The main diagonal element of $(\mathbf{X'X})^{-1}$ corresponding to β_2 is $C_{22} = 0.00000123$, so the t statistic (3.29) becomes

$$t_0 = \frac{\hat{\beta}_2}{\sqrt{\hat{\sigma}^2 C_{22}}} = \frac{0.01438}{\sqrt{(10.6239)(0.00000123)}} = 3.98$$

Since $t_{0.025,22} = 2.074$, we reject $H_0\colon \beta_2 = 0$ and conclude that the regressor x_2 (distance) contributes significantly to the model given that x_1 (cases) is also in the model. This t test is also provided in the Minitab output (Table 3.4), and the P value reported is 0.001. ■

We may also directly determine the contribution to the regression sum of squares of a regressor, for example, x_j, given that other regressors $x_1(i \neq j)$ are included in the model by using the **extra-sum-of-squares method**. This procedure can also be used to investigate the contribution of a **subset** of the regressor variables to the model.

Consider the regression model with k regressors

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

where y is $n \times 1$, $\mathbf{X}$ is $n \times p$, $\boldsymbol{\beta}$ is $p \times 1$, $\boldsymbol{\varepsilon}$ is $n \times 1$, and $p = k + 1$. We would like to determine if some subset of $r < k$ regressors contributes significantly to the regression model. Let the vector of regression coefficients be partitioned as follows:

$$\boldsymbol{\beta} = \left[\frac{\boldsymbol{\beta}_1}{\boldsymbol{\beta}_2}\right]$$

where $\boldsymbol{\beta}_1$ is $(p - r) \times 1$ and $\boldsymbol{\beta}_2$ is $r \times 1$. We wish to test the hypotheses

$$H_0\colon \boldsymbol{\beta}_2 = \mathbf{0}, \quad H_1\colon \boldsymbol{\beta}_2 \neq \mathbf{0} \tag{3.30}$$

The model may be written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon} \tag{3.31}$$

where the $n \times (p - r)$ matrix $\mathbf{X}_1$ represents the columns of $\mathbf{X}$ associated with $\boldsymbol{\beta}_1$ and the $n \times r$ matrix $\mathbf{X}_2$ represents the columns of $\mathbf{X}$ associated with $\boldsymbol{\beta}_2$. This is called the **full model.**

For the full model, we know that $\hat{\boldsymbol{\beta}} = (\mathbf{X'X})^{-1}\mathbf{X'y}$. The regression sum of squares for this model is

$$SS_R(\boldsymbol{\beta}) = \hat{\boldsymbol{\beta}}'\mathbf{X'y} \ (p \text{ degrees of freedom})$$

and

$$MS_{Res} = \frac{\mathbf{y'y} - \hat{\boldsymbol{\beta}}'\mathbf{X'y}}{n - p}$$

TEST ON $\beta_0 \dots \beta_k$ T-test

To find the contribution of the terms in $\boldsymbol{\beta}_2$ to the regression, fit the model assu
that the null hypothesis H_0: $\boldsymbol{\beta}_2 = \mathbf{0}$ is true. This **reduced model** is

$$\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon} \qquad (3.3$$

The least-squares estimator of $\boldsymbol{\beta}_1$ in the reduced model is $\hat{\boldsymbol{\beta}}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y}$. Th
regression sum of squares is

$$SS_R(\boldsymbol{\beta}_1) = \hat{\boldsymbol{\beta}}_1'\mathbf{X}_1'\mathbf{y}\,(p - r \text{ degrees of freedom}) \qquad (3.3$$

The regression sum of squares due to $\boldsymbol{\beta}_2$ given that $\boldsymbol{\beta}_1$ is already in the model is

$$SS_R(\boldsymbol{\beta}_2|\boldsymbol{\beta}_1) = SS_R(\boldsymbol{\beta}) - SS_R(\boldsymbol{\beta}_1) \qquad (3.3$$

with $p - (p - r) = r$ degrees of freedom. This sum of squares is called the **extra su
of squares due to $\boldsymbol{\beta}_2$** because it measures the increase in the regression sum o
squares that results from adding the regressors $x_{k-r+1}, x_{k-r+2}, \dots, x_k$ to a model th
already contains $x_1, x_2, \dots, x_{k-r}$. Now $SS_R(\boldsymbol{\beta}_2|\boldsymbol{\beta}_1)$ is independent of MS_{Res}, and the nu
hypothesis $\boldsymbol{\beta}_2 = \mathbf{0}$ may be tested by the statistic

$$F_0 = \frac{SS_R(\boldsymbol{\beta}_2|\boldsymbol{\beta}_1)/r}{MS_{\text{Res}}} \qquad (3.35$$

If $\boldsymbol{\beta}_2 \neq \mathbf{0}$, then F_0 follows a noncentral F distribution with a noncentrality paramet
of

$$\lambda = \frac{1}{\sigma^2}\boldsymbol{\beta}_2'\mathbf{X}_2'\left[\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\right]\mathbf{X}_2\boldsymbol{\beta}_2$$

This result is quite important. If there is multicollinearity in the data, there are situ-
ations where $\boldsymbol{\beta}_2$ is markedly nonzero, but this test actually has almost no powe
(ability to indicate this difference) because of a near-collinear relationship betwee
$\mathbf{X}_1$ and $\mathbf{X}_2$. In this situation, λ is nearly zero even though $\boldsymbol{\beta}_2$ is truly important. Thi
relationship also points out that the maximal power for this test occurs when $\mathbf{X}_1$ an
$\mathbf{X}_2$, are orthogonal to one another. By orthogonal we mean that $\mathbf{X}_2'\mathbf{X}_1 = \mathbf{0}$.

If $F_0 > F_{\alpha,r,n-p}$, we reject H_0, concluding that at least one of the parameters in $\boldsymbol{\beta}$
is not zero, and consequently at least one of the regressors $x_{k-r+1}, x_{k-r+2}, \dots, x_k$ in $\mathbf{X}$
contribute significantly to the regression model. Some authors call the test in (3.35)
a **partial F test** because it measures the contribution of the regressors in $\mathbf{X}_2$ give
that the other regressors in $\mathbf{X}_1$ are in the model. To illustrate the usefulness of thi
procedure, consider the model

$$y = \beta_0 + \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \varepsilon$$

The sums of squares

$$SS_R(\beta_1|\beta_0, \beta_2, \beta_3), \quad SS_R(\beta_2|\beta_0, \beta_1, \beta_3), \quad SS_R(\beta_3|\beta_0, \beta_1, \beta_2)$$