Bachelor of Technology in Mechanical Engineering

ME326 – Minor Project

Finite-Element Analysis of Functionally Graded Thick Cylinder under Pressure and Rotation

Submitted by

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1. Introduction

1.1. Functionally Graded Materials

Functionally graded materials (FGMs):

- Composite materials with continuously varying material composition from one surface to another surface as shown in Figure 1.
- Non-homogeneous materials with varying material properties and microstructures

Such material properties can be described by a function *f*(*x*).

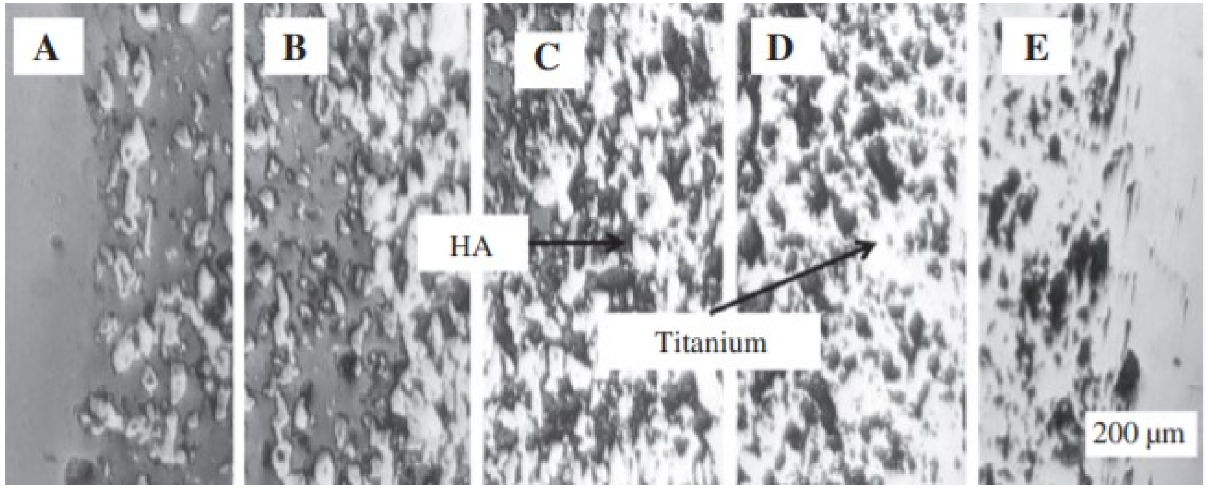


Figure 1. Microstructure of a cross section of
HA-Ti FGM having a continuous composition gradient

Advantages of FGM

- Reduced maximum stress
- Improved surface properties
- Protection against severe environments
- Improved bonding strength of the interfacial zones
- Reduced residual stress

1. Introduction contd.

1.2. Applications

Functionally Graded Materials have applications in the following technology fields:

1. ***Aerospace Field*** : Using FGMs high temperature resistance, thermal shock resistance, thermal fatigue resistance and corrosion resistance can be applied to heat-resistant surface of space shuttle and aircraft engine parts.

2. ***Nuclear Energy Field*** : FGMs show great superiority in the construction materials of nuclear furnaces and inner wall materials of nuclear furnaces, which greatly protects the safety of the nuclear industry.

3. ***Machine Tool Technology***: Thermo-mechanical stress concentration is significantly relaxed by adding an FGM layer between the steel shank and the ceramic tip in case of lathe cutting tools.

4. ***Electromagnetic Field*** : FGMs gradient structure has the piezoelectric gradient function and can be used to make electromagnetic shielding materials, ceramic filters, ultrasonic oscillators.

5. ***Energy Sector*** : FGMs provide thermal barrier, thus used as protective coating on turbine blades in gas turbine. And prevents the emitter from cracking in high temperature working environment of 1860°C and greatly reduces the thermal stress of the system.

2. Analytical Modelling

Analytical model is computational in nature, it represents the system in terms of mathematical equations that specify parametric relationships between associated parameter values as a function of some system parameter.

Problem Statement:

- The geometry of hollow cylinder is defined using the cylindrical coordinate system with coordinates x, θ and r denoting the axial, circumferential and radial coordinates, respectively as shown in Figure 2.
- The inner and outer radius is denoted by "a" and "b" respectively, Poisson's ratio ν is assumed to be constant.
- Radial stress σ_r and hoop stress σ_θ are to be determined under pressure load and rotation.
- Since the applied loads and material properties are independent of θ, mechanical solutions are assumed to be axisymmetric.

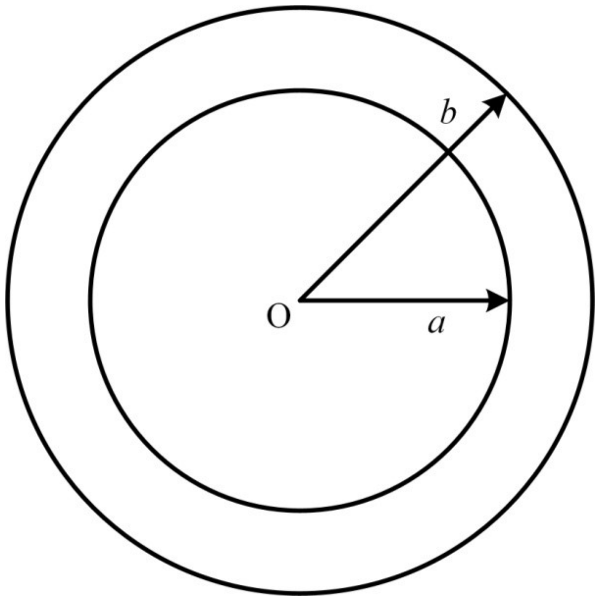


Figure 2. Cross- sectional view of an axisymmetric long cylinder

The following cases are discussed:

1. Stationary isotropic cylinder
2. Rotating isotropic cylinder
3. Stationary FGM cylinder - *power* and *exponential* functions
4. Rotating FGM cylinder - *power* and *exponential* functions

2. Analytical Modelling contd.

2.1. Elastic analysis of a stationary isotropic cylinder

Stress and displacement analysis of a cylinder which is free to expand in the axial direction can be treated as generalized plane strain problem.

The equilibrium equation is given as

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1)$$

The governing differential equation for the stationary isotropic cylinder is obtained by combining the equilibrium and the compatibility equation, and takes the form as follows

$$r\frac{d^2\sigma_r}{dr^2} + 3\frac{d\sigma_r}{dr} = 0 \qquad (2)$$

Applying the boundary conditions:

$$\sigma_r = -P_i \quad \text{at } r = a \qquad (3i)$$

$$\sigma_r = -P_o \quad \text{at } r = b \qquad (3ii)$$

The radial and circumferential stresses are given by

$$\sigma_r = \frac{(P_i a^2 - P_o b^2)}{(b^2 - a)^2} + \frac{(P_o - P_i)a^2 b^2}{(b^2 - a^2)r^2} \quad \text{and} \quad \sigma_\theta = \frac{(P_i a^2 - P_o b^2)}{(b^2 - a)^2} - \frac{(P_o - P_i)a^2 b^2}{(b^2 - a^2)r^2} \qquad (4)$$

2. Analytical Modelling contd.

2.2. Elastic analysis of a rotating isotropic cylinder

For the case of a rotating cylinder the equilibrium equation is modified as

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \tag{5}$$

where, ρ is the density of the material and ω is the angular velocity of the rotation
The governing differential equation for the rotating isotropic cylinder case is obtained by the same procedure and takes the following form

$$r\frac{d^2\sigma_r}{dr^2} + 3\frac{d\sigma_r}{dr} = -(3+\nu)\rho\omega^2 r^2 \tag{6}$$

Applying the boundary conditions:

$$\sigma_r = -P_i \qquad \text{at } r = a$$

$$\sigma_r = -P_o \qquad \text{at } r = b$$

The radial and circumferential stresses are given by

$$\sigma_r = \left[\frac{(P_i a^2 - P_o b^2)}{(b^2 - a)^2} - \frac{(3+\nu)}{8}\rho\omega^2(b^2 - a^2)\right] + \frac{1}{r^2}\left[\frac{(P_o - P_i)a^2 b^2}{(b^2 - a^2)} - \frac{(3+\nu)}{8}\rho\omega^2 a^2 b^2\right] - \frac{(3+\nu)}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = \left[\frac{(P_i a^2 - P_o b^2)}{(b^2 - a)^2} - \frac{(3+\nu)}{8}\rho\omega^2(b^2 - a^2)\right] - \frac{1}{r^2}\left[\frac{(P_o - P_i)a^2 b^2}{(b^2 - a^2)} - \frac{(3+\nu)}{8}\rho\omega^2 a^2 b^2\right] - \left[1 - \frac{3(3+\nu)}{8}\right]\rho\omega^2 r^2 \tag{7}$$

2. Analytical Modelling contd

2.3. Elastic analysis of a stationary FGM cylinder

2.3.1. Power Function

In this case the modulus of elasticity is defined as a power function

$$E(r) = E_o r^{\beta} \tag{8}$$

where, E_0 is constant for Modulus of elasticity and β is a constant that controls the behavior of power function

And stress function F is introduced such that $\sigma_r = \dfrac{F(r)}{r}$ and $\sigma_\theta = \dfrac{dF(r)}{dr}$. (9)

The governing differential equation is given as

$$r^2 \frac{d^2F}{dr^2} + (1-\beta) r \frac{dF}{dr} - \left(1 - \frac{\nu\beta}{1-\nu}\right) F = 0 \tag{10}$$

The expression for radial and hoop stresses obtained after applying the boundary conditions are

$$\sigma_r = \frac{-P_i + P_o \left(\frac{a}{b}\right)^{m_2-1}}{a^{m_1-1} - a^{m_2-1} b^{m_1-m_2}} r^{m_1-1} + \left[\frac{-P_o}{b^{m_2-1}} + \frac{P_i - P_o \left(\frac{a}{b}\right)^{m_2-1}}{a^{m_1-1} - a^{m_2-1} b^{m_1-m_2}} b^{m_1-m_2}\right] r^{m_2-1}$$

$$\sigma_\theta = \frac{-P_i + P_o \left(\frac{a}{b}\right)^{m_2-1}}{a^{m_1-1} - a^{m_2-1} b^{m_1-m_2}} m_1 r^{m_1-1} + \left[\frac{-P_o}{b^{m_2-1}} + \frac{P_i - P_o \left(\frac{a}{b}\right)^{m_2-1}}{a^{m_1-1} - a^{m_2-1} b^{m_1-m_2}} b^{m_1-m_2}\right] m_2 r^{m_2-1} \tag{11}$$

where, $m_{1,2} = \dfrac{-\beta \pm \sqrt{\beta^2 - 4(1 - \frac{\nu}{1-\nu})\beta}}{2}$

2. Analytical Modelling contd.

2.3. Elastic analysis of a stationary FGM cylinder contd.

2.3.2. Exponential Function

Similarly, the modulus of elasticity is defined as an exponential function

$$E(r) = E_0 e^{\left(\frac{\beta(r-a)}{b-a}\right)} \tag{12}$$

The resulting differential equation is expressed as

$$\frac{d^2F}{dr^2} + \left[1 - \frac{\beta r}{(b-a)}\right]\frac{1}{r}\frac{dF}{dr} - \left[1 - \frac{\nu}{(1-\nu)r^2}\frac{\beta}{b-a}\right]F = 0 \tag{13}$$

The solution for the above equation assumes the form

$$F(r) = F_1(r) + cF_2(r) \tag{14}$$

F1 and F2 are evaluated by the boundary conditions using numerical differentiation for which Runge-Kutta Method of order four is used. And from that stress distribution is evaluated.

2. Analytical Modelling contd.

2.4. Elastic analysis of a rotating FGM cylinder

2.4.1. Power Function

The modulus of elasticity and density through the wall thickness are assumed to vary as follows

$$E = E_o r^{\beta_1} \quad \text{and} \quad \rho = \rho_o r^{\beta_2} \tag{15}$$

The governing differential equation for the rotating FGM cylinder case is obtained in the form of Navier's equation as

$$r^2 \frac{d^2 u_r}{dr^2} + (1+\beta_1) r \frac{du_r}{dr} + (n\beta_1 - 1) u_r = -\frac{\rho_i \omega^2 a^{\beta_1 - \beta_2}}{E_o C_{11}} r^{\beta_2 - \beta_1 + 3} \tag{16}$$

where, $C_{11} = \frac{(1-\nu)}{(1+\nu)(1-2\nu)}$, $n = \frac{\nu}{(1-\nu)}$ and radial displacement u_r is related to strain as $\varepsilon_r = \frac{du_r}{dr}$ and $\varepsilon_\theta = \frac{u_r}{r}$.

2.4.2. Exponential Function

The modulus of elasticity and density through the wall thickness are assumed to vary as follows:

$$E = E_o r^{\left(\frac{\beta_1 (r-a)}{b-a}\right)} \quad \text{and} \quad \rho = \rho_o r^{\left(\frac{\beta_2 (r-a)}{b-a}\right)} \tag{17}$$

The second order differential equation becomes

$$\frac{d^2 u_r}{dr^2} + \left(\frac{\beta_1}{b-a} + \frac{1}{r}\right)\frac{du_r}{dr} + \left(\frac{n}{r}\frac{\beta_1}{b-a} - \frac{1}{r^2}\right) u_r = -\frac{\rho_o \omega^2}{C_{11} E_o} r^{(\beta_2 - \beta_1)\left(\frac{r-a}{b-a}\right)} \tag{18}$$

The radial displacement can be obtained by numerical differentiation methods, and from that radial and hoop stresses can be evaluated.

3. Finite Element Modelling

Finite Element Modelling (FEM) is an approximation method that subdivides a complex problem space into numerous simpler pieces whose behavior can be described with comparatively simple equations.

Thus, the analytical complexity of thick FGM cylinder can be reduced by using FEM to divide the cylinder into elements as shown in Figure 3.

Objectives of Finite Element Analysis:

1. Isotropic cylinders under: p_i, p_e & ω
2. Comparison of FGM defined by *power* and *exponential* functions
3. FGM cylinders under: p_i, p_e & ω
4. Comparison of isotropic cylinders with FGM cylinders

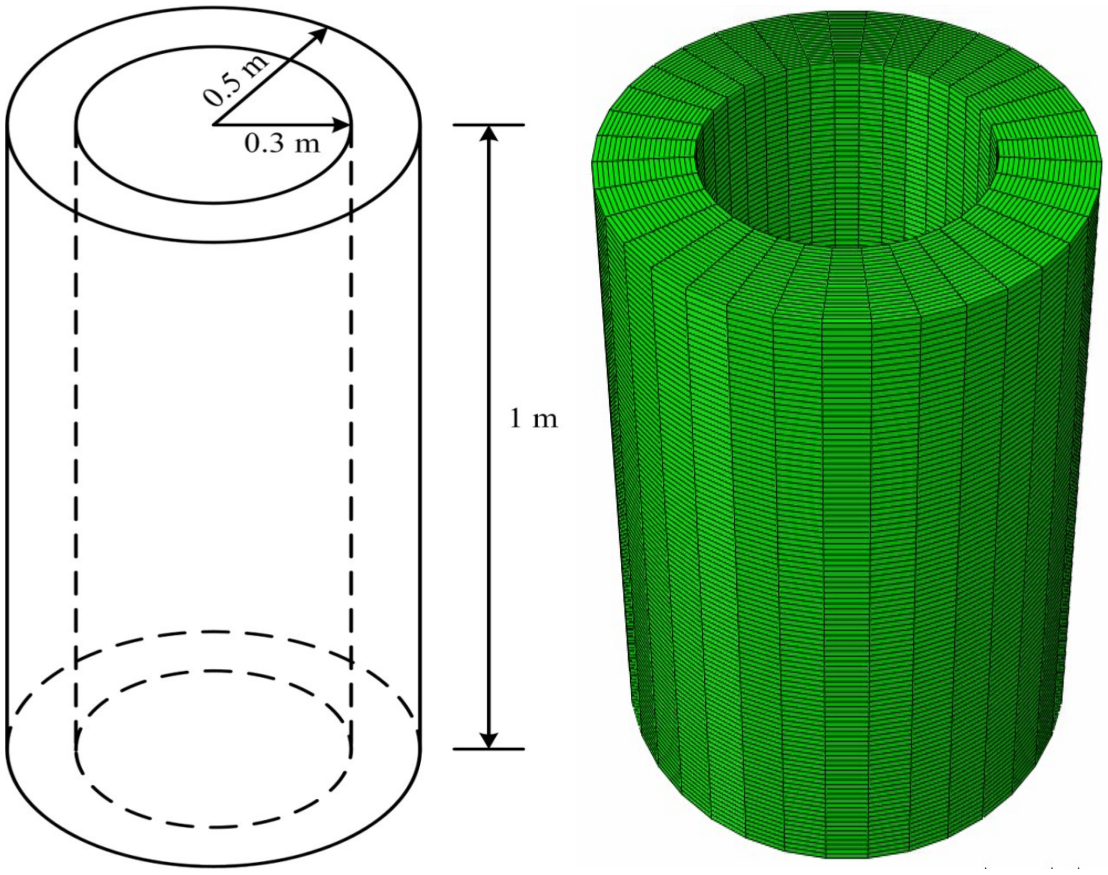


Figure 3. Cylinder divided into elements for FEA

3. Finite Element Modelling contd.

3.1 Geometric Parameters

According to R. Hibbeler, Mechanics of materials 9th edition (2014) the limiting assumption for thick cylinders

$$\frac{\text{internal radius } (r_i)}{\text{thickness } (t)} < 10$$

Hence, the dimensions of the cylinder are assumed as:
internal radius (r_i) = 0.3 m, thickness (t) = 0.2 m & height (h) = 1 m

The FEA of a cylinder can be reduced to the FEA of a rectangle.

- reduces computational time
- reduces the approximation error

An axisymmetric deformable shell part with the geometry as shown in Figure 4 is created in ABAQUS/CAE.

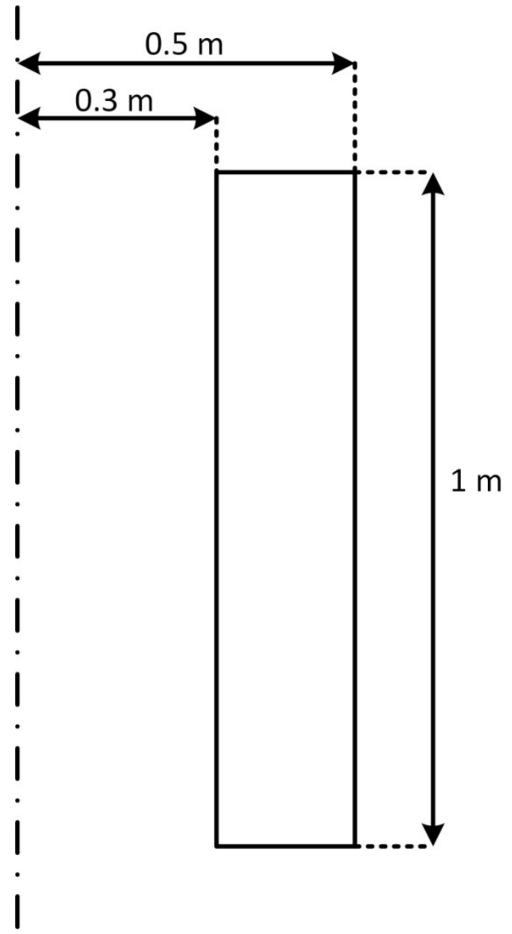


Figure 4. Geometry of the part created for FEA

3. Finite Element Modelling contd.

3.2 Material Properties

Isotropic Material

Isotropic materials have identical material properties in all directions at every given point over the volume of the system.

Material properties for isotropic functions can be directly input into ABAQUS as a metal.

Material properties of **AISI 1045 Medium Carbon Steel**

Properties	**Symbol**	**Value**
Density	ρ	7870 kg/m^3
Modulus of Elasticity	E	200 GPa
Poisson's Ratio	v	0.29

Functionally Graded Material Power Function

Properties of AISI 1045 Steel varying according to *power* function

$$E(r) = E_0 r^{0.1667}$$

Analytical Field is created in ABAQUS to input FGM defined by *power* function.

Modulus of Elasticity, E (in N/mm^2)	**Poisson's Ratio, v**	**Radius, r (in mm)**
517468.0474	0.29	300
523064.1944	0.29	320
528376.08	0.29	340
533433.6551	0.29	360
538262.2574	0.29	380
542883.5233	0.29	400
547316.0837	0.29	420
551576.1025	0.29	440
555677.699	0.29	460
559633.2829	0.29	480
563453.8228	0.29	500

Functionally Graded Material Exponential Function

Properties of AISI 1045 Steel varying according to *exponential* function

$$E(r) = E_0 e^{\left(\frac{r-300}{500-300}\right)}$$

Analytical Field is created in ABAQUS to input FGM defined by *exponential* function.

Modulus of Elasticity, E (in N/mm^2)	**Poisson's Ratio, v**	**Radius, r (in mm)**
200000	0.29	300
221034.1836	0.29	320
244280.5516	0.29	340
269971.7615	0.29	360
298364.9395	0.29	380
329744.2541	0.29	400
364423.7601	0.29	420
402750.5415	0.29	440
445108.1857	0.29	460
491920.6222	0.29	480
543656.3657	0.29	500

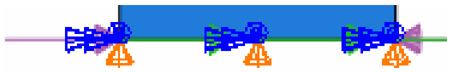
3. Finite Element Modelling contd.

3.3 Load and Boundary Conditions

It is necessary to correctly define the boundary conditions and loads as shown in Figure 5 to extract the accurate results form the finite element solver in ABAQUS.

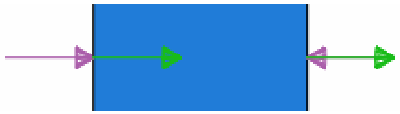
The FEM of a cylinder under axisymmetric assumption, requires only a single boundary condition:

1. YSYMM (U2 = UR1 = UR3 = 0)
   - applied on the lower side of rectangle
   - the y-coordinates on the side remain unchanged



Loads can be applied on the FEM, as per requirements. In this study we have considered the following loads in different combinations:

1. Internal Pressure (p_i)
2. External Pressure (p_e)
3. Rotational Body Force (ω)



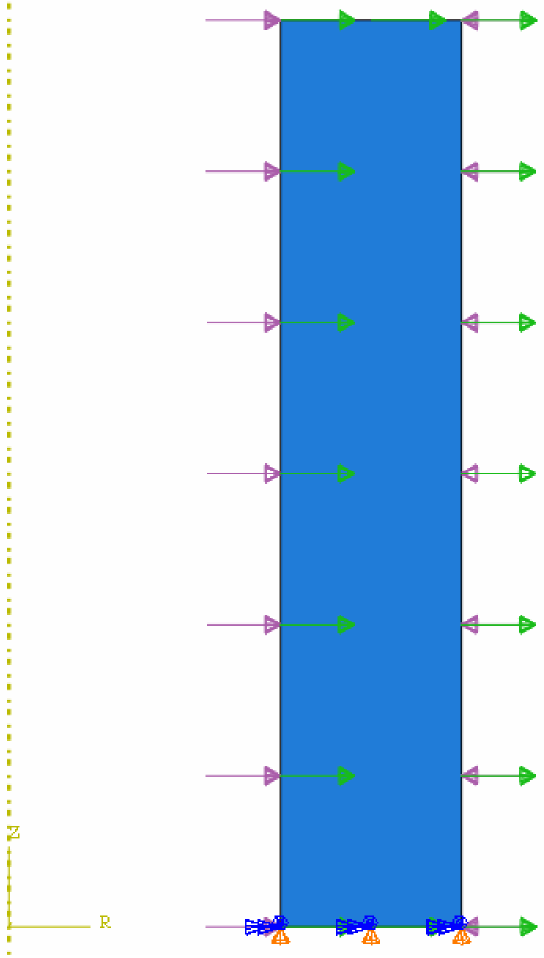


Figure 5. Load (p_i, p_e and ω) and Boundary conditions (Y SYMM)

3. Finite Element Modelling contd.

3.4 Element type and mesh sensitivity

Meshing divides the geometry into finite elements, thereby further reducing the complexity of the rectangle to simpler, smaller rectangular elements.

Element Shape and Type:

- A structured quad element shape is selected to mesh the rectangular geometry.
- A 4-node bilinear axisymmetric quadrilateral (CAX4) element type is selected from the axisymmetric stress family.

Mesh Sensitivity:

The optimum mesh is attained at 67 divisions along the x-axis and 333 divisions along the y-axis.

22311 elements and 22712 nodes as shown in Figure 6.

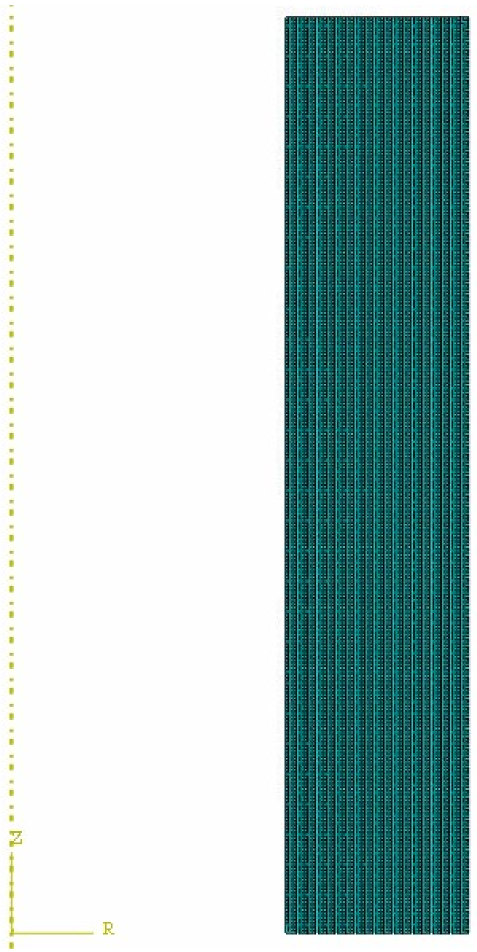


Figure 6. Structured quad CAX4 Mesh

4. Results and Discussion

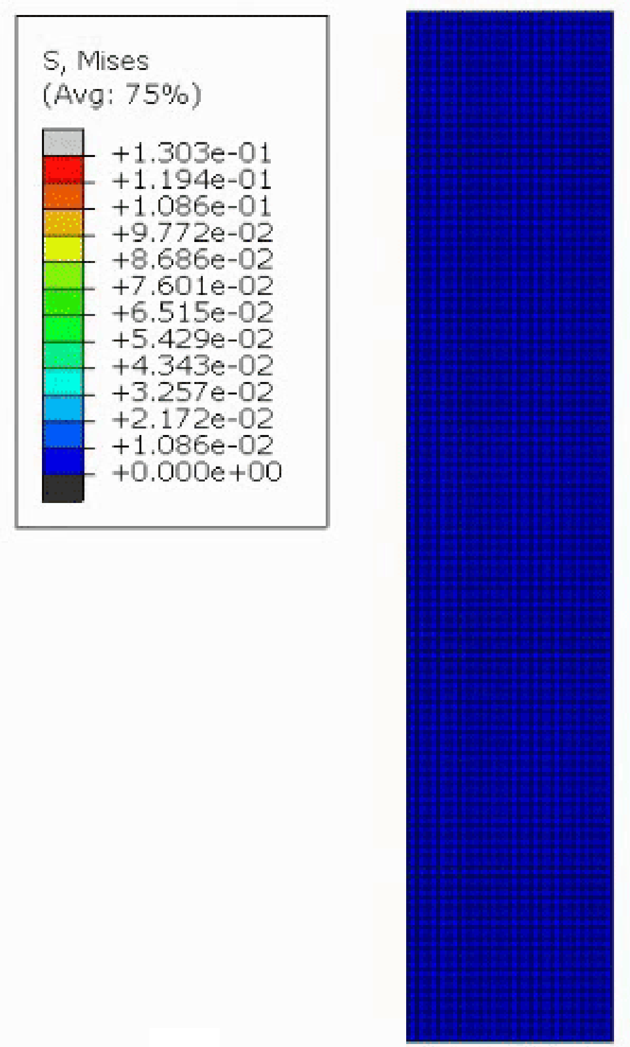
4.1 Results from Analytical Modelling — MATLAB Code

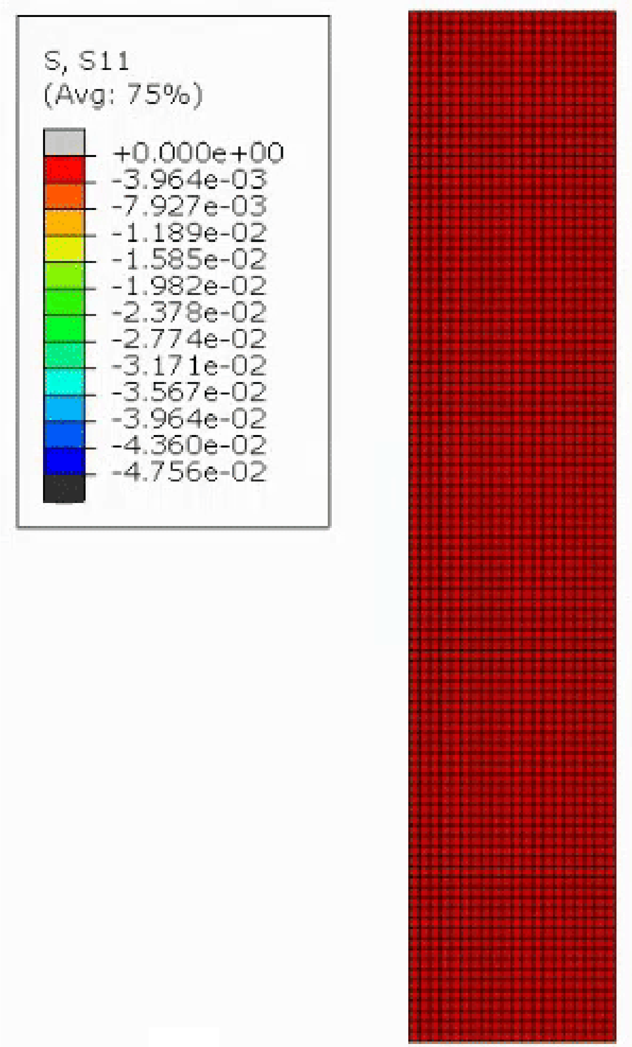
```matlab
%%% Analytical Solution of Isotropic Cylinder under Pressure %%%
%%%%%%%%%%%%%%%%%%%% Geometry and Load %%%%%%%%%%%%%%%%%%%%
a = 300;            % Initial Internal Radius
b = 500;            % Initial Outer Radius
qa = 25;            % Internal Pressure
qb = 10;            % External Pressure
u = 300.091;        % Deformed Internal Radius (Extracted from FEM)
v = 500.07;         % Deformed Outer Radius (Extracted from FEM)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%% Calculating Radial Stress from Analytical Model %%%%%
v_isig_r = zeros(200,1);     % Pre-allocation of memory
i=1;
for r = u:1:v
isig_r = (qa*(a^2)-qb*(b^2))/((b^2)-(a^2))+...
    (((a^2)*(b^2)*(qb-qa))/((b^2)-(a^2)))*(1/(r^2));
v_isig_r(i,1) = isig_r;
i=i+1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%% Calculating Hoop Stress from Analytical Model %%%%%%
v_isig_t = zeros(200,1);     % Pre-allocation of memory
i=1;
for r = u:1:v
isig_t = (qa*(a^2)-qb*(b^2))/((b^2)-(a^2))-...
    (((a^2)*(b^2)*(qb-qa))/((b^2)-(a^2)))*(1/(r^2));
v_isig_t(i,1) = isig_t;
i=i+1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

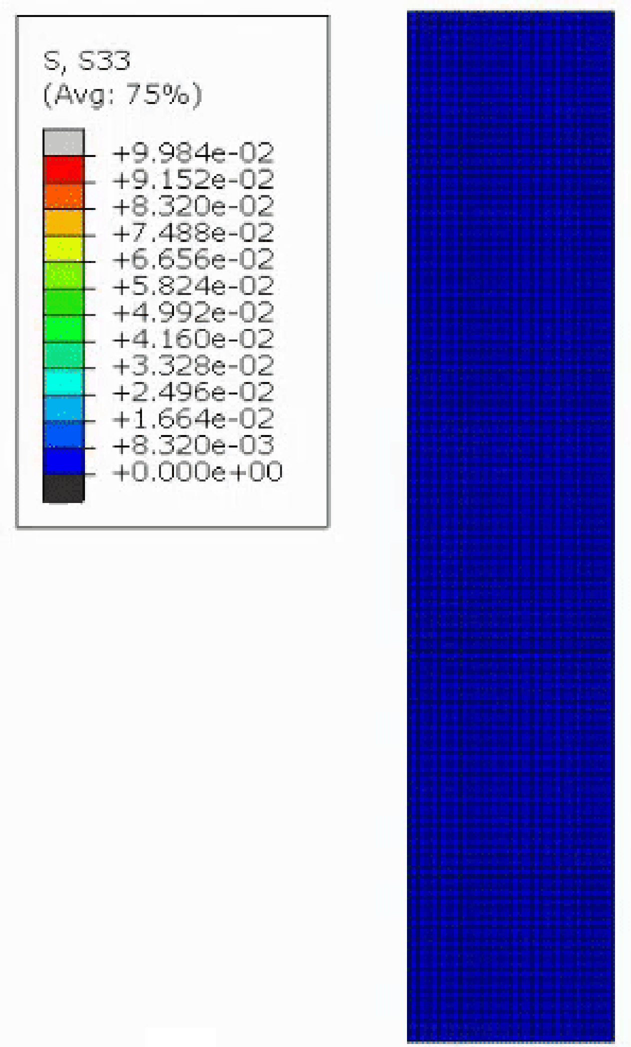
```matlab
%%%%%%%%% Data from Finite Element Analysis (FEA) in ABAQUS %%%%%%%%%%%%
radius = [];            % x-coordinates on a path along x-axis
radial_stress = [];     % S11 values along the same path
hoop_stress = [];                    % S33 values along the same path
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%% Plotting graphs for validation of results %%%%%%%%%%%%%%
%%%%% Vector Transformation to make the size of vectors equal %%%%%
r = u:1:v;
r = r';
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
subplot(1,2,1)
plot(r,v_isig_r)
hold on
plot(r,v_isig_r)
hold on
xlabel('$\textit{r}$', 'FontName', 'Times New Roman', ...
       'FontSize',14,'Color','k', 'Interpreter', 'LaTeX')
ylabel('Radial Stress, \sigma_{r} [MPa]', 'FontName', 'Times New Roman',
'FontSize',12,'Color','k')
subplot(1,2,2)
plot(r,v_isig_t)
hold on
plot(r,v_isig_t)
hold on
xlabel('$\textit{r}$', 'FontName', 'Times New Roman', ...
       'FontSize',14,'Color','k', 'Interpreter', 'LaTeX')
ylabel('Hoop Stress, \sigma_{\theta} [MPa]', 'FontName', 'Times New Roma
n', 'FontSize',12,'Color','k')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

4. Results and Discussion contd.

4.2 Results from Finite Element Modelling — ABAQUS







4. Results and Discussion contd.

4.3 Isotropic Cylinder — P_i

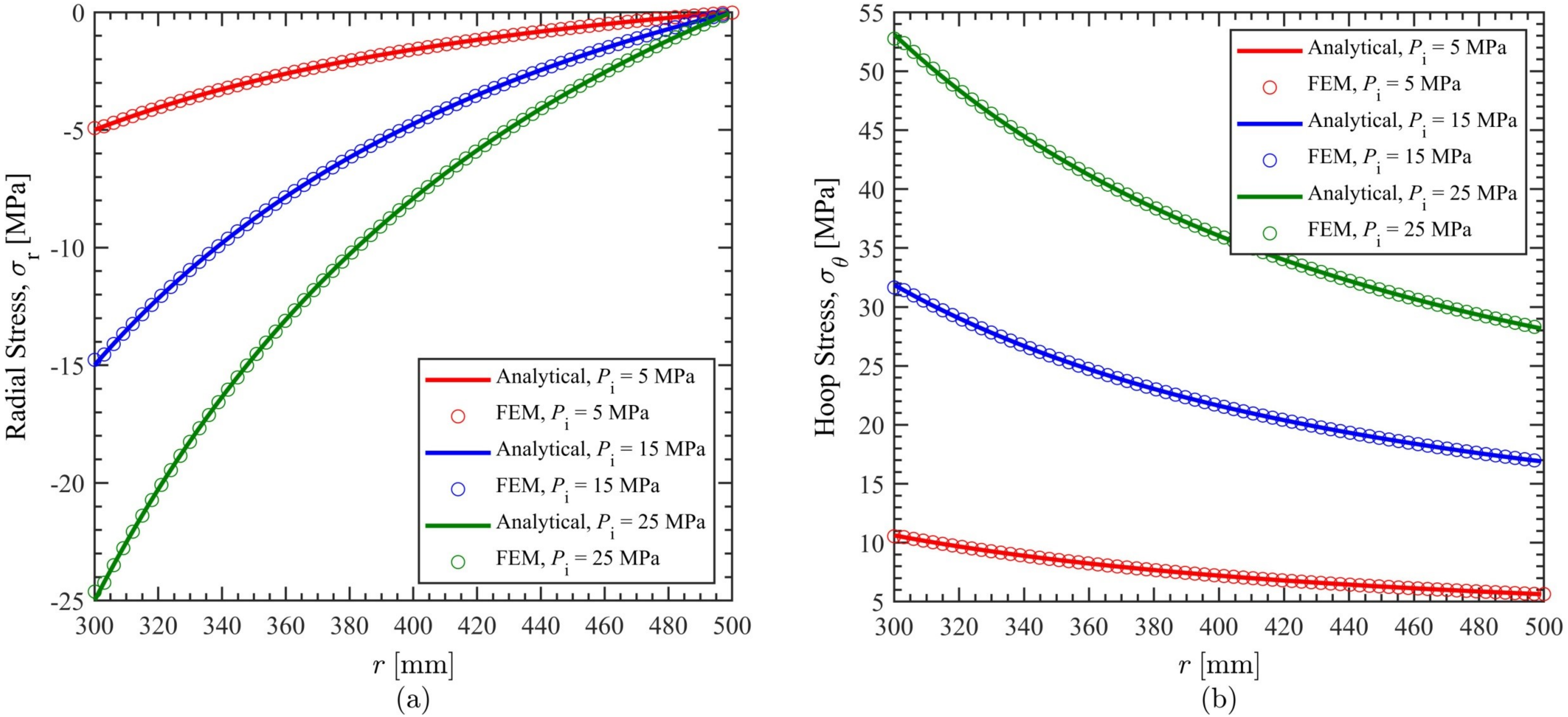


Figure 7. (a) Radial stress distribution (b) Hoop stress distribution at internal pressure, P_i = 5, 15 & 25 MPa and angular velocity, $\omega = 0$

4. Results and Discussion contd.

4.4 Isotropic Cylinder — P_i & P_e

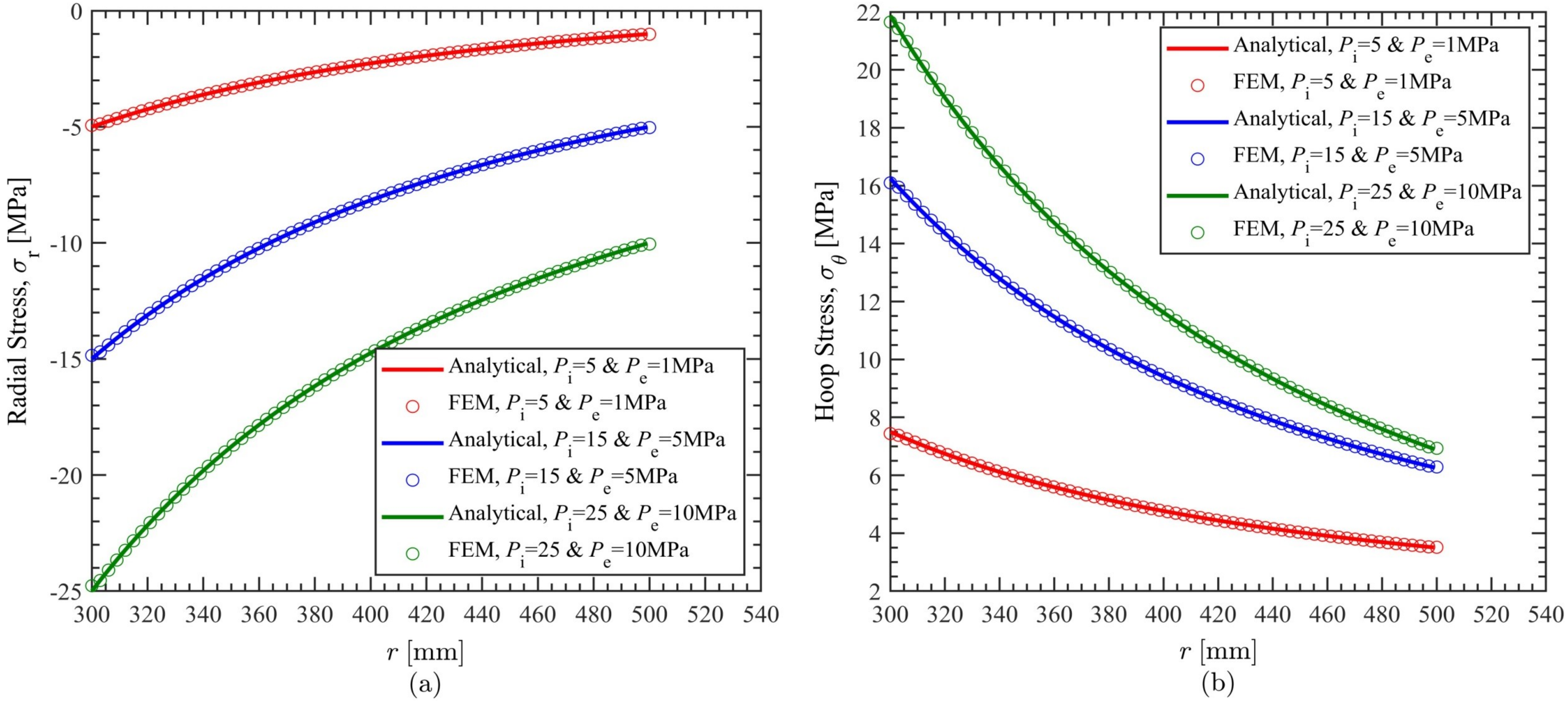


Figure 8. (a) Radial stress distribution (b) Hoop stress distribution at internal pressure, Pi = 5, 15 & 25 MPa with corresponding external pressure, Pe = 1, 5 & 10 MPa

4. Results and Discussion contd.

4.5 Isotropic Cylinder — P_i & ω

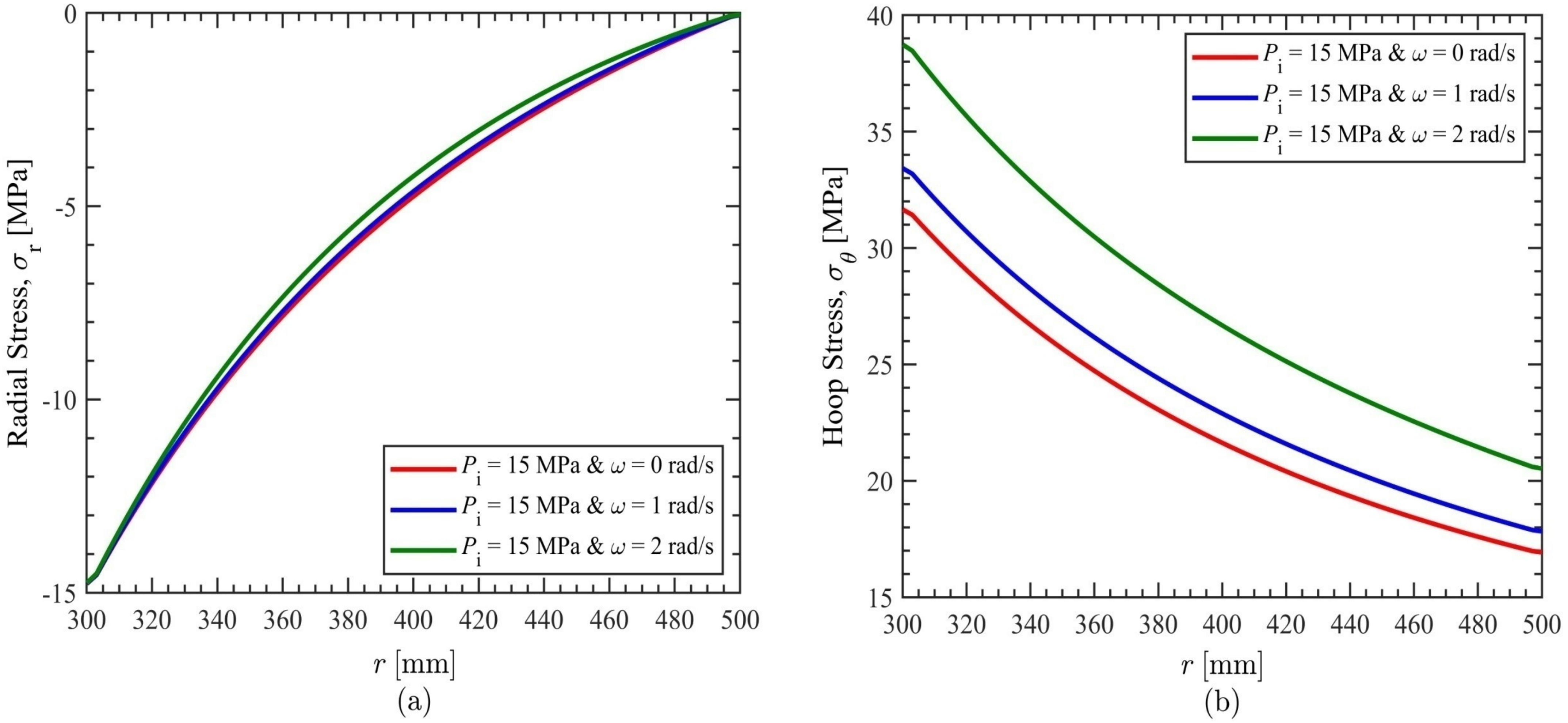


Figure 9. (a) Radial stress distribution (b) Hoop stress distribution at angular velocity, ω = 0, 1 & 2 rad/s with internal pressure, Pi = 15 MPa

4. Results and Discussion contd.

4.6 Comparison of FGM — *Power & Exponential* function

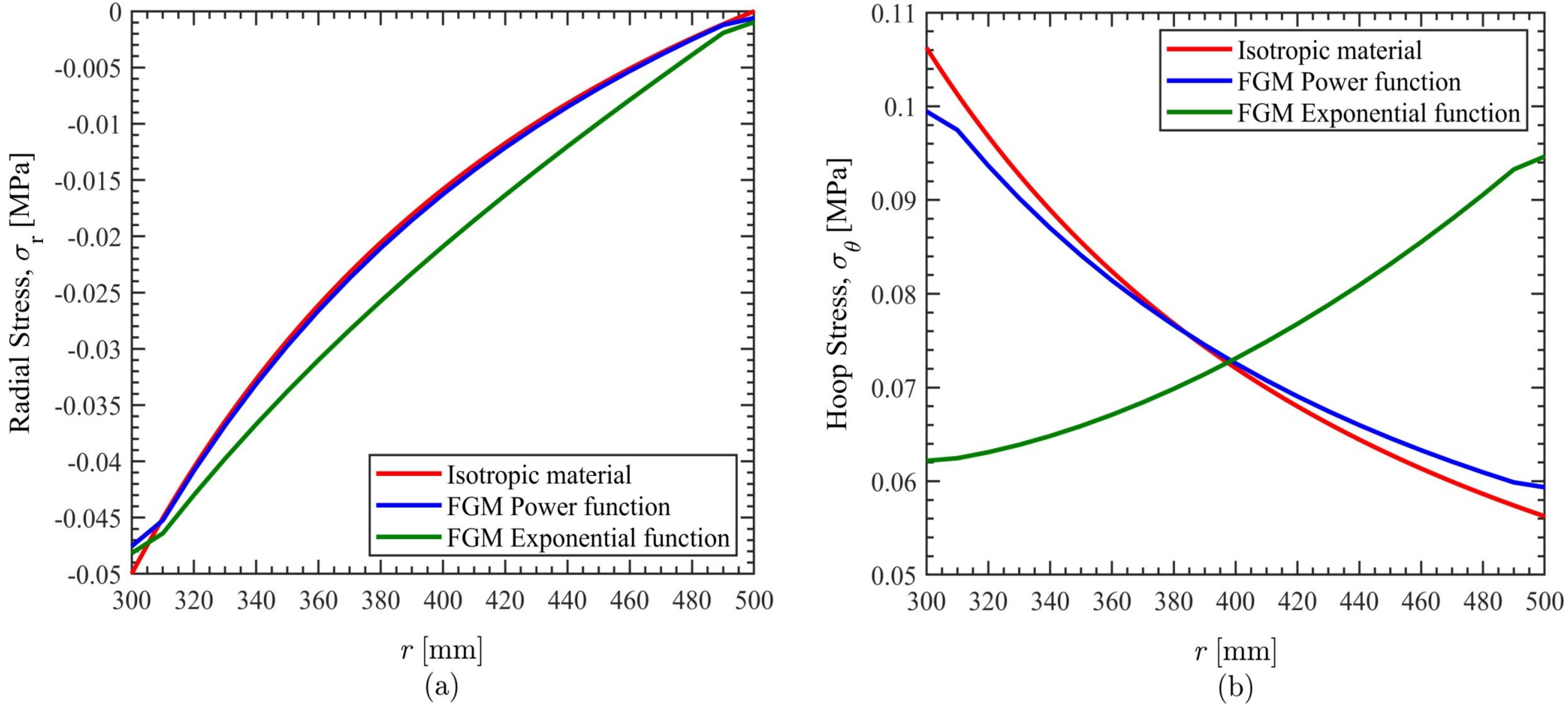


Figure 10. (a) Radial stress distribution (b) Hoop stress distribution for isotropic and FGM cylinder defined by the power and exponential function at internal pressure, Pi = 0.05 MPa

4. Results and Discussion contd.

4.7 FGM Cylinder (*Exponential Function*) — P_i

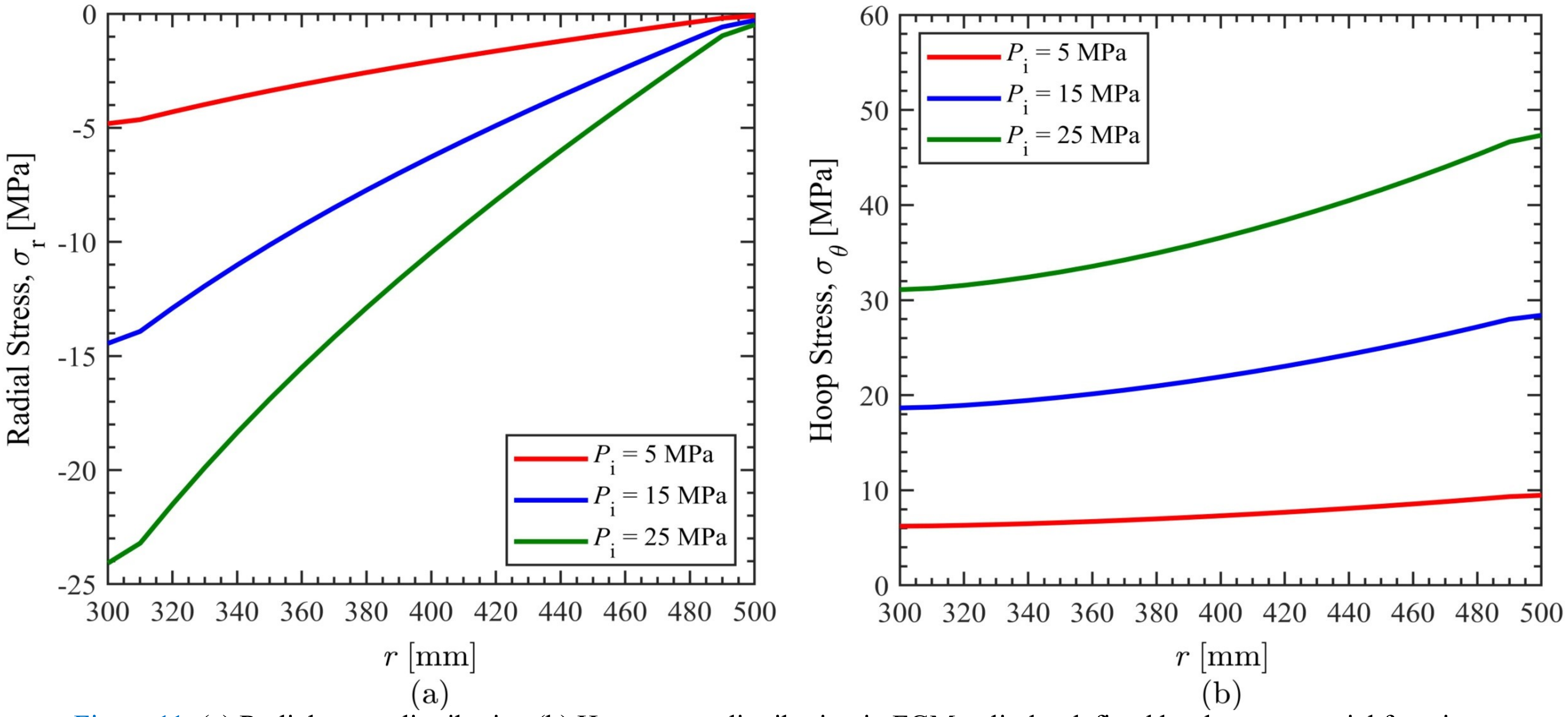


Figure 11. (a) Radial stress distribution (b) Hoop stress distribution in FGM cylinder defined by the exponential function at internal pressure, Pi = 5, 15 & 25 MPa

4. Results and Discussion contd.

4.8 FGM Cylinder (*Exponential Function*) — P_i & P_e

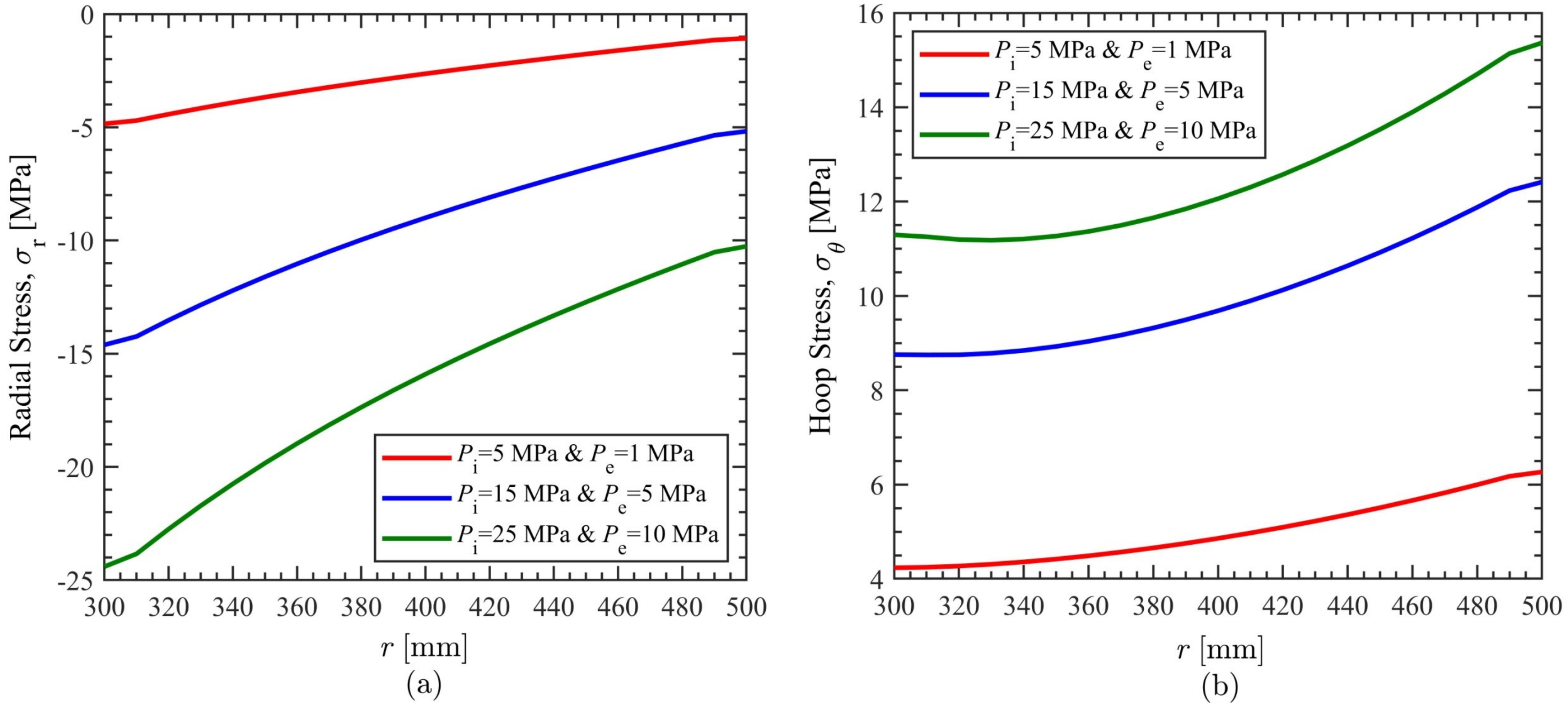


Figure 12. (a) Radial stress distribution (b) Hoop stress distribution in FGM cylinder defined by the exponential function at internal pressure, Pi = 5, 15 & 25 MPa with corresponding external pressure, Pe = 1, 5 & 10 MPa

4. Results and Discussion contd.

4.9 FGM Cylinder (*Exponential Function*) — P_i & ω

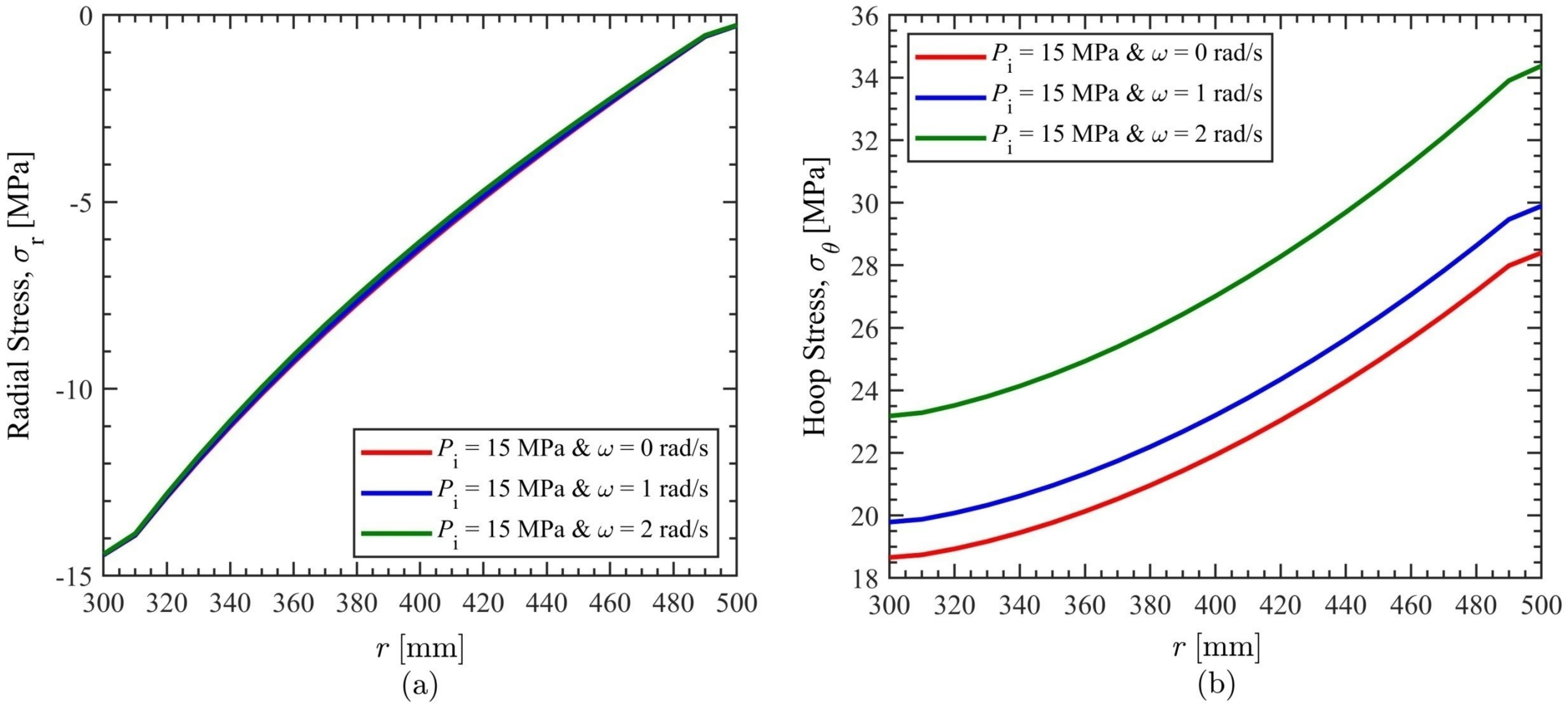


Figure 13. (a) Radial stress distribution (b) Hoop stress distribution in FGM cylinder defined by the exponential function at angular velocity, ω = 0, 1 & 2 rad/s with internal pressure, Pi = 15 MPa

5. Conclusions and Future Study

5.1 Conclusions

Isotropic and FGM cylinders behave differently under the same load.

The stress patterns in the FGM cylinders defined by exponential functions are completely different from that of isotropic cylinders as shown in Figure 14.

It is concluded that FGMs drastically improve the stress distributions and loading capacity of the cylinder as compared to isotropic materials.

Thus, the study of FGMs for possible applications in different fields is necessary.

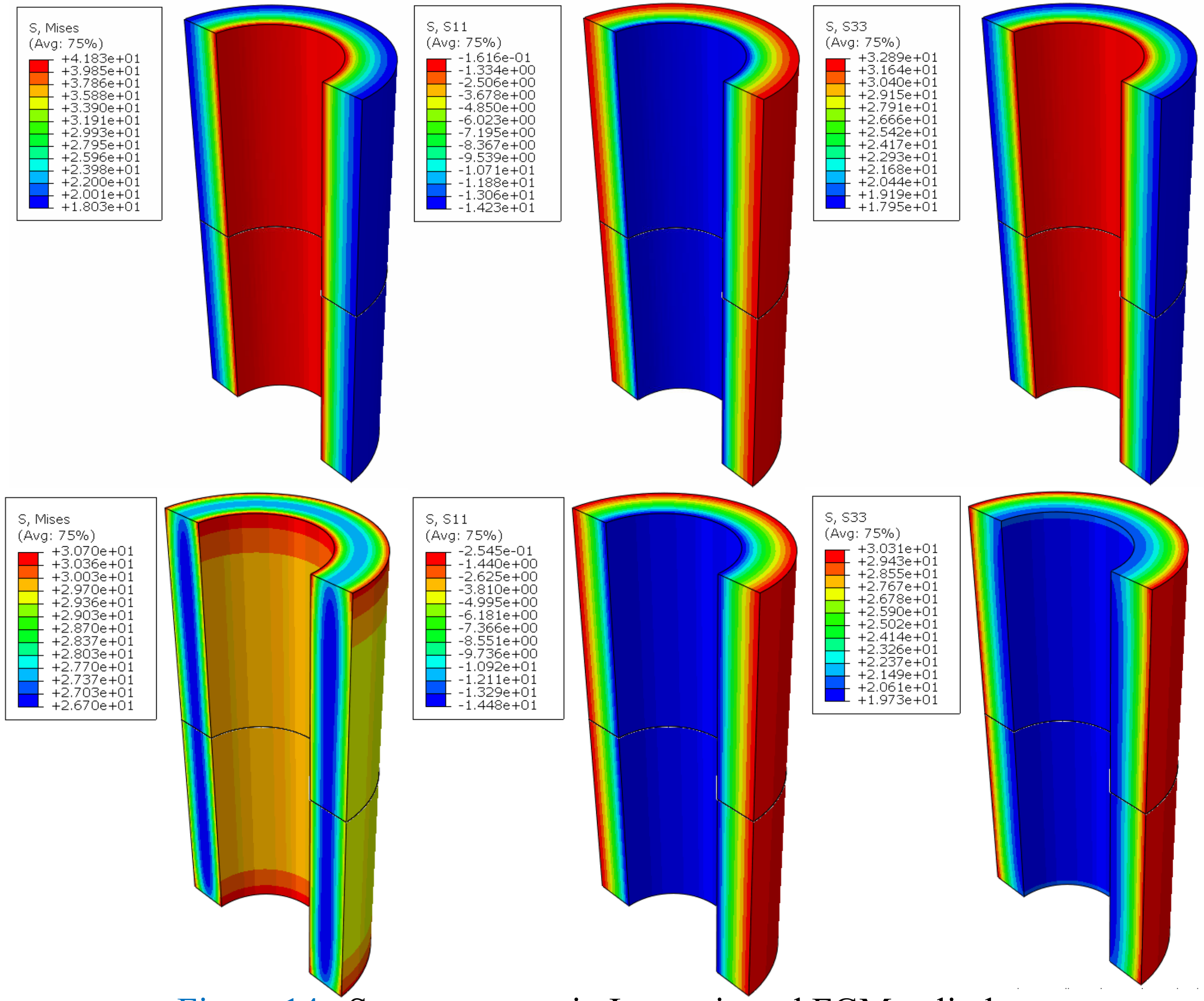


Figure 14. Stress patterns in Isotropic and FGM cylinders

5. Conclusions and Future Study contd.

5.2 Future Study

Minor Project is the basis for our future study –
Analytical (Figure 15) **and Finite Element** (Figure 16) **Modelling of FGM Roller during Hot Rolling**.

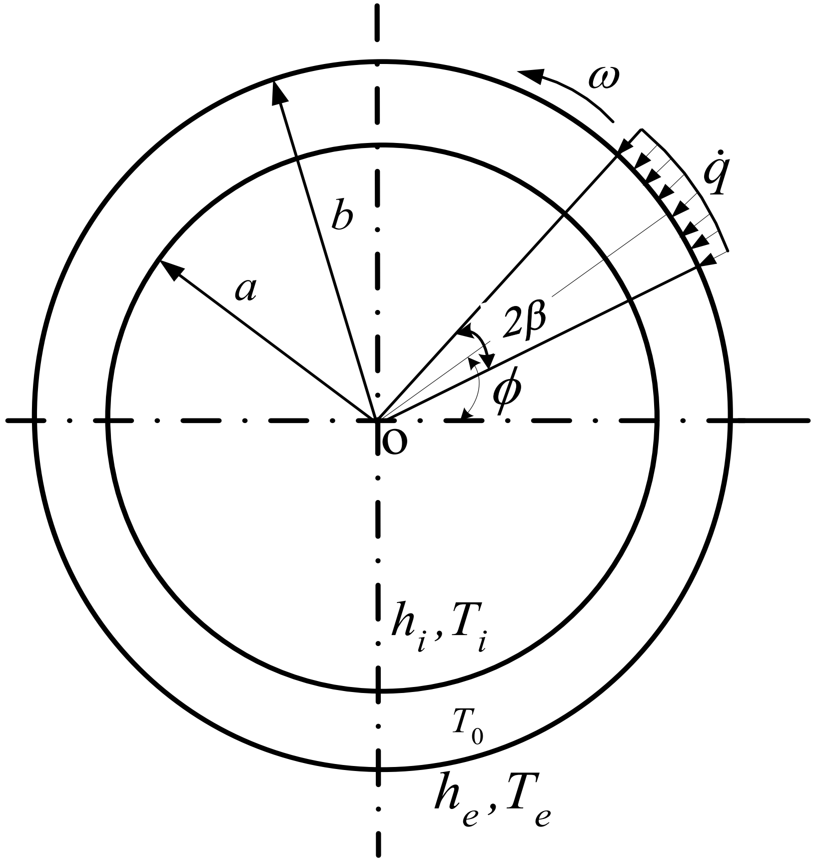


Figure 15. Analytical Modelling of Isotropic Roller during Hot Rolling

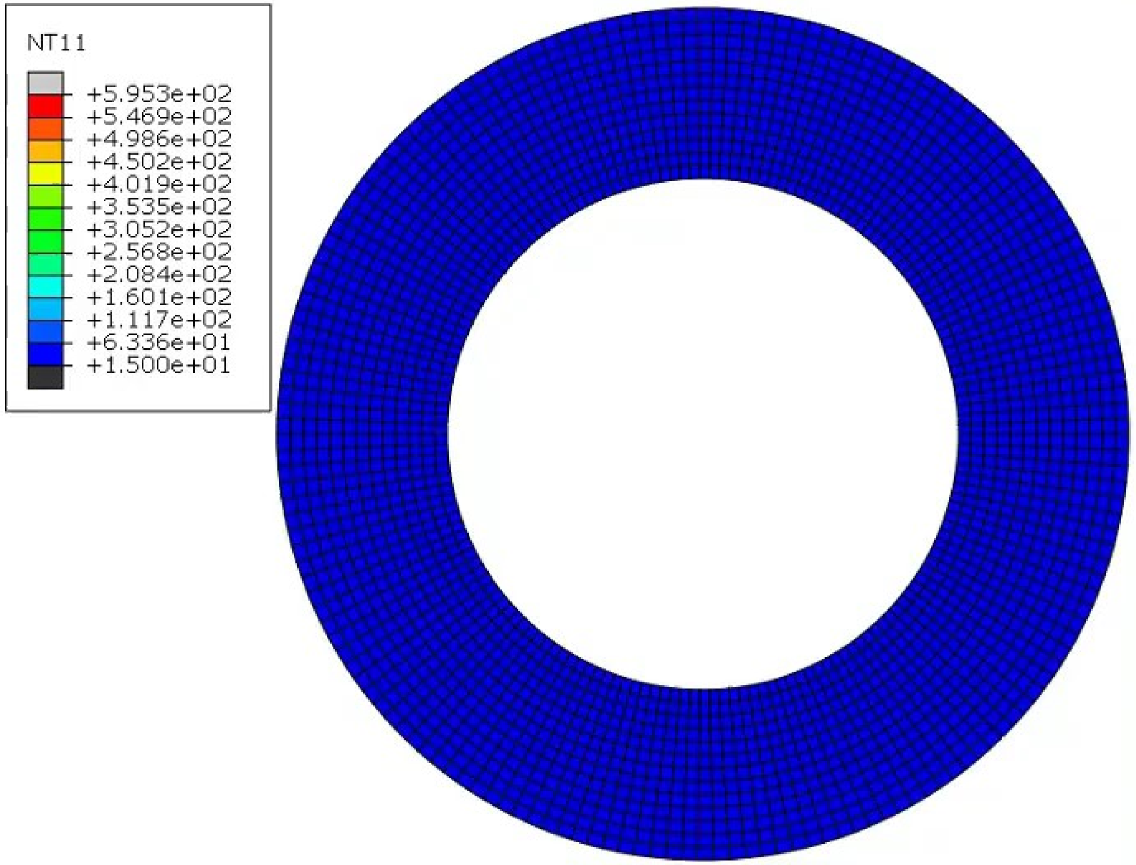


Figure 16. Finite Element Modelling of Isotropic Roller during Hot Rolling