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1. Taylor's polynomial approximating a function is defined as follows

$$P_n(x) = f(a) + \sum_{j=1}^{n} \frac{f^{(j)}(a)(x-a)^j}{j!}$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(j)}(\mu), \qquad \alpha \le x \le \beta \text{ , and } a \le \mu \le x.$$

$$f(x) = P_n(x) + R_n(x).$$

   i. Find the Taylor's polynomial and error for $f(x) = \mathrm{e}^{x}$, $a = 0$, $0 \le x \le 1$

   ii. Find an upper bound for the error. Show that the error converges to zero as n goes to ∞.

Solutions: i. $f^{(j)}(x) = e^x$, $f^{(j)}(0) = 1, ==> P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

Ii $R_n(x) = \frac{x^{n+1}e^{\mu}}{(n+1)!} \le \frac{e}{(n+1)!}$ Which converges to zero as n increases.

2. One wants to solve the equation $x + ln(x) = 0$, whose root $p \approx 0.5$, by iteration, and one chooses among the following iteration formulas:

   I1: $x_{n+1} = -ln\,(x_n)$ *I2:* $x_{n+1} = e^{-x_n}$ *I3*: $x_{n+1} = \frac{x_n + e^{-x_n}}{2}$

   a. Which formula can be used? I2,I3

   b. Which formula *should* be used? I3

   c. Give an even better formula. Newton's method

SOLUTION: $g(x) = -\ln(x)\,, g'(x) = -\frac{1}{x}, g'\left(\frac{1}{2}\right) = -2, \left|g'\left(\frac{1}{2}\right)\right| = 2 > 1. NO$

$$g(x) = \exp(-x)\,, g'(x) = -\exp(-x)\,,$$

$$g'\left(\frac{1}{2}\right) = -\exp\left(-\frac{1}{2}\right), \left|g'\left(\frac{1}{2}\right)\right| = 0.61 < 1\ \ YES$$

$$g(x) = \frac{x + e^{-x}}{2}, g'(x) = \frac{1 - e^{-x}}{2}, g'\left(\frac{1}{2}\right) = \frac{1 - 0.61}{2} = 0.2, \left|g'\left(\frac{1}{2}\right)\right| < 1, YES$$

A better method is $g(x) = x - \frac{x + \ln(x)}{1 + \frac{1}{x}}$, Newton's method.

3. The function $f(x) = x^2 + x - 2$ has two roots in the intervals $[0,3]$ and $[-3,0]$.

   i. What are the roots? Roots: 1,-2

   ii. Perform 2 steps of the bisection method for the root in $[0, 3]$. How many steps will you need so that the error in the n^{th} iteration of the bisection method is less than 0.0001?

$$x_1 = \frac{0+3}{2} = \frac{3}{2}, \qquad f(0) * f\left(\frac{3}{2}\right) < 0, \left[0, \frac{3}{2}\right], x_2 = \frac{3}{4}$$

   iii. Perform 2 steps of Newton's method for both roots starting with x_0=2.

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$$x_0 = 2, x_1 = x_0 - \frac{x_0^2 + x_0 - 2}{2x_0 + 1} = 2 - \frac{4}{5} = \frac{6}{5}, x_2 = \frac{6}{5} - \frac{\left(\frac{6}{5}\right)^2 + \frac{6}{5} - 2}{\frac{12}{5} + 1}$$

$$= \frac{6}{5} - \frac{\frac{36}{25} + \frac{30}{25} - \frac{50}{25}}{\frac{17}{5}} = \frac{6}{5} - \frac{\frac{16}{25}}{\frac{17}{5}} = \frac{1}{5}\left(6 - \frac{16}{17}\right) = \frac{86}{85} = 1.0118$$

iv. Prove that Newton's method converges to the positive root for all $x_0 > -\frac{1}{2}$.

HINT: The error for Newton's method is given by where p is the positive root:

$$|x_{n+1} - p| = |x_n - p|^2\, M_n, \quad M_n = |f''(\sigma_n)|/|2f'(x_n)|$$

First show that error of the next step is less than the error of the previous step for all positive starting points and next show that $(-\frac{1}{2}, 0]$ is mapped to positive by Newton's method.

$$|x_{n+1} - 1| = |x_n - 1|\,|x_n - 1|\,M_n, M_n = \frac{1}{|2x_n + 1|}, \quad ==> \frac{|x_n - 1|}{|2x_n + 1|} < 1 ==>$$

$$-1 < \frac{x_n - 1}{2x_n + 1} < 1\ , ==> \ -2x_n - 1 < x_n - 1 < 2x_n + 1\ ,$$

$$==> 3x_n > 0 \ \ and\ x_n > -2 ==> x_n > 0.$$

So for $x_n > 0, \frac{|x_n - 1|}{|2x_n + 1|} < 1$, implying that the error reduces for each step for all positive initial choices. To prove that this is also true in the interval $\left(-\frac{1}{2}, 0\right)$ we need to prove that $x_1 > 0$. When will $x_1 = x_0 - \frac{x_0^2 + x_0 - 2}{2x_0 + 1} > 0, if\ 2x_0 + 1 > 0\ then$ ➔$2x_0^2 + x_0 - x_0^2 - x_0 + 2 > 0$

$==>$ $x_0^2 + 2 > 0$ always true. Go Backwards for the proof.

v. A fixed point iteration can be derived by re-writing $x^2 + x - 2 = 0$ into its equivalent form $x = 2 - x^2 = g(x)$. Will this iteration converge to any of the two roots? Explain.

$$g'(x) = -2x ==> |g'(1, -2)| > 1\ for\ both\ roots.\ \ NO$$

vi. Another fixed point iteration can be derived by re-writing the equation as $x = \sqrt{2 - x}$. Will this iteration converge to any of the roots?

$$g'(x) = -\frac{1}{2\sqrt{2 - x}} ==> |g'(1, -2)| < 1 \ \ YES.$$

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4. We want to interpolate the following data:

x	0	1	2	3	4
$f(x)$	1	1	7	25	61

i. Determine Newton's divided difference Table.

ii. Give the interpolation polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ using the Table.

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iii. Do these data come from a polynomial? Explain.

iv. What is the error form for $P_2(x)$?

SOLUTION:

x	$F[x]$	$F[,]$	$F[,,]$	$F[,,,]$	
0	1				
1	1	0			
2	7	6	3		
3	25	18	6	1	
4	61	36	9	1	0

$$P_1(x) = 1, P_2(x) = 1 + 3(x-0)(x-1), P_3(x) = 1 + 3x(x-1) + x(x-1)(x-2), P_4(x) = P_3(x)$$

$$Error = |P_2(x) - P_3(x)| = |x(x-1)(x-2)|$$

5. A. There exist a unique polynomial $p(x)$ of degree 2 or less such the $p(0) = 0,\ p(1) = 1, p'(\alpha) = 2$ for any value of $0 \le \alpha \le 1$, except one value of α, say α_0. Determine α_0 and give the polynomial for $\alpha \neq \alpha_0$.

$p(x) = ax^2 + bx + c, p(0) = 0 = c, p(1) = 1 = a + b, p'(\alpha) = 2 = 2a\alpha + b ==> 2 = 2a\alpha + 1 - a ==> 1 = a(2\alpha - 1) ==> a = \frac{1}{2\alpha-1}$ provided $2\alpha - 1 \neq 0 ==>\ \alpha \neq \frac{1}{2}$

$$b = 1 - \frac{1}{2\alpha - 1}.$$

B. Given that a polynomial $g(x)$ interpolates the function $f(x)$ at $x_1, x_2, \dots, x_{n-1}$ and polynomial $h(x)$ interpolates $f(x)$ at the points $x_2, x_3, \dots, x_n$. Prove that the function

$$F(x) = g(x) + \frac{x_1 - x}{x_n - x_1}[g(x) - h(x)]$$

interpolates $f(x)$ at $x_1, x_2, x_3, \dots, x_n$, i.e. $F(x_i) = f(x_i), i = 1,2, \dots, n$.

$$F(x_1) = g(x_1) + \frac{x_1 - x_1}{x_n - x_1}[g(x_1) - h(x_1)] = g(x_1) = f(x_1)$$

$$F(x_n) = g(x_n) + \frac{x_1 - x_n}{x_n - x_1}[g(x_n) - h(x_n)] = h(x_n) = f(x_n)$$

$$F(x_i) = g(x_i) + \frac{x_1 - x_i}{x_n - x_1}[g(x_i) - h(x_i)] = g(x_i) = f(x_i), \quad because\ g(x_i) = h(x_i)$$