QUIZ 3 Practice: April 30 during lecture.

1. We want to solve Ax=b. Solve the system by using Gaussian elimination with partial pivoting for the following linear systems:

   i. $$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

   Find PLU and solve $PLUx = b$

   Solution: $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, p = [1\ 2\ 3].$

   $K = 1, max\{|0|, |-1|, |-2|\} = 2$, interchange row 3 with 1, $p = [3,2,1]$

   $$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

   Multiply the first equation by $\frac{1}{2}$ and subtract from the second to obtain:

   $$\begin{bmatrix} -2 & 0 & 1 \\ (\frac{1}{2}) & 2 & -\frac{1}{2} \\ (0) & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

   Notice that we save the multipliers in the positions of zero's of the matrix.

   $K = 2\ max\{2,1\} = 2$ so No interchange and , $p = [3,2,1]$

   Multiply the second equation with $\frac{1}{2}$ and subtract from the third equation:

   $$\begin{bmatrix} -2 & 0 & 1 \\ (\frac{1}{2}) & 2 & -\frac{1}{2} \\ (0) & (\frac{1}{2}) & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{3}{4} \end{bmatrix}$$

   So

   $$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, U = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

   And to derive P we start with the unit matrix:

   $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the permutation vector: $p = [3,2,1]$. The first row becomes the third and the third becomes the first:

   $$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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Test using Matlab:

```
>> A=[0 1 -1
-1 2 0
-2 0 1]

A =

     0     1    -1
    -1     2     0
    -2     0     1

>> [L U P]=lu(A)

L =

    1.0000         0         0
    0.5000    1.0000         0
         0    0.5000    1.0000


U =

   -2.0000         0    1.0000
         0    2.0000   -0.5000
         0         0   -0.7500


P =

     0     0     1
     0     1     0
     1     0     0

>> P*L*U

ans =

     0     1    -1
    -1     2     0
    -2     0     1

>>
```

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There are two ways of solving $Ax = b$.

a. Perform the elimination steps for the Right Hand Side vector b at the same time you derive PLU. Then perform Backward substitution. For the example above we have:

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{3}{4} \end{bmatrix}$$

$$-\frac{3}{4}x_3 = -\frac{3}{4} ==> x_3 = 1,$$

$$2x_2 - \frac{1}{2}x_3 = \frac{3}{2} ==> 2x_2 = \frac{3}{2} + \frac{1}{2} * 1 = 2 ==> x_2 = 1,$$

$$-2x_1 + 0x_2 + x_3 = -1 ==> -2x_1 = -1 - x_3 = -1 - 1 = -2 ==> \ -2x_1 = -2 ==> x_1 = 1.$$

b. We determine PLU without performing elimination steps on b and then perform the final solver:$PLUx = b$. To solve this equation we split it into two linear systems by making the following substitution:$PLz = b, Ux = z ==> P^T PLz = P^T b$, and since we know that $P^T P = I$ we get $Lz = P^T b, Ux = z$. We now have two systems to solve one using forward substitution and the other backward substitution.

c. $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

We now perform forward substitution: $z_1 = -1$,

$$\frac{1}{2}z_1 + z_2 = 1 ==> z_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$\frac{1}{2}z_2 + z_3 = 0 ==> z_3 = -\frac{3}{4}$

Next Backward substitution—which it was done above.

ii. What is the det(A)? $\det(A) = \det(PLU) = \det(P)\det(L)\det(U) = -1 * 1 * (-2 * 2 * \left(-\frac{3}{4}\right)) = -3$

iii. Find the inverse of A by using Gauss-Jordan method—eliminating both above and below the diagonal at the same time using the extended system:

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$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify your result by using $inv(A)$ in Matlab. Then derive the computational complexity of inverting a general nxn matrix using the Gauss-Jordan method.

Solution: $p = [1,2,3]$,➔ find the maximum and interchange➔$p = [3,2,1]$

$$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

➔

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

➔

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

⇨ $$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

⇨ $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

⇨ $$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{4}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$ ➔ Testing with MATLAB

$>> \ A$

$A \ =$

$$\begin{matrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{matrix}$$

$>> \ inv(A)$

$ans \ =$

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$$\begin{matrix} -0.6667 & 0.3333 & -0.6667 \\ -0.3333 & 0.6667 & -0.3333 \\ -1.3333 & 0.6667 & -0.3333 \end{matrix}$$

2. The Gauss-Jacobi iterative method is defined as follows: $Ax = b ==> [D + (A - D)]x = b ==> Dx = b - (A - D)x ==> x = D^{-1}b - D^{-1}(A - D)x$ so we can derive an iterative method: $x^{k+1} = D^{-1}b - D^{-1}(A - D)x^k$ Write a matlab program that implements this method using Matlab matrix definitions e.g. $D = tril(A) - tril(A, -1)$; $x(k + 1) = inv(D) * (b - (A - D) * x(k))$, you can use as stopping criterion $norm(x(k + 1) - x(k), \gamma), \gamma = 2, or\ \gamma = 1, or\ \gamma = inf$ for the 3 norms that we learned in class, the 2 norm or the 1 norm or the infinity or maximum norm. Apply this method to the following system:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A - D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \ D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$M = D^{-1}(A - D) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

$$x^1 = D^{-1}b - Mx^0, \ x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x^1 = D^{-1}b = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{4} \\ \frac{5}{3} \end{bmatrix}$$

$$x^2 = D^{-1}b + Mx^1 = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{4} \\ \frac{5}{3} \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ \frac{7}{4} \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{4} \\ \frac{5}{3} \end{bmatrix} - \begin{bmatrix} \frac{7}{8} \\ \frac{29}{24} \\ \frac{13}{12} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} \\ \frac{13}{24} \\ \frac{7}{12} \end{bmatrix} \ldots\ldots$$

Norms of M: The max Sum of absolute values of rows is 1-Norm, The max of Sum of absolute values of columns is the maximum or infinite norm.

$$||M||_1 = \max\left\{\frac{1}{4} + \frac{1}{3}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1, ||M||_\infty = \max\left\{\frac{1}{2}, \frac{1}{4} + \frac{1}{2}, \frac{1}{3} + \frac{1}{3}\right\} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1$$

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So the iteration will converge to the solution.

```
>> x=[0
0
0]

x =

     0
     0
     0

>> x=inv(D)*b-M*x

x =

    1.5000
    1.7500
    1.6667

>> x=inv(D)*b-M*x

x =

    0.6250
    0.5417
```

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```
   0.5833

>> x=inv(D)*b-M*x

x =

   1.2292
   1.3021
   1.2778

>> x=inv(D)*b-M*x

x =

   0.8490
   0.8038
   0.8229

>> x=inv(D)*b-M*x

x =

   1.0981
   1.1263
   1.1157
```

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>> x=inv(D)*b-M*x

x =

0.9368

0.9176

0.9252

3. The kij form of Gaussian Elimination without partial pivoting is given by :

```
function [L U a] = kij( a )
m=size(a);
n=m(1);
for k=1:n
    for i=k+1:n
        a(i,k)=a(i,k)/a(k,k);
        for j=k+1:n
            a(i,j)=a(i,j)-a(i,k)*a(k,j);
        end
    end
end
L = tril(a,-1);
U=a-L;
L=L+eye(n);
end
```

What is the complexity of this program: $\sum_{k=1}^{n} 1 \sum_{i=k+1}^{n} 1 \sum_{j=k+1}^{n} 1 = \sum_{k=1}^{n} (n-k)^2 = \sum_{k=1}^{n} k^2 \approx \frac{n^3}{3}$