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Q.1) Cauchy - Riemann equation in the Cartesian form.

Statement :-

In the necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that exist four continous first order partially derivatives

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ and

Satisfying $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$u_x = v_y$$
$$u_y = -v_x$$

Proof :-

Let $w = f(z)$ be given analytic function

i.e $u + iv = f(x+iy)$ —— (1)

$z = x + iy$.

Differentiating eq^n (1) partially with respect to 'x' on b.s

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(x+iy)(1+0)$$

$$= f'(x+iy) \quad \text{—— (2)}$$

again differentiate eq^n (1) w.r.t y

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = f'(x+iy)(i) \quad \text{—— (2)}$$

Substitute eq^n (2) in (3)

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = f'(x+iy)(i)$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(i)$$

$$= i\frac{\partial u}{\partial x} + i^2\frac{\partial v}{\partial x}$$

$$= i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}.$$

equate the real and imaginary parts from both side.

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$$\boxed{\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}} \text{ and } \boxed{\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}}$$

or

$$\boxed{u_x = v_y} \text{ and } \boxed{v_x = -u_y}$$

Q.4

Let $v = \left(r - \frac{1}{r}\right) \sin\theta$

using imaginary part by CR eq^n.

$$v_r = \left(1 + \frac{r}{r^2}\right) \sin\theta .$$

$$v_\theta = \left(r - \frac{1}{r}\right) \cos\theta .$$

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$u_r = \frac{1}{r} v_\theta \; (\text{CR eq}^n)$$

$$f'(z) = e^{-i\theta}\left(\frac{1}{r} v_\theta + i v_r\right)$$

$$f'(z) = e^{-i\theta}\left[\left(1 - \frac{1}{r^2}\right)\cos\theta + i \left(1 + \frac{1}{r}\right)\sin\theta\right] \quad (1)$$

Putting $r = z$ & $\theta = 0$.

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$$f'(z) = 1 - \frac{1}{r^2}$$

$$f(z) = \int\left(1 - \frac{1}{r^2}\right) dz + c$$

$$u = \left(r + \frac{1}{r}\right) \cos\theta$$

imaginary part

$$v = \left(r - \frac{1}{r}\right) \sin\theta$$

$$\boxed{v = \left(r - \frac{1}{r}\right) \sin\theta}$$

Q.

$$\psi = (x^2 - y^2) + \frac{x}{x^2 + y^2}$$

diff^n w.r.t 'x'.

$$\psi_x = 2x + \frac{(x^2 - y^2)(1 - x \cdot 2x)}{(x^2 - y^2)^2}$$

$$= 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

diff^n w.r.t y

$$\psi_y = -2y + \frac{(x^2 + y^2)\,0 - x \cdot 2y}{(x^2 + y^2)^2}$$

$$= -2y - \frac{2xy}{(x^2 + y^2)^2}$$

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consider $f'(z) = \phi_x + i\psi_x$ but

$\phi_x = \psi_y$

$f'(z) = \psi_y + i\psi_x$

putting $x = z$, $y = 0$ we have

$$f'(z) = [\psi_y]_{(z,0)} + i[\psi_x]_{(z,0)}$$

$$f'(z) = 0 + i\left(2z + \frac{-z^2}{(z^2)^2}\right)$$

$$= i\left(2z - \frac{1}{z^2}\right)$$

$$= i\int\left(2z - \frac{1}{z^2}\right) dz + c$$

$$= i\left(z^2 + \frac{1}{z}\right) + c$$

$$\boxed{f(z) = i\left(z^2 + \frac{1}{z}\right) + c}$$

$$\phi + i\psi = i\left\{(x+iy)^2 + \frac{1}{x+iy}\right\} + c$$

$$= i\left\{(x^2 + i^2y^2 + 2xiy) + \frac{x-iy}{(x+iy)(x-iy)}\right\} + c$$

$$\phi + i\psi = i\{(x^2 - y^2) + 2xiy\} + i\left\{\frac{x-iy}{x^2+y^2}\right\} + c$$

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$$\phi + i\psi = i(x^2-y^2) - 2xy + \frac{ix}{x^2+y^2} - \frac{y}{x^2+\ldots} + c$$

$$\therefore \phi + i\psi = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left[x^2-y^2 + \frac{x}{x^2+y^2}\right) + c$$

$$\boxed{\phi = -2xy + \frac{y}{x^2+y^2}}$$

Q. 3 (a).

$w = e^z$

$z = x+iy$

$u+iv = e^{(x+iy)}$

$= e^x e^{iy}$

$= e^x [\cos y + i\sin y$

$u+iv = e^x \cos y + i(\sin y \, e^x)$

$u = e^x \cos y$ & $v = e^x \sin y$ — (i)

add eq^n (i)

$u + v = e^x \cos y + e^x \sin y$

squaring on b.s

$u^2 + v^2 = e^{2x} \cos^2 y + e^{2x} \sin^2 y$

$u^2+v^2 = e^{2x}$ — (a)

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Case 1 :- Let $x = c_1$.

from eqn ⓐ.

$$u^2 + v^2 = e^{2c_1}$$

$$= \text{constant}.$$

$$u^2 + v^2 = r^2$$

$$\boxed{u^2 + v^2 = r^2} \quad \text{--- (2)}$$

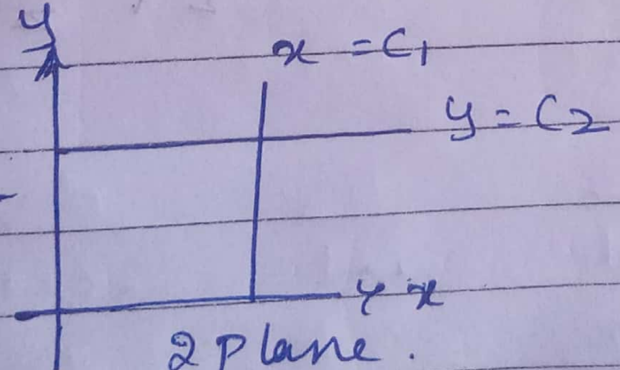
Case II. Let $y = c_2$ = constant

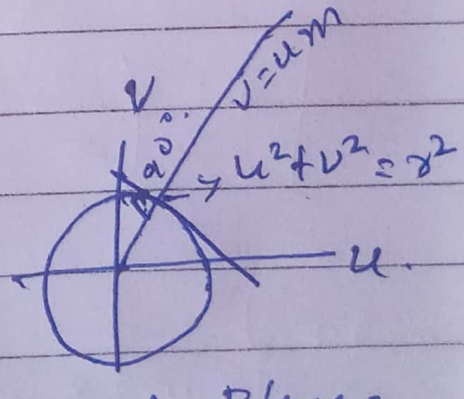
$$\frac{v}{u} = \tan c_2 = m.$$

$$\boxed{v = um} \quad \text{--- (3)}$$

eqn (3) say Straight line passing through the origin w plane.

Sketch :-





b) — ?.

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b) $z = \infty, i, 0 \qquad w = -1, -i, 1$

$$w = \frac{az+b}{cz+d}$$

$$w = \frac{z}{z}\frac{\left(a+\frac{b}{z}\right)}{\left(c+d/z\right)}$$

$$w = \frac{a+\frac{b}{z}}{c+\frac{d}{z}}$$

i) $$-1 = \frac{a+0}{c+0}$$

$$= \frac{a}{c}$$

$$-c = a.$$

$$a+c = 0 \quad \text{—— } (1)$$

ii) $$-i = \frac{a(i)+b}{c(i)+b}$$

$$-i = \frac{a(i)+b}{c(i)+b}$$

$$-i = \frac{ai+b}{ci+d}$$

$$(ci+d)-i = ai+b$$

$$-ci^2-di = ai+b$$

$$ai+b+ci^2+di^2 = 0 \quad \text{—— } (2)$$

$$ai+b+c(-1)+di = 0$$

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$ai + b - c + di$ — (2)

iii) $1 = \frac{a(0)+b}{c(0)+d}$

$1 = \frac{b}{d}$

$d = b$

$b - d = 0$ — (3)

eqⁿ (1) + eqⁿ (2)

$a + c + [ai + b - c + di] = 0$

$(1+i) a + b + di = 0$ — (4)

Sub eqⁿ (3) & (4)

$0 \cdot a + b - d = 0$ — (3)

$(1+i) a + b + di = 0$ — (4)

$\frac{a}{\begin{matrix} 1 & -1 \\ i & i \end{matrix}}$ $\frac{b}{\begin{matrix} 0 & -1 \\ 1+i & -i \end{matrix}}$ $\frac{c}{\begin{matrix} 0 & 1 \\ 1+i & 1 \end{matrix}}$

$\frac{a}{i+1} = \frac{b}{i+1} = \frac{d}{-(1+i)}$

$\frac{a}{1} = \frac{b}{-1}$ $\frac{d}{-1}$

$a = 1, \quad b = -1, \quad d = 1$

$a + c = 0$

$1 + c = 0$

$\boxed{c = -1}$

Sub values a, b, c, d in bil - near.

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$$w = \frac{az+b}{cz+d}$$

$$w = \frac{-1(z)+(-1)}{(-1)(z)+(-1)}$$

$$= \frac{z-1}{-z-1}$$

$$= \frac{-(1-z)}{-(1+z)}$$

$$= \frac{1-z}{1+z} \text{ //.}$$

Let $w = z$

$$z = \frac{1-z}{1+z}$$

$$z(1+z) = 1-z$$

$$z + z^2 = 1-z$$

$$z^2 + 2z = 1$$

$$z^2 + 2z - 1 = 0$$

$a = 1, \quad b = 2, \quad c = -1$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$$

$$z = \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \frac{-2 + \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= \frac{-\cancel{2}(1 \pm \sqrt{2})}{\cancel{2}}$$

$$z = -1 \pm \sqrt{2}$$

$z = -1 + \sqrt{2}$ & $-1 - \sqrt{2}$ are invariant points