LQR Coupling, Intuition, and Steady-State Behavior

1. System Overview

We consider a four-state dynamical system controlled using a Linear Quadratic Regulator (LQR). The state vector is defined as:

x = [position, velocity, angle, angular velocity]■

2. What is Coupling? (Key Concept)

Coupling means that one state directly affects the dynamics of another state. In state-space form, coupling appears as non-zero off-diagonal elements in the system matrix A.

3. Coupling in the A Matrix

The system dynamics are given by:

■ = A x + B u

■■ = −(b■/m)x■ + (L■/m)x■

The term (L■/m)x■ shows that angle directly causes linear acceleration. This is the primary coupling in the system.

4. One-Way Coupling (Very Important)

Angle affects position, but position does NOT affect angle. Mathematically, there are no x■ or x■ terms in the angular dynamics equation.

■■ = (L■/I)x■ − (b■/I)x■

5. Physical Intuition

Imagine a cart with a rotating arm. Tilting the arm produces a horizontal force that moves the cart. However, moving the cart does not automatically rotate the arm. This physical asymmetry explains the one-way coupling.

6. How LQR Thinks (Optimization View)

LQR minimizes the cost function:

J = ∫ (x■Qx + u■Ru) dt

Because control effort is penalized, LQR may accept a small angle error if it helps keep position at zero with less control effort.

7. Why Position Goes to Zero but Angle Does Not

Due to coupling, controlling angle indirectly controls position. Therefore, LQR focuses on eliminating position error and allows angle to settle at a non-zero equilibrium when constant disturbances or step inputs are present.

8. Why Tuning Q Is Not Enough

Increasing the angle weight in Q reduces the steady-state offset but cannot eliminate it completely. This is because LQR has no memory of past errors and cannot correct constant bias.

9. Role of Integral Action (LQI)

By adding integral states of position and angle error, the controller accumulates steady-state error and forces it to zero. This augmented design is known as Linear Quadratic Integral (LQI) control.

10. Final Exam-Ready Takeaway

In coupled systems, LQR may regulate certain states indirectly, allowing steady-state offsets. Integral augmentation is required to guarantee zero steady-state error in all controlled outputs.