1. a) Yes it converges, this is because for $\alpha = 1, 2$. $f'(\alpha) = -1, 1$. and also $f''(x)$ exists. Hence this method converges. To check for the interval, we know $|\alpha - x_0| < \frac{1}{|M|} = \left|\frac{2f'(\alpha)}{f''(\alpha)}\right|$

Here, $f'(x) = 2x-3$ and $f''(x) = 2$. Therefore, for root $\alpha = 1$, $x_0 = 0$ to 2; for root $\alpha = 2$; $x_0 = 1$ to 3.

1b) $g(x) = (x^2+2)/3$, $g'(x) = 2x$. Therefore, $g'(\alpha) = 2, 4$. Since $g'(\alpha) > 1$, this method does not converge.

1c) $g(x) = \sqrt{3x-2}$, $g'(x) = \frac{3}{2x\sqrt{3x-2}}$. At $\alpha = 1$, $g'(\alpha) = \frac{3}{2}$. Hence, it will not converge for $\alpha = 1$.

At $\alpha = 2$. $g'(\alpha) = \frac{3}{8}$. $|g'(\alpha)| < 1$, hence it ~~will~~ converges. In the interval $[1,3]$ it will.

1d) $g(x) = 3 - 2/x$, $g'(x) = 2/x^2$, At $\alpha = 1$, $g'(\alpha) = 2$. but for $\alpha = 2$. $g'(\alpha) = 1/2$. Therefore, ~~at~~ for root $\alpha = 2$ this method will converge. In the interval $[1, 3]$

1e) $g(x) = (x^2-2)/(2x-3)$, $g'(x) = \dfrac{2x}{2x-3}$

$- \dfrac{2(x^2-2)}{(2x-3)^2}$. This will converge as $g'(\alpha)=0$

at $\alpha = 1, 2$. The range is $[1, 3]$

2. The rate of convergence for $e^{-x} - x = 0$ is 6 iterations for $x_0 = 0$. This will not converge for any starting point. Here $f'(x) = -e^{-x} - 1$

$f''(x) = e^{-x}$. Therefore $|M| = |2 + 2e^{x}|$

For it to converge we should have

$|\alpha - x_0| < |2 + 2e^{x_0}|$. For high values of x_0 this will always be true. For negative values of x_0 this will become false.

This is a better method than fixed point iteration and bisection method in this case.

3. For third order we have.

$$f(x_0) + (x - x_0) f'(x_0) + (x - x_0)^2 f''(x_0)$$

$$+ (x - x_0)^3 f'''(x_0) = 0$$

Here, $f(x) = x^2 - a$

So, we have, $x_0^2 - a + (x_1 - x_0)\, 2x + (x - x_0)^2\, 2$

$$+ (x - x_0)^3 . 0 = 0$$

solving for this we get,

$$x_1 = \frac{x_0 (x_0^2 + 3a)}{3x_0^2 + a}$$

Here, the above method converges faster because the error term reduces by the third power. This is better than bisection method where it reduces by 1/2 and newton method where it reduces by quadratic.