MECH&AE: 271A

PROBABILITY & STOCHASTIC PROCESSES IN DYNAMICAL SYSTEMS

Project Report: Calibration of a 1D accelerometer using GPS measurements

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1 Abstract

The report discusses an implementation of the Kalman filter to calibrate an accelerometer's model with measurements from a GPS receiver. The state to be estimated by the filter is associated with a process noise in the form of an independent white-noise sequence. The information form the measurements is also associated with an independent white-noise sequence. Knowledge about the noise statistics, the accelerometer's model and the actual truth model allows to simulate multiple realizations. The designed filter's performance was then verified by performing various tests involving the estimated errors, the error variance, and orthogonal properties. It must be noted that the filter estimates the difference between true values and outputs obtained from accelerometer.

2 Introduction

A vehicle accelerates in an inertial frame of reference with the profile:

$$a(t) = 10sin(\omega t) \qquad m/s^2$$
$$\text{where } \omega = 0.1 \text{ rad/s}$$

This acceleration is sampled at a rate of 200 Hz by an accelerometer that is to be modelled as:

$$a_c(t_j) = a(t_j) + b_a + w(t_j)$$

where w is a white-noise sequence and is sampled at each time from the distribution $N(0, 0.0004(m/s^2)^2)$. The bias b_a has statistics $N(0, 0.01(m/s^2)^2)$. To calibrate the accelerometer, GPS measurements of position and velocity are available at a 5 Hz rate (synchronized with the accelerometer) as:

$$z_{1i} = x_i + \eta_{1i}$$
$$z_{2i} = v_i + \eta_{2i}$$

The initial conditions for the position and velocity have distributions $x_0 \sim N(0, (10m)^2)$ and $v_0 \sim N(100m/s, (1m/s)^2)$. The associated measurement noise are white-noise sequences with statistics:

$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1m^2 & 0 \\ 0 & (4\text{cm/s})^2 \end{bmatrix}\right)$$

3 Theory and Algorithm

It is required to create a discrete time stochastic system to calibrate the accelerometer. This discrete time system must be set up independent of the true acceleration profile. It is then required to implement a minimum variance estimator (the Kalman filter) on the system and test the implementation.

The Kalman filter is derived by minimizing a quadratic error in the estimate. In the case of the process and measurement noises being Gaussian (which is satisfied by our problem), the filter determines the conditional mean of the state given the measurement sequence from the past up until the current time. Thus, the Kalman filter implementation allows us to estimate the state by merging information from a model and a measurement sequence, without precise knowledge of the behaviour of the true state of the system.

Assume that the dynamics of a system and its measurements are modelled as:

$$\begin{aligned} x_{k+1} &= \Phi_k x_k + \Gamma_k w_k, \qquad x_0 \sim N(\bar{x}_0, M_0) \\ z_k &= H_k x_k + v_k \end{aligned}$$

where, k is the sample index, x is the state vector of dimension n, Φ_k is the nxn state transition matrix, Γ is the matrix acting on the w_k process noise, z_k is the measurement obtained at time k, and v_k is the measurement noise.

Note that, w_k and v_k are zero-mean, Gaussian, white noise processes with covariances W_k and V_k, respectively. In addition, w_k, v_k, and x_0 are all independent of each other.

Define $Z_k := \{z_0.....z_{k-1}, z_k\}$ as the measurement history upto time k. The Kalman filter is a set of recursive computations that estimate the state of the system with its conditional mean $\widehat{x}_k := E[x_k|Z_k]$. This is called the *a posteriori* estimate since it is an estimate updated to include information from the measurement at time k. The *a priori* estimate, denoted by $\overline{x}_k := E[x_k|Z_{k-1}]$, describes the propagation of the state between measurements.

Define the error in the *a priori* estimate as $\overline{e}_k$ and the error in the *posteriori* estimate as $\widehat{e}_k$:

$$\begin{aligned} \overline{e}_k &:= x_k - \overline{x}_k \\ \widehat{e}_k &:= x_k - x_k \end{aligned}$$

and define their corresponding conditional covariances as:

$$
\begin{aligned}
M_{k+1} &:= E[\overline{e}_{k+1}\overline{e}_{k+1}^T | Z_k] \\
P_k &:= E[\widehat{e}_k\widehat{e}_k^T | Z_k]
\end{aligned}
$$

The recursive equations of the Kalman filter are:
(1) Propagation of the estimate, forwards in time:

$$
\begin{aligned}
\overline{x}_{k+1} &= \Phi_k \widehat{x}_k \\
M_{k+1} &= \Phi_k P_k \Phi_k^T + \Gamma_k W_k \Gamma_k^T
\end{aligned}
$$

(2) Update of the estimate to $\widehat{x}_k$ at time k due to measurement z_k:

$$
\begin{aligned}
&K_k = M_k H_k^T (H_k M_k H_k^T + V_k)^{-1} \\
&\widehat{x}_k = \overline{x}_k + K_k(z_k - H_k \overline{x}_k) \\
&P_k = (M_k^{-1} + H_k^T V_k^{-1} H_k)^{-1} \\
&\text{but for better computation, rewrite as} \\
&P_k = (I - K_k H_k) M_k (I - K_k H_k)^T + K_k V_k K_k^T
\end{aligned}
$$

K_k is the Kalman gain, which can be viewed as a transformation that determines the relative importance of the measurement and the model.

For the given problem, the truth model is given by:

$$
\begin{aligned}
&a(t) = 10sin(\omega t) \\
&v(t) = v(0) + \frac{10}{\omega} - \frac{10}{\omega}cos(\omega t) \\
&p(t) = p(0) + \left(v(0) + \frac{10}{\omega}\right)t - \frac{10}{\omega^2}sin(\omega t)
\end{aligned}
$$

To calibrate the accelerometer, we construct a stochastic discrete time dynamical system independent of the truth model. In order to do this, integrate the accelerometer output by Euler's formula, knowing that the sampling frequency is 200 Hz,ie, $\triangle t = \frac{1}{200}s$:

$$
\begin{aligned}
&a_c(t_j) = a(t_j) + b + w(t_j) \\
&v_c(t_{j+1}) = v_c(t_j) + a_c(t_j)\triangle t \\
&p_c(t_{j+1}) = p_c(t_j) + v_c(t_j)\triangle t + a_c(t_j)\frac{\triangle t^2}{2}
\end{aligned}
$$

To create the dynamical system, assume that the actual acceleration also integrates by Euler's formula as:

$$v_E(t_{j+1}) = v_E(t_j) + a(t_j)\triangle t$$
$$p_E(t_{j+1}) = p_E(t_j) + v_E(t_j)\triangle t + a_c(t_j)\frac{\triangle t^2}{2}$$

Subtracting the accelerometer computations v_c, and p_c from v_E and p_E, we get the dynamical system with coefficients independent of the true acceleration $a(t)$ as:

$$\begin{bmatrix} \delta p_E(t_{j+1}) \\ \delta v_E(t_{j+1}) \\ b(t_{j+1}) \end{bmatrix} = \begin{bmatrix} 1 & \triangle t & -\frac{\triangle t^2}{2} \\ 0 & 1 & -\triangle t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta p_E(t_j) \\ \delta v_E(t_j) \\ b(t_j) \end{bmatrix} - \begin{bmatrix} \frac{\triangle t^2}{2} \\ \triangle t \\ 0 \end{bmatrix} w(t_j)$$

where

$$\delta p_E(t_0) = p_E(t_0) - p_c(t_0) \sim N(0, M_0^p)$$
$$\delta v_E(t_0) = v_E(t_0) - v_c(t_0) \sim N(0, M_0^v)$$
$$b(0) \sim N(0, M_0^b)$$

For the purposes of our Kalman filter implementation, the measurements are taken as:

$$\delta z = \begin{bmatrix} \delta p(t_i) + \eta^p(t_i) \\ \delta v(t_i) + \eta^v(t_i) \end{bmatrix}$$

where $\delta p(t_i) = p(t_i) - p_c(t_i)$ and $\delta v(t_i) = v(t_i) - v_c(t_i)$. Note that measurements provide no information of the bias.

4 Implementation for the problem

For the given problem, we have:

$$\triangle t = \frac{1}{200}\text{s}$$
$$\Phi = \begin{bmatrix} 1 & \triangle t & -\frac{\triangle t^2}{2} \\ 0 & 1 & -\triangle t \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} \frac{\triangle t^2}{2} \\ \triangle t \\ 0 \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$M_0 = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$
$$V = \begin{bmatrix} 1 & 0 \\ 0 & 0.0016 \end{bmatrix}$$

The fact that the sampling frequency of the acceleormeter is 200 Hz and that of the GPS is 5 Hz, leads to the following implementation to compute the estimate $\delta\widehat{x}_{t_j}$ and the filter covariance P_{t_j} as:

- After obtaining the measurement at time t_j, compute the Kalman gain K_{t_j}, the *posteriori* estimate $\delta\widehat{x}_{t_j}$, and the filter covariance P_{t_j}.
- The next measurement time is $t_{j+1} = t_j + 40\triangle t$. So the estimated $\delta\widehat{x}_{t_j}$ and its error covariance P_{t_j} have to be propagated for 40 time-steps, before the update to $\delta\widehat{x}_{t_{j+1}}$ and $P_{t_{j+1}}$ with information from the new measurement.
- This is done by applying Φ to the estimate $\delta\widehat{x}_{t_j}$ for 40 times, to reach at $\delta\overline{x}_{t_{j+1}}$.
- P_{t_j} is first propagated by a time-step of $\triangle t$ into $M_{t_{j+\triangle t}} = \Phi P_{t_j}\Phi^T + \Gamma W \Gamma^T$
- This covariance at time $(t_j + \triangle t)$ is now propagated for 39 more time-steps without measurements, to reach time $(t_j + \triangle t) + 39\triangle t = t_j + 40\triangle t = t_{j+1}$. This propagation occurs as $M_{k+1} = \Phi M_k \Phi^T + \Gamma W \Gamma^T$.
- repeat from step 1.

5 Results and Performance

It must be remembered that the Kalman gain K_k and the error covariance P_k can be calculated *apriori*, without any measurement data. The Kalman gain and the error covariance at all measurement times were computed and stored before proceeding with the Monte Carlo simulation that computed an ensemble of 3x10^5 realizations.

The state of the stochastic system is denoted by $\delta x(t_j) = \begin{bmatrix} p(t_j) - p_c(t_j) \\ v(t_j) - v_c(t_j) \\ b(t_j) \end{bmatrix}$.

The estimate obtained from the filter is denoted by $\delta \widehat{x}(t_j) = \begin{bmatrix} \widehat{p}(t_j) - p_c(t_j) \\ \widehat{v}(t_j) - v_c(t_j) \\ \widehat{b}(t_j) \end{bmatrix}$.

Of the 3x10^5 realizations, the realisation numbered 266148 was randomly chosen to plot the state $\delta x(t_j)$ and the estimate $\delta \widehat{x}(t_j)$ (Figures 5.1, 5.2, 5.3). Notice how the bias $\widehat{b}$ takes more time to approach the true bias b. This is because the bias has no explicit measurements of the bias are made. The velocity and position estimates are close to the actual states.

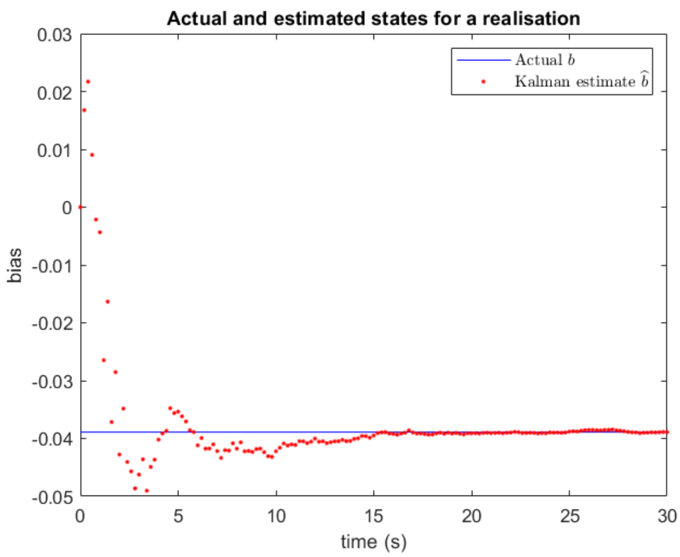


Figure 5.1: Actual and estimated bias for a realization.

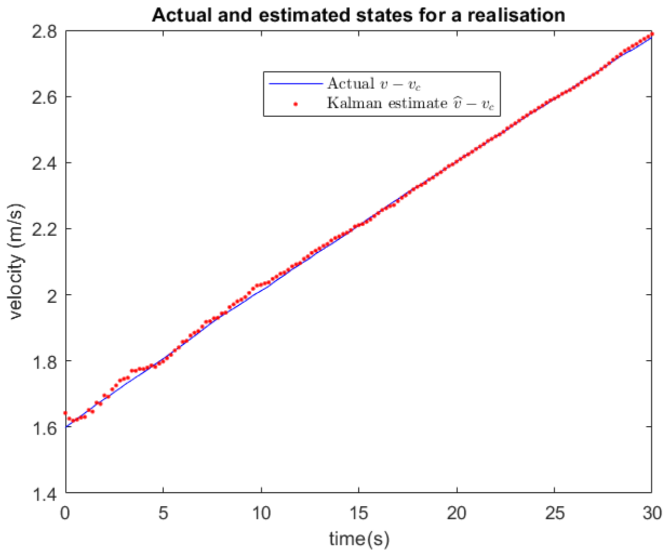


Figure 5.2: Actual and estimated velocity state for a realization.

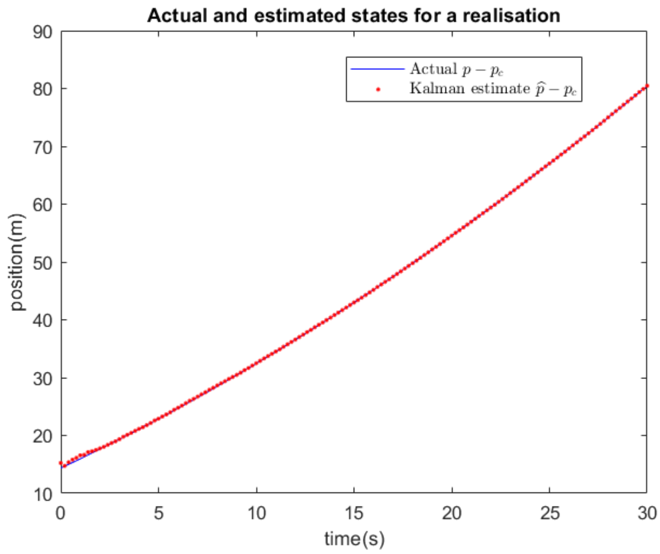


Figure 5.3: Actual and estimated position state for a realization.

The *posteriori* estimation error $e(t_j)$ for the same 226148 th realization is plotted in figures 5.4, 5.5, and 5.6. Notice that, $e(t_j) = \delta x(t_j) - \delta\widehat{x}(t_j) = \begin{bmatrix} p(t_j) - \widehat{p}(t_j) \\ v(t_j) - \widehat{v}(t_j) \\ b(t_j) - \widehat{b}(t_j) \end{bmatrix}$. Thus, the *posteriori* estimation error $e(t_j)$ also computes the difference between true quantities of p, v, b and the estimates $\widehat{p}, \widehat{v}, \widehat{b}$.

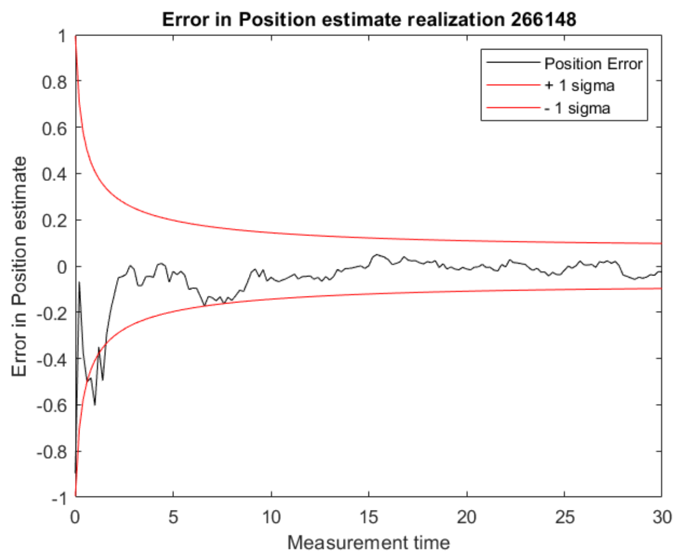


Figure 5.4: Error in estimate of the position, $p(t_j) - \widehat{p}(t_j)$, for a realization.

The error in the Kalman filter's estimate of position at all measurement times is reasonably within the 1σ bound predicted by the filter variance element P_{11}.

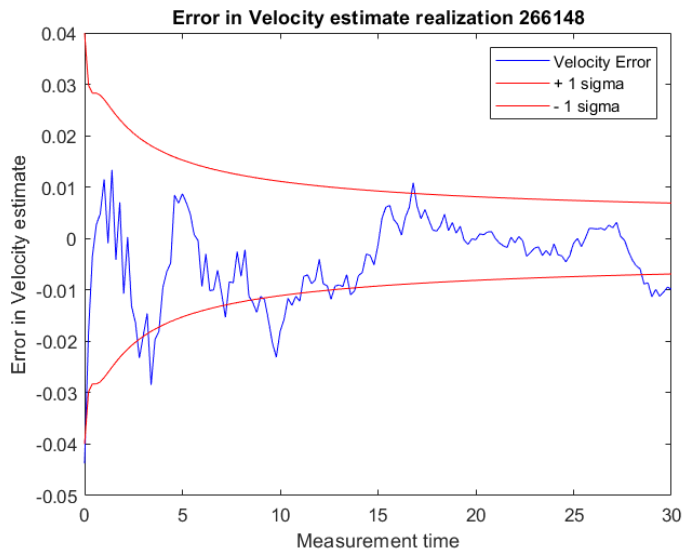
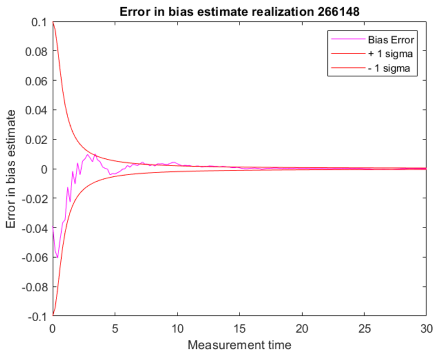
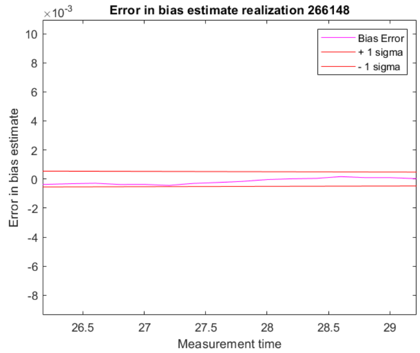


Figure 5.5: Error in estimate of the velocity, $v(t_j) - \widehat{v}(t_j)$, for a realization.

The error in the Kalman filter's estimate of velocity at all measurement times is reasonably within the 1σ bound predicted by the filter variance element P_{22}.



(a) Error in estimate of bias,$b(t_j) - \widehat{b}(t_j)$



(b) Error in estimate of bias, magnified

The error in the Kalman filter's estimate of bias at all measurement times is reasonably within the 1σ bound predicted by the filter variance element P_{33}. The bounds go to zero at larger times and hence the zoomed image is shown. Note that the bias has been estimated without any measurements.

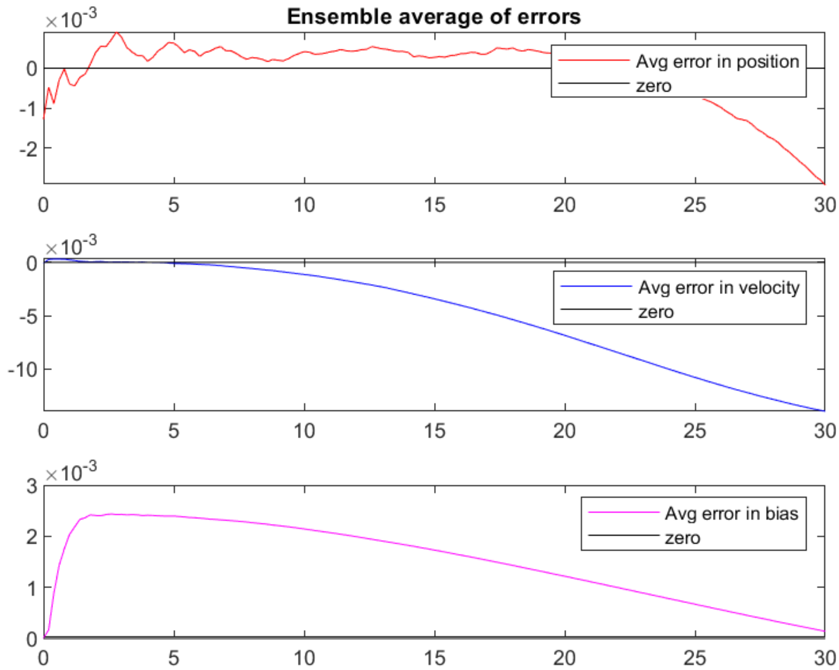


Figure 5.7: Computed the average of the error in the estimate across all realizations.

The error in the estimate, in a realization l is given by, $e^l(t_j) = \delta x^l(t_j) - \delta\widehat{x}^l(t_j)$. The ensemble average of the error, $e^{ave}(t_j)$ is computed as:

$$e^{ave}(t_j) = \frac{1}{N_{ave}} \sum_{l=1}^{N_{ave}} e^l(t_i).$$

The computed average error (plotted in figure 5.7) must be close to zero for estimates of p, v, b. The elements of $e^{ave}(t_j)$ vector stay close to zero, but do not exactly equal zero. A probable reason for this behaviour is that the state of the system $\delta x(t_j) = \begin{bmatrix} p(t_j) - p_c(t_j) \\ v(t_j) - v_c(t_j) \\ b(t_j) \end{bmatrix}$ which was estimated by the Kalman filter, approximated p, v by an Euler formula. However, the measurements $\delta z = \begin{bmatrix} \delta p(t_i) + \eta^p(t_i) \\ \delta v(t_i) + \eta^v(t_i) \end{bmatrix}$ use the true value of p, v obtained by integrating the acceleration profile: $a(t) = 10sin(\omega t)$. This causes the error in the estimate to oscillate.
The following figure (figure 5.8) was obtained on computing the ensemble average of $5\text{x}10^4$ realizations, over a longer time duration from $[0s, 120s]$, instead of the regular $[0s, 30s]$.

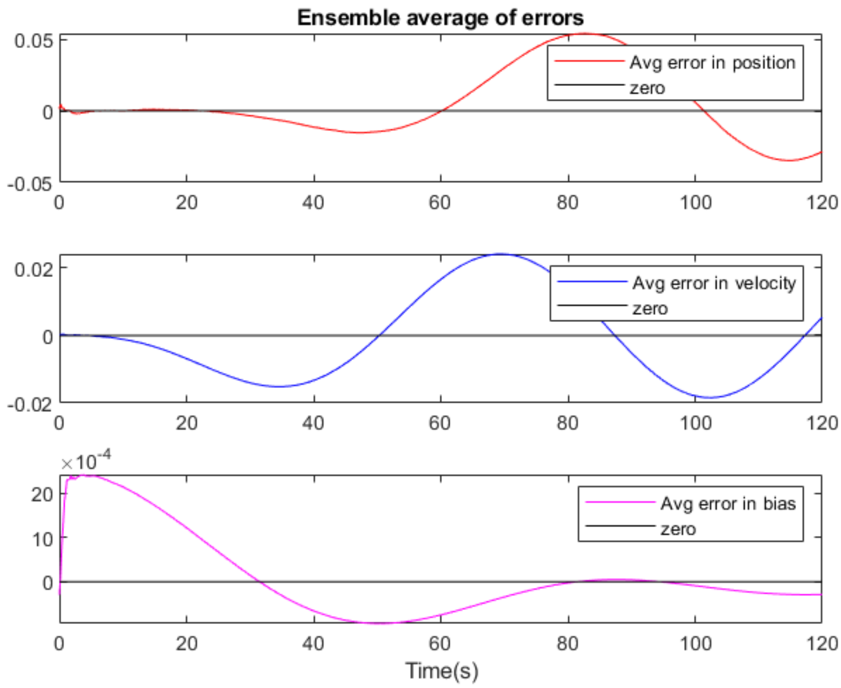


Figure 5.8: Computed the average error in the estimate over $[0s, 120s]$.

On correct implementation, at all times, the elements of the filter covariance must match with those of the simulated error covariance, $P^{ave}(t_i)$, calculated as:$P^{ave}(t_i) = \frac{1}{N_{ave}-1}\sum_{l=1}^{N_{ave}}[e^l(t_i) - e^{ave}(t_i)][e^l(t_i) - e^{ave}(t_i)]^T$. Figures 5.9 and 5.10.

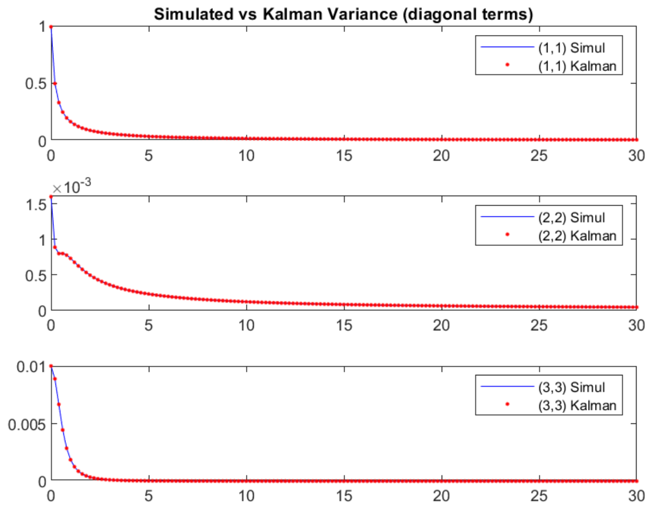


Figure 5.9: Comparing diagonal terms of Simulated ensemble average variance $P^{ave}(t_i)$ and filter covariance

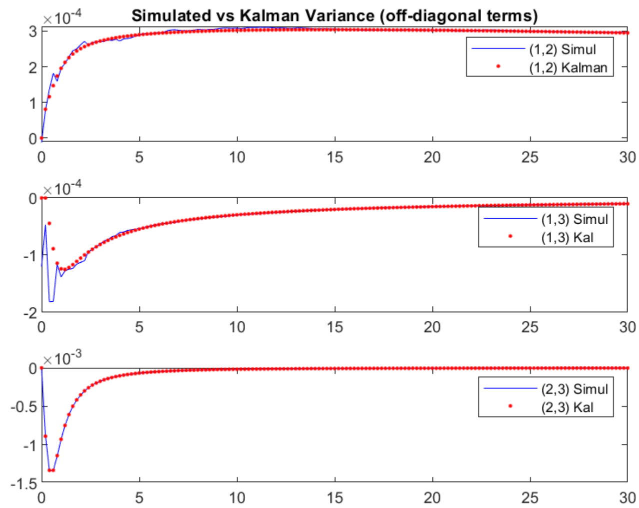


Figure 5.10: Comparing off-diagonal terms of Simulated ensemble average variance $P^{ave}(t_i)$ and filter covariance

The orthogonal properties verified are $E[e_k \widehat{x}_k^T] = 0$ and $E[r_k r_j^T] = 0 \; \forall \; j < k$. Note, r_k is the residual, given by $r_k = z_k - H\overline{x}_k$.

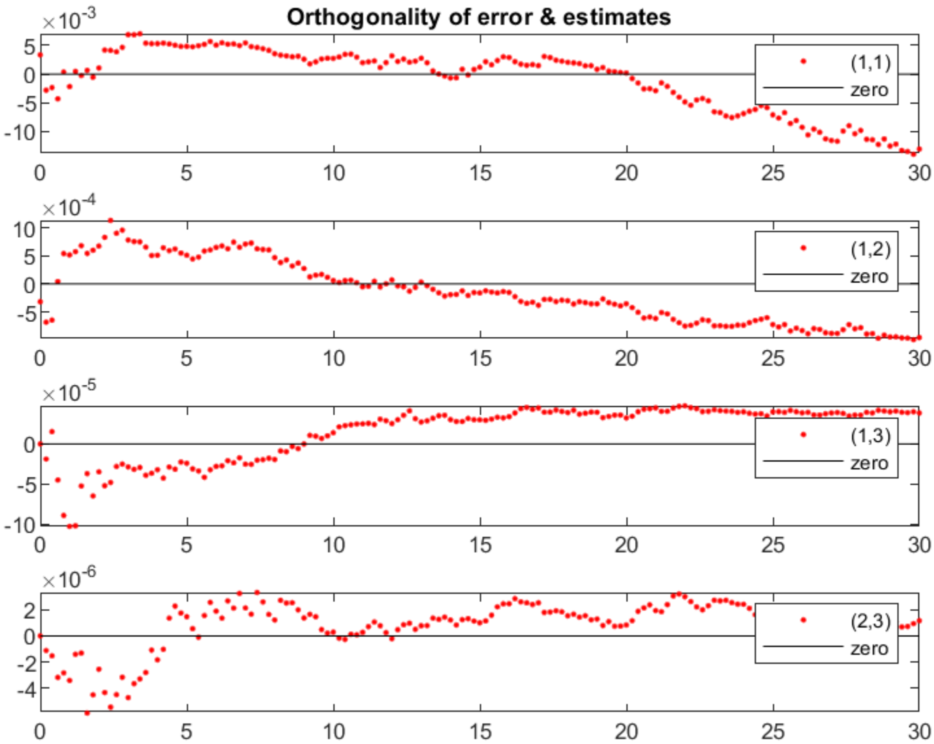


Figure 5.11: Orthogonality of the error and its estimate: elements of the matrix $E[e_k \widehat{x}_k^T]$

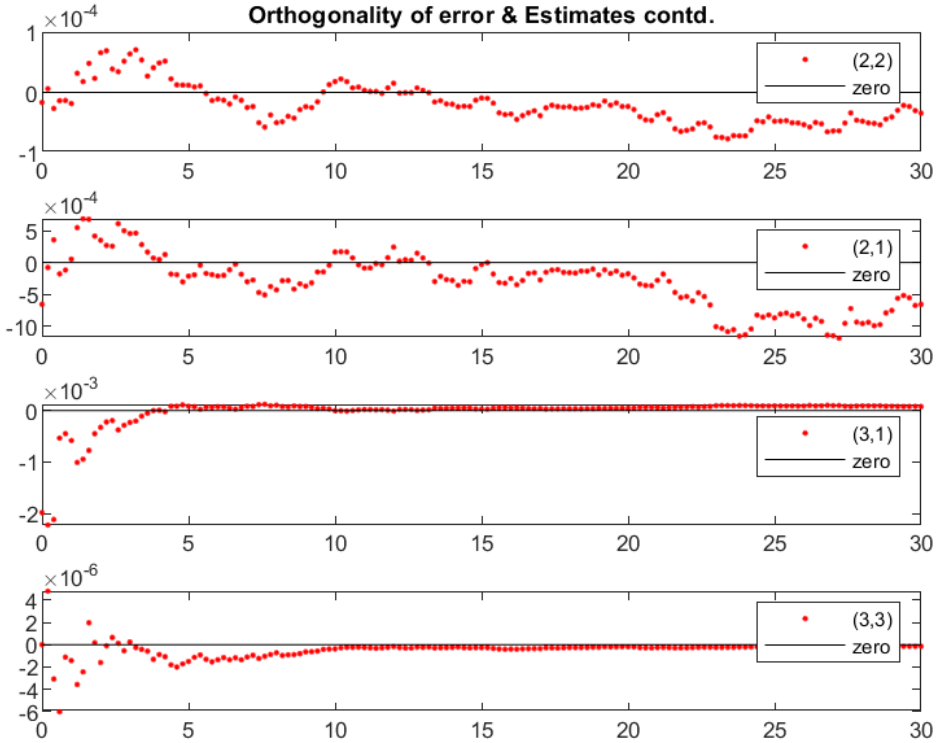


Figure 5.12: Orthogonality of the error and its estimate: other elements of the matrix $E[e_k\hat{x}_k^T]$

It is required that the matrix $E[r_k r_j^T] = 0 \forall j < k$. This was verified for $k = 23.8$ seconds, and $j = 5.8$ seconds to be:

$$E[r_k r_j^T] = \begin{bmatrix} 0.5523 \times 10^{-6} & 0.0529 \times 10^{-6} \\ -0.0058 \times 10^{-6} & -0.0006 \times 10^{-6} \end{bmatrix} \quad \text{where } k = 23.8 \text{ s, and } j = 5.8 \text{ s}$$

6 Conclusions

The Kalman filter was successfully implemented by first developing a discrete time stochastic system and used the *apriori* information to calculate the Kalman gains and the conditional error variance. Measurements of only postion and velocity were provided, but the filter could reasonably estimate position, velocity, and bias too. During implementation, attention must be provided to synchronize information from GPS measurements at 5 Hz, and the model whose frequency is 200 Hz.
A Monte Carlo simulation computed an ensemble of $3\text{x}10^5$ realizations. This was

used to verify the filter's covariance matrix, the average of the estimated error, and the states for a randomly chosen realization were plotted. Finally, orthogonality tests were performed to verify the orthogonality between the estimate and its error, and between residuals at different times. All the tests were shown to satisfy the theoretical properties, making the implementation a success.