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1. Understand Figure 3.6 Page 103 from the book. Shown in class.
2. A sufficient condition to prove convergence of an iterative method $x_n = g(x_{n-1}), n = 1,2,\ldots$ is to show that the error at the n^{th} step is less than the error at $n-1$ step, i.e. $|x_n - p| = |x_{n-1} - p| M_n, \quad M_n < 1,$ in some interval that contains the root p. However this condition is not necessary since convergence might also occur outside that interval as long as the sequence x_n monotonically winds up in the interval(see HW-2). Answer the following questions: The quadratic equation has two roots $2x^2 - x - 1 = 0$

   i. Find the roots using the quadratic equation

      ANSWER: $x = 1, x = -\frac{1}{2}$

   ii. The equation can be rewritten as $x = 2x^2 - 1 = g(x)$. Will this iteration converge to any of the two roots?[Since you know the roots you can use the QUICK test to answer this question $|g'(p)| < 1$]

      ANSWER: $g'(x) = 4x,\ \left|g'\left(-\frac{1}{2}\right)\right| = 2 > 1,\ |g'(1)| = 4 > 1$ it will not converge in general. [unless you select the roots or a value that jumps to the exact root, i.e. $x_0 = 1$ or $x_0 = 0$, which jumps to $x = 1$.] Verify: $x_0 = 2,\ x_1 = 7, x_2 = 97\ldots$ or $x_0 = 1.001, x_1 = 1.0040, x_2 = 1.0160, x_3 =$1.0647,…,

   iii. Another way of rewriting the equation is $x = \sqrt{\frac{x+1}{2}} = g(x)$. Will this iteration converge to a root?

      $g'(x) = 1/(2\sqrt{2(x+1)})$ which is $|g'(x)| < 1 ==> 1/(2\sqrt{2(x+1)}) < 1 \quad ==> \frac{1}{8} < (x+1) ==> -\frac{7}{8} < x$ so this iteration will converge for all starting points in that interval.

      $x_0 = -\frac{1}{2}, x_1 = \frac{1}{2}, x_2 = 0.8660, x_3 = 0.9659, x_4 = 0.9914, x_5 = 0.9979\ldots$.

      [It converges linearly since $g'(1) = \frac{1}{4} \neq 0$ In other words the error in the next iteration is dividing the error of the previous iteration by four.]

      Which root? 1

      What is the interval of convergence? $-\frac{7}{8} < x$

   iv. Derive Newton's method for this equation.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{2x^2 - x - 1}{4x - 1} = \frac{4x^2 - x - 2x^2 + x + 1}{4x - 1}$$

$$= \frac{2x^2 + 1}{4x - 1}$$

$$g'(x) = \frac{1}{2} - \frac{9}{2(4x-1)^2}$$

      Notice the $g'(1) = 0$. This is expected since for Newton's method it is always true and it results in quadratic convergence "close" to the root.

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Now having $|g'(x)| < 1$=➔ $|\frac{1}{2} - \frac{9}{2(4x-1)^2}| < 1$ ➔-1<$\frac{1}{2} - \frac{9}{2(4x-1)^2} < 1$➔

➔$-\frac{3}{2} < -\frac{9}{2(4x-1)^2} < \frac{1}{2}$➔$3 > \frac{9}{(4x-1)^2}$ ➔ $(4x-1)^2 > 3$➔$4x - 1 > \sqrt{3}$

➔$x > \frac{\sqrt{3}+1}{4} = 0.6830$ or $x < \frac{1-\sqrt{3}}{4} = -0.1830$.

v. Will newton's method converge for all initial starting points?

For $x > 0.6830$ will converge to 1 while for $x < -0.1830$ will converge to $-\frac{1}{2}$. For example: $x_0 = .75, x_1 = 1.0625, x_2 = 1.0024$… Notice the quadratic convergence. And if you select $x_0 = -.75$ it will converge to the negative root. The question is if it will also converge in the interval $-0.1830 < x < 0$ and $0 < x < 0.6830$. You can easily show that selecting any point it will jump to the convergence intervals concluding that the method converges for any initial starting point. Check it out with few points.

3. The function $f(x) = x^2 + x - 2$ has two roots in the intervals $[0,3]$ and $[-3,0]$.
   i. What are the roots?
   ii. Perform 3 steps of the bisection method for the root in $[0, 3]$. How many steps will you need so that the error in the n^th iteration of the bisection method is less than 0.0001?
   iii. Perform 3 steps of the Regula Falsi iteration.
   iv. Perform 2 steps of Newton's method for both roots staring with x_0=2 and x_0=-2. How many steps will you need to get to 0.0001 error in Newton's method.
   v. A fixed point iteration can be derived by re-writing $x^2 + x - 2 = 0$ into its equivalent form $x = 2 - x^2 = g(x)$. Will this iteration converge to any of the two roots? Explain.
   vi. Another fixed point iteration can be derived by re-writing the equation as $x = \sqrt{2 - x}$. Will this iteration converge to any of the roots?

   SOLUTION SIMILAR TO 3

4. Given the iteration $x_{n+1} = \frac{x_n(x_n^2+3)}{3x_n^2+1} = g(x_n)$.
   i. What is the original equation that we are trying to solve?

   $$x = \frac{x(x^2+3)}{3x^2+1} ==> 3x^3 + x = x^3 + 3x ==> 2x^3 - 2x = 0 ==>$$

   $f(x) = x(x^2 - 1) = 0$ with roots $\pm 1, 0$.

   ii. Can you propose two alternatives to the above method that converge to the root?

   Add an $x + x = \frac{x(x^2+3)}{3x^2+1} + x ==> x = \frac{x^3+3x+3x^3+x}{2(3x^2+1)} = \frac{x(4x^2+4)}{2(3x^2+1)} = \frac{2x(x^2+1)}{3x^2+1}$

   We have $g'(x) = \frac{2(3x^4+1)}{(3x^2+1)^2}$, implying $g'(\pm 1) = \frac{1}{2} < 1$. Thus it converges linearly around the root.

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Another method is to start from $x^3 - x = 0$, for all $x \neq 0$ we get $x^2 - 1 = 0$ ➔ $x = \frac{1}{x}$ ➔ $x = \frac{x+\frac{1}{x}}{2}$ which is Newton's method for $f(x) = x^2 - 1$

iii. What is Newton's method for this equation.

$$g(x) = x - \frac{x^3 - x}{3x^2 - 1}$$

iv. We know that for Newton's method the convergence is quadratic, i.e. $g'(p) = 0$. What is the order of convergence for the iteration above.

$$g'(x) = \frac{3(x^2 - 1)^2}{(3x^2 + 1)^2}$$

Notice that $g'(\pm 1) = 0, g'(0) = 3 > 1$ so this is telling us that the convergence is at least quadratic for ± 1 but also it does not converge to the zero root. Next $g''(x) = \frac{48x(x^2-1)}{(3x^2+1)^3}$, $g''(\pm 1)$= 0. So the convergence is at least cubic. You can show that $g^{iv}(\pm 1) \neq 0$ implying that the convergence is cubic.(HINT use Matlab diff function to derive the derivatives!)