# HW-4-with solutions

1. Another form of the error for Trapezoidal rule can be given by The General Euler McLaurin formula is defined by ,

$$\int_a^b f(x)dx = h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a)+f(b)) - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} h^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) + R$$

Where the Bernoulli numbers are given by

$B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}, \ldots$

and

$h = \frac{b-a}{n},\quad x_i = a + ih,\ i = 0, \ldots, n.$

a. Use the Bernoulli Generating equation to verify few of the numbers above: $\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}$ where $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

b. Compute $\sum_{i=1}^{n} i^2$ and for $\sum_{i=1}^{n} i^4$ using the above formula for $a = 0, b = n,\ f(x_i) = f(i)$.

SOLUTIONS:

a. $\frac{1}{1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} = B_0 + \frac{B_1 t}{1!} + \frac{B_2 t^2}{2!} + \cdots ==> 1 = B_0 + B_1 t + \frac{B_2 t^2}{2} + .. + \frac{B_0 t}{2} + \frac{B_1 t^2}{2} + \frac{B_2 t^3}{4} + \cdots . + \frac{B_0 t^2}{6} + \cdots . = B_0 + \left(B_1 + \frac{B_0}{2}\right)t + \left(\frac{B_2}{2} + \frac{B_1}{2} + \frac{B_0}{6}\right)t^2 + \cdots . ==>$ $B_0 = 1, B_1 + \frac{B_0}{2} = 0, \frac{B_2}{2} + \frac{B_1}{2} + \frac{B_0}{6} = 0, \ldots ==> B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots$

b. $\int_0^n x^2 dx = \sum_{i=0}^{n} i^2 - \frac{1}{2}(n^2 + 0) - \frac{1}{2*6}(2n - 0) ==> \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \sum_{i=0}^{n} i^2$

c. $\int_0^n x^4 dx = \sum_{i=0}^{n} i^4 - \frac{1}{2}(n^4) - \frac{1}{12}(3n^3) + \frac{1}{30} * \frac{1}{4!}(4*3*2n) = \frac{n^5}{5} ==>$ $\sum_{i=0}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$

2. The mean value Theorem for integrals from Calculus sates that $\int_a^b w(x)g(x)dx = w(\mu)\int_a^b g(x)dx$ provided that $g(x)$ does not change sign in the interval of integration. Use this theorem and the interpolation error to show that :

$$\int_0^1 f(x)dx - \frac{1}{2}[f(0) + f(1)] = -\frac{1}{12} f''(\sigma)$$

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SOLUTION: $\int_0^1 f(x)dx - \frac{1}{2}[f(0)+f(1)] = \int_0^1 [f(x) - P_1(x)] = \int_0^1 \frac{f''(\theta)x(x-1)}{2} dx = \frac{1}{2}\int_0^1 f''(\theta)x(x-1)dx = \frac{1}{2} f''(\sigma)\int_0^1 x(x-1)dx = \frac{1}{2} f''(\sigma)[\frac{x^3}{3} - \frac{x^2}{2}]0..1 = \frac{1}{2} f''(\sigma)[\frac{1}{3} - \frac{1}{2}] = -\frac{1}{12} f''(\sigma)$

3. The equation from 1 can be rewritten as follows:

$$I = \int_a^b f(x)dx = T(h) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} h^{2k} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)$$
$$= T(h) + A_1 h^2 + A_2 h^4 + A_3 h^6 + \cdots$$

Where

$T^{(0)}(h) = \ h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a) + f(b)) = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) \ + f\,(x_n)$ is the trapezoidal rule. We will derive Romberg's integration. The basic idea is that if you know the error or a form of the error you can get a better approximation of your integration. Let us assume that we have computed $T^{(0)}(h)$ and $T^{(0)}\left(\frac{h}{2}\right)$ then we have

$$I = T^{(0)}(h) + A_1 h^2 + A_2 h^4 + A_3 h^6 + \cdots$$
$$I = T^{(0)}\left(\frac{h}{2}\right) + A_1\left(\frac{h}{2}\right)^2 + A_2\left(\frac{h}{2}\right)^4 + A_3\left(\frac{h}{2}\right)^6 + \cdots$$

Multiply the second equation by $2^2$ we get

$$2^2 I = 2^2 T^{(0)}\left(\frac{h}{2}\right) + A_1 h^2 + 2^2 A_2\left(\frac{h}{2}\right)^4 + 2^2 A_3\left(\frac{h}{2}\right)^6 + \cdots$$

Subtracting the two equations above will eliminate $A_1 h^2$ factor:

$$2^2 I - I$$
$$= 2^2 T^{(0)}\left(\frac{h}{2}\right) + A_1 h^2 + 2^2 A_2\left(\frac{h}{2}\right)^4 + 2^2 A_3\left(\frac{h}{2}\right)^6 - T^{(0)}(h) - A_1 h^2$$
$$- A_2 h^4 - A_3 h^6 + \cdots = 2^2 T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h) + A_2^* h^4 + \cdots.$$

Or equivalently: $= T^{(1)}(h) + A_2^* h^4 + \cdots$ . where $T^{(1)}(h) = \frac{4\,T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h)}{4-1}$ so we can say that the error for $O(h^4)$ which is more accurate than the error of trapezoidal rule $O(h^2)$. This new rule is called the corrected trapezoidal rule. The following table shows that trapezoidal rule and corrected Trapezoidal rule for different $h$:

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| $O(h^2)$ | $O(h^4)$ | $O(h^6)$ | $O(h^8)$ |
|---|---|---|---|

| | | | |
|---|---|---|---|
| $T^0(h)$ | | | |
| $T^0(\frac{h}{2})$ | $T^{(1)}(h) = \dfrac{4\,T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h)}{4-1}$ | | |
| $T^0(\frac{h}{2^2})$ | $T^{(1)}\left(\frac{h}{2}\right) = \dfrac{4\,T^{(0)}\left(\frac{h}{2^2}\right) - T^{(0)}\left(\frac{h}{2}\right)}{4-1}$ | $T^{(2)}(h) = \dfrac{4^2\,T^{(1)}\left(\frac{h}{2}\right) - T^{(1)}(h)}{4^2-1}$ | |
| $T^0(\frac{h}{2^3})$ | $T^{(1)}\left(\frac{h}{2^2}\right) = \dfrac{4\,T^{(0)}\left(\frac{h}{2^3}\right) - T^{(0)}\left(\frac{h}{2^2}\right)}{4-1}$ | $T^{(2)}(h) = \dfrac{4^2\,T^{(1)}\left(\frac{h}{2^2}\right) - T^{(1)}\left(\frac{h}{2}\right)}{4^2-1}$ | $T^{(3)}(h) = \dfrac{4^3\,T^{(2)}\left(\frac{h}{2}\right) - T^{(2)}(h)}{4^3-1}$ |
| | | | |
| | | | |

i. Show how we derive the $O(h^6)$ formula in the table by using $I = T^{(1)}(h) + A_2^* h^4 + \cdots$ and the same idea we used for deriving the $O(h^4)$ formula.

SOLUTION: $I = T^{(1)}(h) + A_2^* h^4 + \cdots ==> I = T^{(1)}\left(\frac{h}{2}\right) + A_2^*\left(\frac{h}{2}\right)^4 + \cdots ==>$ $4^2 I = 4^2 T^{(1)}\left(\frac{h}{2}\right) + A_2^* h^4 ==> subtract: (4^2-1)I =$ $4^2 T^{(1)}\left(\frac{h}{2}\right) - T^{(1)}(h)\ \ solve\ to\ get\ O(h^6)$

ii. Show how we derive the $O(h^8)$ formula in the table by using $I = T^{(2)}(h) + A_3^* h^6 + \cdots$ and the same idea we used for deriving the $O(h^4)$ formula.

$$I = T^{(2)}(h) + A_2^* h^6 + \cdots ==> I = T^{(2)}\left(\frac{h}{2}\right) + A_2^*\left(\frac{h}{2}\right)^6 + \cdots ==> 4^3 I$$
$$= 4^3 T^{(2)}\left(\frac{h}{2}\right) + A_2^* h^6 ==> subtract: (4^6-1)I$$
$$= 4^3 T^{(2)}\left(\frac{h}{2}\right) - T^{(2)}(h)\ \ solve\ to\ get\ O(h^8)$$

iii. Write a MATLAB program to compute the trapezoidal method $T^{(0)}(h)$. Remember that $n = \frac{b-a}{h}$ the number of subintervals in the trapezoidal rule $T^{(0)}(h) =$ $h\sum_{i=0}^{n} f(x_i) + B_1 h(f(a) + f(b)) = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i)\ + f\,(x_n)$

and $x_i = a + ih, i = 0{:}\,n$. Submit your program along with your table results.

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```
function integral=trapezoidal(a,b,h,index_f)
n=(b-a)/h;

% Initialize for trapezoidal rule.
sumend = (f(a,index_f) +f(b,index_f))/2;
sum = 0;

for i=1:1:n-1
   sum = sum + f(a+i*h,index_f);
end
integral = h*(sumend + sum);


function f_value = f(x,index)
%
% This defines the integrand.

switch index
case 1
    f_value = x;
case 2
    f_value = x^7;

end
```

iv. Use the table to derive approximation for $\int_0^1 x^7 dx = \frac{1}{8}$ starting with $h = 1 - 0 = 1$

The table formulae are also known as Romberg's integration.

| $O(h^2)$ | $O(h^4)$ | $O(h^6)$ | $O(h^8)$ |
|---|---|---|---|
| 0.500000000000000 | | | |
| 0.253906250000000 | 0.171875000000000 | | |
| 0.160339355468750 | 0.129150390625000 | 0.126302083333333 | |
| 0.134043693542480 | 0.125278472900390 | 0.125020345052083 | 0.125000000000000=1/8 |

4. The method of undetermined coefficients assumes that the error in a quadrature is zero for a given set of functions and then determines the coefficient by solving a system of linear equations. Apply this method to the following integrals. Determine

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A, B ,C and c so that the formula is exact when $f(x)$ is polynomial of the highest possible degree.

a.

$$\int_0^1 f(x)dx \approx A\,f(0) + Bf\left(\frac{1}{2}\right) + C\,f(1)$$

b.

$$\int_0^1 \sqrt{x}\,f(x)dx \approx Af(c) + Bf(1)$$

Determine the error of the above integration formulae.

c. Use the above formulae to estimate the integral for $f(x) = x^3$.

SOLUTION:a. :$f(x) = 1, x, x^2$ $A + B + C = 1,\ \frac{B}{2} + C = \frac{1}{2}, \frac{B}{4} + C = \frac{1}{3}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{12} \end{bmatrix} \rightarrow C = \frac{1}{6}, B = \frac{4}{6}, A = \frac{1}{6}$$

Error:

$\int_0^1 f(x)dx - \frac{1}{6}[f(0) + 4f\left(\frac{1}{2}\right) + f(1)] = Df''' + Ef'^{v} + \cdots +$

Set $f(x) = x^3$

$$\int_0^1 x^3 dx - \frac{1}{6}[0 + 4\left(\frac{1}{2}\right)^3 + 1] = D3! ==> \frac{1}{4} - \frac{1}{6}\left[\frac{3}{2}\right] = 6D ==> 0 = D$$

Next we try $f(x) = x^4$

$$\int_0^1 x^4 dx - \frac{1}{6}[0 + 4\left(\frac{1}{2}\right)^4 + 1] = E4! ==> \frac{1}{5} - \frac{1}{6} * \frac{5}{4} = 24E ==> -\frac{1}{5*6*4} = 24E ==> E$$

$= 2880$

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B

$\int_0^1 \sqrt{x}\, dx \approx A + B = \frac{2}{3},\ \ Ac + B = \frac{2}{5},$

$Ac^2 + B = \frac{2}{7} ==> A(1-c) = \frac{4}{15}, \qquad A(c-c^2) = \frac{4}{35} ==> c = \frac{\frac{4}{35}}{\frac{4}{15}} = \frac{15}{35}, \qquad A =$

$\frac{\frac{4}{15}}{\frac{20}{35}} = \frac{35*4}{15*20} = \frac{35}{75}, B = \frac{45}{225}$

Error: $\int_0^1 \sqrt{x}\ x^3 dx - \frac{35}{75} * \left(\frac{15}{35}\right)^3 - \frac{45}{225} = 3!\, D ==> D = \frac{\left[\frac{2}{9} - \frac{35}{75} * \left(\frac{15}{35}\right)^3 - \frac{45}{225}\right]}{3!} = -0.0024$

Error== $-0.0024 f'''(\mu)$

C. $\int_0^1 x^3 dx \approx \frac{1}{6}\left(0 + 4\left(\frac{1}{2}\right)^3 + 1\right) = \frac{1}{4}$ Exact because the error is zero for this function.

$$\int_0^1 \sqrt{x}\ x^3 dx \approx \frac{35}{75} * \left(\frac{15}{35}\right)^3 + \frac{45}{225} = 0.236734693877551$$

$$exact\ error = \left|\frac{2}{9} - 0.236734693877551\right| = 0.014512471655329$$

$error\ from\ formula = 0.0024 * 3! = 0.0144$

Formula AGREE