MIDTERM name___________________

1. Taylor's polynomial approximating a function is defined as follows

$$P_n(x) = f(a) + \sum_{j=1}^{n} \frac{f^{(j)}(a)(x-a)^j}{j!}$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(j)}(\mu), \qquad \alpha \le x \le \beta\, , and\ a \le \mu \le x.$$

$$f(x) = P_n(x) + R_n(x).$$

   i. Derive the Taylor's polynomial for $f(x) = \sin(x)$, $a = 0$. The derivatives $\frac{d(\sin(x))}{dx} = \cos(x)$, $\frac{d(\cos(x))}{dx} = -\sin(x)$.
   ii. Derive an error bound using $\max|R_n(x)|$.
   iii. How many steps n will it take for the method to achieve an $error \le 10^{-6}$.
   iv. What is the best way to evaluate the expression $\frac{e^x - 1}{x}$ when x is near zero. Explain your approach.
   v. What is the largest positive integer number represented in 32 bit arithmetic?

2. A. Derive Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1, \dots$ by using second order Taylor's polynomial $f(x) \cong P_2(x)$ and assuming that $P_2(x_{n+1}) = 0$ and $a = x_n$.

   B. Use the error in Taylor's polynomial

$$f(x) = P_2(x) + R_2(x)$$

   to show that $p - x_{n+1} = (p - x_n)^2[-\frac{f''(c_n)}{2f'(x_n)}]$ where $f(p) = 0$, p is the root. Explain why Newton's method always converges "near" the root.

   C. We want to find the roots $p = \pm\sqrt{3} = \pm\ 1.732050807568877$ of the function $f(x) = x^2 - 3 = 0$. Give Newton's iteration for this function $f(x) = x^2 - 3 = 0$. Perform two steps of the iteration starting with $x_0 = 2$.
   How many more steps will it take to achieve that accuracy given above for $p = \pm\sqrt{3}\ \ (i.e. eps = 10^{-16})$.

   D. Perform two steps for $f(x) = x^2 - 3 = 0$ using the bisection method in the interval [0,2]. Which method is faster Newton's or bisection ? Explain.

3. You have to choose between the following 3 fixed point iterations:

   i. $x_{n+1} = \frac{3}{x_n}$ ii. $x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n})$ iii. $x_{n+1} = \frac{x_n(x_n^2+9)}{3x_n^2+3}$

   a. Which one converges to the root locally(i.e. near the root always converges).
   b. Which one will you select and why?
   c. What is the rate of convergence for each fixed point iteration.
   d. Perform 3 steps for each convergent method starting from $x_0 = 2$ and compare the error for each step with the exact roots $p = \pm\sqrt{3} = \pm\ 1.732050807568877$

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4. A. Determine the Newton's form for the interpolating polynomial for the data set: $\{(-1,5),(0,1),(1,1),(2,11)\}$, where each pair represents the points $(x_i, f_i), i = 0{:}3$.
   i. Determine the finite difference table first.
   ii. Determine the polynomial $P_3 = f_0 + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$.

   B. As generalized interpolation problem, find the cubic polynomial $q(x)$ for which

$$q(0) = -1, q'(0) = 4, \qquad q(1) = -1, q'(1) = 4.$$

   C. For an interval $[a,b]$ define $h = \frac{b-a}{n}$ and the evenly spaced points
   $x_j = a + jh, \quad j = 0,1,\dots,n.$
   Consider the polynomial

$$\Omega_n(x) = (x-x_0)(x-x_1)\dots(x-x_n).$$

   Show that $\quad |\Omega_n(x)| \le n!\,h^{n+1}, \quad a \le x \le b.$

5. We want to determine $\int_a^b f(x)dx = A\,f(x_0) + Bf''(\mu), \quad a \le \mu \le b$ so it is exact for polynomials of highest possible degree, e.g. $1, x, x^2$ .... Type equation here.
   i. Determine A and x_0 .
   ii. Determine the parameter B in the error $Bf''(\mu)$.
   iii. What is the name of the integration method you just derived.
   iv. The composite form is derived by applying the above method to the following formula:

$$I(f) = \int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \dots + \int_{a+(n-1)h}^{b} f(x)dx$$

   where $h = \frac{b-a}{n}$ and $x_j = a + jh, \quad j = 0,1,\dots,n.$

   a. Approximate each integral the summation above, for example $\int_a^{a+h} f(x)dx \approx A\,f(x_0)\ etc$, to derive the composite integration formula $R(f,h)$?
   b. What is the error for the composite formula $R(f,h)$? HINT: Find the summation of all errors $Bf''(\mu)$.

   v. Use Romberg's integration for the Trapezoidal rule to integrate $I(f) = \int_0^1 x^4 dx$. Start with $h = 1$ and complete Romberg's extrapolation Table. How many divisions of h does it take to get the exact answer in the Table.

6. EXTRA CREDIT QUESTION: Derive a similar table to Romberg's table for $I(f) \cong R(f,h)$ integration described in problem 5.