Solutions Quiz 1.

# QUIZ-1 LastName________________________FirstName__________________

1. Taylor's polynomial approximating a function is defined as follows

$$P_n(x) = f(a) + \sum_{j=1}^{n} \frac{f^{(j)}(a)(x-a)^j}{j!}$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(j)}(\mu), \qquad \alpha \le x \le \beta \text{ , and } a \le \mu \le x.$$

$$f(x) = P_n(x) + R_n(x).$$

i. Find the Taylor's polynomial and error for $f(x) = \cos(x)$, $a = 0$, $0 \le x \le \frac{\pi}{2}$.

$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n$, when $n$ gets even values, otherwise 0

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$R_n(x) = \frac{(-1)^m}{(m)!} x^m$, where $m = n+1$ if $n$ even or $m = n+2$ if $n$ odd

ii. Find an upper bound for the error. Show that the error converges to zero as $n$ goes to $\infty$.

$$\max|R_n(x)| = \max\left|\frac{(-1)^m}{(m!)} x^m\right| \le \frac{1}{m!}\left(\frac{\pi}{2}\right)^m$$

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$$\lim_{m\to\infty}\left(\frac{1}{m!}\left(\frac{\pi}{2}\right)^m\right) = \lim_{m\to\infty}\frac{\pi^m}{2\cdot m!} = \frac{\infty}{\infty}, \text{ so } \lim_{m\to\infty}\frac{\frac{1}{(m+1)!}\left(\frac{\pi}{2}\right)^{m+1}}{\frac{1}{m!}\left(\frac{\pi}{2}\right)^m} =$$

$$= \lim_{m\to\infty}\frac{m!}{(m+1)!}\cdot\frac{\pi}{2} = \lim_{m\to\infty}\frac{\pi}{2(m+1)!} = 0$$

2. a. Draw a graph for $f(x) = x^2 - 2$.

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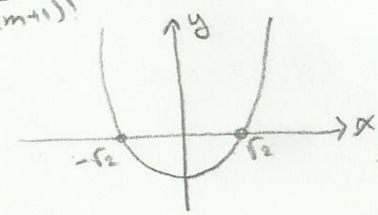


b. Prove that there exist a root in the interval [0, 2].

10 $f(0) = 0 - 2 < 0$
$f(2) = 4 - 2 > 0$
therefore there is at least one root between $[0, 2]$

c. Compute two steps for each of the following methods for the positive root.

i. Bisection Method in the interval [0,2]

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$\underline{n=1}$ $[0,2]$ $c=1$
$f(0) = 0 - 2 = -2 < 0$
$f(2) = 4 - 2 = 2 > 0$
$f(1) = 1 - 2 = -1 < 0$

$\underline{n=2}$ $[1, 2]$ $c = 1.5$
$f(1) = 1 - 2 = -1 < 0$
$f(2) = 4 - 2 = 2 > 0$
$f(1.5) = 0.25$

ii. Newton's method starting at $x_0 = 2$

$f'(x) = 2x$, $f(2) = 2$, $f'(2) = 4$

15 $\underline{n=1}$ $x_1 = 2 - \frac{2}{4} = 1.5$

$\underline{n=2}$ $x_2 = 1.5 - \frac{(1.5)^2 - 2}{3} = 0.91$

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3. Prove that the Bisection method requires $p \cong \log_2 \frac{b-a}{\varepsilon}$ steps for the error $|x_p - \alpha| \leq \epsilon$ where $a \leq \alpha \leq b,\ \alpha = unique\ root.$

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When we have done n iterations:

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$$

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \qquad (1)$$

If root is $\alpha$ then:

$$|\alpha - x_n| \leq x_n - a_n = b_n - x_n = \frac{1}{2}(b_n - a_n) \overset{(1)}{=} \frac{1}{2^n}(b-a)$$

$$\frac{1}{2^n}(b-a) \leq \varepsilon$$

$$\frac{(b-a)}{e} \leq 2^n$$

$$n \cong \log_2 \frac{b-a}{\varepsilon}$$