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Sliding mode control of a discrete system

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Abstract Conventional sliding mode control designed on the basis of a continuous system is known to be robust to the plant uncertainties A realized digital system, however, not only yields chattering, but also may become unstable by a long sampling interval This paper presents a stable discrete sliding mode control insensitive to the choice of sampling interval and not yielding chattering The control system is designed on the basis of a discrete Lyapunov function and a sufficient condition of the control gain to make the system stable is given Contrary to the continuous case, the derived switching plane of the control law is different from the sliding mode, and in its neighborhood, the control law is given by the linear state feedback Simulations show the effectiveness of the proposed method

Keywords Sliding mode control, sampling interval, discrete Lyapunov function

1. Introduction

Sliding mode control has been studied since the early sixties by the name of variable structure system [1]. The designed control system is known to be robust to the uncertainties of the controlled plant, and many researches have been done, as shown in the references of the survey paper of Utkin [2]. The relation to LQ optimal and high gain control has been clarified by Utkin, Young, Kokotovic [3,4] Until now these researches studied only a continuous system which is proved to be more robust against uncertainty when a larger gain is employed. Its implementation by a digital computer, however, requires a certain sampling interval which brings not only chattering along the predesigned sliding mode but also possible instability with a large gain. The problem of chattering is its chaoticness which is difficult to analyze. Examples of chattering due to the sampling interval can be seen in Figure 1 in Section 5 which shows that only overfast sampling should be used. Fast sampling poses a limitation on its applications to mechanical systems such as robotics [5–8], and it has not been used in practical process control, since it may cause fatigue of actuators

In spite of the use of digital computers, sliding mode control has not been studied except in the author's paper [9] which gave the basic idea. Drakunov and Utkin independently propose an idea of discrete sliding mode control based on a different principle from this paper. Their idea is based on the contraction mapping. This paper extends and revises the idea of [8] to give a design procedure for a discrete system with uncertainty The control law used in the switching coefficient type, which is derived on the basis of a discrete Lyapunov function. The derived switching surface is different from the sliding surface, which contradicts continuous sliding mode control, and in a neighborhood along the sliding mode, there is a region where linear control is used Outside of the region, feedback coefficients are changed The proposed discrete sliding mode control avoids-not only chattering but also assures stability Section 2 reviews continuous sliding mode control, and Section 3 gives a discrete sliding mode control which sufficiently assures the stability with the gain in a given bound. Section 4 discusses robustness in the presence of uncertainty. Chapter 5 presents examples to show the validity of the proposed method.

2. Sliding mode control for continuous systems

In this section, continuous sliding mode control with switching coefficients is briefly reviewed The linear continuous system considered in the paper has a single input and is represented by

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where u is a scalar, x is an n-state vector, and y is a p-output vector The sliding mode

$$S = \{ x \mid Gx = 0 \} \tag{2}$$

is given so that the system is stable as long as the state remains on the hyperplane (2). The equivalent control law to keep the state on (2) is known to be given by

$$Gx = GAx + GBu = 0. \tag{3}$$

Equation (3) yields

$$u = F_{\mathrm{eq}} x \tag{4}$$

where

$$F_{\mathrm{eq}} = -(GB)^{-1} GA. \tag{4$'$}$$

The sliding mode S is chosen so that the closed loop system under the feedback (4) is stable on S, i.e.,

$$x = (A + BF_{\mathrm{eq}})x, \tag{5a}$$

$$Gx = 0 \tag{5b}$$

This gives that x is stable. In the design of sliding mode control, the sliding mode is designed firstly. Secondly, the control to transfer the state to the sliding mode is designed. Let

$$s = Gx \tag{2$'$}$$

Then a Lyapunov function is defined by

$$V = \tfrac{1}{2} s^2. \tag{6}$$

The derivative of v is written as

$$\dot{V} = s\dot{s} = s(GAx + GBu). \tag{7}$$

If u is chosen as

$$u = Fx, \tag{8a}$$

$$F = F_{\mathrm{eq}} + F_D, \tag{8b}$$

where the i-th element of F_D, f_i, is

$$f_i = \begin{cases} f_i^+ \ (>0) & \text{for } s(GB)x_i < 0, \\ 0 & \text{for } s(GB)x_i = 0, \\ f_i^- \ (<0) & \text{for } s(GB)x_i > 0, \end{cases} \tag{9}$$

then $\dot{V}$ is negative and the state of the system is transferred on the hyperplane, and only overfast sampling makes the state stay on it.

3. Sliding mode control for discrete system

The control law (8) is now realized by a digital computer The control is given at every sampling instant $k\Delta$, where Δ is the sampling period In digital control, the input u has a constant value between sampling

$$u(t) = u_k, \quad k\Delta \le t < (k+1)\Delta. \tag{10}$$

So it is known that the state at the sampling instant is given by

$$x_{k+1} = \Phi x_k + \Gamma u_k, \tag{11a}$$

$$y_k = C x_k, \tag{11b}$$

where

$$x_k = x(k\Delta), \qquad y_k = y(k\Delta),$$

$$\Phi = \mathrm{e}^{A\Delta}, \qquad \Gamma = \int_0^{\Delta} \mathrm{e}^{A\tau}\, \mathrm{d}\tau\, B.$$

Similar to a continuous system, s_k is defined as

$$s_k = G x_k. \tag{12}$$

G is designed so that the state staying on $s_k = 0$ for all k is stable. Such G is determined from the following lemma.

Lemma 1. *The equivalent control law for the system* (11) *such that the state rests on the hyperplane of* $s_k = s_{k+1}$ *for all* k *is given by*

$$u_k = F_{\mathrm{eq}} x_k \tag{13}$$

where

$$F_{\mathrm{eq}} = -(G\Gamma)^{-1} G(\Phi - I). \tag{13$'$}$$

Proof. On $s_k = s_{k+1}$,

$$s_k = s_{k+1} = s_{k+2} = \cdots.$$

This gives

$$Gx_{k+1} = G\Phi x_k + G\Gamma u_k = G x_k$$

which yields (13)

From Lemma 1, the hyperplane, equivalently G, is determined from Theorem 1.

Theorem 1. *The hyperplane, equivalently* G, *should be determined so that the system*

$$x_{k+1} = \left[\Phi - \Gamma(G\Gamma)^{-1} G(\Phi - I)\right] x_k, \tag{14a}$$

$$G x_k = 0, \tag{14b}$$

is stable.

Since (14) is rewritten as

$$\begin{bmatrix} \Phi - zI & \Gamma \\ G & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} = 0 \tag{14$'$}$$

where $x_{k+1} = zx_k$, G can be determined for a given set of stable z, i.e., $|z| < 1$ In the second step, the control law to transfer the state on the hyperplane should be determined. Let V_k and Δs_k be defined as

$$V_k = \tfrac{1}{2} s_k^2 \tag{15}$$

and

$$\Delta s_{k+1} = s_{k+1} - s_k. \tag{16}$$

Then the control law to decrease V_k is given by the following lemma

Lemma 2. *If the control satisfies*

$$s_k \Delta s_{k+1} < -\tfrac{1}{2}(\Delta s_{k+1})^2 \quad \textit{for } s_k \neq 0, \tag{17}$$

then

$$V_{k+1} < V_k. \tag{18}$$

Proof. From (16),

$$s_{k+1} = s_k + \Delta s_{k+1}.$$

Squaring both sides yields

$$s_{k+1}^2 = s_k^2 + 2 s_k \Delta s_{k+1} + (\Delta s_{k+1})^2$$

Substituting (17), (18) is derived

The control law to satisfy (17) should be studied. In this paper, the following type of control law is considered

$$u_k = (F_{eq} + F_D) x_k, \tag{19}$$

where F_{eq} is given by (13)$'$ and F_D is discontinuous control law Substituting (19) into (11),

$$\Delta s_{k+1} = G\Gamma F_D x_k \tag{20}$$

So the variable structure control law to make the system stable is given by the following theorem.

Theorem 2. *If the sliding mode satisfies the conditions of Theorem* 1, *and the control law is chosen by* (19), *where the absolute value of the i-th element of* F_D, f_i, *is constant for all time and the same for all* i, *i.e.*,

$$|f_i^+| = |f_i^-| = f_0, \quad i = 1, 2, \ldots, n,$$

then the control law

$$f_i = \begin{cases} f_0 & \textit{for } (G\Gamma) s_k x_{k_i} < -\delta_i, \\ 0 & \textit{for } -\delta_i \leq (G\Gamma) s_k x_{k_i} \leq \delta_i, \\ -f_0 & \textit{for } (G\Gamma) s_k x_{k_i} > \delta_i, \end{cases} \tag{21}$$

makes the system stable, where x_{k_i} *is the i-th element of* x_k *and* δ_i *is defined as*

$$\delta_i = \tfrac{1}{2} \sum_{j=1}^{n} |x_{k_i}| \, |x_{k_j}| \, f_0 (G\Gamma)^2 \tag{22}$$

with the amplitude of f_0 *limited by*

$$0 < f_0 < \left| \frac{2}{G\Gamma \sum_{j=1}^{n} |t_{1_j}|} \right|, \tag{23}$$

t_{1_i} *being the i-th element of* t_1 *satisfying* $Gt_1 = 1$, $Gt_i = 0$, $t_i \perp t_j$ $(i \neq j)$.

Proof. Substituting (8), (4)$'$ into (1), Δs_{k+1} is written as

$$\Delta s_{k+1} = G\Gamma F_D x_k \tag{24}$$

Therefore

$$\begin{aligned} s_k \Delta s_{k+1} &= s_k (G\Gamma) F_D x_k \\ &= s_k (G\Gamma) [f_1, f_2, \ldots, f_n] x_k \\ &= s_k (G\Gamma) \sum_{i=1}^{n} f_i x_{k_i}. \end{aligned} \tag{25}$$

If any one of the f_i's is chosen as f_0 or $-f_0$ by (21), then

$$\begin{aligned} s_k \Delta s_{k+1} &\leq -\sum_{i=1}^{n} \delta_i |f_i| \\ &\leq -\tfrac{1}{2}(G\Gamma)^2 \left(\sum_{i=1}^{n} |f_i| \, |x_{k_i}| \right)^2 \\ &< -\tfrac{1}{2}(\Delta s_{k+1})^2 \end{aligned} \tag{26}$$

From Lemma 2, $V_{k+1} < V_k$ is derived. And if $|s_k x_{k_i}| < \delta_i$ $(s_k \neq 0)$ for all i, then $f_i = 0$ $(i = 1, \ldots, n)$, and $\Delta s_{k+1} = 0$, which yields (14a) and

$$s_{k+1} = s_k \neq 0 \tag{27}$$

$|s_k x_{k_i}| \leq \delta_i$ tells us that Gx_k is nearly equal to zero and x_k is decreasing from Theorem 1

This case considered in more detail, let $s_k \neq 0$ and $|G\Gamma s_k x_{k_i}| < \delta_i$ ($i = 1, \ldots, n$). If all f_i's are considered to be chosen 0 ($i = 1, 2, \ldots, n$), then

$$\Delta s_{k+1} = 0$$

which induces

$$s_k = s_{k+1} \neq 0.$$

Under the above conditions, $V_k = V_{k+1}$.

Letting

$$x_k = T\bar{x}_k$$

where $[t_1, t_2, \ldots, t_n]$ and t_i are chosen to satisfy

$$Gt_1 = 1,$$

$$t_i \in \operatorname{Ker} G, \quad i = 2, 3, \ldots, n,$$

then from Theorem 1, the state variable on $s_k = 0$ satisfies $\bar{x}_{k_i} \to 0$ ($i = 2, \ldots, n$) Therefore

$$s_k = Gx_k \to \bar{x}_{k1}.$$

When substituted into $(G\Gamma)s_k x_{k_i}$,

$$(G\Gamma)s_k x_{k_i} \to (G\Gamma)t_{1_i}\bar{x}_{k1}^2, \; (k \to \infty), \quad i = 1, \ldots, n.$$

On the other hand,

$$\delta_i = \tfrac{1}{2} \sum_{j=1}^{n} (G\Gamma)^2 |x_{k_i}| \, |x_{k_j}| f_0$$

will approach to

$$\tfrac{1}{2}(G\Gamma)^2 \bar{x}_{k1}^2 \sum_{j=1}^{n} |t_{1_i}| \, |t_{1_j}| f_0. \tag{28}$$

Since the amplitudes of all f_i's satisfying (23) are the same, (28) is less than

$$(G\Gamma)|t_{1_i}|\bar{x}_{k1}^2.$$

This tells us that

$$|G\Gamma s_k x_{k_i}| > \delta_i \qquad \text{as } k \to \infty,$$

which contradicts the condition

Therefore the state cannot continue to stay under $\Delta s_{k+1} = 0$ Therefore $s_k \to 0$ as $k \to 0$, and so $x_k \to 0$ as $k \to \infty$.

This theorem tells us that the sliding mode control of a discrete system is different from that of a continuous system in that the switching surface is different from the sliding mode hyperplane and there exists a switching region along the sliding mode The proposed control has three different feedback coefficients, which gives better performance.

4. Consideration of robustness

In the previous discussion, the system is known exactly The sliding mode control has been known to be robust to the uncertainty of the controlled plant. In this section, the design is done under plant uncertainty

The plant is known to be represented by

$$x_{k+1} = \Phi_0 x_k + \Gamma u_k, \tag{1a$'$}$$

$$y_k = Cx_k. \tag{1b$'$}$$

The control system is designed on the basis of the above model. The actual system matrix Φ is related to Φ_0 by

$$\Phi = \Phi_0 + \Delta\Phi \tag{29}$$

and both (1) and (1)$'$ are assumed to be stabilized on $\{s_k\} = 0$. Since the equivalent control F_{eq} is designed by

$$F_{eq} = -(G\Gamma)^{-1}G(\Phi_0 - I), \tag{13$'$}$$

$$\Delta s_{k+1} = G(\Phi_0 - I)x_k + G\Gamma F_{eq} x_k + G\Delta\Phi x_k + G\Gamma F_D x_k, \tag{30}$$

where the uncertainty of the system matrix, $\Delta\Phi$, is assumed to be represented by

$$\Delta\Phi = \Gamma D, \tag{31a}$$

$$D = [d_1, d_2, \ldots, d_n], \quad |d_i| < \bar{d} \; (i = 1, \ldots, n) \tag{31b}$$

Substituting (31) into (30),

$$\Delta s_{k+1} = G\Gamma(F_D + D)x_k$$

In this case, Theorem 2 is modified as follows.

Theorem 3. *If the plant system matrix Φ has the uncertainty $\Delta\Phi$ satisfying* (31), *and on $s_k = 0$, the plant is stable with the equivalent control, then the i-th element f_i of the control law F_D satisfying*

$$f_i = \begin{cases} f_0 & \text{for } (G\Gamma)s_k x_{k_i} < -\delta_i', \\ 0 & \text{for } -\delta_i' \leq (G\Gamma)s_k x_{k_i} \leq \delta_i', \\ -f_0 & \text{for } (G\Gamma)s_k x_{k_i} > \delta_i', \end{cases} \tag{32}$$

makes the system stable, where

$$\delta_i' = \tfrac{1}{2}(G\Gamma)^2 \sum_{j=1}^{n} |x_{ki}|\,|x_{kj}|\left(f_0 + \bar{d}\right) \tag{33}$$

and f_0 *is satisfying*

$$\bar{d} < f_0 < \left| \frac{2}{G\Gamma \sum_{j=1}^{n} |t_{1j}|} \right| - \bar{d} \tag{34}$$

This theorem is easily derived from Theorem 2. It tells us that the amplitude of the uncertain control should be smaller in order to stabilize the uncertain system, which is quite contrary to results of continuous systems. The switching region becomes larger as the uncertainty increases.

5. Example

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = [1 \quad 0]x,$$

$$s = [1 \quad 1]x$$

In this example, the equivalent input is given by

$$u_{\mathrm{eq}} = [1 \quad 1]x.$$

So the sliding mode control of the continuous system is given by

$$u = \{[1 \quad 1] + [f_2 \quad f_2]\}x$$

where

$$f_i = \begin{cases} f_0 & \text{if } sx_i < 0, \\ 0 & \text{if } sx_i = 0, \\ -f_0 & \text{if } sx_i > 0 \end{cases} \quad (i = 1, 2).$$

The above sliding mode control makes the system stable. A simulation of the above control system with the initial condition $x(0) = [1 \quad 0]^{\mathrm{T}}$ is given in Figure 1a and 0.1 in Figure 1b. The phase plane $(x_1(t), x_2(t))$ of the latter case is given in Figure 1c. In this case, it is found that there exists chaotic chattering along the switching surface which coincides with the sliding surface. When the system is sampled with the sampling interval 0.1, the following system is derived:

$$x_{k+1} = \begin{bmatrix} 9\,9532116\,\mathrm{E}-0\,1 & 9\,04837418\,\mathrm{E}-0.2 \\ -9\,04837418\,\mathrm{E}-0.2 & 8.14353676\,\mathrm{E}-0\,1 \end{bmatrix} x_k + \begin{bmatrix} 4\,67884016\,\mathrm{E}-0.3 \\ 9\,04837418\,\mathrm{E}-0.2 \end{bmatrix} u_k,$$

$$y_k = [1 \quad 0]x_k.$$

Let

$$s_k = [1 \quad 1]x_k.$$

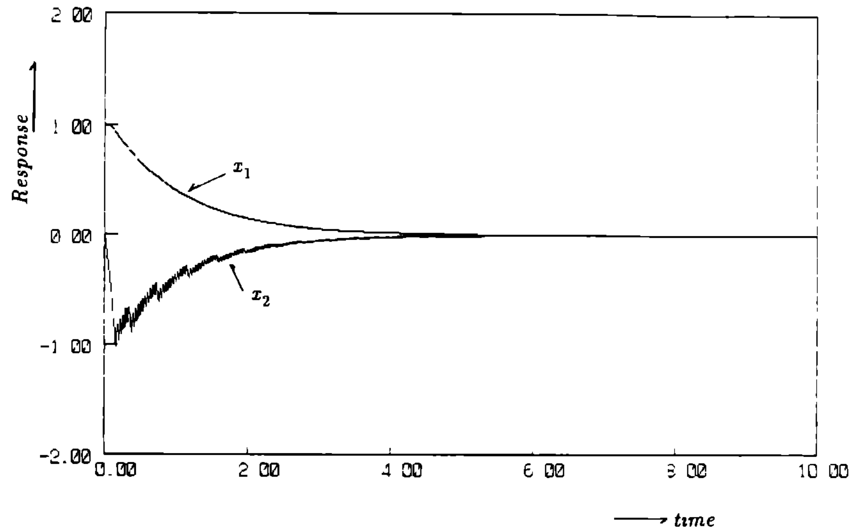

Fig 1a. Response of continuous sliding mode control with sampling interval 0.02

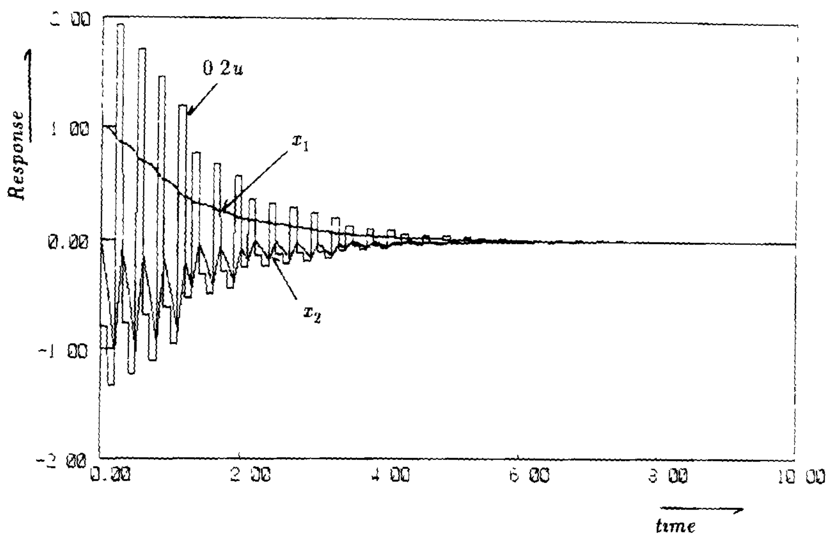

Fig 1b Response of continuous sliding mode control with sampling interval 0 1

Then

$$u_{k\,eq} = [1 \quad 1]x_k$$

So the discrete sliding mode control u_k is given by

$$u_k = \{[1 \quad 1] + [f_1 \quad f_2]\}x_k$$

where the f_i's are chosen by (21). The upper bound of f_0 is about 20 so f_0 is chosen to be 5.

The response for $f_0 = 5$ is shown in Figure 2a and the phase plane is in Figure 2b The control input is not slugging in comparison with Figure 1b. Further, the switching region of (21) to make $d_i = 0$ is shown in Figure 3. In case uncertainty of the system exists with upper bound of the disturbance $\bar{d} = 1$ $(i = 1, 2)$, and the system is supposed to be modeled to give the equivalent control $u_{eq} = 0$, then the response for

$$u_k = [f_1 \quad f_2]x_k$$

and $\bar{d} = 1$, $f_0 = 5$ with sampling interval 0.1 is shown in Figure 4 This figure tells us that even without using the equivalent control, satisfactory results can be obtained by considering the uncertainty In the proposed control, large feedback coefficients are given outside the switching region, and in the neighborhood of the sliding mode, the equivalent linear control is used.

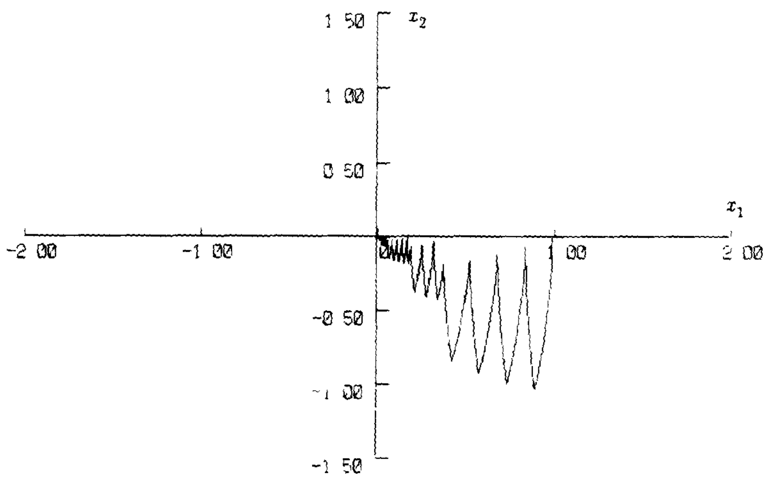

Fig. 1c. Phase plane of continuous sliding mode with sampling interval 0 1

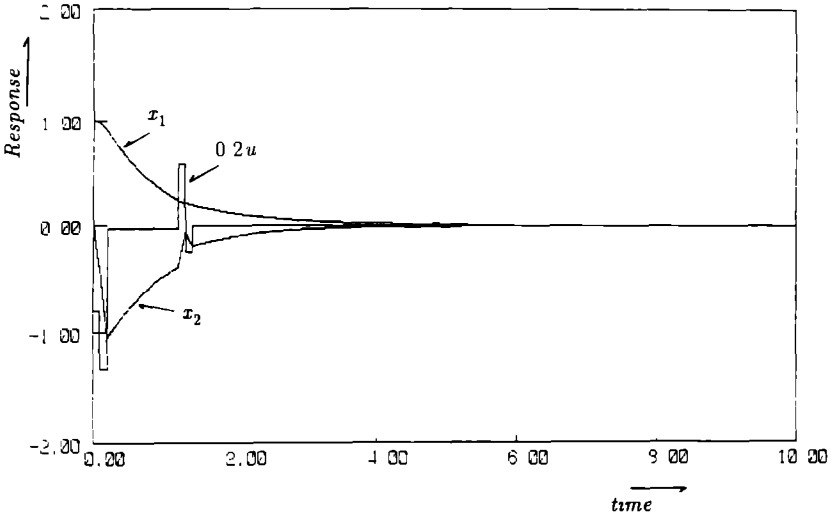

Fig 2a Response of discrete sliding mode control with sampling interval 0 1

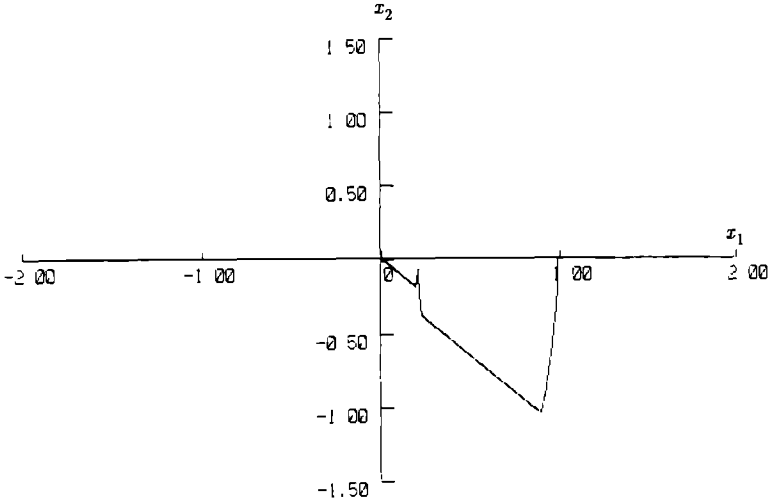

Fig. 2b. Phase plane of discrete sliding mode with sampling interval 0 1.

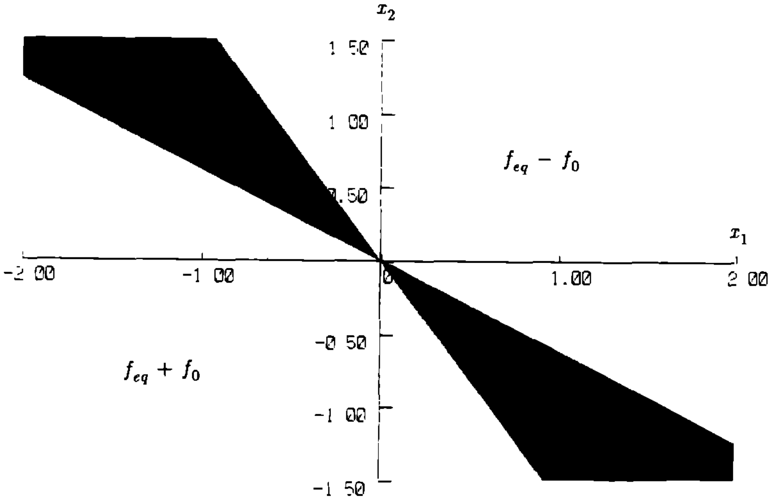

Fig 3. Switching region of discrete sliding mode control for $f_0 = 5$, sampling interval 0 1 and $\bar{d} = 0$.

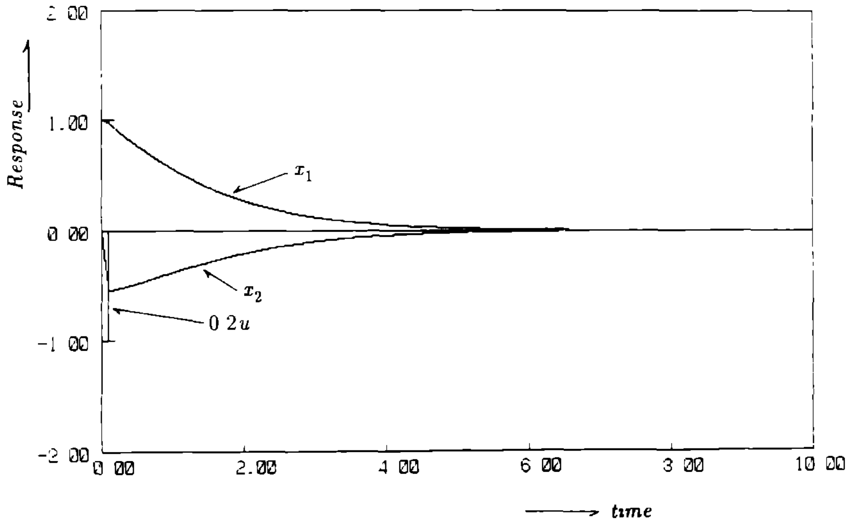

Fig 4 Response of discrete sliding mode control with sampling interval 0 1 and $\bar{d} = 1$

6. Conclusion

This paper presents a new design method of sliding mode control for a discrete system, which is found to be an extension of the results obtained for a continuous system The control coefficient takes values of three levels. Many features of the sliding control for a discrete system are found that are quite different from those obtained for continuous systems, such as that the control input does not switch so frequently. The results which eliminate overfast sampling may bring the sliding mode control into practice even for the control of chemical processes with a long sampling interval One application to self-tuning control is under study [11]. The author appreciates the help in the computations of Mr M. Morisada of Canon Co

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