09/03/2020

ME397 - HW2

SRINATH. T.

st34546

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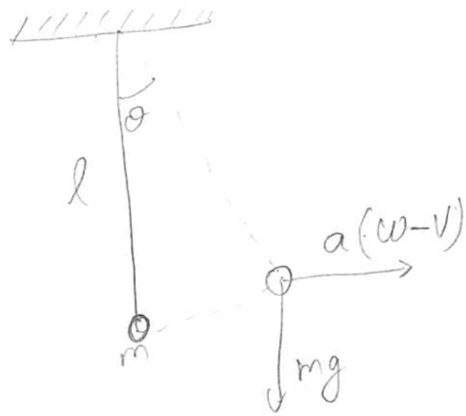
Q4.18.



h = angular momentum

$\dot{\theta} = h/ml^2,$

$\dot{h} = -mgl\theta + a(w - l\dot{\theta}) = -mgl\theta - \frac{ah}{ml} + aw$

$$\begin{bmatrix} \dot{\theta} \\ \dot{h} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/ml^2 \\ -mgl & -a/ml \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} \theta \\ h \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ \cancel{\frac{a}{ml}} \end{bmatrix}}_{G} aw$$

$E(w(t), w(t+\tau)) = b\delta(\tau)$

$\Rightarrow \langle w, w \rangle = b.$

$dX = FXdt + Gd\beta,$

$\dot{P} = FP + PF^T + GQG^T$

$\dot{P} = 0$ for stationary operation

$$\Rightarrow 0 = FP + PF^T + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} q \quad , \quad q = b.$$

$$P = \begin{bmatrix} \langle h\, h \rangle & \langle h\, \theta \rangle \\ \langle h\, \theta \rangle & \langle \theta\, \theta \rangle \end{bmatrix}$$, hence we need P_{22}.

$$FP + PF^T + \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} = 0$$ [P is the solution to Lyapunov equation]

$$\begin{bmatrix} 0 & \frac{1}{ml^2} \\ -mgl & -\frac{a}{ml} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & -mgl \\ \frac{1}{ml} & -\frac{a}{ml} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = 0$$

$$\frac{P_{12} + P_{21}}{ml} + 0 = 0 \Rightarrow P_{21} = -P_{12}$$

$$\frac{P_{22}}{ml} - mgl\, P_{11} - \frac{aP_{12}}{ml} = 0$$

$$-mgl \cdot P_{11} - \frac{a}{ml} P_{21} + \frac{P_{22}}{ml} = 0$$

$$\cancel{-mgl\, P_{12}} - \frac{a}{ml} P_{22} \cancel{-mgl\, P_{21}} - \frac{a\, P_{22}}{ml} + b = 0$$

$\frac{ml^2}{2a}$

$$\Rightarrow \frac{2ml\, g\, P_{22}}{a} = b$$

Contd. at last $\Rightarrow P_{22} = \frac{ab}{2mgl} \Rightarrow \langle \theta\, \theta \rangle = \frac{ab}{2mgl}$

$$x = l\theta \Rightarrow \boxed{\langle l\theta\, l\theta \rangle = \frac{abl}{2mg}}$$

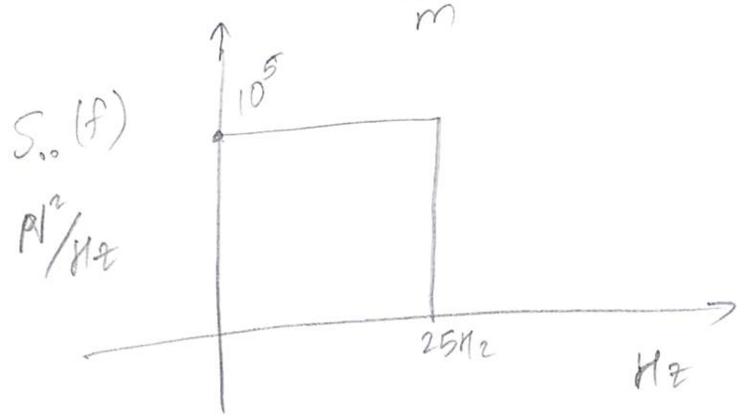
Q3. $m = 60\,kg$, $\omega_n = 4\,Hz$, damping coeff $= \xi$, $\underline{\underline{\omega_n = 2\pi \times 4 = 8\pi}}$

$$\Rightarrow \quad G(s) = \frac{1/60}{\omega_n^2 + 2\xi\omega_n s + s^2}$$

$$\Rightarrow \quad G(\omega) = \frac{1/60}{16 + 2\xi\omega_n i\omega - \omega^2}$$

$$|G(\omega)|^2 = \frac{1/60^2}{((8\pi)^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2} = \frac{1}{m^2(\omega_n^2 - \omega^2)^2 + 4m^2\xi^2\omega^2\omega_n^2}$$

$$\frac{F}{m}(s) = G(s)\,\cancel{x(s)}$$



$$\Rightarrow \omega \rightarrow [0 \rightarrow 2\pi \times 25]$$

$$\Rightarrow \quad |x^2(\tau)| = \int_0^{50\pi} \frac{S_{00}(f)}{2\pi} |G(\omega)|^2 e^{-i\omega\tau}\, d\tau$$

$$|x^2(\tau)| = \frac{10^5}{2\pi} \int_0^{50\pi} \frac{e^{-i\omega\tau}\, d\omega}{60^2((64\pi^2 - \omega^2)^2 + 256\pi^2\xi^2\omega^2)}$$

$$\text{at } \tau = 0, \quad \langle x^2 \rangle = \frac{10^5}{2\pi} \int_0^{50\pi} \frac{d\omega}{60^2((64\pi^2 - \omega^2)^2 + 256\pi^2\xi^2\omega^2)}$$

Integration using MATLAB gives,

for, $\boxed{\xi = 0.5\ , \quad \langle x^2 \rangle = 4.37\times10^{-4}}$ X

$\boxed{\xi = 0.1\ , \quad \langle x^2 \rangle = 2.2\times10^{-3}}$ X

not correct $\langle x^2 \rangle$ See Soln

code -5

we can assume 'x' to have a gaussian distribution

$$\Rightarrow \quad p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}} = f(x)$$

$$P(x > 0.132) = \int_{0.132}^{\infty} f(y)\,dy = \frac{1}{\sqrt{2\pi}\sigma} \int_{0.132}^{\infty} e^{-y^2/2\sigma^2}\,dy$$

for $\xi = 0.5$, $\sigma = \sqrt{0.000437} = 0.0209$

$$P(x > 0.132) = \frac{1}{\sqrt{2\pi}\times0.0209} \times \int_{0.132}^{\infty} e^{-y^2/0.0008}\,dy. \simeq 2.14\times10^{-10} \text{ [done in MATLAB]}$$
$$= 0$$

for $\xi = 0.1$, $\sigma = \sqrt{0.0022} = 0.0469$

$$P(x > 0.132) = \frac{1}{\sqrt{2\pi}\times0.0469} \int_{0.132}^{\infty} e^{\left(\frac{-y^2}{0.0044}\right)}\,dy \simeq 0.0024 \text{ [done in MATLAB]}$$

(b) From class approximation method, $\langle x^2 \rangle = \frac{10^5}{2\pi} \times |G_7(\omega_n)|^2 \Delta\omega_r = \frac{10^5}{4\xi\omega_n^3 m^2}$

$\boxed{\text{for } \xi = 0.5, \ \langle x^2 \rangle = 8.75\times10^{-4}}$

$\Rightarrow \sigma = 2.96\times10^{-2}$

$0.132 \approx 4.5\sigma$

From z-tables

$P(x > 4.5\sigma = 0.132) \approx 10^{-6} = 0$ ✓

$\boxed{\text{for } \xi = 0.1, \ \langle x^2 \rangle = 0.0044}$

$\sigma = 0.0663$

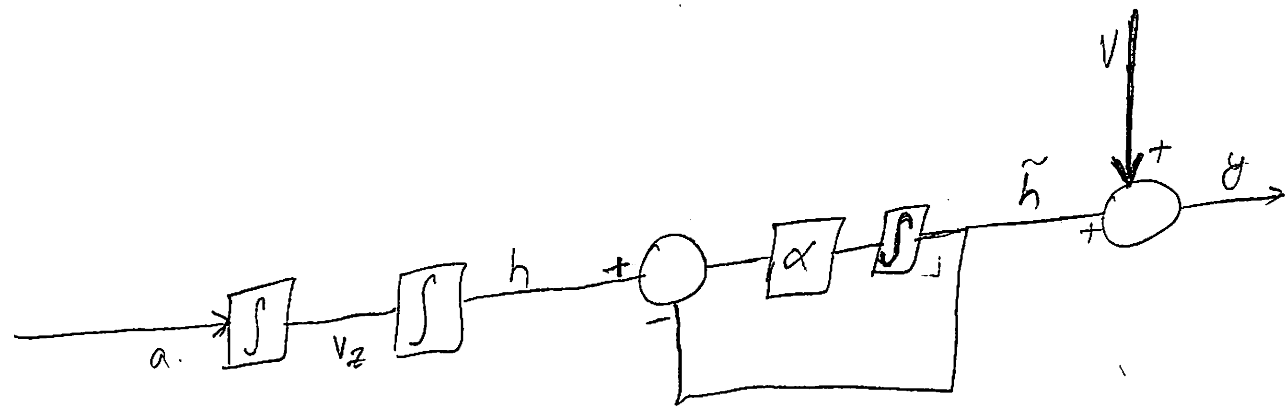
$0.132 \approx 2\sigma$

From z-tables

$P(\cancel{x > 2\sigma} = 0.132) = \frac{0.0228}{2}$

$= 0.0114$

Q 4.28



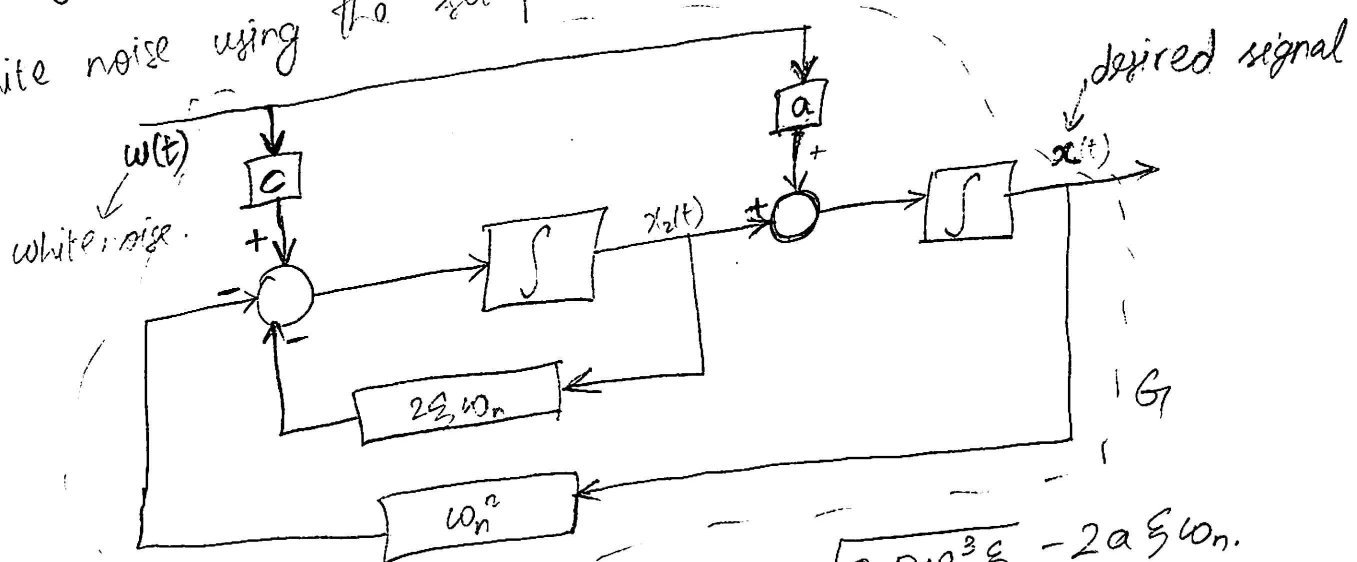
$a = w + x$

$\langle x(t), x(t+\tau) \rangle = p\, e^{(-\xi\omega_n|\tau|)} \cos\left(\omega_n\sqrt{1-\xi^2}\,|\tau|\right)$

(1) $V = b\,v$, where v is a Gaussian random variable of mean 0 & variance 1

$\Rightarrow \langle V\, V \rangle = b^2$.

clearly x is a 2nd order Markov process which can be generated from white noise using the setup below



where, $a = \sqrt{2\,p\,\omega_n\xi}$, $C = \sqrt{2\,p\,\omega_n^3\xi - 2a\xi\omega_n}$.

$$\tilde{h}(s) = \frac{s}{s+\alpha} h$$

$$\dot{h} = V_z$$

$$\dot{V}_z = a_z = w + x$$

$$\begin{bmatrix} \cancel{\dot{\tilde{h}}} \\ \cancel{\dot{h}} \\ \cancel{\dot{V}_z} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{F} \begin{bmatrix} \tilde{h} \\ h \\ V_z \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{G_x} a_w(t)$$

→ input a_w

$$\tilde{h} = H \begin{bmatrix} \tilde{h} \\ h \\ V_z \end{bmatrix} + v(t) + n(t) \quad ; H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

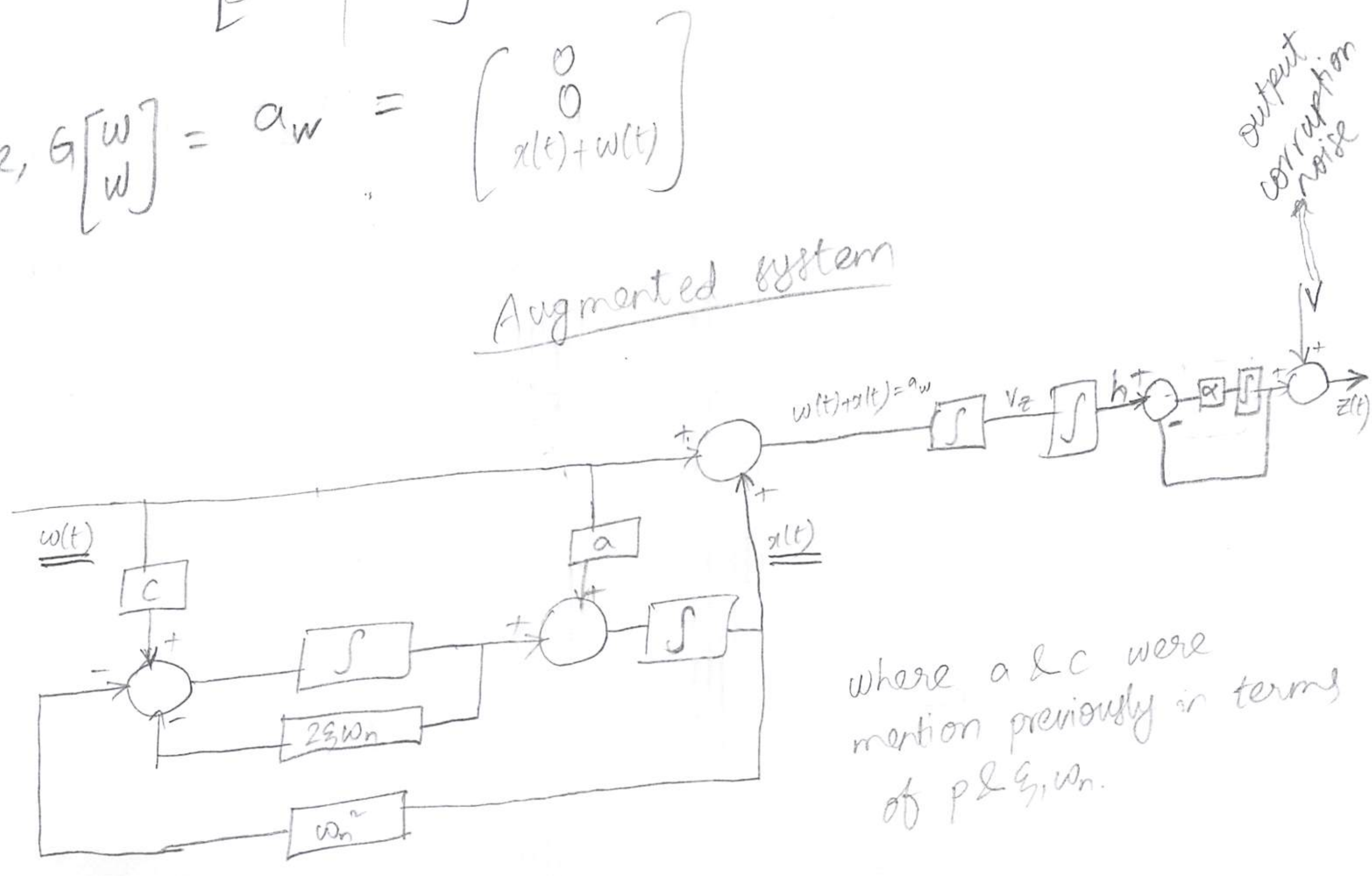
→ output noise corruption.

$$G = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & G_f(t) \\ 0 & \end{array}\right] \begin{bmatrix} w(t) \\ w(t) \end{bmatrix}$$

where G_f = Filter Gain matrix.

$$\text{Hence, } G\begin{bmatrix} w \\ w \end{bmatrix} = a_w = \begin{bmatrix} 0 \\ 0 \\ x(t) + w(t) \end{bmatrix}$$

<u>Augmented system</u>

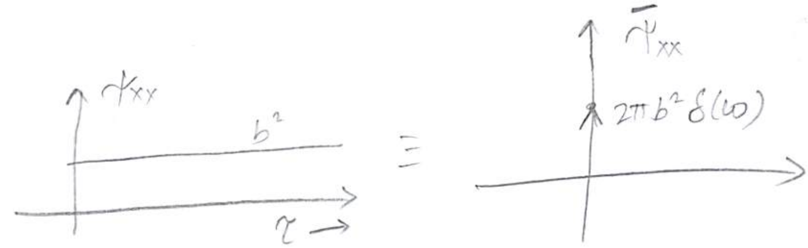


where a & c were mention previously in terms of p & ξ, ω_n.

The output is given by

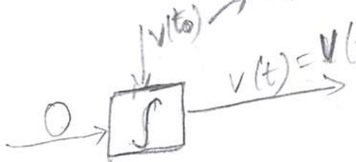
$$z(t) = H\begin{bmatrix} h \\ v_2 \end{bmatrix} + v(t) = \tilde{h}(t) + v(t), \quad H(s) = \frac{\alpha}{s+\alpha}.$$

(1) $v(t) = b \rightarrow$ bias: $\longrightarrow$



$\Rightarrow \langle v(t)\ v(t) \rangle = b^2.$

Model



integrator
giving constant output $\cancel{b}$

~~sin~~ $\dot{h}_b = 0$ is a state equation.

~~$\dot{h}_b = 0$ is a state~~

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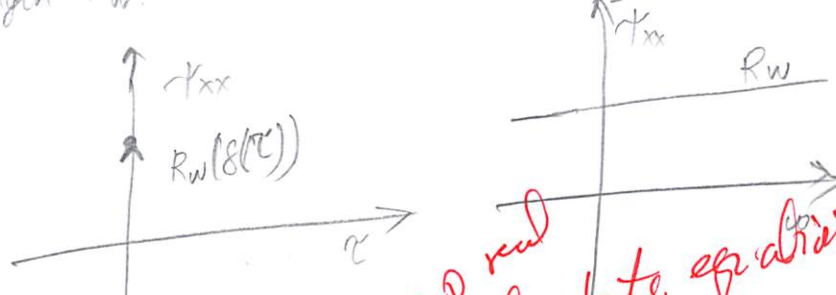
(2) White Gaussian noise of strength R_W.

$$V(t) = \frac{\omega(t)}{\sqrt{q_W}} \times \sqrt{R_W}.$$



State space eqs. for Filter:

$$\dot{x}_f = 0, \quad v(t) = \frac{\omega(t) \times \sqrt{R_W}}{\sqrt{q_W}}$$

No real ~~need~~ state equations

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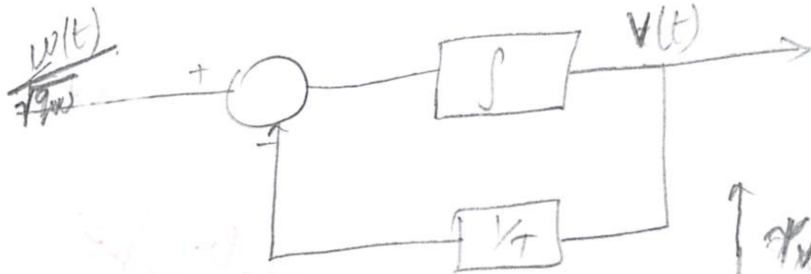
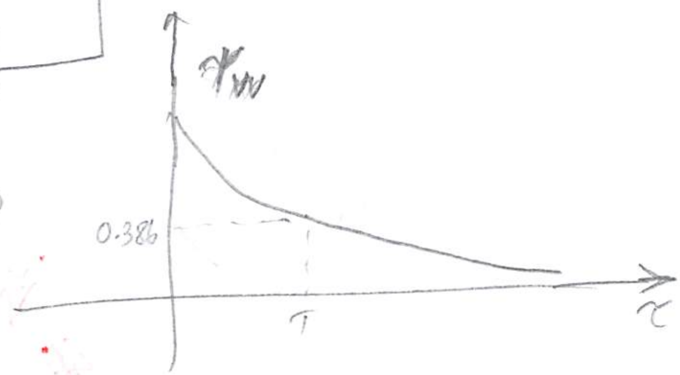
PTO.

(3) exponentially time correlated noise of mean square value 'S' & correlation time 'T':

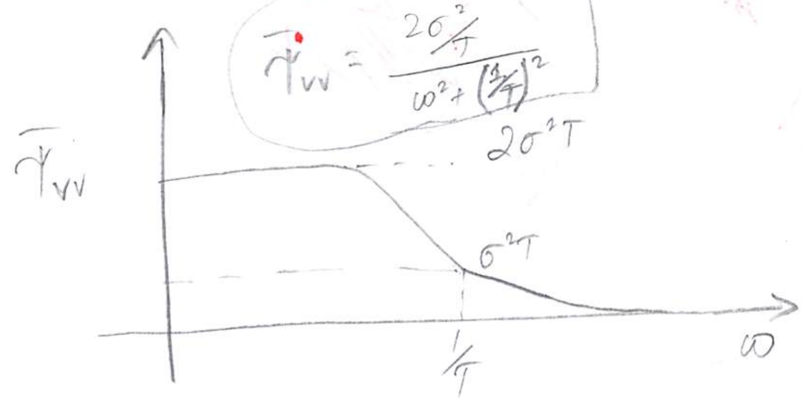
This can be expressed as <u>1st order Markov process</u>



$$\langle v(t)\, v(t+\tau)\rangle = \sigma^2 e^{-|t|/\tau} \Rightarrow$$



$$\Phi_{vv} = \frac{2\sigma^2/T}{\omega^2 + (1/T)^2}$$



<u>State space eqs for filter:-</u>

$$\begin{bmatrix} \dot{x}_{1f} \\ \dot{x}_{2f} \end{bmatrix} = \begin{bmatrix} -1/T & 0 \\ 1 & -b \end{bmatrix} \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$

$$v(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix}$$

$v(t)$ → corruption noise