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1. Taylor's polynomial approximating a function is defined as follows

$$P_n(x) = f(a) + \sum_{j=1}^{n} \frac{f^{(j)}(a)(x-a)^j}{j!}$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(j)}(\mu), \qquad \alpha \le x \le \beta \text{ , and } a \le \mu \le x.$$

$$f(x) = P_n(x) + R_n(x).$$

i. Derive the Taylor's polynomial for $f(x) = \sin(x)$, $a = 0$. The derivatives $\frac{d(\sin(x))}{dx} = \cos(x)$, $\frac{d(\cos(x))}{dx} = -\sin(x)$.

Solution: $P_n(x) = \sum_{j=1}^{\frac{n-1}{2}} \frac{(-1)^j (x)^{2j+1}}{(2j+1)!}$ 5pts

ii. Derive an error bound using $\max |R_n(x)|$.

$|R_n(x)| \le \frac{|x|^{n+2}}{(n+2)!}$ 5pts

iii. How many steps n will it take for the method to achieve in $[0,1]$ an $error \le 10^{-6}$.

$|R_n(x)| \le \frac{|x|^n}{(n+2)!} < 10^{-6}$, Assume the interval $[0,1]$, then $(n+2)! > 10^6, n = 10.$ 5pts

iv. What is the best way to evaluate the expression $\frac{e^x-1}{x}$ when x is near zero. Explain your approach.

$\frac{e^x-1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{n-1}}{n!}$ 5pts

v. What is the largest positive integer number represented in 32 bit arithmetic?

$+1111111111111111111111111111111_2 = 2^{30} + 2^{29} + \cdots + 2^0 = 2^{31} - 1$ 5pts

2. A. Derive Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,\ldots$ by using second order Taylor's polynomial $f(x) \cong P_1(x)$ and assuming that $P_1(x_{n+1}) = 0$ and $= x_n$.

$P_1(x) = f(a) + (x-a)f'(a)$ 5pts

$0 = f(x_n) + (x_{n+1} - x_n)f'(x_n)$ is Newton's method.

The error is given from

$f(p) = 0 = f(x_n) + (p - x_n)f'(x_n) + \frac{(p-x_n)^2 f''(\mu)}{2} = f(x_n) + (p - x_{n+1} + x_{n+1} - x_n)f'(x_n) + \frac{(p-x_n)^2 f''(\mu)}{2}$ implying

$(p - x_{n+1})f'(x_n) + \frac{(p-x_n)^2 f''(\mu)}{2} = 0$

So we can derive the formula in B below:

B. Use the error in Taylor's polynomial

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$$f(x) = P_2(x) + R_2(x)$$

to show that $p - x_{n+1} = (p - x_n)^2[-\frac{f''(c_n)}{2f'(x_n)}]$ where $f(p) = 0$, p is the root. Explain why Newton's method always converges "near" the root.

$p - x_{n+1} = (p - x_n)^2\left[-\frac{f''(c_n)}{2f'(x_n)}\right] = (p - x_n)(p - x_n)M_n, where\ M_n = \left[-\frac{f''(c_n)}{2f'(x_n)}\right].$ 5pts

So if M_n is bounded then $|(p - x_n)M_n| < 1$ when the iteration is close to the root implying that it always converges locally(i.e. when the iteration is close to the root) 5pts

C. We want to find the roots $p = \pm\sqrt{3} = \pm\ 1.732050807568877$ of the function $f(x) = x^2 - 3 = 0$. Give Newton's iteration for this function $f(x) = x^2 - 3 = 0$. Perform two steps of the iteration starting with $x_0 = 2$.

$x_1 = 2 - \frac{2^2-3}{2*2} = \frac{7}{4}$=1.75, $x_2 = \frac{7}{4} - (\left(\frac{7}{4}\right)^2 - 3)/(2 * \frac{7}{4})$= 1.732142857142857 5pts

$error = 9.2 * 10^{-5}$ since the convergence is quadratic it will take two more steps: 5pts

How many more steps will it take to achieve that accuracy given above for $p = \pm\sqrt{3}$ $(i.e. eps = 10^{-16})$.

D. Perform two steps for $f(x) = x^2 - 3 = 0$ using the bisection method in the interval $[0,2]$. Which method is faster Newton's or bisection ? Explain.

$[0,2], [1.5,2], [1.5,1.75]$ 5pts

Newton's method is faster since it is quadratic vs linear of bisection method 5pts

3. You have to choose between the following 3 fixed point iterations:
   i. $x_{n+1} = \frac{3}{x_n}$ ii. $x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n})$ iii. $x_{n+1} = \frac{x_n(x_n^2+9)}{3x_n^2+3}$
   a. Which one converges to the root locally(i.e. near the root always converges).
      i. Does not converge since $|g'(\sqrt{3})|$=1 ii. Newton's method converges, iii. Halleys method converges 5pts each=15
   b. Which one will you select and why? Iii 5pts
   c. What is the rate of convergence for each fixed point iteration. Ii quadratic iii cubic 5pts
   d. Perform 3 steps for each convergent method starting from $x_0 = 2$ and compare the error for each step with the exact roots $p = \pm\sqrt{3} = \pm\ 1.732050807568877$

      Ii $x_0 = 2$, $x_1 = 1.75, x_2 = 1.750000000000000$

      iii.$x_0 = 2, x_1 = 1.733333333333333, x_2 = 1.732050807744482$ 5pts each=10 total
4. A. Determine the Newton's form for the interpolating polynomial for the data set: $\{(-1,5), (0,1), (1,1), (2,11)\}$, where each pair represents the points $(x_i, f_i), i = 0{:}3$.
   i. Determine the finite difference table first. 10pts

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-1	5			
0	1	-4		
1	1	0	2	
2	11	10	5	1

ii. Determine the polynomial $P_3 = f_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$.
polynomial $P_3 = 5 - 4 * (x+1) + 2 * (x+1)(x) + (x+1)(x)(x-1) = x^3 + 2x^2 - 3x + 1$. 5pts

B. As generalized interpolation problem, find the cubic polynomial $q(x)$ for which 10 pts

$$q(0) = -1, q'(0) = 4, \qquad q(1) = -1, q'(1) = 4.$$

$$q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$q(0) = -1 ==> a_0 = -1, q'(0) = 4 ==> a_1 = 4,$$

$$q(1) = -1 = -1 + 4 + a_2 + a_3 ==> a_2 + a_3 = -4$$

$$q'(1) = 4 = 4 + 2a_2 + 3a_3 ==> 2a_2 + 3a_3 = 0$$
$$a_3 = 8, a_2 = -12$$

$$q(x) = -1 + 4x - 12x^2 + 8x^3$$

C. For an interval $[a, b]$ define $h = \frac{b-a}{n}$ and the evenly spaced points
$x_j = a + jh, \quad j = 0, 1, \dots, n.$
Consider the polynomial

$$\Omega_n(x) = (x - x_0)(x - x_1) \dots (x - x_n).$$

Show that $|\Omega_n(x)| \leq n!\, h^{n+1}, \quad a \leq x \leq b.$

The largest value is obtained if x in the first or last subintervals. Then $x - x_0 \leq h, x - x_1 \leq 2h, \dots x - x_n \leq nh$. Plug in above to prove the inequality. 10pts

5. We want to determine $\int_a^b f(x)dx = A\, f(x_0) + Bf''(\mu), \quad a \leq \mu \leq b$ so it is exact for polynomials of highest possible degree, e.g. $1, x, x^2$ .... Type equation here.

i. Determine A and x_0 . $b - a = A$, $(b^2 - a^2)/2 = Ax_0$

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + Bf''(\mu), \quad a \leq \mu \leq b$$

Set $f(x) = x^2$

$$\frac{b^3 - a^3}{3} = \frac{(b-a)(a+b)^2}{4} + 2B$$

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We use the trick set $a = 0$ and solving $\frac{b^3}{3} = \frac{b^3}{4} + 2B ==> B = \frac{b^3}{24}$

Replacing $b \to b - a$ we get $B = \frac{(b-a)^3}{24}$ 10pts total; 5 for each uknown

ii. Determine the parameter B in the error $Bf''(\mu)$. $B = \frac{(b-a)^3}{24}$ 5pts

iii. What is the name of the integration method you just derived. Midpoint 5pts

iv. The composite form is derived by applying the above method to the following formula:

$$I(f) = \int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-1)h}^{b} f(x)dx$$

where $h = \frac{b-a}{n}$ and $x_j = a + jh, \quad j = 0,1,\ldots,n.$

a. Approximate each integral the summation above, for example $\int_a^{a+h} f(x)dx \approx A\,f(x_0)\ etc$, to derive the composite integration formula $R(f,h)$?

$R(f,h) = hf\left(a+\frac{h}{2}\right) + hf\left(a+\frac{3h}{2}\right) + \cdots + hf(a + \frac{(2n-1)h}{2})$ 10pts

b. What is the error for the composite formula $R(f,h)$? HINT: Find the summation of all errors $Bf''(\mu)$.

$$E(f,h) = \frac{h^3}{24}f''(\mu_1) + \frac{h^3}{24}f''(\mu_2) + \cdots + \frac{h^3}{24}f''(\mu_n) = \frac{h^3}{24}nf''(\mu) = \frac{(b-a)h^2f''(\mu)}{24}$$

10pts

v. Use Romberg's integration for the Trapezoidal rule to integrate $I(f) = \int_0^1 x^4dx$. Start with $h = 1$ and complete Romberg's extrapolation Table. How many divisions of h does it take to get the exact answer in the Table.

10pts

$T^0(h)$			
$T^0(\frac{h}{2})$	$T^{(1)}(h) = \frac{4\,T^{(0)}\left(\frac{h}{2}\right) - T^{(0)}(h)}{4-1}$		
$T^0(\frac{h}{2^2})$	$T^{(1)}\left(\frac{h}{2}\right) = \frac{4\,T^{(0)}\left(\frac{h}{2^2}\right) - T^{(0)}\left(\frac{h}{2}\right)}{4-1}$	$T^{(2)}(h) = \frac{4^2\,T^{(1)}\left(\frac{h}{2}\right) - T^{(1)}(h)}{4^2-1}$	
$T^0(\frac{h}{2^3})$	$T^{(1)}\left(\frac{h}{2^2}\right) = \frac{4\,T^{(0)}\left(\frac{h}{2^3}\right) - T^{(0)}\left(\frac{h}{2^2}\right)}{4-1}$	$T^{(2)}(h) = \frac{4^2\,T^{(1)}\left(\frac{h}{2^2}\right) - T^{(1)}\left(\frac{h}{2}\right)}{4^2-1}$	$T^{(3)}(h) = \frac{4^3\,T^{(2)}\left(\frac{h}{2}\right) - T^{(2)}(h)}{4^3-1}$

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$\frac{1}{2}$=0.5			
$\frac{9}{32}$=0.2813	$\frac{10}{48}$=.2083		
$\frac{113}{512}$=0.2207	$\frac{77}{384}$=.2005	$\frac{16*\frac{77}{384}-\frac{10}{48}}{15}=\frac{1}{5}$=.2000	

6. EXTRA CREDIT QUESTION: Derive a similar table to Romberg's table for $I(f) \cong R(f,h)$ integration described in problem 5. Same as Rombersgs 10pts