norm, 1 or ∞.
   d. For the matrix A above find an $A = LU$

l =

1.0000	0	0
0.5000	1.0000	0
0.5000	0.3333	1.0000

u =

2.0000	1.0000	1.0000
0	1.5000	0.5000
0	0	1.333

   e. Solve $Ax = b$, where , $b = [4,0,1]^T$ Using $x = A^{-1} * b$

ans =

2.7500

-1.2500
-0.2500

f. Solve $Ax = b$, where $b = [4,0,1]^T$, using $Ly = b$, $Ux = y$.

ans =

y=

4.0000

-2.0000

-0.3333

x =

2.7500

-1.2500

-0.2500

g. For a general nxn system which method is faster in terms of operations (e) or (f). Explain by deriving the number of operations for each approach.

LU is faster $\frac{n^3}{3}$ $vs\ n^3$

2. The normal equations $A^T Ab = A^T y$ minimize $E(b_1, b_2, \dots, b_n) = \sum_{i=1}^{m}(P(x_i) - y_i)^2$ where $P(x) = b_1P_1(x) + b_2P_2(x) + \cdots + b_nP_n(x)$, $n \le m$.

a. For the data x=[1 2 3 4] and y=[1 8 27 64] and the basis functions $\{1, x, x^2\}$ solve the system above and find the solution b.

A =

1 1 1

1 2 4

1 3 9

1 4 16

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```
>> B=A'*A

B =

    4    10    30
   10    30   100
   30   100   354

>> c=A'*y

c =

        100
        354
       1300

>> inv(B)*c

ans =

   10.5000
  -16.7000
    7.5000
```

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b. Can you guess what is the function that generates the above data? x^3

c. Now assume the basis functions is $\{1, x, x^2, x^3 - 2x + 1\}$ can you guess the solution of the above system without writing down the system and solving it. Explain the reason behind your solutions.

$$b_1 + b_2 x + b_3(x^3 - 2x + 1) = x^3$$

$$b_1 + b_3 + (b_2 - 2b_3)x + b_3 x^3 = 0 + 0x + 1x^3$$

$$b_1 + b_3 = 0, b_2 - 2b_3 = 0\,, b_3 = 1. ==> b_1 = -1, b_2 = 2\,, b_3 = 1$$

3. Consider the iterative solution of $Ax = b$ with

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

a. Show that the system has a unique solution .

b. Derive the Gauss-Jacobi method and prove that it will converge for any initial value x_0.

```
>> D=[4 0
0 4]

D =

     4     0
     0     4

>> L=[0 0
1 0]

L =

     0     0
     1     0

>> U=[0 1
0 0]

U =

     0     1
```

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```
     0     0

>> GJ=-inv(D)*(L+U)

GJ =

         0   -0.2500
   -0.2500         0


>> cj=inv(D)*[5;5]

cj =

    1.2500
    1.2500


>> x=[0;0]

x =

     0
     0

>> x=cj+GJ*x

x =

    1.2500
    1.2500

>> x=cj+GJ*x

x =

    0.9375
    0.9375
```

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```
>> x=cj+GJ*x

x =

    1.0156
    1.0156

>> x=cj+GJ*x

x =

    0.9961
    0.9961
```

```
>> D=[4 0
0 4]

D =

     4     0
     0     4

>> L=[0 0
1 0]

L =
```

EXAM-2

```
     0     0
     1     0

>> U=[0 1
0 0]

U =

     0     1
     0     0

>> GS=-inv(L+D)*U

GS =

         0   -0.2500
         0    0.0625

>> cs=inv(L+D)*[5 ;5]

cs =

    1.2500
```

EXAM-2

```
    0.9375

>> x=[0;0]

x =

     0
     0

>> x=cs+GS*x

x =

    1.2500
    0.9375

>> x=cs+GS*x

x =

    1.0156
    0.9961

>> x=cs+GS*x
```

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x =

1.0010

0.9998

>> x=cs+GS*x

x =

1.0001

1.0000

>>

c. Derive the Gauss-Seidel method and prove that it will converge for any initial value x_0 .
   Both norms of iteration matrix is less than one.
d. Perform 4 iteration of Gauss-Seidel and Gauss-Jacobi methods with $b = [5\ 5]^T$. How fast do they converge? (Ratio of the error between consecutive steps). .25 and .0625

>> x=cs+GS*x

>> x=[0 ;0]

x =

0

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```
     0


>> ex=[1;1]


ex =


     1
     1


>> x=cs+GS*x


x =


    1.2500
    0.9375


>> s1=norm(ex-x)


s1 =


    0.2577


>> x=cs+GS*x
```

EXAM-2

```
x =

   1.0156
   0.9961

>> s2=norm(ex-x)

s2 =

   0.0161

>> s2/s1

ans =

   0.0625

>> x=cs+GS*x

x =

   1.0010
   0.9998
```

EXAM-2

```
>> s3=norm(ex-x)

s3 =

   0.0010

>> s3/s2

ans =

   0.0625

>>
```

e. Consider the iteration

$$x^{k+1} = b + \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x^k$$

where α is some real constant. Find all α for which the above method converges for all initial starting values x^0.

$$\left\| \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\| < 1 ==> -\frac{1}{3} < \alpha < \frac{1}{3}$$

4. A. For a symmetric matrix A, is it always the case that $||A||_1 = ||A||_\infty$? Prove it or disprove it. YES symmetric implies the summations are the same.

B. True or False with a Proof: If A is any $n \; x \; n$ matrix and P is any permutation matrix of the same size then (a) $PA = AP$ NO (b) $PA = AP^T$ NO (c) $PP^T = P$ NO(d) $P^T P = I$ YES.

```
>> P=[0 1
1 0]
```

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```
P =

     0     1
     1     0

>> P'*P

ans =

     1     0
     0     1

>> A=[-1 2
3 4]

A =

    -1     2
     3     4

>> P*A

ans =

     3     4
    -1     2

>> A*P

ans =

     2    -1
     4     3

>> P*A

ans =
```

EXAM-2

```
    3     4
   -1     2

>> P*A'

ans =

    2     4
   -1     3

>>
```

C. Can every nonsingular $n\ x\ n$ matrix be written as a product $A = LU$, where L is a lower triangular and U is an upper triangular matrix? (b) If so what is an algorithm for accomplishing this? If not, give a counterexample to illustrate.

NO example

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} * \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}$$

$$u_1 = 0, u_2 = 1, u_1 * l = 1 \ \ impossible!!$$

D. Find a $PLU = A$ for the following matrix.

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

```
a =

    4     1     1
    1     2     0
    0     2     3

>> [L U P]=lu(a)

L =

   1.0000         0         0
```

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```
        0    1.0000         0
   0.2500    0.8750    1.0000


U =

   4.0000    1.0000    1.0000
        0    2.0000    3.0000
        0         0   -2.8750


P =

     1     0     0
     0     0     1
     0     1     0

>>
```