Finite Element Method (ME5130)

(Project report)

FEM Analysis of Buckled Beams

Group D

Abstract

In this report we will be looking into the numerical solution of buckling load for supported beams. Complete numerical analysis of buckling is performed by Finite Element Method technique. The solution will be compared with exact analytical solution. Code needed to perform analysis is written in MATLAB

Notations

$B \rightarrow$ first/second derivative of shape function matrix
C or E $\rightarrow$ Young's Modulus
$F \rightarrow$ vector of internal forces
$g \rightarrow$ vector of displacement gradients
$H \rightarrow$ Zero - one matrix
$K_o \rightarrow$ linear stiffness matrix
$K_T \rightarrow$ tangent stiffness matrix
$K_u \rightarrow$ matrix of initial displacement
$K_\sigma \rightarrow$ matrix of initial stress
$P \rightarrow$ external load
$P^* \rightarrow$ reference load
$Q \rightarrow$ vector of degree of freedom
U $\rightarrow$ potential due to internal stresses
$u \rightarrow$ vector of displacement
W $\rightarrow$ work done by external force and tractions.
$\phi \rightarrow$ shape function matix
$\Gamma \rightarrow$ first/second derivative of shape function matrix
$\sigma \rightarrow$ stress vector
$\varepsilon \rightarrow$ strain vector
$\Pi \rightarrow$ net potential enegy
$\lambda \rightarrow$ scalar load parameter (load factor)

Introduction

Column Buckling is a curious and unique subject in a way that the failure is not directly related to the strength of material. Buckling is a sudden change in shape of a structural component under critical load. It may occur even if the stresses that develop in structure are well below the yielding stress. Buckling usually occurs when long slender columns are loaded axially in compression. A few examples where buckling happens is - Spokes of bicycle wheels, railway tracks, pipes/pressure vessels etc.

In solid mechanics, buckling falls under the category of structural stability analysis. A distinguised feature of structural stability analysis is that non linear equations are used and incremental formulation is needed. Special case where we make assumptions for linear analysis is buckling. Incremental equations imply the analysis of adjacent state of equillibrium.

In this report we will first provide a theoretical background on Euler buckling. We will then proceed to Finite element formulation. This is split into 2 stages-

Stage 1: Computation of fundamental state of equillibrium
Stage 2: Computation of first critical state associated with appearance of first critical point at the fundamental path of equillibrium

Analytical Solution

Euler Bernoulli Beam Bending Equation ⇒

$$EI\frac{d^2y}{dx^2} = M$$

Beam Bending Equation is used to calculate Deflection(y) using Bending Moment(M), Young's Modulus(E) and Area moment of Inertia(I). By using Boundary conditions and deflection at end point of beam we calculate Critical load(P_{Cr}) for each boundary condition.

(i) Clamped - Free ('c-f')

$$EI\frac{d^2y}{dx^2} = M = P(a - y)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{P}{EI}y + \frac{P}{EI}a$$

After solving the differential equation we get:

$$y = c_1 sin(\sqrt{\frac{P}{EI}}x) + c_2 cos(\sqrt{\frac{P}{EI}}x) + a$$

Boundary conditions:

$$x = 0 \rightarrow y = 0; \qquad x = 0 \rightarrow \frac{dy}{dx} = 0$$

Substituting boundary conditions we get:

$$c_2 = -\frac{Pa}{EI}; \qquad c_1 = 0$$

At $x = L \rightarrow y = a$ thus

$$-\frac{Pa}{EI} cos(\sqrt{\frac{P}{EI}}L) + a = a$$

$$\Rightarrow cos(\sqrt{\frac{P}{EI}}L) = 0 \Rightarrow \sqrt{\frac{P}{EI}}L = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$$

For 1st Mode we take the first non zero value i.e. $\frac{\pi}{2}$

$$\Rightarrow P_{Cr} = \frac{\pi^2 EI}{4L^2}$$

(ii) Clamped - Clamped ('c-c')

$$EI\frac{d^2y}{dx^2} = M = M_o - Py$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_o}{EI}$$

After solving the differential equation we get:

$$y = c_1 sin(\sqrt{\frac{P}{EI}}x) + c_2 cos(\sqrt{\frac{P}{EI}}x) + \frac{M_o}{P}$$

Boundary conditions:

$$x = 0 \rightarrow y = 0; \qquad x = 0 \rightarrow \frac{dy}{dx} = 0; \qquad x = L \rightarrow y = 0$$

Substituting boundary conditions we get:

$$c_2 = -\frac{M_o}{P}; \qquad c_1 = 0$$

At $x = L \to y = 0$ thus

$$-\frac{M_o}{P} cos(\sqrt{\frac{P}{EI}} L) + \frac{M_o}{P} = 0$$

$$\Rightarrow cos(\sqrt{\frac{P}{EI}} L) = 1 \Rightarrow \sqrt{\frac{P}{EI}} L = 0, 2\pi, 4\pi \ldots$$

For 1st Mode we take the first non zero value i.e. 2π

$$\Rightarrow P_{Cr} = \frac{4\pi^2 EI}{L^2}$$

(iii) Clamped - Supported ('c-s')

$$EI \frac{d^2y}{dx^2} = M = -Py + H(l - x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI} y = \frac{H}{EI}(l - x)$$

After solving the differential equation we get:

$$y = c_1 sin(\sqrt{\frac{P}{EI}} x) + c_2 cos(\sqrt{\frac{P}{EI}} x) + \frac{H}{P}(l - x)$$

Boundary conditions:

$$x = 0 \to y = 0; \qquad x = 0 \to \frac{dy}{dx} = 0; \qquad x = L \to \frac{dy}{dx} = 0$$

Substituting boundary conditions we get:

$$c_2 = -\frac{Hl}{P}; \qquad c_1 = \frac{H}{P}\sqrt{\frac{EI}{P}}$$

At $x = L \to y = 0$ thus

$$\frac{H}{p}\sqrt{\frac{EI}{P}} \sin\left(\sqrt{\frac{P}{EI}} L\right) - \frac{HL}{P} \cos\left(\sqrt{\frac{P}{EI}} L\right) = 0$$

$$\Rightarrow tan(\sqrt{\frac{P}{EI}}L) = 0 \Rightarrow \sqrt{\frac{P}{EI}}L = 0, 4.5\,, 10.907\ldots$$

For 1st Mode we take the first non zero value i.e. 4.5

$$\Rightarrow P_{Cr} = 20.25\frac{EI}{L^2}$$

(iv) Supported - Supported ('s-s')

$$EI\frac{d^2y}{dx^2} = M = -Py$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

After solving the differential equation we get:

$$y = c_1 sin(\sqrt{\frac{P}{EI}}x) + c_2 cos(\sqrt{\frac{P}{EI}}x)$$

Boundary conditions:

$$x = 0 \to y = 0; \qquad x = L \to y = 0$$

Substituting boundary conditions we get:

$$c_2 = 0; \qquad c_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

At $x = L \to y = 0$ we have

$$c_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

but $c1 \neq 0$ (because y(x) = 0 if $c_1 = 0$)

$$\Rightarrow sin(\sqrt{\frac{P}{EI}}L) = 0 \Rightarrow \sqrt{\frac{P}{EI}}L = 0, \pi\,, 2\pi\ldots$$

For 1st Mode we take the first non zero value i.e. π

$$\Rightarrow P_{Cr} = \pi^2\frac{EI}{L^2}$$

FEM Formulation

Matrix equations for discreet system

Let us consider a deformable, three dimensional body which is under action of force f and traction t. Let the displacement vector $\mathbf{u} = (u(x,y,z), v(x,y,z), w(x,y,z))$ be a continuous differntiable function of x,y,z co-ordinates.

The vector of strains $\varepsilon = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ is related to gradient of displacement as-

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} u_{,x} + \frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2) \\ v_{,x} + \frac{1}{2}(u_{,y}^2 + v_{,y}^2 + w_{,y}^2) \\ w_{,x} + \frac{1}{2}(u_{,w}^2 + v_{,w}^2 + w_{,w}^2) \\ (u_{,y} + v_{,x}) + (u_{,y}u_{,x} + v_{,y}v_{,x} + w_{,y}w_{,x}) \\ (v_{,z} + w_{,y}) + (u_{,z}u_{,y} + v_{,z}v_{,y} + w_{,z}w_{,y}) \\ (w_{,x} + u_{,z}) + (u_{,x}u_{,z} + v_{,x}v_{,z} + w_{,x}w_{,z}) \end{bmatrix} \tag{1}$$

where $u_{,x} = \frac{\partial u}{\partial x}$

The rhs can be split into linear and non linear strain as

$$\varepsilon_i = \varepsilon_i^L + \varepsilon_i^N \tag{2}$$

which can also be wriiten in matrix form as

$$\varepsilon_i^L + \varepsilon_i^N = l_i^T g + \frac{1}{2} g^T H_i g \tag{3}$$

where g is vector of displacement gradients

$$g_{(9\times1)} = (u_{,x}v_{,x}w_{,x}u_{,y}v_{,y}w_{,y}u_{,z}v_{,z}w_{,z})^T \tag{4}$$

l_i and H_i are zero one matrices of dimension (9×1) and (9×9) respectively which selects appropriate components of gradient vector g

Transitioning into discreet system, we make an approximation over global domain of whole structure. Feilds of displacements and their gradients are approximated as linear combinations of "shape functions" as:

$$\begin{aligned} u &= \phi(X)Q \\ g &= \Gamma(X)Q \end{aligned} \tag{5}$$

where $\phi(X)$ is shape function. $\Gamma(X)$ is first derivative of shape function. Being consistent with the linear and non linear part we have,

$$\varepsilon = B^L Q + \frac{B^N}{2} Q \tag{6}$$

Comparing with equation (3) we get,

$$\begin{aligned} B_i^L &= l_i^T \Gamma \\ B_i^N = g^T H_i \Gamma &= Q^T \Gamma^T H_i \Gamma \end{aligned} \tag{7}$$

Potential Energy Formulation

Potential Energy is defined as-

$$\Pi = U - W = \frac{1}{2}\int_V \sigma^T \varepsilon \, dV - \int_V u^T f \, dV - \int_{S_t} u^T t \, ds \tag{8}$$

Potential due to internal stress is

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon \, dV \tag{9}$$

We know, $\sigma = E\varepsilon$. Thus

$$\begin{aligned} U &= \frac{1}{2}\int_V \varepsilon^T E \varepsilon \, dV \\ &= \frac{1}{2}\int_V (Q^T (B^L)^T + \frac{1}{2} Q^T (B^N)^T) E (B^L Q + \frac{B^N}{2} Q) \, dV \\ &= \frac{1}{2}\int_V Q^T [(B^L)^T E B^L + \frac{1}{2}(B^L)^T E B^N + \frac{1}{2}(B^N)^T E B^L + \frac{1}{4}(B^N)^T E B^N] Q \, dV \\ &= Q^T [\frac{K_0}{2} + \frac{K_1}{6} + \frac{K_2}{12}] Q \end{aligned} \tag{10}$$

where,

$$K_0 = \int_V (B^L)^T E B^L \, dV \tag{11}$$

$$K_1 = \frac{3}{2}\int_V [(B^L)^T E B^N + (B^N)^T E B^L]\, dV \tag{12}$$

$$K_2 = \frac{3}{2}\int_V (B^N)^T E B^N\, dV \tag{13}$$

Now, Work done due to external forces and tractions

$$\begin{aligned} W &= \int_V u^T f\, dV + \int_{S_t} u^T t\, ds \\ &= Q^T(\int_V \phi^T f\, dV + \int_{S_t} \phi^T t\, ds) = Q^T P \end{aligned} \tag{14}$$

Where,

$$P = \int_V \phi^T f\, dV + \int_{S_t} \phi^T t\, ds \tag{15}$$

Substituting these expressions into potential, Π

$$\Pi = Q^T[\frac{K_0}{2} + \frac{K_1}{6} + \frac{K_2}{12}]Q - Q^T P \tag{16}$$

We now apply Principle of Virtual Work. It states that internal virtual work done by equillibrium stresses should be equal to external virtual work done by forces. But we now perform the incremental formulation.

These equations have to be valid not only in the current configuration but also in the adjacent configuration. Therefore in adjacent state, Principle of Virtual Work is written as,

$$\int_V \delta\Delta\epsilon^T(\sigma + \Delta\sigma)\, dV = \int_V \delta\Delta u^T(f + \Delta f)\, dV + \int_{S_t} \delta\Delta u^T(t + \Delta t)\, ds \tag{17}$$

The strain increments can be written as,

$$\Delta\varepsilon_i = \varepsilon_i(g + \Delta g) - \varepsilon_i(g) \tag{18}$$

From equation(3) we know,

$$\begin{aligned} \varepsilon_i &= B_i Q = l_i^T \gamma Q + \frac{1}{2} g^T H_i \gamma Q \\ &= l_i^T g + \frac{1}{2} g^T H_i g \end{aligned} \tag{19}$$

Thus equation(18) can be wriiten as,

$$\begin{aligned} \Delta\varepsilon &= l_i^T(g + \Delta g) + \frac{1}{2}(g + \Delta g)^T H_i (g + \Delta g) - l_i^T g - \frac{1}{2} g^T H_i g \\ &= l_i^T \Delta g + \frac{1}{2} g^T H_i \Delta g + \frac{1}{2}(\Delta g)^T H_i (\Delta g) + \frac{1}{2}(\Delta g^T) H_i g \end{aligned} \tag{20}$$

As defined in equation(5), $g = \gamma Q$ and $\Delta g = \gamma \Delta Q$. Substituting in above eq(20)

$$\begin{aligned}\Delta\varepsilon_i &= l_i^T\gamma\Delta Q + \frac{1}{2}g^T H_i\gamma\Delta Q + \frac{1}{2}\Delta Q^T\gamma^T H_i\gamma\Delta Q + \frac{1}{2}\Delta Q^T\gamma^T H_i\gamma Q \\ &= B_i^L\Delta Q + B_i^N\Delta Q + \frac{1}{2}\Delta Q^T\gamma^T H_i\gamma\Delta Q\end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}\delta\Delta\dot{\varepsilon}_i &= B_i^L\delta\Delta Q + B_i^N\delta\Delta Q + \frac{1}{2}\Delta Q^\top\gamma^\top H_i\gamma\delta\Delta Q + \frac{1}{2}\Delta Q^T\gamma^T H_i\gamma\delta\Delta Q \\ &= \left[B_i^L + B_i^N + \Delta Q^\top\gamma^\top H_i\gamma\right]\delta\Delta Q\end{aligned} \tag{22}$$

Stress increments can be wriiten as,

$$\Delta\sigma_i = C\Delta\epsilon_i = C\left[B_i^L + B_i^N + \frac{1}{2}\Delta Q^\top\gamma^\top H_i\gamma\right]\Delta Q \tag{23}$$

Hence from eq (22) and (23)

$$\begin{gathered}\delta\Delta\epsilon_i^T(\sigma_i + \Delta\sigma_i) = \\ (\delta\Delta Q)^\top\left[\left(B_i^L\right)^\top + \left(B_i^N\right)^\top + \gamma^\top H_i\gamma\Delta Q\right]\left[\sigma_i + \left[CB_i^L + CB_i^N + \frac{C}{2}\Delta Q^\top\gamma^\top H_i\gamma\right]\Delta Q\right]\end{gathered} \tag{24}$$

Consider:

$$\begin{aligned}A &= (B_i^L)^T + (B_i^N)^T \\ X &= \gamma^T H_i\gamma\Delta Q \\ Y &= C\left[B_i^L + B_i^N\right]\Delta Q \\ Z &= \frac{C}{2}\Delta Q^\top\gamma^\top H_i\gamma\Delta Q\end{aligned} \tag{25}$$

Degrees of ΔQ in each term $A \to 0, X \to 1, \sigma \to 0, Z \to 2, Y \to 1$

Substituting eq(25) into eq(24),

$$\begin{aligned}\delta\Delta\epsilon_i^T(\sigma_i + \Delta\sigma_i) &= (\delta\Delta Q)^\top[A + X]\left[\sigma_i + Y + Z\right] \\ &= (\delta\Delta Q)^\top[A\sigma + AY + AZ + X\sigma + XY + XZ]\end{aligned} \tag{26}$$

Neglecting terms containing ΔQ second order

Terms that survive:$A\sigma, X\sigma, AY$

$$\Rightarrow \delta\Delta\epsilon_i^T(\sigma_i + \Delta\sigma_i) = (\delta\Delta Q)^\top\left[A\sigma + X\sigma + AY\right] \tag{27}$$

Substituting A, X , Y from eq(25),

$$= (\delta\Delta Q)^\top \left[\left[\left(B_i^L\right)^\top + \left(B_i^N\right)^\top\right]\sigma + \left[\gamma^T H_i\sigma_i\gamma + \left[\left(B_i^L\right)^\top + \left(B_i^N\right)^\top\right] C\left[B_i^L + B_i^N\right]\right]\Delta Q\right]$$

$$\Rightarrow \delta\Delta\epsilon_i^T(\sigma_i + \Delta\sigma_i) = (\delta\Delta Q)^T[[(B_i^L)^T + (B_i^N)^T]\sigma + [(B_i^L)^T CB_i^L + (B_i^N)^T CB_i^L + (B_i^L)^T CB_i^N + (B_i^N)^T CB_i^N + \gamma^T H_i\sigma_i\gamma]\Delta Q] \tag{28}$$

Substituting in LHS of eq(17),

$$\Rightarrow \int_V \delta\Delta\varepsilon^\top \cdot (\sigma + \Delta\sigma)dV = \int_V (\delta\Delta Q)^\top[[(B_i^L)^\top + (B_i^N)^\top]\sigma + [(B_i^L)^\top C(B_i^L) + (B_i^L)^\top CB_i^N + (B_i^N)^\top CB_i^L + (B_i^N)^\top CB_i^N + \gamma^\top H_i\sigma_i\gamma]\Delta Q]dV \tag{29}$$

Thus the above equation is the virtual internal work done by stresses. Writing it in a compact form:

$$\delta\Delta W^{int} = \delta\Delta Q^\top\left[F + \left[K_0 + K_u + K_\sigma\right]\Delta Q\right] \tag{30}$$

Now comparing eq(29) with eq(30):

$$F = \int_V \left(B_i^L\right)^\top + \left(B_i^N\right)^\top \sigma dV \tag{31}$$

$$K_0 = \int_V \left(B_i^L\right)^\top C\left(B_i^L\right) dV \tag{32}$$

$$k_u = \int_V \left[\left(B_i^L\right)^\top C\left(B_i^N\right) + \left(B_i^N\right)^\top CB_i^L + \left(B_i^N\right)^\top CB_i^N\right] dV \tag{33}$$

$$K_\sigma = \int_V \gamma^\top H_i\sigma_i\gamma dV \tag{34}$$

$$S = H_i\sigma_i = \sum_i H_i\sigma_i \Rightarrow S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ z_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{35}$$

Splitting K_u into linear and non linear form:

$$\begin{aligned} K_u &= K_{u_1} + K_{u_2} \\ K_{u_1} &= \int_v [(B_i^L)^T C(B_i)^N + ((B_i)^N)^T CB_i^L]dV \\ K_{u_2} &= \int_v (B_i^N)^T C(B_i)^N dV \end{aligned} \tag{36}$$

where F is the vector of internal forces, K_0 is linear matrix, K_σ is matrix of initial stresses and K_u is matrix of initial displacements.

The RHS of eq(17) corresponds to the virtual work of loads and increment.

$$\delta \Delta W^{(ext)} = \delta \Delta Q^T (P + \Delta P) \tag{37}$$

where P is same as in eq(15) :

$$P = \int_V \phi^T f \, dV + \int_{S_t} \phi^T t \, ds \tag{38}$$

Assuming that $\delta \Delta Q$ are linearly independent, using eq(17) we equate eq(30) and eq(37) and get,

$$(K_0 + K_\sigma + K_u)\Delta Q = \Delta P + R \tag{39}$$

where $R = P - F$

P is the external force and F is internal force. At equillibrium, P = F. Hence at equillibrium R = 0

Eq(39) at equillibrium is:

$$K_T \Delta Q = \Delta P \tag{40}$$

where $K_T = K_0 + K_\sigma + K_u$

Solution for Beam Element

From Bernoulli Euler Hypothesis

$$\varepsilon = -w_{,xx} y + \frac{1}{2} w_{,x}^2 \tag{41}$$

$$\varepsilon = \begin{bmatrix} 0 \\ -w_{,xx} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} w_{,x}^2 \\ 0 \end{bmatrix} = \varepsilon^L + \varepsilon^N \tag{42}$$

We know,

$$\varepsilon = l^T g^L + \frac{1}{2} (g^N)^T H g^N \tag{43}$$

comparing with above equation:

$$\begin{aligned} g^L &= \{w_{,xx}\} \\ g^N &= \{w_{,x}\} \\ l &= [-1] \quad and \quad H = [1] \end{aligned} \tag{44}$$

The shape function we are using are Hermitian cubic polynomials-

$$\phi = \begin{bmatrix} 1-3\xi^2+2\xi^3 & l\xi(1-\xi)^2 & \xi^2(3-2\xi) & -l\xi^2(1-\xi) \end{bmatrix} \tag{45}$$

where $\xi = x/l$
For representation, let $\phi = \begin{bmatrix} f_1 & g_1 & f_2 & g_2 \end{bmatrix}$

For a beam element, there are 2 degrees of freedom - transverse displacement and rotation. We are considering a two noded beam element. Thus the degree of freedom matrix becomes -

$$Q = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} \tag{46}$$

Element displacement and rotation is written as -

$$w = \phi Q \tag{47}$$

$$\theta = \phi^{'} Q \tag{48}$$

we know from eq(5), $g = \Gamma Q$

$$\Rightarrow g^L = \Gamma^L Q \tag{49}$$

and from eq(44),

$$\omega_{,xx} = \Gamma^L Q \tag{50}$$

but $w = \phi Q$

$$\Rightarrow \Gamma^L = \phi^{''} = \begin{bmatrix} f_1^{''} & g_1^{''} & f_2^{''} & g_2^{''} \end{bmatrix} \tag{51}$$

Similarly,

$$g^N = \Gamma^N Q$$

and $w_{,x} = \Gamma^N Q$
But $w = \phi a$

$$\Rightarrow \Gamma^N = \phi^{'} = \begin{bmatrix} f_1^{'} & g_1^{'} & f_2^{'} & g_2^{'} \end{bmatrix} \tag{52}$$

Also from eq(7), $B^L = l^T \Gamma^L$ and $B^N = (g^N)^T H \Gamma^N$

$$\Rightarrow B^L = [-1]\phi^{''} = \begin{bmatrix} -f_1^{''} & -g_1^{''} & -f_2^{''} & -g_2^{''} \end{bmatrix} \tag{53}$$

and

$$B^N = w_{,x} \Gamma^N = w_{,x} \begin{bmatrix} f_1^{'} & g_1^{'} & f_2^{'} & g_2^{'} \end{bmatrix} \tag{54}$$

Substituting eq(53) in eq(32)

$$K_o = \int_0^l (B^L)^T E B^L A \, dx = EA \int_0^l (B^L)^T B^L \, dx \tag{55}$$

$$K_o = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l^e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix} \tag{56}$$

In our case, there is a constant axial force acting along the bar axis

$$\Rightarrow N(x) = constant = N \tag{57}$$

Therefore stress matrix from eq(35) will just be defined because of one force $\rightarrow$ N

$$K_\sigma = \int_V (\Gamma^N)^T S \Gamma^N \, dV = \int_0^{l_e} (\Gamma^N)^T S A \Gamma^N \, dx$$

$$\Rightarrow K_\sigma = N \int_0^{l_e} (\Gamma^N)^T \Gamma^N \, dx \tag{58}$$

Substituting from eq(52) we get,

$$K_\sigma = N \begin{bmatrix} \frac{6}{5l_e} & \frac{1}{10} & -\frac{6}{5l_e} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{15}l_e & -\frac{1}{10}l_e & -\frac{1}{30}l_e \\ -\frac{6}{5l_e} & -\frac{1}{10} & \frac{6}{5l_e} & -\frac{1}{10} \\ \frac{1}{10} & \frac{1}{30}l_e & -\frac{1}{10} & \frac{2}{15}l_e \end{bmatrix} \tag{59}$$

Eigen Value formulation

In problems of structural mechanics, load vector is defined in the form:

$$P = \lambda P^* \tag{60}$$

$\lambda \rightarrow$ Scalar Load Parameter(Load Factor)
$P^* \rightarrow$ Load Reference Factor
(Note that any variable containing * as superscript indicates that quantity being defined in reference state)

Consider, a $\Delta\lambda_1$ is chosen such that P is just infinitesimally smaller than the buckling load.

Thus, $\Delta P^- = (\Delta\lambda_1 P^*)$
From eq(40),

$$\Rightarrow (K_T)(\Delta Q^-) = \Delta P^- \tag{61}$$

Where, ΔQ^- represents the displacement of the beam just before buckling load.
Consider another $\Delta\lambda_2$, such that P is just infinitesimally larger than the buckling load.
Thus, $\Delta P^+ = \Delta\lambda_2 P^*$
Again from eq(40),

$$\Rightarrow (K_T)(\Delta Q^+) = \Delta P^+ \tag{62}$$

Where, ΔQ^+ represents the displacement of the beam just after the Buckling load.

Subtracting equation (61) from equation (62)

$$(K_T)(\Delta Q^+ - \Delta Q^-) = (\Delta\lambda_2 - \Delta\lambda_1)P^* \tag{63}$$

Note that, the change in load is infinitesimally small whereas change in displacement is much significant once the Buckling critical load has been crossed

Thus eq(63) turns out to be,

$$\Rightarrow (K_T)(\Delta Q^+ - \Delta Q^-) = 0 \tag{64}$$

The change in displacement $(\Delta Q^+ - \Delta Q^-)$ is significant. Hence,

$$\mathbf{det(K_T) = 0} \tag{65}$$

Note that,

$$K_T = K_o + K_\sigma + K_u$$

$$\Rightarrow |K_o + K_\sigma + K_u| = 0 \tag{66}$$

Now we simplify the above equation.

From eq(60), $P = \lambda P^*$,

Since stress is proportional to applied load, $\Rightarrow \sigma = \lambda\sigma^*$

And displacement is also proportional to load, $\Rightarrow u = \lambda u^*$

From eq(5), $u = \phi Q$, $g = \gamma Q$

Since $u = \lambda u^*$,

$$\begin{aligned} \Rightarrow Q &= \lambda Q^* \\ \Rightarrow g &= \lambda g^* \end{aligned} \tag{67}$$

From eq(7),

$$\begin{aligned} B^L &= l^T \gamma \Rightarrow (B^L)^* = B^L \\ B^N &= g^T H \gamma \Rightarrow (B^N) = \lambda (g^*)^T H \gamma = \lambda (B^N)^* \end{aligned} \tag{68}$$

From eq(32),

$$(K_o)^* = \int_v ((B^L)^*)^T C (B^L)^* dV = K_o \tag{69}$$

From eq(33),

$$\begin{aligned} K_u &= \int_v [((B^L)^*)^T C \lambda (B^N)^* + \lambda ((B^N)^*)^T C B^L + \lambda^2 ((B^N)^*)^T C (B^N)^*] dV \\ &= \lambda [K^*_{u_1}] + \lambda^2 [K^*_{u_2}] \end{aligned} \tag{70}$$

From eq(34), $K_\sigma = \int_v \gamma^T S \gamma dV$
As S $\rightarrow$ Stress Matrix, where S = λS^* we get

$$K_\sigma = \lambda K^*_\sigma \tag{71}$$

Substituting all these in eq(66) we get,

$$det(K_o + \lambda (K^*_\sigma + K^*_{u_1}) + \lambda^2 K^*_{u_2}) = 0 \tag{72}$$

This is called quadratic eigen value problem. Neglecting the quadratic terms gives us linearized buckling problem. Thus neglecting λ^2 terms,

$$\Rightarrow |K_o + \lambda (K^*_\sigma + K^*_{u_1})| = 0 \tag{73}$$

There is another formulation in buckling theory called Initial Buckling Problem (classical buckling problem) where, $K^*_{u_1} = 0$. Thus,

$$\Rightarrow |K_o + \lambda K^*_\sigma| = 0 \tag{74}$$

Sometimes K_G^* called Geometric stiffness matrix is used to define the above equation. It is defined as-

$$K_G^* = \begin{cases} K_\sigma^*, & \text{initial buckling.} \\ K_\sigma^* + K_{u_1}^*, & \text{linear buckling.} \end{cases} \tag{75}$$

$$|K_o + \lambda K_G^*| = 0 \tag{76}$$

Upon solving we get vector of λ. The first value corresponds to first mode, second to second mode and so on. Since we are interested in 1st mode, we get $P_{cr} = \lambda_1 P^*$ where λ_1 is the first eigen value

Results

To solve eq(76), MATLAB is used. The following are the constant values choosen as inputs-

E = 210 GPa, I = $20\mu m^4$, L = 10m

The number of elements are 15

We have taken reference load $P^* = 1$ so that the eigen value is the critical load itself.

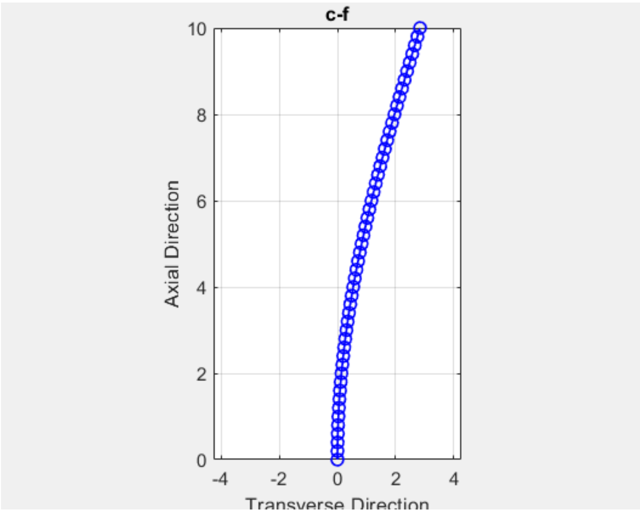
Upon solving, the solution is:

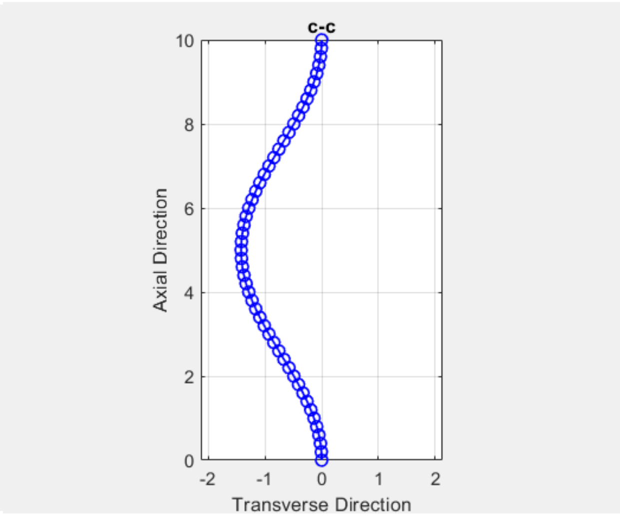
Result			
Constraint type	Theretical Solution(in N)	FEM Solution(in N)	Error (in percent)
c-f	103630.846211	103630.846326	$1.1 * 10^{-7}$
c-c	1658093.539383	1658094.113194	$3.5 * 10^{-5}$
c-s	845966.091522	848010.676163	0.24168
s-s	414523.384846	414523.393842	$2 * 10^{-6}$

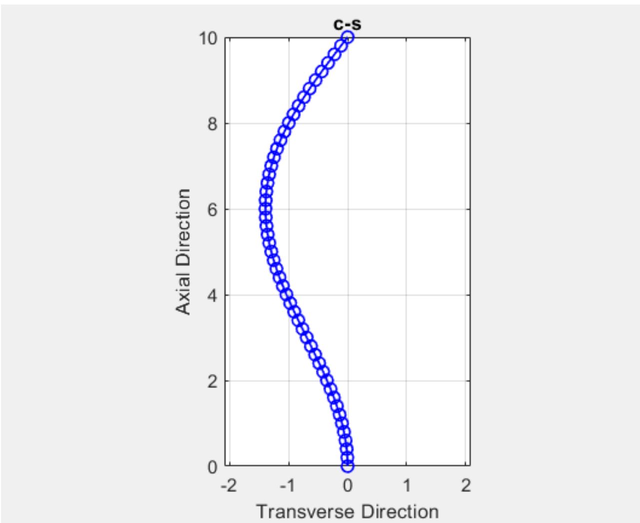
where c-s is clamped free, c-c is clamped clamped, c-s is clamped-supported and s-s is supported-supported.

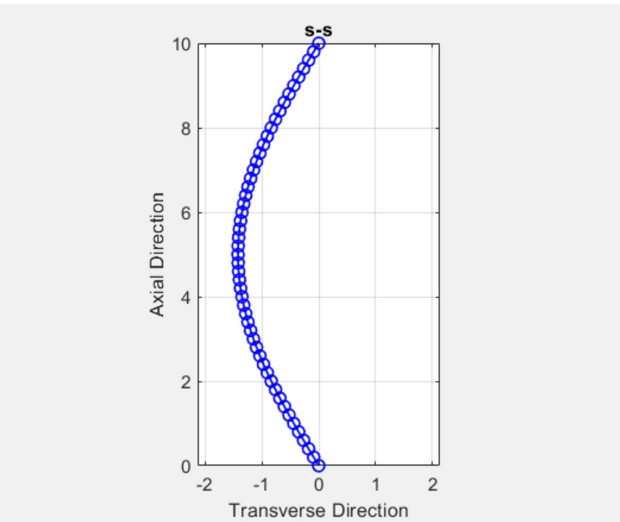
The error observed is very less. FEM method for 1-D beam buckling problem is very accurate.

Plotting the buckled shape-


c-f
Axial Direction
Transverse Direction


c-c
Axial Direction
Transverse Direction


c-s
Axial Direction
Transverse Direction

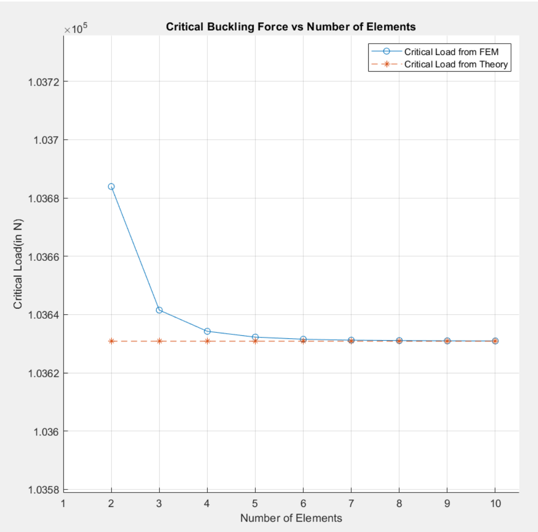

s-s
Axial Direction
Transverse Direction

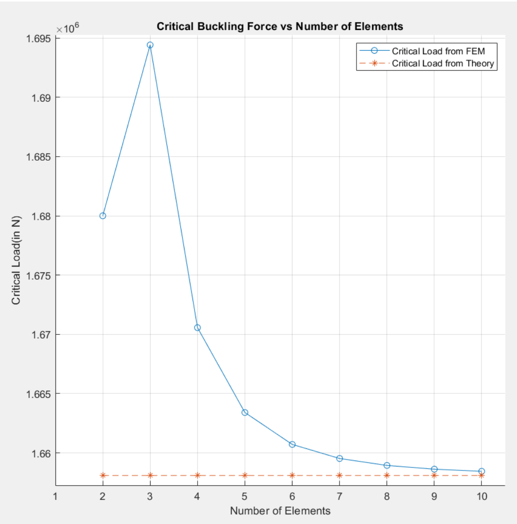
Convergence Study:

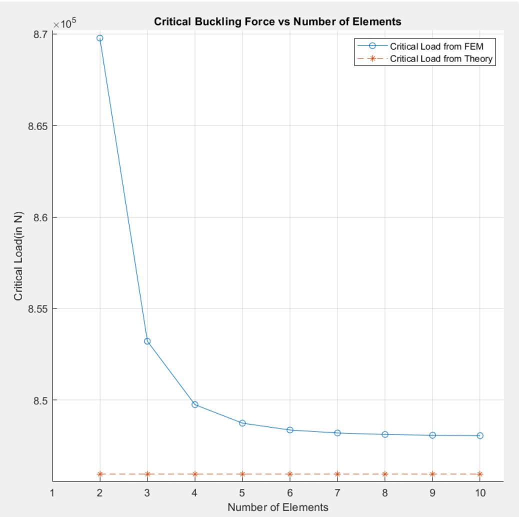
Finding out if the model converges or not is very important in FEM. Even though we checked our analysis with analytical results, it is essential that we find the rate of convergence.

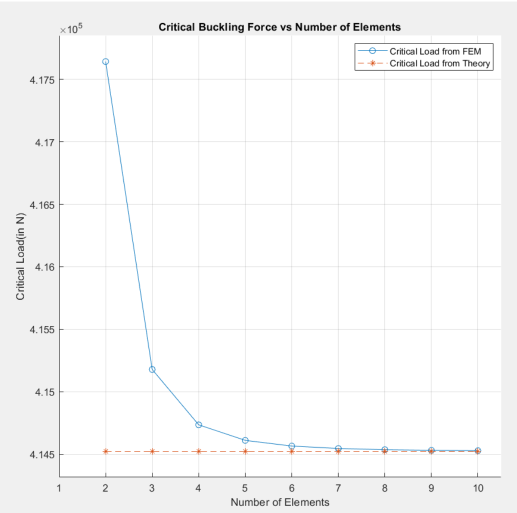
Buckling is a unique problem where we are concerned about the load applied and not the displacement. Thus the convergence study will be applied on critical load (P_{cr}).

Following are the graphs of number of elements vs P_{cr} for 4 boundary conditions-









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