HW-2 with solutions

1. Taylor's polynomial approximating a function is defined as follows

$$P_n(x) = f(a) + \sum_{j=1}^{n} \frac{f^{(j)}(a)(x-a)^j}{j!}$$

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(j)}(\mu), \qquad \alpha \le x \le \beta \text{ , and } a \le \mu \le x.$$

$$f(x) = P_n(x) + R_n(x).$$

i. The tangent line at the point x_0 is the first degree Taylor's polynomial $P_1(x) = f(x_0) + f'(x_0)(x - x_0)$ that has as a root the next point $x_1, i.e. P_1(x_1) = 0$,. First Derive Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$$

using the above assumption and then show that the error is given by:

$$|x_{n+1} - p| = |x_n - p|^2\, M_n, \quad M_n = |f''(\sigma_n)|/|2f'(x_n)|$$

ii. Newton's Matlab program is given in Sakai. Use that program to solve the following equations

a. $x^2 - 2 = 0$. This method derives the square root of 2! Use the above error to show that the method converges for every $x_0 \neq 0$. Verify your results by running the program. Modify the program so that you print the results in a file as you did in your first HW-1.

SOLUTION: $f'(x_n) = 2x_n,\ f''(\sigma_n) = 2,\ M_n = \left|\frac{1}{x_n}\right|,\quad \left|x_{n+1} - \sqrt{2}\,\right| = \left|x_n - \sqrt{2}\,\right| \left|\frac{x_n - \sqrt{2}}{x_n}\right|$ so to converge the Error at the n+1 step must be less than the error at the nth step. This will be true if $\left|\frac{x_n-\sqrt{2}}{x_n}\right| < 1 ==> -1 < \frac{x_n-\sqrt{2}}{x_n} < 1 ==> -x_n < x_n - \sqrt{2} < x_n ==> -2x_n < -\sqrt{2} < 0 ==> x_n > \frac{\sqrt{2}}{2}\ if\ x_n > 0\ or\ similarly\ x_n < -\frac{\sqrt{2}}{2}\ if\ x_n < 0$. So as long as you select x_0 as above Newton's method will converge. Now to prove that it also converges in the interval $0 < x_0 \le \frac{\sqrt{2}}{2}$ for the positive root we will need to show that $x_1 > \frac{\sqrt{2}}{2}$ in other words even though x_0 is outside the guarantee interval of convergence as long as the second iteration is mapped into the convergence interval it will still converge. To show this we compute $x_1 = x_0 - \frac{x_0^2-2}{2x_0} = x_0 - \frac{x_0}{2} + \frac{1}{x_0} = \frac{x_0}{2} + \frac{1}{x_0} > \frac{1}{x_0} \ge \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. The proof is the same for the negative root.

b. $1 - e^x = 0$. Find the solution using Newton's program. Does it converge for every initial starting point?

SOLUTION: $f'(x) = -e^x, f''(x) = -e^x,\ M_n = \frac{1}{2}$, So $|x_{n+1} - p| = \frac{|x_n-p|^2}{2}$ and the error $|x_{n+1} - p| = |x_n - p|\frac{|x_n-p|}{2}$ to have convergence we must have $\frac{|x_0|}{2} < 1$

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since we know that the root is $p = 0$. This implies that $-2 < x_0 < 2$. To prove if it also converges or diverges if the initial starting value is not in that interval we must prove that x_1 is mapped in the above interval eventually. Let us select $|x_0| \geq 2$. Then $x_1 = x_0 - \frac{(1-e^{x_0})}{(-e^{x_0})} = x_0 - 1 + \frac{1}{e^{x_0}}$. We will show that for $x_0 \geq 2$ $x_1 < x_0,\ \ x_2 < x_1 \ldots$ $implying\ that\ it\ will\ converge\ to\ 0$. We have that $-1 + \frac{1}{e^{x_0}} < 0$, $x_0 \neq 0 ==> x_0 - 1 + \frac{1}{e^{x_0}} < x_0 ==> x_1 < x_0$. If $x_0 \leq -2$, then we can easily show that $x_1 > 0$ and therefore we can conclude that Newton's method will converge for every starting point. $x_0 = -2, x_1 = 4.3891, x_2 = 3.4015, x_3 = 2.4348, x_4 = 0.7406\ x_5 = \ 0.2174, x_6 = 0.0220 \ldots$

2. Modify the newton.m program by using Matlab graphical capabilities so that you can visualize Newton's method at each step. Remember that the tangent line is given by the first degree Taylor's polynomial.

SOLUTION:
```
function root = newtongraphics(x0,error_bd,max_iterate)
    syms x real;
    syms z real;
    f = x^2-1;
    p=z^2-1+2*z*(x-z);


format short e
error = 1;
it_count = 0;
while abs(error) > error_bd && it_count <= max_iterate


    grid
    ezplot(f,[0,2])
    hold on
    ezplot(subs(p,'z',x0),[0,2])
    grid
    fx =  subs(f,'x',x0);
    dfx = subs(diff(f,x,1),'x',x0);
    if dfx == 0
        disp('The derivative is zero.  Stop')
        return
    end
    x1 = x0 - fx/dfx;
    error = x1 - x0;
%   Internal print of newton method. Tap the carriage
%   return key to continue the computation.
    iteration = [it_count x0 fx dfx error]


    pause
    x0 = x1;
    it_count = it_count + 1;
```

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```
end

if it_count > max_iterate
    disp('The number of iterates calculated exceeded')
    disp('max_iterate.  An accurate root was not')
    disp('calculated.')
else
    format long
    root = x1
    format short e
    error
    format short
    it_count
end
```

3. Derive a new method similar to Newton's by selecting the second order Taylor's polynomial and finding the next point so that $P_2(x_1) = 0$. Derive the error. Is this a better method than Newton's method? Explain.

SOLUTION: $P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2}$ We determine the next step by setting the polynomial equal to zero. $f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(x_0)(x_1-x_0)^2}{2} = 0$. This is a quadratic equation whose solution is given by $x_1 - x_0 = \frac{-f'(x_0)\pm\sqrt{(f'(x_0))^2 - \frac{4f''(x_0)f(x_0)}{2}}}{2\left(\frac{f''(x_0)}{2}\right)} = \frac{-f'(x_0)\pm\sqrt{(f'(x_0))^2 - 2f''(x_0)f(x_0)}}{f''(x_0)}$.

Let's see how this method works for $f(x) = x^2 - 1,\ f'(x) = 2x, f''(x) = 2$ so we have $x_1 = x_0 + \frac{-2x_0\pm\sqrt{4x_0^2 - 4(x_0^2-1)}}{2} = x_0 - x_0 \pm 1 = \pm 1$ and the method finds the solution in one step. Let's consider $f(x) = 1 - e^x, f'(x) = -e^x,\ f''(x) = -e^x$

$$x_1 = x_0 - \frac{e^{x_0} \pm \sqrt{e^{2x_0} + 2e^{x_0}(1 - e^{x_0})}}{e^{x_0}}$$

For this iteration to converge the square root must be positive:$e^{2x_0} + 2e^{x_0}(1 - e^{x_0}) > 0$ implying that $x_0 < \log(2) = 0.6931$. Let's start with $x_0 = .5$. Then if we select the + it diverges but the – it converges very fast(cubic convergence)!!!$x_0 = .5, x_1 = -0.0384\ , x_2 = 9.1863e - 06, x_3 = \ 5.6412e - 19$!!