SIGNALS AND SYSTEMS

Signals:

Every country has its own frequency band width.

TRAI in india decides the frequency band for diff. subscribers.

In a India 800 - 1800 MHz.

There is a source for every signal.

Every signal is governed by - amplitude, frequency, phase.

TRAI: Telecommunication regulatory authority of India.

Function generator, generates different types of signals or waves.

Source generates signal and what has to be done by the signal is determined by system.

eg. Electricity is signal, fan is system.

Signal definition:

A signal is defined as a function of one or more independent variable carrying information about the behaviour or nature of the physical phenomenon.

eg: velocity of car is function of time

speech is function of acoustic pressure and time

picture is a function of intensity of brightness

in two special co-ordinates and

EM waves as a function of space co-ord. and time

$$E(x,y,z,t) = E_{0x} e^{-\alpha z} \cos(\omega t - \beta z)$$

signal E depends on x, y, z, t

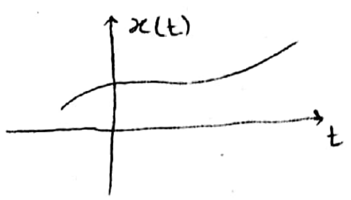
x, y, z, t are independent variables

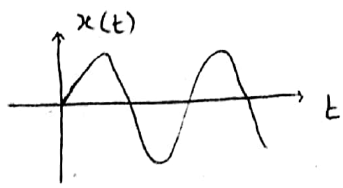
Signal is a mathematical reaction for any kind of physical phenomenon.

Classification of signals:

I: 1. Continuous

A signal $x(t)$ is continuous time signal if the independent variable 't'. is continuous and the signal is defined for all values of 't'.





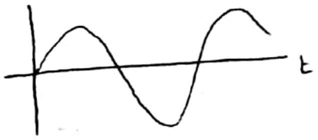
There can be an interval of independent variable, but it must be continuous. eg: ECG

Where $x[n]$ is called the samples and the time interval b/w the samples is called the sampling interval or sampling time.

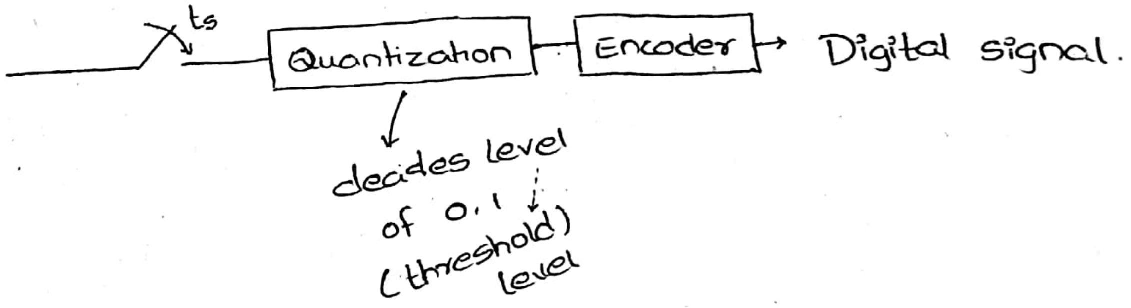
Classification II :

1. Analog 2. Digital

1. A signal is said to be analog signal if it is <u>continuous</u> and its magnitude have any <u>finite</u> value.

eg 

A <u>discrete</u> time signal is said to be digital if its samples can have <u>finite</u> no. of distinct values.



Classification III :

1. Deterministic 2. Random

1. Deterministic signals are those signals which can be represented in graphical or mathematical form

eg: $x(t) = \sin t$, $x(t) = u(t)$ → unit step fnc. $x(t) = 1 ; t \geq 0$, $x(t) = 0 ; t < 0$

Random signals are those signals which can be defined by probabilistic expressions.

eg: Noise signal, earthquake signal, thermal noise

↳ by interference ↳ internally generated in all electronic devices.

Classification IV:

1. Periodic 2. Non-periodic

i) For Continuous time:

If $x(t) = x(t+nT)$ for all t where T is a positive constant called fundamental time period and $f = \frac{1}{T}$ is called the fundamental frequency.

Non-Periodic signals donot satisfy the above math. eqn.

NOTE:

If $x_1(t)$ and $x_2(t)$ are periodic signals with their fundamental time periods T_1 and T_2 then,

$x(t) = x_1(t) + x_2(t)$ will be a periodic signal if,

$T = mT_1 = nT_2$ where m, n are integers $\frac{T_1}{T_2} = \frac{n}{m}$: rational no.

and the fundamental time period will be:

$T = LCM(T_1, T_2)$.

eg: $x(t) = \cos\frac{t}{3} + \sin\frac{t}{4}$

$T_1 = \frac{2\pi}{1/3} = 6\pi$

$T_2 = \frac{2\pi}{1/4} = 8\pi$

$\frac{T_1}{T_2} = \frac{3}{4}$ = rational $\therefore$ x(t) is periodic

$T = LCM(6\pi, 8\pi) = 24\pi$

$x(t) = \cos t + 2\sin\sqrt{2}t$

individually periodic

but x(t) is not periodic $\because$ $\frac{T_1}{T_2}$ = irrational

no syncronisation

ii, For discrete time:

$x[n] = x[n+N]$ for all n, where N is an integer

and the smallest value of N is called fundamental time period.

$\Omega = \frac{2\pi}{N}$ is called the fundamental angular frequency.

NOTE:

If two or more discrete time periodic signals are added, then the composite signal will always be a periodic signal

$x_1[n] \longrightarrow N_1$ $x_2[n] \longrightarrow N_2$

$x[n] = x_1[n] + x_2[n]$; $\frac{N_1}{N_2} \rightarrow$ always rational

$N = LCM\{N_1, N_2\}$: Fundamental time periodic

NOTE:

For composite exponential signal, $x[n] = e^{j\omega_0 n}$ is periodic if $\frac{\omega_0}{2\pi} = \frac{m}{N}$; m: integer

Check for periodic : replace n by n+N

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \, e^{j\omega_0 N}$$

$$= e^{j\omega_0 n} \, e^{j\omega_0 \cdot \frac{2\pi m}{\omega_0}} = e^{j\omega_0 n} \, e^{j2\pi m}$$

for m = 1 it is $e^{j\omega_0 n}$ = periodic

we can have many m values

Eg : $x[n] = e^{j\frac{2\pi n}{3}} + e^{j\frac{3\pi}{4}n}$

$N_1 = m_1 \frac{2\pi}{\omega_0} = m \cdot \frac{2\pi}{2\pi/3} = 3m_1$ — for fundamental time period ; $m_1 = 1$ $\therefore N_1 = 3$

$N_2 = m_2 \frac{2\pi}{\omega_0} = m \cdot \frac{2\pi}{3\pi/4} = \frac{8m_2}{3}$ ↘ $m_2 = 3$ $\therefore N_2 = 8$.

($\because$ N has to be integer)

$N = LCM(3, 8) = 24$

$\therefore$ Signal is periodic with periodicity 24

Classification II :

1. Even signal (Mirror symmetry)
2. Odd signal (anti-symmetric)

For a continuous time signal $x[t]$ will be a mirror signal if;

$$x[t] = x[-t]$$

and an odd signal if; $x(t) = -x(-t)$

Any continuous time signal $x(t)$ can be split into even and odd part of the signal.

$$\rightarrow x(t) = \underbrace{x_e(t)}_{\text{even}} + \underbrace{x_o(t)}_{\text{odd part}}$$

$$x(-t) = x_e(-t) + x_o(-t)$$

$$\rightarrow x(-t) = x_e(t) - x_o(t)$$

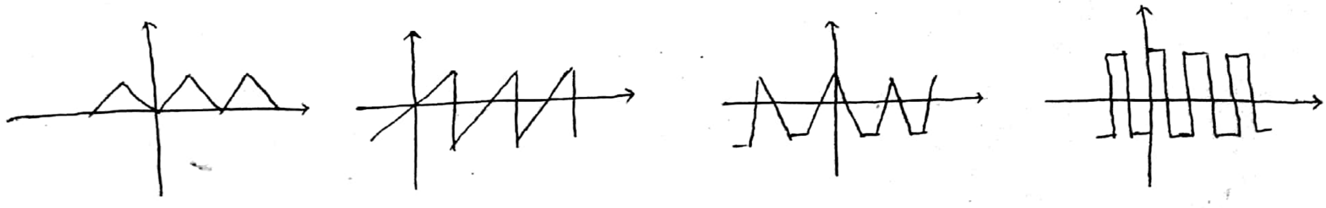
$$\therefore \quad x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

For discrete time signals $x[n]$, the even and odd parts can be written as,

$$x_e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

$$x_o[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$$



even odd even odd

NOTE:

If the given signal, $x(t) = a(t) + jb(t)$ is a complex signal, then signal is said to conjugate symmetric if

$x(-t) =$ complex conjugate of $x(t)$.

$$x(t) = a(t) + jb(t) \rightarrow x^*(t) = a(t) - jb(t)$$

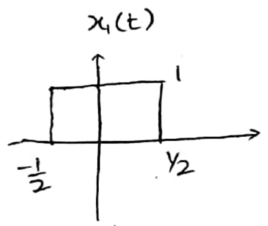
$$x(-t) = a(-t) + jb(-t) \equiv x^*(t) = a(t) - jb(t)$$

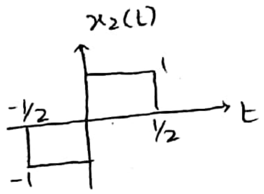
$\therefore$ $a(-t) = a(t)$ — even

$b(-t) = -b(t)$ — odd

$\therefore$ For a complex valued signal to be symmetric, the real part should be even symmetric and imaginary part should be odd symmetric.

Q:





$x(t) = x_1(t) + i\, x_2(t)$

Comment on $x(t)$.

Here the real part, $x_1(t)$ is even and odd imaginary part $x_2(t)$ is odd.

$\therefore$ $x(t)$ is conjugate symmetric.

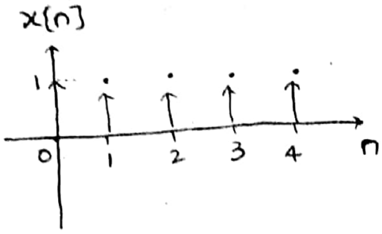
Q: Find the even and odd part of the signal,

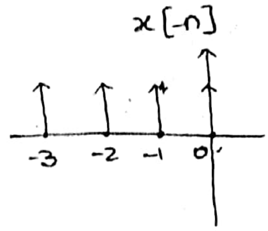
$x[n] = u[n]$

$x[n]$ is defined only for $n \geq 0$

$u[n] = 1, \; n \geq 0$

and $n = 1, 2, 3, \ldots$



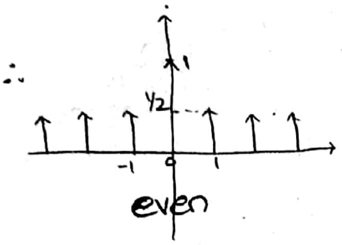


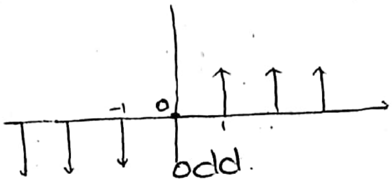
At once there is only one signal, as, if $n \geq 0$ $x[-n]$ doesn't exist and if $-n \geq 0$, $x[n]$ doesn't exist.

$$x_e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

$$\frac{1}{2} x[n] + \frac{1}{2} x[-n]$$

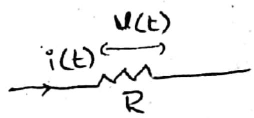
$\therefore$





Q: Consider a resistance of R for applied voltage V. Then the instantaneous power following through the resistance

$$P(t) = i^2(t) R = \frac{v^2(t)}{R}$$



For $R = 1$: $P(t) = i^2(t) = v^2(t)$ This is normalised instantaneous power.

In signals and system, it is a custom to represent normalised instantaneous power of a signal $x(t)$ by;

$$P(t) = x^2(t)$$ regardless of whether the signal is voltage or current signal.

The total energy of a signal $x(t)$ is given by:

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

and the average power is given by:

$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)^2| dt$$

For a discrete signal:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_{avg} = \lim_{N \to \infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2$$

NOTE:

* For a signal to be energy signal, its energy must be finite. i.e. $0 < E < \infty$

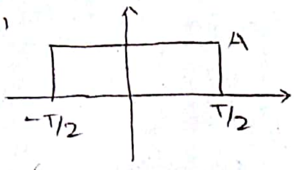
  For a signal to be power signal, its power must be finite i.e. $0 < P_{avg} < \infty$

* All the energy signals have $P_{avg} = 0$ whereas all the power signals have $E \to \infty$ (infinite E)

* The periodic and random signals are usually Power signals.

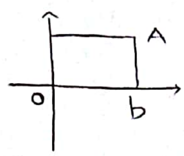
* The signals which are both deterministic and non-periodic are usually energy signals.

Q: Find energy of the given signals.

i) 

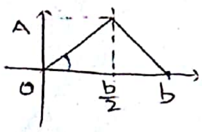
$$E = \int_{-T/2}^{T/2} A^2 dt \neq \frac{t}{3} A^3 \Big|_{-T/2}^{T/2}$$

$$= A^2 \int dt = A^2 T$$

ii) 

$$E = \int_0^b A^2 dt$$

$$= A^2 b$$

iii) 

$y = \frac{2A}{b} t$

$(b,0)\ (\frac{b}{2}, A)$

$\frac{y}{x-b} = \frac{2A}{-b}$

$y = \frac{2A}{b}(b-t)$

$= 2A - \frac{2At}{b}$

$E = E_1 + E_2$

$$E_1 = \int_0^{b/2} \left(\frac{2A}{b} t\right)^2 dt$$

$$= \frac{4A^2}{b^2} \cdot \frac{1}{3} \cdot \frac{b^3}{8} = \frac{A^2 b}{6}$$

$$E_2 = \int_{b/2}^{b} \left(2A - \frac{2At}{b}\right)^2 dt$$

$$= 4A^2 \int \left(1 + \frac{t^2}{b^2} - \frac{2t}{b}\right) dt$$

$$\cancel{= 4A^2 \left(\frac{b}{2} + \frac{1}{b^2}\left(\frac{3b^2}{4}\right) - \frac{2}{b}\cdot\frac{b}{2}\right)}$$

$$\cancel{= 2A^2 b + 3A^2 - 4A^2}$$

$$\cancel{= 2A^2 b - A^2}$$

$$4A^2 \left[t + \frac{1}{b^2}\cdot\frac{t^3}{3} - \frac{2}{b}\cdot\frac{t^2}{2}\right]_{b/2}^{b}$$

$$4A^2 \left[\frac{b}{2} + \frac{1}{3b^2}\cdot\frac{7b^3}{8} - \frac{1}{b}\cdot\frac{3b^2}{4}\right]$$

$$= 4A^2 b \left[\frac{1}{2} + \frac{7}{24} - \frac{3}{4}\right] = \frac{4A^2 b}{24}\cdot[12+7-18]$$

$$= \frac{A^2 b}{6}$$

$$\therefore E = E_1 + E_2 = \frac{A^2 b}{6} + \frac{A^2 b}{6} = \frac{A^2 b}{3}$$

Q: Test whether signal is energy or power signal,

$x[n] = u[n]$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} u[n]^2 \rightarrow \infty \qquad E \rightarrow \infty$$

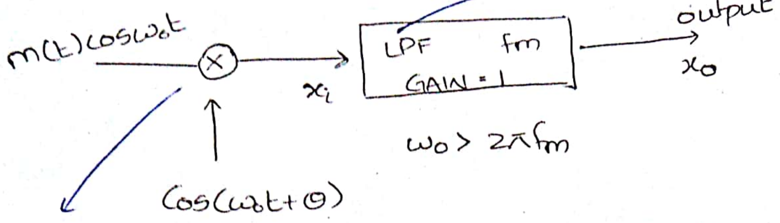
$$P_{avg} = \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{n=0}^{N} u[n]^2 \neq \lim_{N\to\infty} \frac{1}{(2N+1)} \frac{N(N+1)(2N+1)}{6}$$

$$= \lim_{N\to\infty} \frac{N}{2N+1} = \frac{1}{2}$$

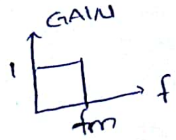
P is finite and $E \rightarrow \infty$

$\therefore$ Power signal

Q. Find the output power of the given diagram, if the power of $m(t)$ is P_m.



low pass filter.

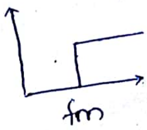


$f \to (0, f_m)$

at f_m GAIN = 0

$\omega_0 > 2\pi f_m$

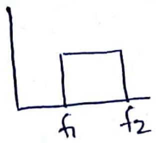
multiplier
(signal multiplication)



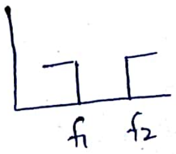
high pass filter

$\rightarrow$ $x_i = m(t)\cos\omega_0 t\cos(\omega_0 t+\theta)$

$= m(t)\cdot\frac{1}{2}[\cos\theta + \cos(2\omega_0 t+\theta)]$



Band of freq.



Band reject filter.

no freq. component
$\therefore$ DC

double freq.
ie. $2f_m$
So LPF doesn't allow this signal to pass $\because (0, f_m)$

$x_0(t) = \frac{1}{2} m(t)\cos\theta$

$\left(\frac{1}{2T}\int_{-T}^{T} \equiv \frac{1}{T}\int_0^T\right)$

$x_0 = \frac{1}{2} m(t)\cos\theta$

$P_{avg} = \lim_{T\to\infty} \frac{2}{2T}\int_0^T |x_0(t)|^2 dt = \lim_{T\to\infty}\frac{2}{2T}\int_0^T \frac{1}{4}\cos^2\theta\, |m(t)|^2 dt$

$= \lim_{T\to\infty} \frac{\cos^2\theta}{\cancel{4\cos^2\theta}}\cdot \underbrace{\frac{2}{2T}\int_0^T |m(t)|^2 dt}_{\text{Power of } m(t)} = \frac{1}{4}\cos^2\theta \cdot \lim_{T\to\infty}\frac{2}{2T}\int_0^T |m(t)|^2 dt$

$= \frac{\cancel{P_m}}{\cancel{4\cos^2\theta}}$ $\quad \frac{P_m}{4}\cdot\cos^2\theta$

Q: $x(t) = 8\cos(200\pi t - \frac{\pi}{2}) + 4\sin(100\pi t)$. $8\sin 200\pi t + 4\sin 100\pi t$

$f = \frac{\omega}{2\pi} = 100$

$|x(t)|^2 = 64\sin^2 200\pi t + 16\sin^2 100\pi t + 64\sin 200\pi t \sin 100\pi t$

$64(\cos() - \cos())$

both are periodic for one T.

$\Sigma = 0$.

$$P_{avg} = \lim_{T\to\infty} \frac{2}{2T} \int_0^T 64\left(\frac{1-\cancel{\cos 400\pi t}^{\,0}}{2}\right) + 16\left(\frac{1-\cancel{\cos 200\pi t}^{\,0}}{2}\right) dt$$

$$= \lim_{T\to\infty} \frac{2}{2T} \int_0^T (32+8)\,dt = 40$$

Q: $x(t) = e^{-a|t|}$ Find energy of signal.

$$E = \lim_{T\to\infty} \int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-\infty}^{\infty} e^{-2at} dt = \lim_{T\to\infty} \left.\frac{e^{-2at}}{-2a}\right|_{-\infty}^{\infty}$$

$$= \lim_{T\to\infty} -\frac{1}{2a}\left[e^{-2at} - e^{2at}\right] = \frac{1}{2a} \lim_{T\to\infty} \left[e^{2at} - e^{-2at}\right]$$

$$= \int_{-\infty}^{0} e^{2at} + \int_0^{\infty} e^{-2at} dt = \frac{1}{2a}(e^{2at})_{-\infty}^{0} + \frac{1}{-2a}(e^{-2at})_0^{\infty} = \frac{1}{2a} + \frac{1}{2a} = \frac{1}{a}$$

Q: $x(t) = t\,u[t]$ Test whether signal is power or energy.

$$P_{avg} = \lim_{t\to\infty} \frac{1}{(2t+1)} \Sigma t^2$$

$$= \lim_{t\to\infty} \frac{t(t+1)(2t+1)}{6\cdot(2t+1)} \to \infty$$

$E = \Sigma t^2 u[t]^2 = \sum_{t=0}^{\infty} t^2 \to \infty$.

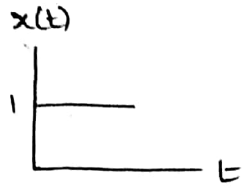
$\therefore$ Both $E, P \to \infty$ Neither energy nor power signal.

Representation of signal :-

1) 1. Continuous time signal –

$u(t) = 1 \quad t \geq 0$

$\quad\quad\;\; 0 \quad t < 0$



unit step function

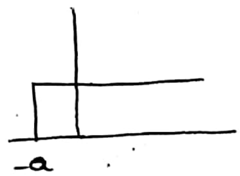
Step function

$A u(t) = x(t)$

$u(t-a)$

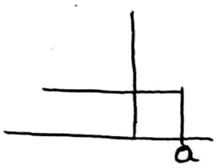
$t-a \geq 0$ i.e. $t \geq a$

$u(t-a) - u(t-b)$



$u(t+a)$

$t+a \geq 0$ i.e. $t \geq -a$



$u(-t+a)$

$-t+a \geq 0$

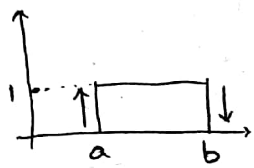
$t \leq a$

$u(-t-a)$

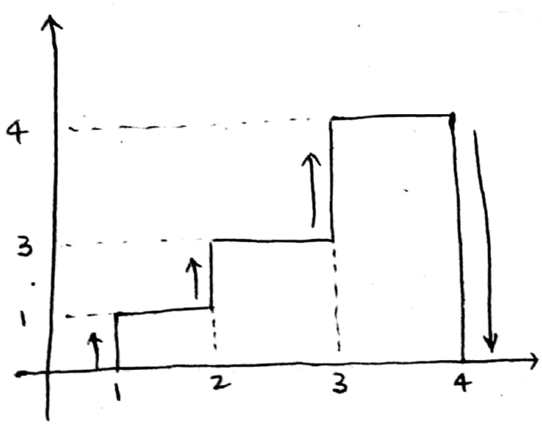
$-t-a \geq 0 \quad t \leq -a$

Moving up +ve func.

" down -ve "



$u(t-a) - u(t-b)$



$u(t-1) + 2u(t-2) + u(t-3) - 4u(t-4)$

eg: Stabilizer, battery in cellphone, watch.

2. Unit Impulse function: (dirac delta function)

$\delta(t) = \infty, \ t=0$
$ = 0, \ t \neq 0$

(Theoretical)

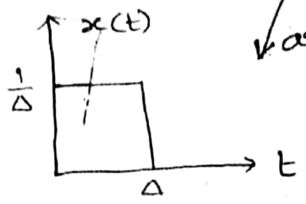
$\delta(t) = 1, \ t=0$
$ = 0, \ t \neq 0$

(Practical)

$\Rightarrow \int_{-\infty}^{\infty} \delta(t) = 1$

area under the curve.

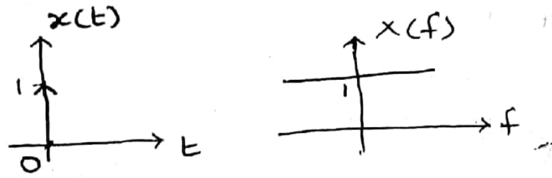
as $\Delta \to 0$



$x(t) = \frac{1}{\Delta}\left[u(t) - u(t-\Delta)\right]$

$\delta(t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\left[u(t) - u(t-\Delta)\right]$

as Δ ↓es i.e. as width ↓es height ↑es $\Rightarrow$ width $\to 0$, height $\to \infty$

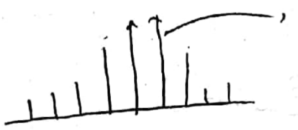


Time domain Freq. domain

Fourier transform

All signals are available in time domain, and we have to transform them to freq. domain.

∴ In time domain $x(t)$ is only at one position but when transformed to freq. domain, it spreads (∴ easy to work on)



we are concerned about these higher magnitude signals.

Properties of $\delta(t)$:

1. $\int_{-\infty}^{\infty} \delta(t-t_0)\,\phi(t)\,dt = \phi(t_0)$: Sampling property.

2. $\delta(at) = \frac{1}{|a|}\delta(t)$

3. $\delta(t) = \delta(-t)$ → even func.

4. $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

5. $\int_{-\infty}^{\infty} \delta(t) = 1$

6. $\delta(t) = \frac{d}{dt} u(t)$

7. Any continuous time signal can be represented in terms of $\delta(t)$.

$$x(t) = \int_{-\infty}^{\infty} \overbrace{x(\tau)\,\delta(t-\tau)\,d\tau}$$

common area of $x(\tau), \delta(t-\tau)$.

$x(t) = x(t) * \delta(t)$ — convolution (signal remains same)

$\int_{-\infty}^{\theta} x(\tau)\,\delta(t-\tau)\,dt$

$x(t) * \delta(t)$

$x(t)$ $\delta(t)$

$x(\tau)$ $\delta(\tau)$

$\delta(-\tau)$

$*$ $\delta(t-\tau)$

common area.

Q: $\underbrace{\int_{0}^{\infty} e^{(t-2)}\,\delta(2t-4)\,dt}$ $= \int_{0}^{\infty} e^{(t-2)} \cdot \frac{1}{2}\,\delta(t-2)\,dt = \frac{1}{2} \cdot e^{(2-2)} = \frac{1}{2}$.

$\phi(t)$, $t_0 = 2$

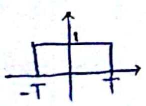
Q: $\int_{-\infty}^{\infty} (t^2 + \cos\pi t)\,\delta(t-1)\,dt = \phi(t_0) = 1 + \cos\pi = 0$.

$t_0 = 1$

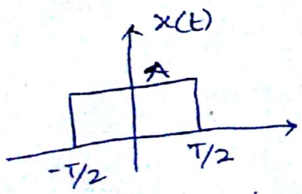
Q: $\int_{-1}^{3} (2t+1)\,\delta(t-4)\,dt = 0$

$t_0 = 4$ (not in limit) but $-1 < t < 3$.

3. Rectangular pulse:



$x(t) = \text{rect}\left(\frac{t}{2T}\right)$

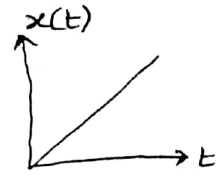


$x(t) = A\,\text{rect}\left(\frac{t}{T}\right)$

(4) Ramp function:

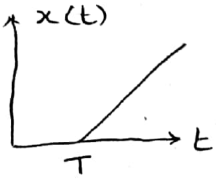
$x(t)=0$

(i)



$$x(t) = \boxed{t\,u(t)} = r(t) \;; \quad t \geq 0$$
$$= 0 \;; \quad t < 0$$

$(t-T)u(t)$

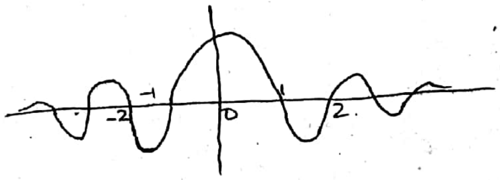


$$x(t) = r(t-T) \quad t \geq T$$
$$= 0 \quad t \leq T$$

$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$

5. Sinc function:

$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

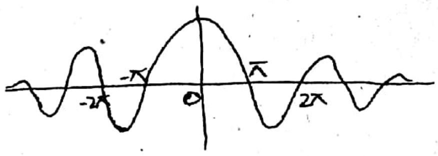


Even function.

Has zero crossing points at $x = \pm m$ (integers $m = 1, 2, 3, \cdots$)

6. Sampling function:

$$S_a(x) = \frac{\sin x}{x}$$



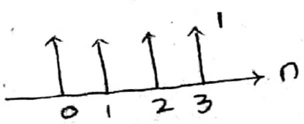
zero crossing points

at $x = \pm n\pi$ $(n = 1, 2, 3, \cdots)$

2) Discrete time signal:

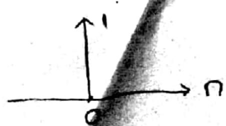
1. Unit step function:

$$u[n] = 1 \;; \quad n \geq 0$$
$$= 0 \;, \quad n < 0$$

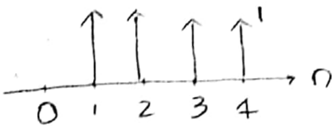


2. Unit impulse function:

$$\delta[n] = 1 \;; \quad n = 0$$
$$= 0 \;; \quad n \neq 0$$



$u[n-1]$ →



$\therefore \quad \delta[n] = u[n] - u[n-1]$

$$u[n] = \sum_{r=0}^{\infty} \delta[n-r]$$

Properties of $\delta[n]$:

1 $\quad x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

2 $\quad$ Any discrete time signal can be represented in terms of impulse function.

$$x[n] = \sum_{k=-\infty}^{\infty} x[n]\delta[n-k] = x[n] * \delta[n]$$

3 Sinusoidal discrete time signal.

$x[n] = A\cos[\Omega n + \phi]$

For $x[n]$ to be periodic, $x[n] = x[n+N]$

$\Omega N = 2\pi m$ — $\Omega = 2\pi \underbrace{\left(\frac{m}{N}\right)}$, rational $\quad m, N \in$ Integers

All sinusoidal discrete signals are not periodic

Q: $x[n] = 2\cos[2\pi n]$

$\Omega = 2\pi = 2\pi \cdot \frac{m}{N}$

$\frac{m}{N} = 1$

periodic

Q: $x[n] = 5\sin[2n]$

$\Omega = 2 = 2\pi \frac{m}{N}$

$\frac{m}{N} = \frac{1}{\pi}$

non-periodic

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Operation on Signals

1. Operation on dependent variable:

a. Amplitude scaling.

$y(t) = x(t)$ $\quad$ $y[n] = x[n]$

$y(t) = a\,x(t)$ $\quad$ $y[n] = a\,x[n]$

if $a > 1$ — Amplification

$0 < a < 1$ — Attenuation

eg: Transformer, speaker, stabilizer, ironing (for diff materials diff. temp.)

b. Addition:

$x_1(t)$, $x_2(t)$

$y(t) = x_1(t) + x_2(t)$

eg: movie (audio + video)

c. Multiplication:

$y(t) = x_1(t)\,x_2(t)$ radio - 93.5 fm → MHz.

eg: $P = V(t)\,i(t)$

purpose is to change the spectrum of signal. Original signal is presewed in the process. To presewe we use carrier signal (f_c) and, f_m- modulating signal ($f_c > f_m$)

- modulation - signals generated normally lie in very low frequency range. So they need to be transfered to high freq to transmit to large distances using antenna
- antenna's height depends on λ. ($h \propto \lambda$)

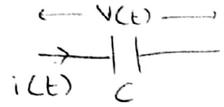
microstep antenna - present in cell phone

Yagi uda antenna - used in earlier T.V antenna

d. Differentiation:

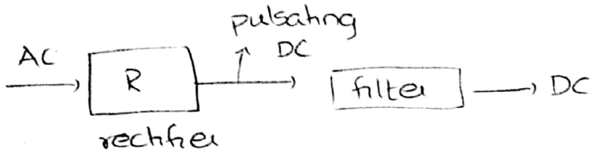
$$y(t) = \frac{d}{dt} x(t)$$

eg: $a = \frac{dv}{dt}$



$i(t) = C \frac{dv}{dt}$

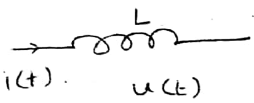
$u(t) = L \frac{di}{dt}$



e. Integration:

$$y(t) = \int_{-\infty}^{\infty} x(t)\,dt$$

eg



$i(t) = \frac{1}{L} \int_{-\infty}^{L} u(t)\,dt$

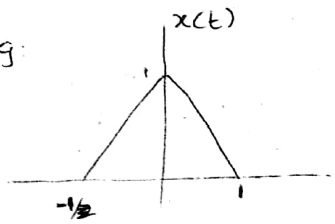
2. Operation on independent variables:

Time scaling: $y(t) = x(at)$

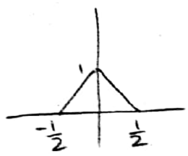
a. Compression – $a > 1$

Expansion – $0 < a < 1$

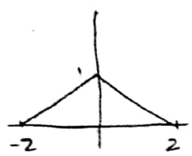
eg:



$x(2t)$:



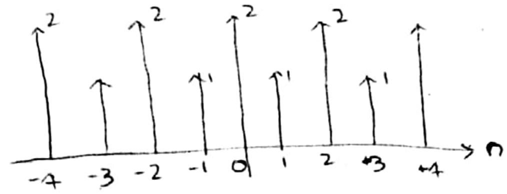
$x(t/2)$



& (fast forward) ↓ compression

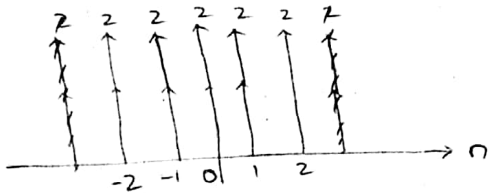
& (slow motion) ↓ expansion

$x[n]$



$n \to -4$ to 4

$x[2n]$



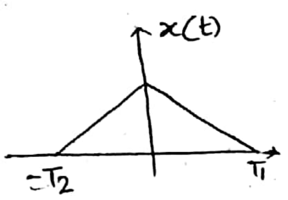
$2n \to -2$ to 2

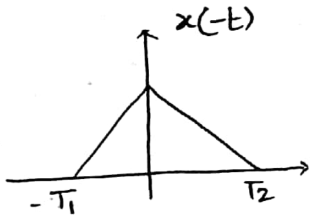
b. Reflection:

If $y(t) = x(-t)$ then $y(t)$ is the reflected version of $x(t)$

If $x(t)$ is an even signal, then

$$y(t) = x(-t) = x(t)$$





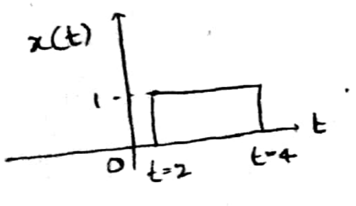
reflection about vertical axis.

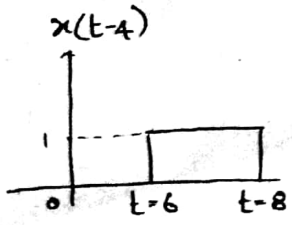
c. Time shifting:

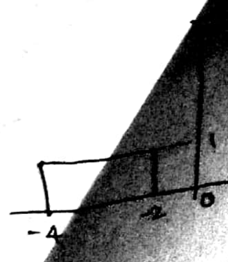
$$y(t) = x(t - t_0)$$

if, $t_0 > 0$: signal shifts to right

$t_0 < 0$: left side shift







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Note

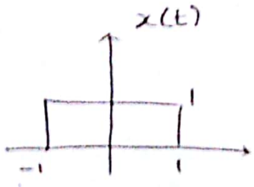
Whenever it is required to have time scaling and time shifting simultaneously, then we prefer to do time shifting first

eg $y(t) = x(at-b)$

$y(0) = x(-b)$

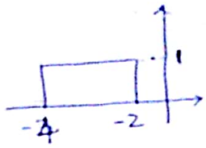
$y(\frac{b}{a}) = x(0)$

$x(t)$

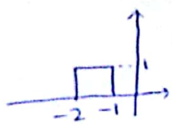


$x(2t+3) = ?$

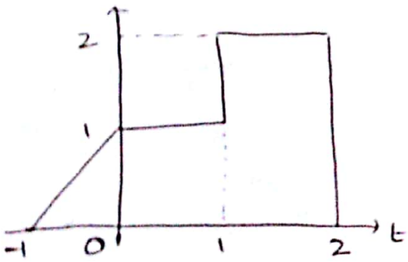
$x(t+3) =$



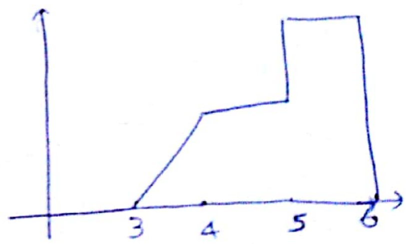
$x(2t+3)$



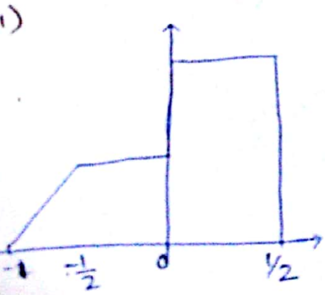
Q.



i, $x(t-4)$

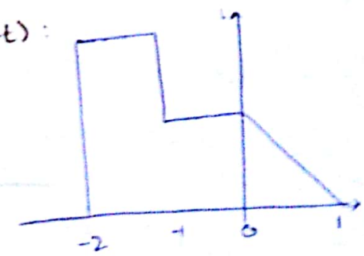


ii, $x(2t+1)$

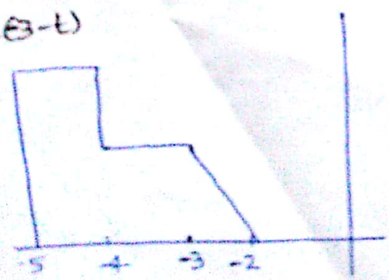


iii, $x(3-t)$

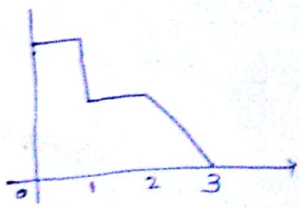
$x(-t)$:



$x(3-t)$



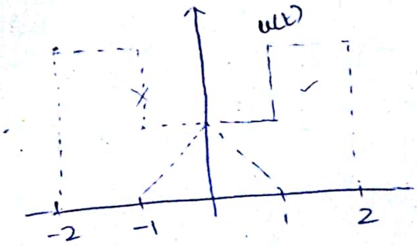
iv, $x(-t+2)$



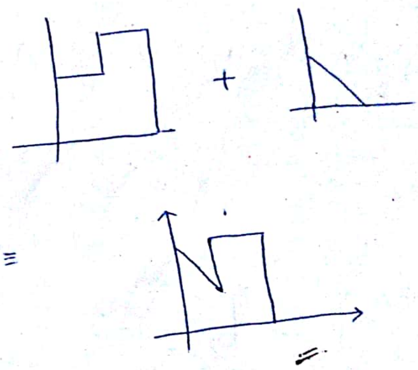
$x(t-t_0)$: $t_0>0$: right
$t_0<0$: left

$x(-t-t_0)$: After taking reflection,
$t_0>0$: left
$t_0<0$: right

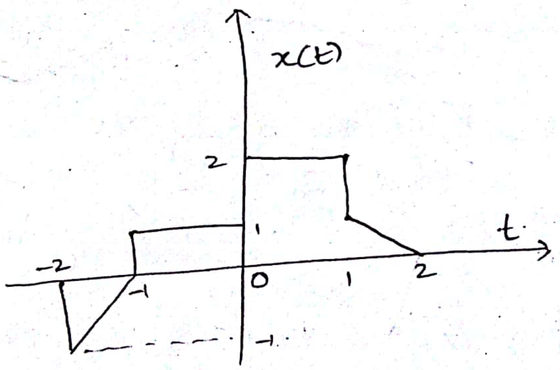
(v) $[x(t)+x(-t)]u(t)$:



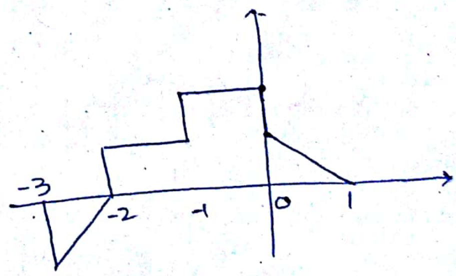
$x(t)\,u(t)$ $\quad$ $x(-t)u(t)$



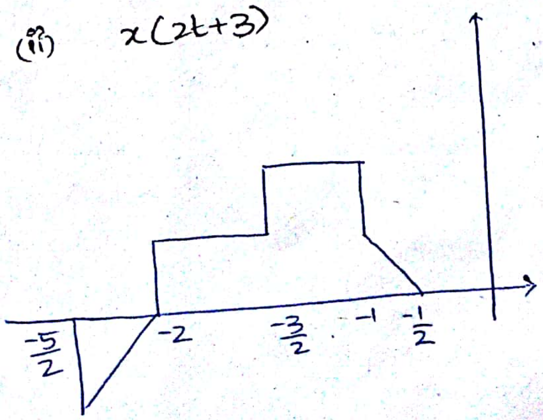
Q:



i) $x(t+1)$

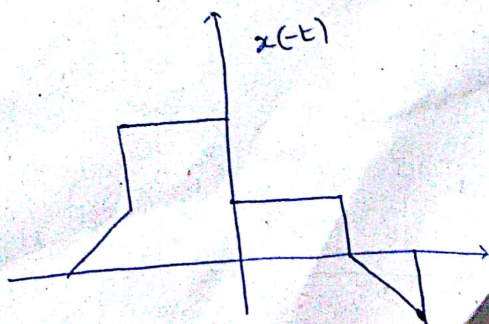


(ii) $x(2t+3)$

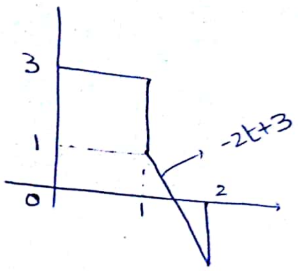


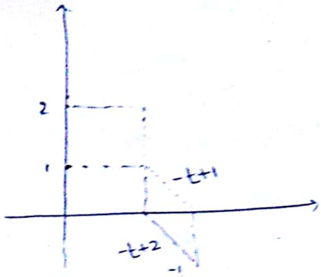
(iv) $(x(t)+x(-t))\,u(t)$

$x(-t)$:



$(x(t)+x(-t))\,u(t)$

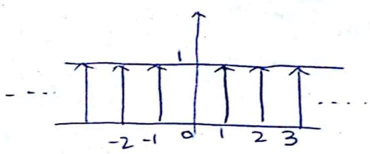


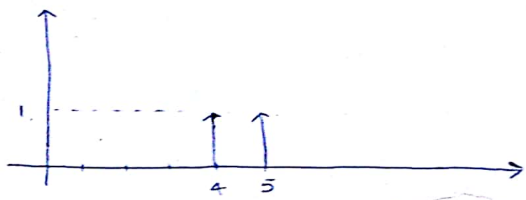


Q: Find M and N_0; such that $x[n] = u[Mn - N_0]$

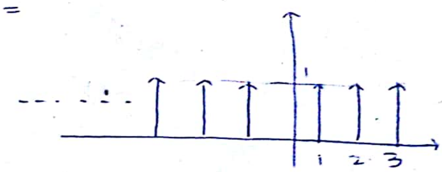
and $x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$ $\quad \delta[n-4] + \delta[n-5] + \cdots$

$x_1[n]$ $\quad$ $x_2[n]$





$x_1 - x_2 =$

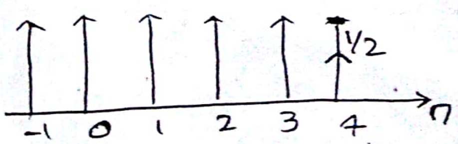


$= u\{$

$x[n] \equiv u[-n+3]$

$M = -1, \quad N_0 = -3$

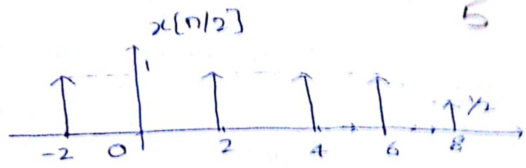
Q:



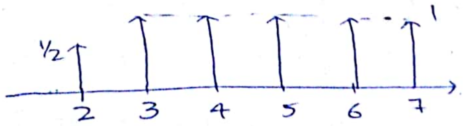
$x[n]$

(i) $x[n/2]$

$y[n] = x[n/2]$ possible $n = 0, 2, 4$, -2

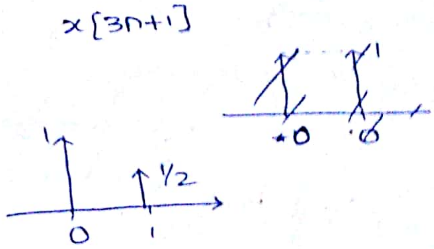


(ii) $x[-n+6]$

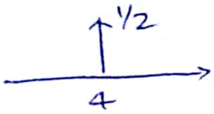


(iii) $x[3n+1]$

~~$y[n] = x[3n]$~~



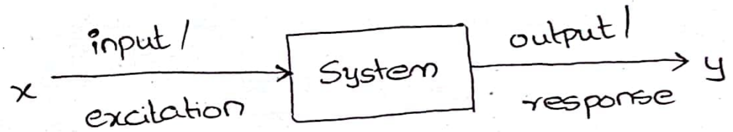
(iv) $x[n]\,\delta[n-4]$



Systems and classification of systems :

→ A system is a mathematical model of physical process that relates the input or excitation signal to the output or response signal.

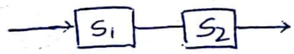
→ If x, y are the input and output signal then the system is viewed as a transformation or mapping of x into y.



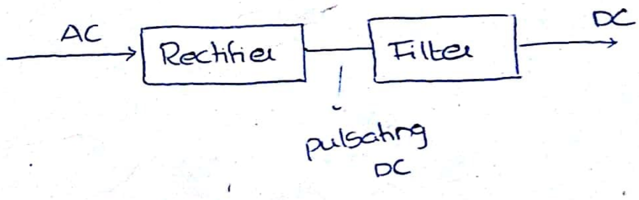
$$y = T\{x\}$$

1. Connection of systems :

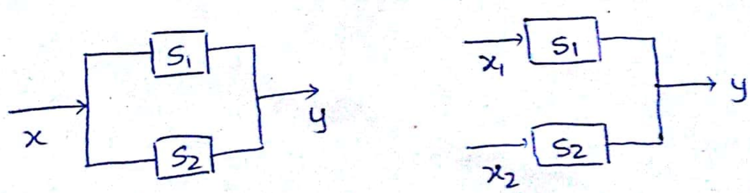
i, Systems in series or Cascade :



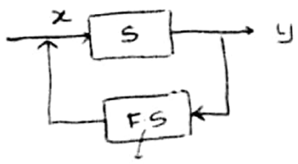
eg.



ii, Parallel :



iii, Feedback:



feedback system

eg: Air conditioner (we can set a particular temp.)

Oscillator (source for electronic signals of diff freq and shape)

Function generator

Classification of system:

1. System with and without memory:

→ A system is said to be memoryless if its output at any time depends only on the input at that same time otherwise the system possesses memory

→ Memoryless systems are also Static system

→ System with memory is dynamic system

eg: $y(t) = (t+1)^2 x(t)$ – static ∵ it becomes a constant at a particular t & hence depends only on x(t)

$y(t) = e^{-3t} x(t)$ – static

$y(t) = \frac{d}{dt} x(t)$ – dynamic ∵ $y(t=2)$ depends on $x'(t=2)$ and not on $x(t=2)$ (x' is diff. from x) at same t

$y[n] = x[n] + x[n-1] + x[n-2]$ – dynamic

eg: $x(t) = \cos t$
$y(t) = -\sin t$
$= \cos(t + \frac{\pi}{2})$
$y(0) = \cos(\frac{\pi}{2})$
diff times

$y[n] = x[2n]$

$y[n] = x[n^2]$ — dynamic

$y[n] = x[-n]$

Resistor — memoryless

Capacitor, inductor — memory

Invertible system:

→ If input of the system can be recovered from the system output.

→ For an invertible system, the output will be distinct for distinct input.

$y(t) = 2x(t)$ — invertible

$y(t) = x^2(t)$ — non-invertible

$y(t) = |x(t)|$ — non-invertible

$y(t) = x(t-3)$ — invertible

* $y[n] = n\,x[n]$ — non-invertible

($\because$ at $n=0$, we donot have a signal).

Causality / Causal system:

↳ (if there is a cause then there is effect)

→ If the output at any time depends on values of input at that time or in the past.

→ Causal systems are also called non-anticipative in the sense that the system output doesnot anticipate future value of input.

→ All memoryless systems are Causal.

Note:

1. A system is said to be non-causal if it is not causal
2. A system is said to be anti-causal if we can make a non-causal system to a causal system, by putting $-t$ or $-n$ instead of t or n

eg: $y[n] = x[n] - x[n-1]$ – Causal

$y[n] = x[-n]$ – non-causal

$y[n] = x[2n]$ – non-causal

$y[n] = x[n^2]$ – non-causal

$y[n] = x(t)\cos(t+1)$ – causal ($\because \cos(t+1)$ becomes constant at a time t)

$y(t) = \sin x(t)$ – causal

$y(t) = x(\sin t)$ – non-causal ($y(\frac{\pi}{2}) = x(1)$ diff t)

Stability:

A system is said to be BIBO stable if finite input leads to finite output.

$|x(t)| \le k_1$ then $|y(t)| \le k_2$

$y[n] = a^n u[n]$ – $a>1$; unstable $\quad 0<a<1$; stable

$y(t) = x^2(t)$ – stable ($\because x(t)$ is bounded, $x^2(t)$ is bounded)

$y(t) = t\,x(t)$ – unstable

$y[n] = 3^n u[-n+10]$ – stable

$\downarrow$

$$\sum_{n=-\infty}^{10} 3^n = \text{finite}$$

Time invariance :

If a time delay or time advance of a input signal leads to an identical time shift in the output signal

The characteristics of time invarient systems donot change with time.

$x(t) \longrightarrow y(t)$

$x(t \pm t_0) \longrightarrow y(t \pm t_0)$

$y(t) = t x(t)$ - time varient

$y(t) = x^2(t)$ - time invarient

$y(t) = x(t) \cos\omega t$ - time varient

$y[n] = n x[n]$ - time varient

$y(t) = x(2t)$ - time varient (if we get $x(2t - t_0)$ then TIV but we get $x(2t - 2t_0)$ so TV)

Linearity :

If system follows principal of superposition.

additivity   homogenity

For a linear system, if $y_1(t)$ is the response due to $x_1(t)$ and $y_2(t)$ is response due to $x_2(t)$ then;

if $a x_1(t) + b x_2(t)$ is input then $a y_1(t) + b y_2(t)$ is output

where a, b are complex constants

$y(t) = t x(t)$ - linear

$y[n] = 2x[n] + \underline{\underline{3}}$ - non-linear.

$y(t) = x(t) \cos t$ - linear

$y(t) = \sin(x(t))$ - non-linear

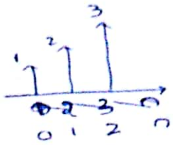
Q1 Express the sequence in terms of scale and shifted version of unit step.

$x[n] = 1 \;, \; n=0$

$\quad = 2 \;; \; n=1$

$\quad = 3 \;; \; n=2$

$\quad = 0 \;; \;$ otherwise



$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

$$x[n] = u[n] - u[n-1] + 2\,[u[n-1] - u[n-2]) + 3\,[u[n-2] - u[n-3]]$$

$$= u[n] + u[n-1] + u[n-2] - 3u[n-3]$$

Q2: A sequence $x[n]$ with normalised energy ϕ has even part $x_e[n] = (\frac{1}{2})^{|n|}$. Find the energy in odd part of signal.

$x[n] = x_e[n] + x_o[n]$

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \sum_{-\infty}^{\infty} |x_e[n]|^2 + \sum_{-\infty}^{\infty} |x_o[n]|^2 + 2\sum_{-\infty}^{\infty} x_o[n]\, x_e[n]$$

0 odd x even = odd signal

$$\therefore \; \phi = \sum_{-\infty}^{\infty} \frac{1}{2^{2|n|}} + x.$$

$$x + 1 + \left(\frac{1}{4} + \frac{1}{4^2} + \cdots\right)2 = 1 + 2\cdot\frac{\frac{1}{4}\left(1 - \frac{1}{4^n}\right)}{1 - \frac{1}{4}} = 1 + \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

$$= \frac{5}{3} - \frac{2}{3\cdot 4^n}$$

Linear time invarient system (LTI system) :-

↳ If system is linear and time invarient.

The characteristic of LTI system is completely described by the impulse response of the system.

The knowledge of impulse response of LTI system allows us to findout the output for anyother input.

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$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$

$\delta(t) \dashrightarrow \boxed{\text{LTI}} \dashrightarrow h(t)$

$h(t)$ — Impulse response of system

$\delta(t) \rightarrow \boxed{\quad} \rightarrow h(t)$

$\delta(t-T) \rightarrow \boxed{\quad} \rightarrow h(t-T)$ (Time invarience)

$x(T)\,\delta(t-T) \rightarrow \boxed{\quad} \rightarrow x(T)\,\delta(t-T)$ (Homogenity)

$\underbrace{\int_{-\infty}^{t} x(T)\,\delta(t-T)\,dT}_{= x(t)} \rightarrow \boxed{\quad} \rightarrow \int_{-\infty}^{t} x(T)\,h(t-T)\,dT = x(t) * h(t)$: convolution

(additivity)

Impulse signal – bad signal ($\because$ momentary)

So we use impulse signals to test the system.

(worst condition is impulse)

Any signal can be represented as an impulse signal.

Step response of LTI system :

$x(t) = u(t)$

$$y(t) = s(t) = \int_{-\infty}^{t} x(T)\,h(t-T)\,dT = \int_{-\infty}^{t} u(T)\,h(t-T)\,dT$$

defined only for $T \geq 0$

$$= \int_{0}^{t} u(T)\,h(t-T)\,dT$$

only at $T=0$

$$= \int_{0}^{t} h(t-T)\,dT$$

$$= \int_{-\infty}^{t} h(T)\,u(t-T)\,dT$$

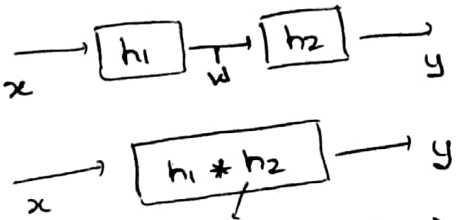
only at $t=T$

$$= \int_{-\infty}^{t} h(T)\,dT$$

Properties of convolution

1. $x(t) * h(t) = h(t) * x(t)$ - Commulative

2. Associative -



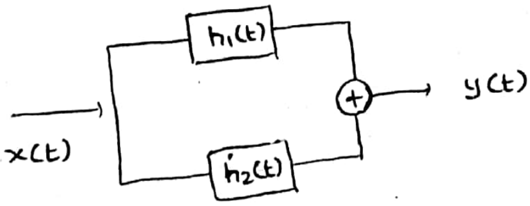
$h(t) = h_1(t) * h_2(t)$

$w(t) = x(t) * h_1(t)$

$y(t) = w(t) * h_2(t)$

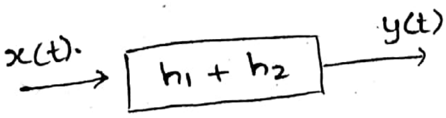
$y(t) = x(t) * (h_1(t) * h_2(t))$

3. Distributive -



$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$

$y(t) = x(t) * \{h_1(t) + h_2(t)\}$

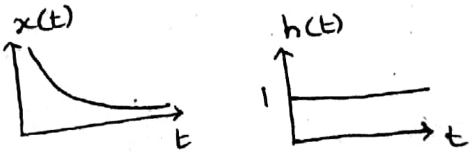


** Q: Find the response of LTI system if,

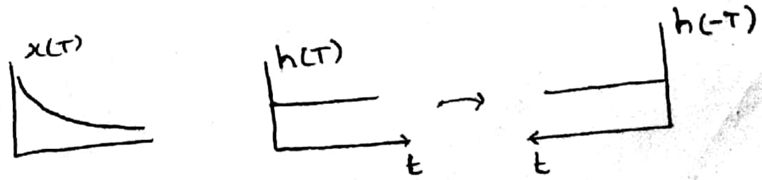
$x(t) = e^{-at} u(t)$

$h(t) = u(t)$

i, Signal in t



ii, Signal in τ



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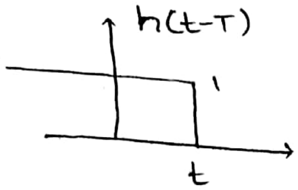
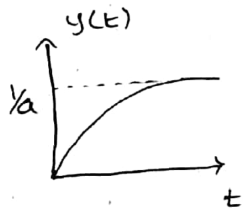
Flip any signal (reflection):



(iv) Roll 2^{nd} signal over first.

$$y(t) = \int_0^t e^{-at}(1)\,dt = \frac{1-e^{-at}}{a}$$

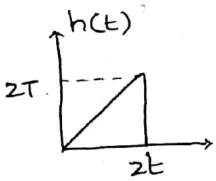
output



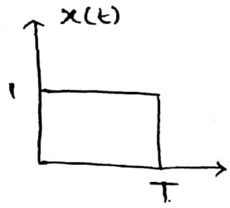
Q: $x(t) = 1, \quad 0 \le t \le T$

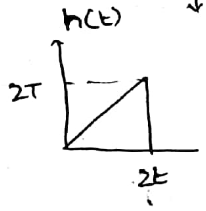
$= 0, \quad$ otherwise

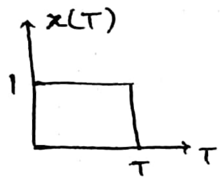
$h(t) =$

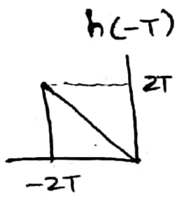


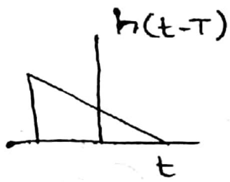
$y = \left(\frac{T}{t}\right)t$











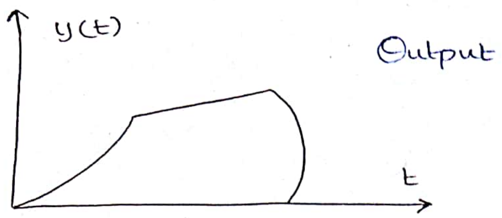
$$y(t) = \int_{-\infty}^{t} x(T)\,h(t-T)\,dt$$

for $t<0$; $\quad y(t) = 0$

for $0<t<T$; $\quad y(t) = \int_0^t 1\cdot(t-T)\,dT = \frac{t^2}{2}$

for $T<t<2T$; $\quad y(t) = \int_0^T (t-T)\,dT = Tt - \frac{T^2}{2}$

for $2T<t<3T$; $\quad y(t) = \int_{t-2T}^{T} (t-T)\,dT = -\frac{t^2}{2} - Tt + \frac{3}{2}T^2$



Properties of continuous time LTI system:

1. Memory less and memory based system:

If a memoryless system is linear and time invarient the output will be in the form of system responce

$y(t) = k\,x(t)$.

$\downarrow$

$h(t) = k\,\delta(t) \longrightarrow$ $h(t) = 0 \; ; \; t \neq 0$

$h(t) \neq 0 \; ; \; t = 0$

2. Causality:

For a LTI system to be causal, that means the system will respond only after the application of input.

$h(t) = 0 \; ; \; t < 0$

$h(t) \neq 0 \; ; \; t \geq 0$

For an arbitrary input, $y(t) = \int_{-\infty}^{t} h(\tau)\,x(t-\tau)\,d\tau$

$= \int_{-\infty}^{t} x(\tau)\,h(t-\tau)\,d\tau$

Limits for causality are 0-t

3. Stability:

A System is said to be BIBO stable if the output is bounded for bounded input.

$x(t) \le k_1 \quad y(t) \le k_2$

$$y(t) = \int_{-\infty}^{t} |h(\tau)||x(t-\tau)|\,d\tau$$

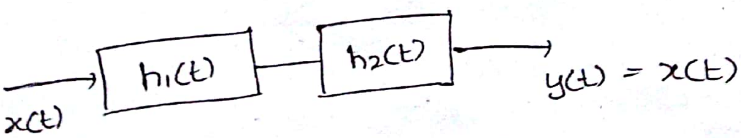
$$k_2 \le \int_{-\infty}^{t} h(\tau) \cdot k_1 \cdot d\tau$$

$$\int_{-\infty}^{t} h(\tau)\,d\tau < \infty$$

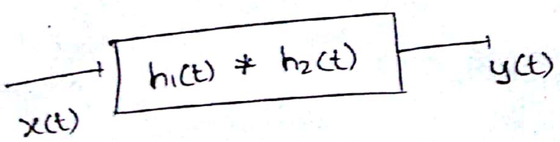
For a LTI system to be stable, its impulse response must be absolutely integrable.

4. Invertibility:

A LTI system is said to be invertible if a system exist such that when connected in series to the original system, the output is equal to the input of the first system.



$\Rightarrow$



For $y(t) = x(t)$; $h_1(t) * h_2(t) = \delta(t)$.

Q: If impulse response of a system is $\delta(t-t_0)$. Find the " " the corresponding inverse system.

$h_1(t) = \delta(t-t_0)$ $\qquad$ $h_1(t) * h_2(t) = \delta(t)$

$h_2(t) = ?$

$h_2(t) = \delta(t+t_0)$

* $x(t) * h(t) = y(t)$

$x(t-t_0) * h(t) = y(t-t_0)$

$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$

Q: If $x(t) = u(t)$ and $h(t) = u(t)$ find $y(t)$.

$x(\tau), h(\tau)$

$y(t) = x(t) * h(t)$ $\qquad$ reflection of $x(\tau) = u(-\tau)$

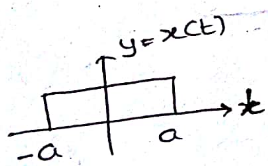
$= u(t) * u(t)$

$y(t) = \int \cancel{x(t)u(t-\tau)d\tau}$ $\qquad$ $\int_{-t}^{t} u(\tau)u(t-\tau)d\tau = \int_0^t u(t)\,d\tau$

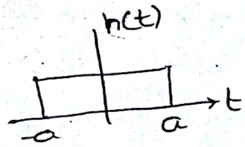
$= tu(t) = r(t)$

ramp.

Q: Convolve



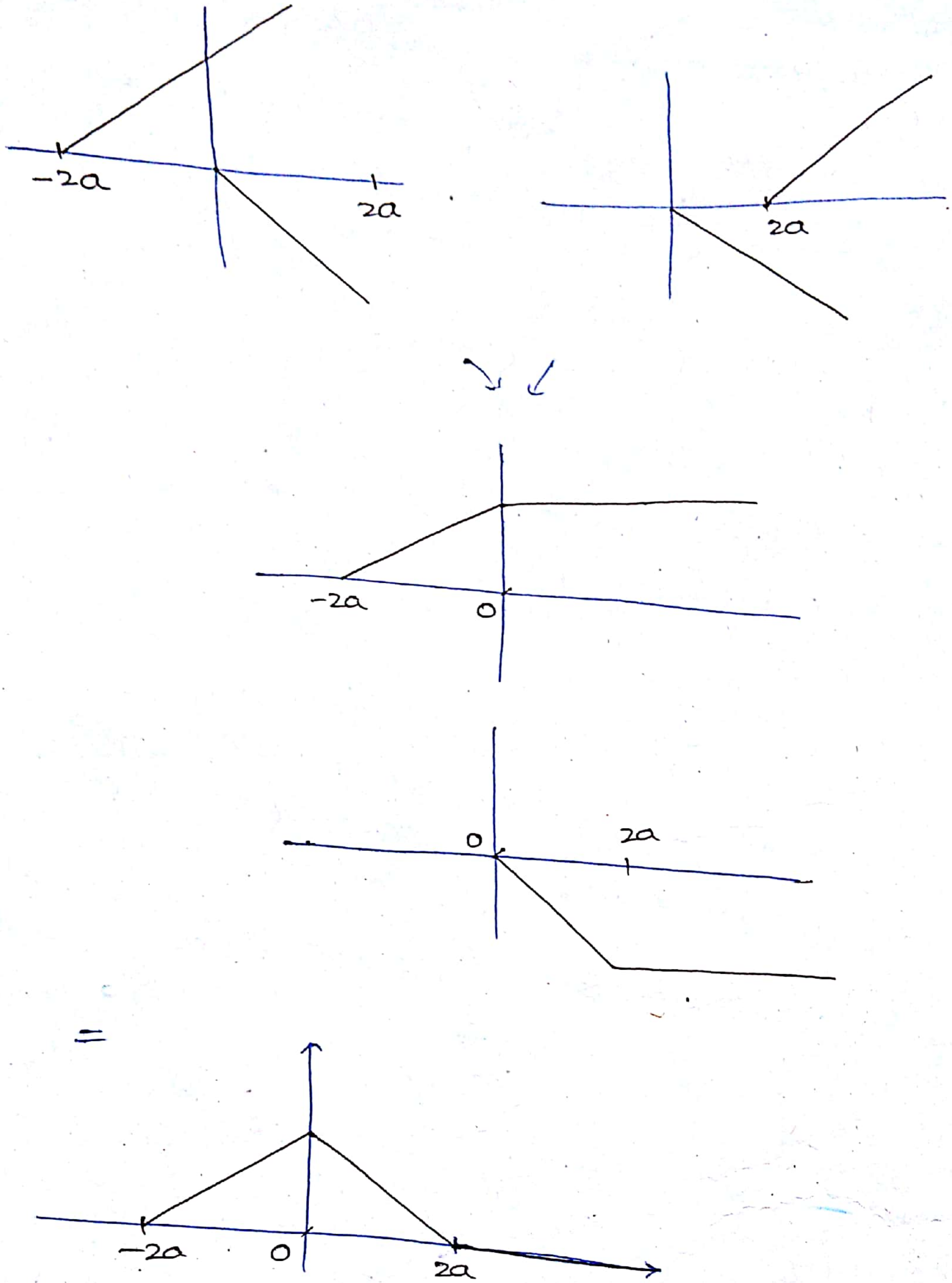
*



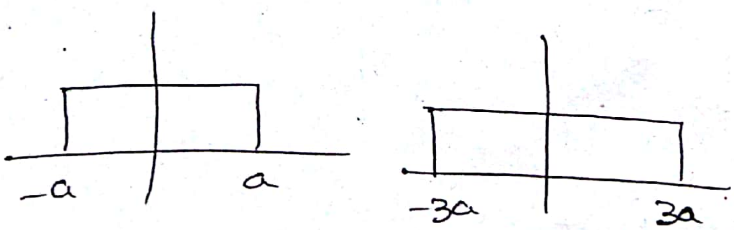
* $u(t+\alpha) * u(t+\beta) = r(t+\alpha+\beta)$

$[u(t+a) - u(t-a)] * [u(t+a) - u(t-a)]$

$= r(t+2a) - r(t) - r(t) + r(t-2a)$

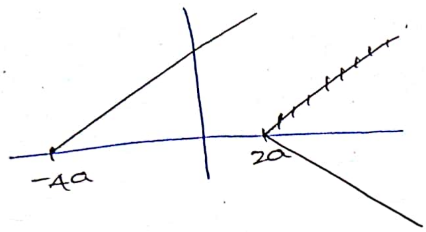


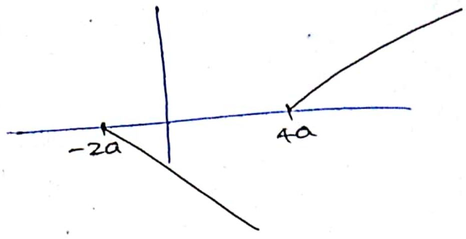
Q:

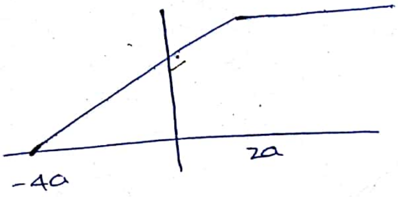


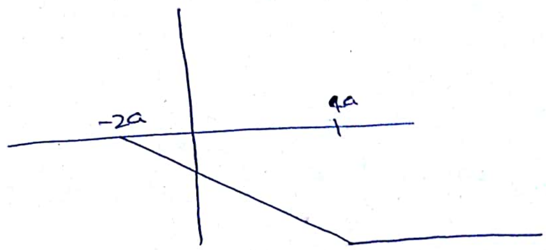
$$[u(t+a) - u(t-a)] * [u(t+3a) - u(t-3a)]$$

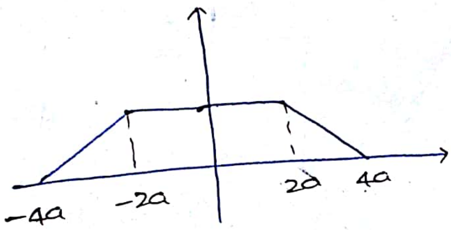
$$r(t+4a) - r(t-2a) - r[t+2a) + r(t-4a)$$







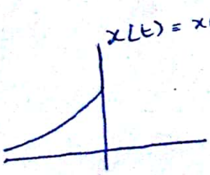


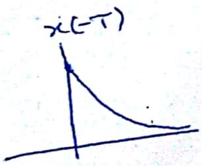


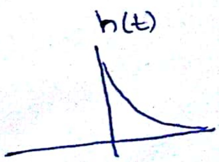
Important relations :-

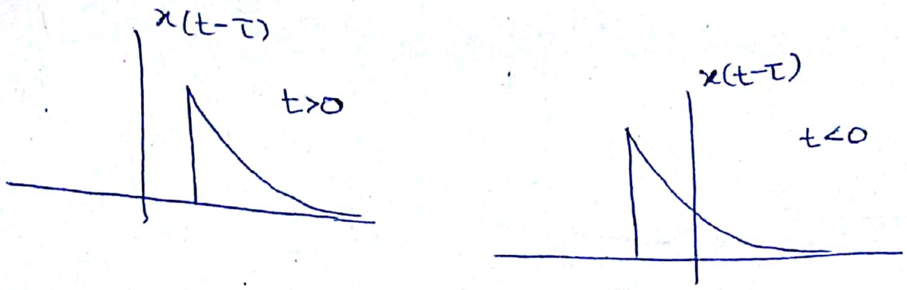
1. $x(t) * h(t) = y(t)$ $\qquad u(t) * u(t) = r(t)$
2. $x(t) * h'(t) = y'(t)$ $\qquad u(t) * \delta(t) = u(t)$
3. $x'(t) * h(t) = y'(t)$ $\qquad \delta(t) * u(t) = u(t)$
4. $x'(t) * h'(t) = y''(t)$ $\qquad \delta(t) * \delta(t) = \delta(t)$

Q: $x(t) = e^{\alpha t} u(-t)$ $\quad h(t) = e^{-\alpha t} u(t)$ Find $y(t)$.





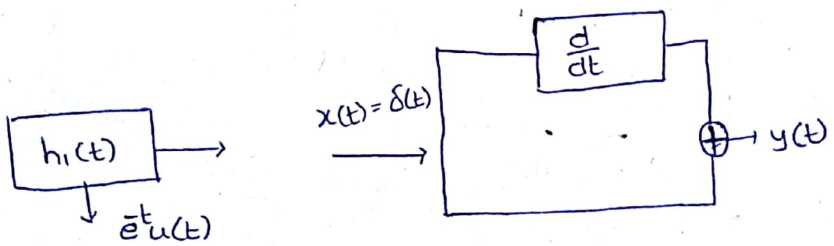




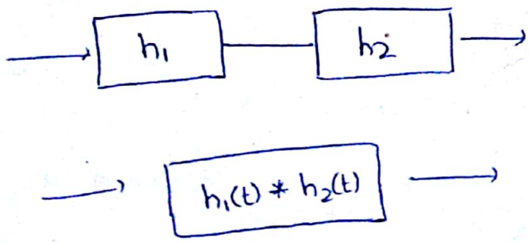
$$y(t) = \int_0^\infty e^{-\alpha\tau} e^{\alpha(t-\tau)}\, d\tau \neq \int_0^\infty \cancel{e^{-2\alpha t}}\, d\tau \neq \frac{\cancel{e^{-2\alpha t}}}{\cancel{-2\alpha}}\Bigg|_0^\infty = \frac{e^{\alpha t}}{2\alpha}$$

$$y(t) = \int_t^\infty e^{-\alpha\tau} e^{\alpha(t-\tau)}\, d\tau = \frac{e^{-\alpha t}}{2\alpha}$$

Q1:



Find the overall impulse response of the system if these 2 systems are connected in cascade.



$$h_2(t) = x(t) + \frac{d}{dt}(x(t))$$
$$= \delta(t) + \delta'(t).$$

$$h(t) = h_1(t) * [\delta(t) + \delta'(t)]$$
$$= h_1(t) + h_1'(t)$$
$$= e^{-t}u(t) + [-e^{-t}u(t) + e^{-t}\delta(t)]$$
$$= e^{-t}\delta(t)$$
$$= \delta(t) \quad [\because \delta(t) \text{ is valid only at } t=0;\ e^{0} = 1]$$

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Discrete Time LTI System :

$\xrightarrow{\delta[n-k]}$ $\qquad$ $\xrightarrow{h[n-k]}$

$\xrightarrow{x[k]\delta[n-k]}$ $\qquad$ $\xrightarrow{x[k]\, h[n-k]}$

$\xrightarrow{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$ $\qquad$ $\xrightarrow{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}$

$= x[n]$ $\qquad$ $= y[n]$

$$\therefore \quad y[n] = \sum x[k]h[n-k] = x[n] * h[n]$$

NOTE :-

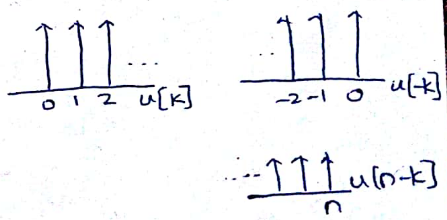
① Step response of an LTI discrete time signal :

$x[n] = u[n]$ $\qquad$ $y[n] = s[n]$ ↳ unit step

$$\sum_{-\infty}^{\infty} h[k]\, x[n-k] = \sum_{-\infty}^{\infty} h[k]\, u[n-k]$$

$$= \sum_{-\infty}^{n} h[k]$$

$$= s[n]$$



* $u[n] - u[n-1] = \delta[n]$

$s[n] - s[n-1] = h[n]$

② It follows all properties of convolution same as that of continuous time system.

Properties of discrete time LTI system:

1. Memoryless:

$$y[n] = k\,x[n]$$

$$\downarrow$$

$$h[n] = k\,\delta[n]$$

$$h[n] = 0 \quad ; \quad n \neq 0$$

$$\neq 0 \quad ; \quad n = 0$$

2. Causality:

$$h[n] = 0 \quad ; \quad n < 0$$

$$y[n] = \sum_{k=0}^{\infty} x[n-k]\,h[k]$$

$\because$ h(k) is valid only for $n \geq 0$.

3 Stability:

$$\sum_{k=-\infty}^{n} x[n]\,h[n-k] = y[n]$$

BIBO stable:

$$y(b) = \sum x[n-k]\,h[k]$$

$$= \sum_{-\infty}^{\infty} |h[k]|\,|x[n-k]| \quad : \quad \sum_{-\infty}^{\infty} |h[k]| < \infty$$

The impulse response should be absolutely summable for system to be BIBO stable.

4. Invertibility :

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

if

$$x[n] \rightarrow \boxed{h[n]} \rightarrow \boxed{h_1[n]} \rightarrow y[n] = x[n]$$

$$x[n] \rightarrow \boxed{h[n] * h_1[n]} \rightarrow y[n] = x[n]$$

possible only when $h[n] * h_1[n] = \delta[n]$

Q: If $h(n) = \alpha^n u[n]$ Check causality and stability of system.

$h[n] = \alpha^n u[n]$

Causal:-

$h[n] = 0$ for $n<0$

$u[n] = 0$, $n<0$

$\therefore h[n] = 0$

Stability $\longrightarrow$ $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

$= \sum_{0}^{\infty} \alpha^n < \infty$ for $0<\alpha<1$

$\therefore$ Causal and, stable for $0<\alpha<1$

Properties of convolution:

1. $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$= \sum h[k]\, x[n-k]$

Convolution follows commutative, associative and distributive properties.

Note:

Graphically the convolution of two finite length sequences of length l_1 and l_2 , their convolution will be of length $(l_1 + l_2 - 1)$

eg:- Finite length seq. $x[n] = \{1,2,3,4\}$

and not $u[n]$

* 2. If $x_1[n]$ contains non-zero value in the interval $[M_1, N_1]$

and $x_2[n]$ " " " " " $[M_2, N_2]$

then the convolution of $x_1[n]$, $x_2[n]$ contains non-zero

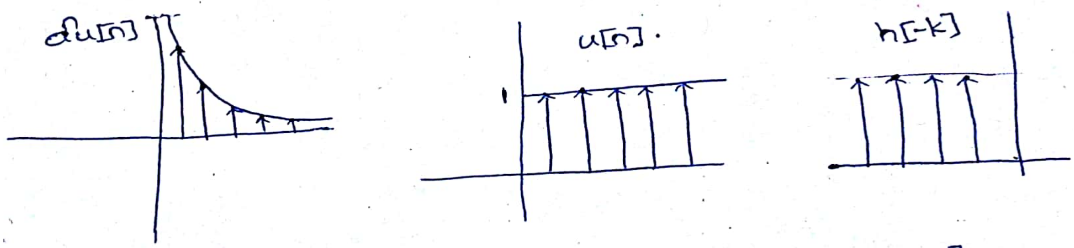
value in interval $[M_1+M_2 \quad N_1+N_2]$

Q: $x[n] = a^n u[n]$ and $h[n] = u[n]$; $0<a<1$ Find

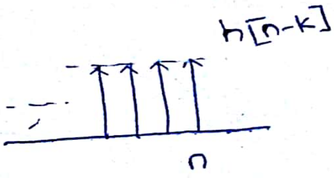
convolution.

$$x[n] * h[n] = a^n u[n] * u[n]$$

$$= \sum a^n u[n] u[n-k]$$



if $n<0$, no multiplication.



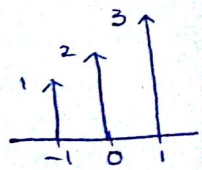
so, $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$

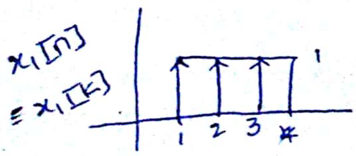
Q: $x_1[n] = u[n-1] - u[n-4]$ $\quad x_2[n] = 1$; $n=-1$

2 ; $n=0$

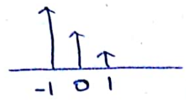
3 ; $n=1$

0 ; otherwise

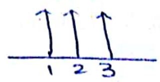




$x_2[n-k]$ — $n>0$, $n<0$

$x_2[-k] =$ 

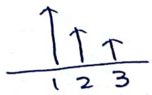
$x_1[k]$



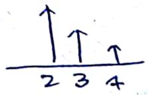
i) $n=0$; $y[0] = 1$

ii) $n=1$; $y[1] = \cancel{0}\ 2+1$
$= 3$

$n=2$; $y[2] = 3+2+1$
$= 6$



$n=3$; $y[3] = 3+2$
$= 5$

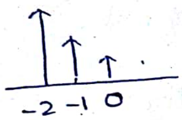


$n=4$; $y[4] = 3$

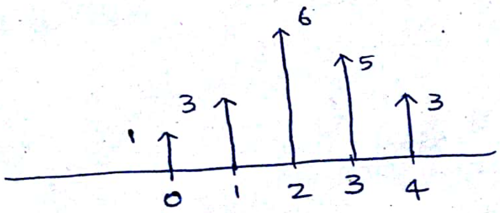
$n=5$; $y[5] = 0$

$\therefore$ $n \geq 5$ $y[n] = 0$

iii) $n=-1$; $y[-1] = 0$



$\therefore$ $n<0$; $y[n]=0$

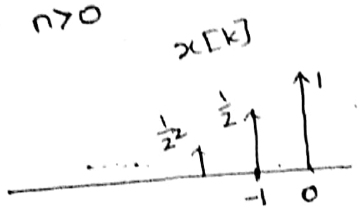
$y[n] =$ 

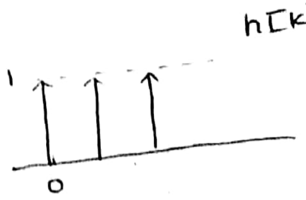
① $l_1 = 3$, $l_2 = 3$; $y[n] \equiv l = 6-1 = 5$

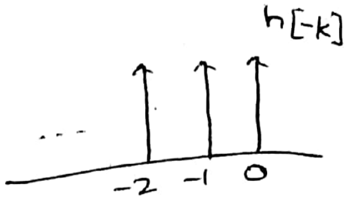
② $[M_1, N_1] = [1,3]$, $[M_2, N_2] = [-1,1]$; $y[n] \equiv [0,4]$

Q: $x[n] = 2^n u[-n]$ $\quad h[n] = u[n]$

i, $n > 0$







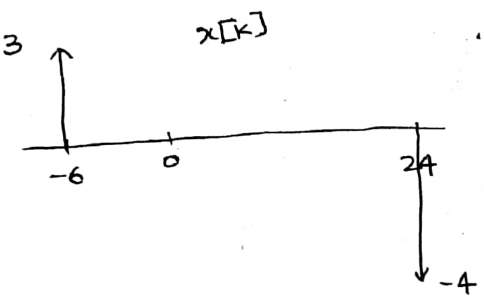
$n = 0$; $\quad y[0] = 1$

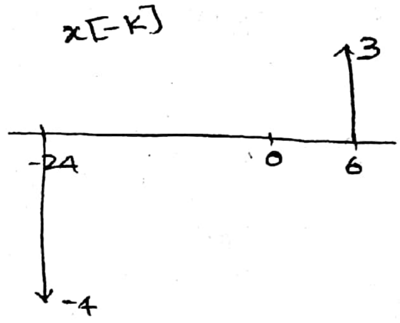
$n=1$; $\quad n > 0$; $\quad \sum_{-\infty}^{0} 2^n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \underset{n \to \infty}{2 - 2^{-n}} = 2.$

ii,

$n < 0$; $\quad \sum_{-\infty}^{n} 2^n = \sum 2^n \quad \overset{(n=-k)}{\frac{1}{2^k} + \frac{1}{2^{k-1}} + \dots} = 2 - \frac{1 - \frac{1}{2^k}}{1 - \frac{1}{2}} = 2 - (2 - 2^{-k+1}) = 2^{-k+1} = 2^{n+1}$

Q: First non-zero value of a finite length sequence occurs at $n = -6$. $x[-6] = 3$ and last non-zero value occurs at $n = 24$. $x[24] = -4$. What is index of first and last non-zero values, in the convolution $y[n] = x[n] * x[n]$ and what are its values.





For multiplication;

$3 \times 3 = y[-12] = 3 \times 3 = 9$ $\quad$ (x[-12-k])

$-4 \times -4 = y[48] = 16.$

$-12, 48$ — index

$9, 16 \rightarrow$ values.

Tabulation method of finding convolution:

This method is used only for finite length sequences.

For eg: $x_1[n] = \{0, 1, 1, 1\}$ — N_1 (↑ at 0)

$x_2[n] = \{1, 2, 3\}$ — N_2 (↑ at 2)

$x_2[n]$ \ $x_1[n]$	0	1	1	1
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

add

$y[n] = \{0, 1, 3, 6, 5, 3\}$ (↑ at 3)

Overall product terms $= N_1 N_2$

" sum terms $= (N_1 - 1)(N_2 - 1)$

Q: $x_1[n] = 0.5n\,[u[n] - u[n-6]]$

$x_2[n] = 2\sin\frac{n\pi}{2}\,(u[n+3] - u[n-4])$

$x_1 \rightarrow n = 0, 1, 2, 3, 4, 5$ ↑↑↑↑↑↑

$x_1[n] = 0, 0.5, 1, 1.5, 2, 2.5$

$x_2 \rightarrow n = -3, -2, -1, 0, 1, 2, 3$

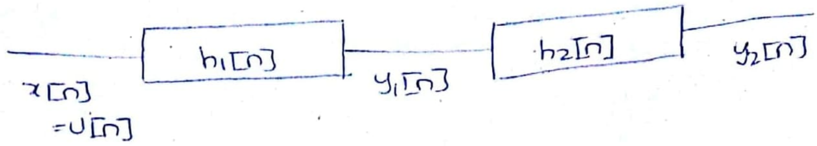
$x_2[n] = +2, 0, -2, 0, 2, 0, -2$ (↑ at 0)

$x_2[n]$ \ $x_1[n]$	0	0.5	1	1.5	2	2.5
2	0	1	2	3	4	5
0	0	0	0	0	0	0
-2	0	-1	-2	-3	-4	-5
0	0	0	0	0	0	0
2	0	1	2	3	4	5
0	0	0	0	0	0	0
-2	0	-1	-2	-3	-4	-5

$y[n] = \{0, 1, 2, 2, 2, 3, -2, -3, 2, 2, -4, -5\}$ (↑ at 2)

Total no of sum terms $= (N_1 - 1)(N_2 - 1)$

Q: Compute the values of $y_1[4]$ and $y_2[4]$ for :

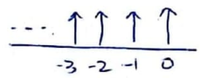


$h_1[n] = \left(1 - \frac{1}{2^n}\right)u[n]$ $\qquad h_2[n] = u[n-1] - u[n-3]$

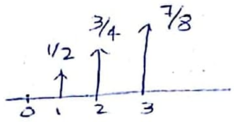
A $\quad y_1[n] = h_1[n] * x[n] = \left(1 - \left(\frac{1}{2}\right)^n\right)u[n] * u[n]$

↓

$x[-n] =$

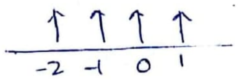


$h_1[n] =$



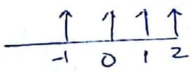
$x[-n+k]$

$k = 1$



$y_1[n] = \frac{1}{2}$

$k = 2$



$y_1[n] = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$

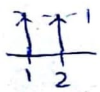
$k=3 \qquad \frac{1}{2} + \frac{3}{4} + \frac{7}{8} = \frac{17}{8} \qquad\qquad k=4$

$$\sum_{k=0}^{1}\left(1 - \left(\tfrac{1}{2}\right)^k\right) = (1+n) - \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right) \qquad (\text{at } n=0,\ \Sigma = 0)$$

$$= (n+1) - \frac{1\left(1 - \frac{1}{2^{n+1}}\right)}{\left(1 - \frac{1}{2}\right)} = n - 1 + \frac{1}{2^n}.$$

$$y_1[4] = 4 - 1 + \left(\tfrac{1}{2}\right)^4 = \frac{49}{16}$$

$y_2[n] = y_1[n] * h_2[n] \qquad h_2[n] =$



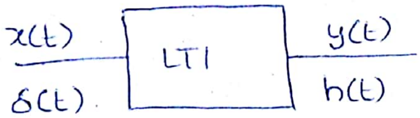
$= \delta[n-1] + \delta[n-2]$

$$= y_1[n] * (\delta[n-1] + \delta[n-2]) = y_1[n-1] + y_1[n-2]$$

$$y_2[4] = y_1[3] + y_1[2] = 3 - 1 + \frac{1}{2^3} + 2 - 1 + \frac{1}{2^2} = 3 + \frac{1}{8} + \frac{1}{4} = \frac{27}{8} = 3.375.$$

Fourier Representation of signals :-

Sinusoidal steady state response of LTI system :-



$e^{j\omega t}$
↓
complex sinusoid

response corresponding to $e^{j\omega t}$ → sinusoidal steady state response

$$y(t) = \int_{-\infty}^{\infty} h(t)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

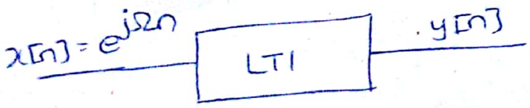
we can recover input from output

freq of output signal = input signal

The steady state response of LTI system will have same freq as input signal. Since $H(j\omega)$ is a function of freq, therefore it is called the frequency response of system.

$$y(t) = |H(j\omega)|\, e^{i(\omega t + \arg(H(j\omega)))}$$

Steady state analysis for discrete system :-



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\Omega(n-k)}$$

$$y[n] = e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}$$

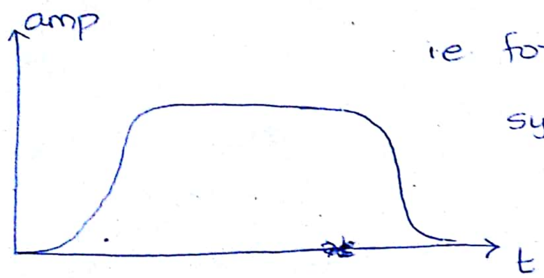
$$= e^{j\Omega n} H(e^{j\Omega}) \qquad H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}$$

The output of the system is a complex sinusoid of the same freq as input, multiplied by a complex no., where $H(e^{j\Omega})$ is a function of frequency Ω. Therefore, this is also known as the frequency response of discrete time system

↳ ∵ it is fnc of freq.

→ Input signal is preserved in output signal.

eg:- Amplifier – we need freq response



ie for diff frequency behaviour of system is different

[All devices cannot be used at every frequency].

Frequency response of system:-

If any device gives good response at a particular freq range, then that is known as freq. response of system.

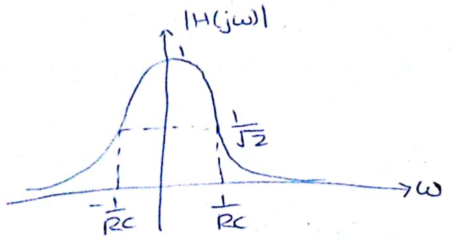
<u>Q</u>: The impulse response of system $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$ Find frequency response of system.

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = \int_{0}^{\infty} \frac{1}{RC} e^{-\tau/RC} e^{-j\omega\tau} d\tau = \frac{1}{RC} \int_{0}^{\infty} e^{-(\frac{1}{RC}+j\omega)\tau} d\tau.$$

$$= \frac{1}{RC} \cdot \frac{e^{-(\ )\tau}\Big|_0^{\infty}}{(\ \ )} = \frac{1}{j\omega RC+1}$$

magnitude response: $|H(j\omega)| = \left| \frac{1-j\omega RC}{1+\omega^2R^2C^2} \right| = \frac{1}{(1+\omega^2R^2C^2)} \sqrt{1+\omega^2R^2C^2}$

This system is made of R, C $= \frac{1}{\sqrt{1+\omega^2R^2C^2}}$



* Bandwidth

- range of frequency over which response of system is very good.

This is bandwidth frequency filter. — Bandpass filter.

Phase response → $\frac{\cancel{\text{zero}}\ \text{phase}}{\tan^{-1}(\omega RC)}$ $= \tan^{-1}(\omega RC)$

$= -\tan^{-1}\left(\frac{1}{\omega RC}\right)$

$= 0\ \text{phase} - \tan^{-1}(\omega RC)$

Q: Impulse response of discrete time system is

$h[n] = \frac{1}{2}[\delta[n] + \delta[n-1]]$ Find freq. response of system and plot the magnitude response.

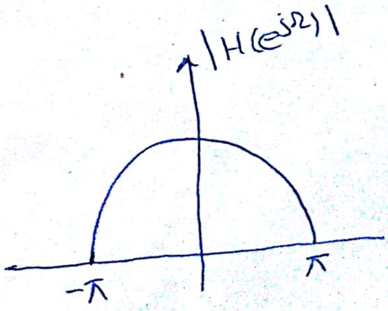
A: $H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \frac{1}{2}[\delta[n] + \delta[n-1]] e^{-j\Omega k}$

$= \frac{1}{2}\left[\sum(\delta[n]e^{-j\Omega k} + \delta[n-1]e^{-j\Omega k})\right] = \frac{1}{2}(1+e^{-j\Omega})$

$= \frac{1}{2}(1+\cos\Omega - j\sin\Omega) = \frac{1}{2}(2\sin^2\frac{\Omega}{2} - j\cdot 2\sin\frac{\Omega}{2}\cos\frac{\Omega}{2})$

$= \cos\frac{\Omega}{2} e^{j\Omega/2}$

$|H(e^{j\Omega})| = \left|\cos\frac{\Omega}{2} e^{j\Omega/2}\right| = \cos\frac{\Omega}{2}$



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Fourier find most of the signals in nature as periodic. Hence whatever system we make must be tested against periodic signals. But all periodic signals need not be sinusoid. Fourier series is the method of conversion of periodic signals to sinusoidal periodic.

i.e. Periodic signals = Superposition of many complex sinusoid.

Note :-

i, The study of signal and system using sinusoidal representation is termed as fourier analysis. We can represent a signal as a weighted ~~representat~~ superposition of complex sinusoids and if such a signal is applied to a linear system, then the output will be weighted superposition of the system response to each complex sinusoids.

$$y[n] = e^{j\Omega n} \cdot \underbrace{H(e^{j\Omega})}_{\text{weighted}}$$

The weight describes the signal as function of frequency.

eg:- music signal - combination of diff. musical instruments playing at diff. frequency.

(ii) Weight associated with sinusoids of given frequency represents the contribution of that sinusoid to overall signal

2. There are 4 fourier representation of different class of ~~systems~~ signals.

i. Continuous time Fourier series } For periodic signals.
ii. Discrete time Fourier series
iii. Continuous time Fourier transform } Aperiodic signals.
iv. Discrete time Fourier transform

Continuous time Fourier series:

→ Representing a periodic signal as a weighted superposition of complex sinusoids.

→ Since the weighted superposition must have the same period as the signal, therefore each sinusoid in the superposition must have same time period as the signal.

→ This implies that the frequency of each sinusoids must be an integer multiple of the signals fundamental frequency.

$$\hat{x}(t) = \sum_{k} A(k) e^{jk\omega_0 t}$$

$A(k)$: weight applied to k^{th} complex sinusoid.

Since complex sinusoids $e^{jk\omega_0 t}$ with distinct freq. $k\omega_0$ are always distinct.

Hence there are potentially an infinite number of distinct term in the series and are approximated by,

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} A(k)\, e^{jk\omega_0 t}$$

Discrete time Fourier series :-

$$e^{jk\omega_0(n+N)} = e^{jk\omega_0 n} \quad \left(\because N = \frac{2\pi}{\omega_0}\right)$$

∴ There are only N distinct complex sinusoids of the form $e^{jk\omega_0 n}$ is capable of representing a discrete time periodic signal.

$$\therefore \quad \hat{x}[n] = \sum_{k\langle N\rangle} A(k)\, e^{jk\omega_0 n}$$

$k\langle N \rangle \rightarrow$ index of k

when k is ranging over any N successive values.

eg:- $k = 0 - (N-1)$

$\left(-\frac{N}{2}\right) - \left(\frac{N}{2}-1\right)$

Continuous time Fourier series :

1. Exponential Fourier series -

$$x(t) = \sum_{k=-\infty}^{\infty} a(k)\, e^{jk\omega_0 t} \quad \text{where} \quad a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$a(k) \rightarrow$ weight

2. Trigonometric or Quadratic form of Fourier series:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t$$

where, $a_0 = \frac{1}{T}\int_0^T x(t)\,dt$ $\quad a_k = \frac{2}{T}\int_0^T x(t)\cos k\omega_0 t\,dt$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin k\omega_0 t\,dt$$

3. Polar form of fourier series :-

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t - \phi_k)$$

where, $C_0 = a_0$, $\quad C_k^2 = a_k^2 + b_k^2$

$$\phi_k = \tan^{-1}\frac{b_k}{a_k}$$

Q. Find the complex exponential fourier series for,

$x(t) = \cos\omega_0 t$

$x(t) = \sin\omega_0 t$

$x(t) = \cos(2t + \frac{\pi}{4})$

$x(t) = \cos 4t + \sin 6t$

$$x(t) = \sum_{k=-\infty}^{\infty} a(k)\, e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$

i) $x(t) = \cos\omega_0 t$ — periodic. $= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$a_k = \frac{1}{T}\int_0^T \cos\omega_0 t \, e^{-jk\omega_0 t}\, dt = \sum a_k e^{jk\omega_0 t} \updownarrow$$

$$\cancel{\frac{1}{T}\int_0^T}$$

$a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}$

ii) $x(t) = \sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$

$$\updownarrow$$

$\sum a_k e^{jk\omega_0 t}$. $\quad a_1 = \frac{1}{2j} \quad a_{-1} = -\frac{1}{2j}$

iii) $x(t) = \cos(2t + \frac{\pi}{4})$

$$= \frac{1}{\sqrt{2}}\cos 2t - \frac{1}{\sqrt{2}}\sin 2t.$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(e^{j2t} + e^{-j2t}) - \frac{1}{2j}(e^{j2t} - e^{-j2t})\right]$$

$$\updownarrow$$

$$\sum a_k e^{jk\omega_0 t} = \sum a_k e^{j2\omega_0 kt}$$

$$a_1 = \frac{1}{2\sqrt{2}}(1+j)$$

$$a_{-1} = \frac{1}{2\sqrt{2}}(1-j)$$

(iv) $\cos 4t + \sin 6t$

$\frac{\pi}{2} \qquad \frac{\pi}{3}$. $\quad T = \text{LCM}\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = \pi$

$$\omega_0 = \frac{2\pi}{T} = 2.$$

$$x(t) = \left[\frac{e^{j4t} + e^{-j4t}}{2} + \frac{e^{j6t} - e^{-j6t}}{2j}\right]$$

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2

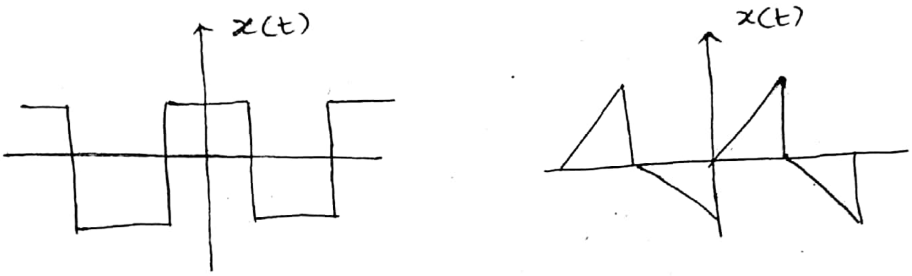
$$x(t) = \sum a_k e^{jk\cdot 2t}$$

$$a_2 = \frac{1}{2} \qquad a_{-2} = \frac{1}{2} \qquad a_3 = \frac{1}{2j} \qquad a_{-3} = \frac{-1}{2j}$$

Half wave symmetry or Odd half wave symmetry :-

If $x(t) = -x(t+\frac{T}{2})$ for a periodic signal

eg:



Even and odd symmetry :

Even: $x(t) = x(-t)$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} x(t)\cos k\omega_0 t\,dt$$

$$b_k = 0$$

Odd: $x(t) = -x(-t)$

$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{T}\int_0^{T/2} x(t)\sin k\omega_0 t\,dt$$

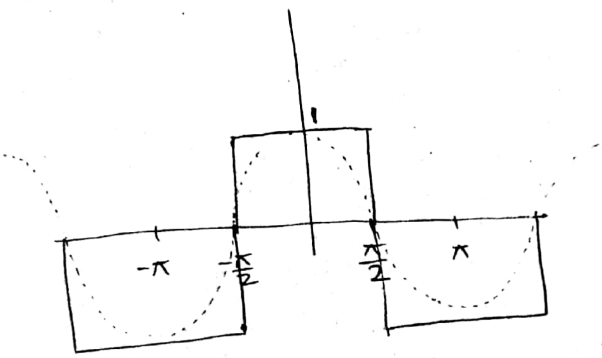
Fourier series coefficient	Even	Odd	Half wave symmetry
a_0	MBNZ	0	0
a_k $\{k = 1, 3, 5, 7, \dots\}$	MBNZ	0	MBNZ
a_k $\{k = 2, 4, 6, 8, \dots\}$	MBNZ	0	0
b_k $\{k = 1, 3, 5, 7, \dots\}$	0	MBNZ	MBNZ
b_k $\{k = 2, 4, 6, 8, \dots\}$	0	MBNZ	0

→ In half wave symmetry even harmonics donot exist whereas only odd harmonics exist.

Q: $x(t) = \text{Sgn}(\cos t)$ The fourier series coefficients of the given signal possess what kind of symmetry?

$\text{sign}(t) = 1, \ t > 0$
$\qquad\quad = -1, \ t < 0$

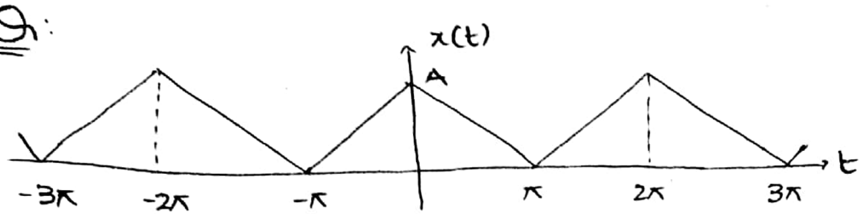
$x(t) =$



Even and Half wave.

→ $a_k \to k = 1, 3, 5 \dots$
$b_k = 0$
$a_0 = \text{MBNO}/0$

Q:



Find F.S.C.

Even symmetric signal so $b_k = 0$

$(0, A)$ $(\pi, 0)$

$x(t) \Rightarrow$ $\frac{y-A}{x} = \frac{-A}{\pi}$

$y = A - \frac{Ax}{\pi}$

$= A\left(1 - \frac{x}{\pi}\right)$

$T = 2\pi$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$= \frac{2}{T}\int_0^{\pi} A\left(1 - \frac{t}{\pi}\right)dt$$

$$= \frac{2}{T}\left(A\pi - \frac{A}{2\pi}\cdot\pi^2\right)$$

$$= \frac{2}{2\pi}\cdot A\left(\pi - \frac{\pi}{2}\right) = \frac{A}{2}$$

$\omega_0 = \frac{2\pi}{T} = 1$

$$a_k = \frac{4}{T}\int_0^{T/2} x(t)\cos k\omega_0 t\,dt$$

$$= \frac{4}{T}\int_0^{\pi} A\left(1 - \frac{t}{\pi}\right)\cos k\omega_0 t\,dt$$

$$= \frac{2A}{\pi}\left[\int \cos k\omega_0 t\,dt - \frac{1}{\pi}\int t\cos k\omega_0 t\,dt\right]$$

$$-\frac{1}{\pi}\left[\cancel{\cos k\omega_0 t\cdot\frac{t^2}{2}}\Big|_0^{\pi} - \int\right.$$

$$-\frac{1}{\pi}\left[t\cdot\frac{\sin k\omega_0 t}{k\omega_0}\Big|_0^{\pi} - \int\frac{\sin k\omega_0 t}{k\omega_0}\right]$$

$$= \frac{2A}{\pi}\cdot\left[\frac{\sin k\omega_0 t}{k\omega_0}\Big|_0^{\pi} - \frac{1}{\pi}\left[0 + \frac{1}{k\omega_0}\cdot\frac{\cos k\omega_0 t}{k\omega_0}\Big|_0^{\pi}\right]\right]$$

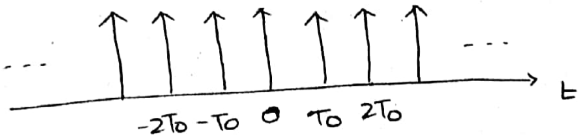
$$= \frac{2A}{\pi}\left[0 - \frac{(-1)^k - 1}{\pi k^2}\right] = \frac{2A}{(\pi k)^2}\left(1 - (-1)^k\right)$$

Note:

It has hidden half wave symmetry, because it contains only odd harmonics.

Q: If $\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$ Determine complex fourier series.

train of impulses



time period = T_0

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_0^T x(T) e^{-jk\omega_0 t}\, dt$$

$$\cancel{= \frac{1}{T_0}\int_0^{T_0} 1 \cdot e^{-jk\omega_0 t}\, dt} \qquad \left(\omega_0 = \frac{2\pi}{T_0}\right)$$

$$\cancel{= \frac{1}{T_0} \cdot -e^{-jk\omega_0 t}\Big|_0^{T_0}}$$

$$\cancel{= \frac{-1}{T_0}\left[e^{-2\pi jk} - 1\right]}$$

$$\cancel{= \frac{1 - e^{-2\pi jk}}{T_0}}$$

$$\cancel{x(t) = \sum_{k=\infty} \frac{1 - e^{-2\pi jk}}{T_0} \cdot e^{jk\omega_0 t}}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{e^{jk\omega_0 t}}{T_0}$$

$-\frac{T_0}{2}$ to $\frac{T_0}{2}$; only one impulse

at $t=0$, $\delta(t) = 1$.

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) \cdot e^{-jk\omega_0 t}\, dt$$

area under the curve = 1.

$$\cancel{= \frac{1}{T_0} \cdot T_0 \cdot 1} \qquad = \frac{1}{T_0} \cdot 1$$

$$a_k = \frac{1}{T_0}$$

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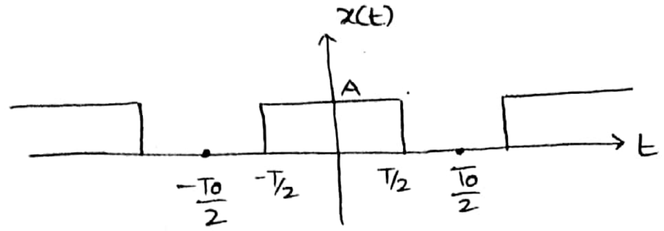
Q: Find the exponential Fourier series and plot the magnitude spectrum for F.S.C.



Period = To.

$$a_k = \frac{1}{T_0}\int_{-T/2}^{T/2} A \cdot e^{-jk\omega_0 t}\,dt$$

$$= \frac{A}{T_0}\left(-e^{-jk\omega_0 t}/jk\omega_0\right)_{-T/2}^{T/2}$$

$$= \frac{A\cancel{jk\omega_0}}{T_0}\,\frac{\left(e^{+jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\frac{T}{2}}\right)}{jk\omega_0} = \frac{A_0}{k\omega_0}\left(\frac{e^{jk\omega_0 T/2} - e^{-jk\omega_0 \frac{T}{2}}}{jT_0\cancel{\ }}\right.$$

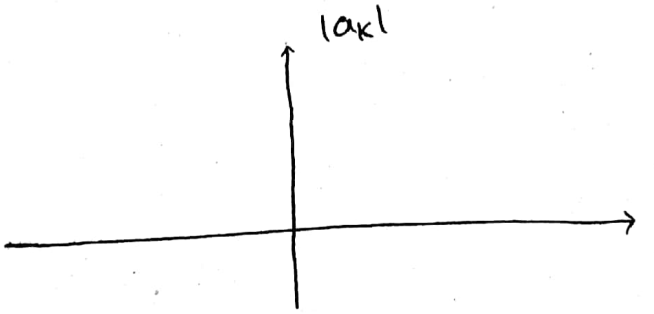
$\omega_0 = \frac{2\pi}{T_0}$

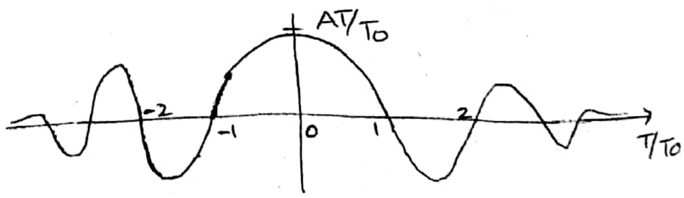
$$= \cancel{\frac{A}{k\omega_0}\sin k\omega_0 T/T_0} = \frac{A}{k\pi}\sin\frac{k\pi T}{T_0}$$

$$a_k = \frac{A}{k}\cdot\frac{\sin\pi\left(\frac{kT}{T_0}\right)}{\pi} = \frac{A}{k}\times\frac{kT}{T_0}\times\frac{\sin\pi\left(\frac{kT}{T_0}\right)}{\pi\cdot\frac{kT}{T_0}}$$

$$= \frac{AT}{T_0}\,\text{sinc}\,\frac{kT}{T_0} \qquad \left[\text{sinc}\,x = \frac{\sin\pi x}{\pi x}\right]$$

Magnitude spectrum:-





$\omega = 2\pi f$

$f = \frac{\omega}{2\pi} = \frac{1}{T_0}$

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Properties of complex Fourier series:

If $x(t)$ and $y(t)$ are the 2 periodic signals with time period T and their Fourier series coefficients are a_k and b_k.

1. Linearity:

If $z(t) = \alpha x(t) + \beta y(t)$ then fourier series coefficient of $z(t) = \alpha a_k + \beta b_k$

2. Time shifting property:

$x(t-t_0)$ will have fourier series coefficient $e^{-jk\omega_0 t_0} \cdot a_k$

3. Scaling:

$x(\alpha t)$ has the same fourier series coefficent a_k.

4. Frequency shifting:

$e^{jk_0\omega_0 t} \cdot x(t)$ will have fourier series coefficient a_{k-k_0}

5. Time reversal:

$x(-t)$ has F.S.C a_{-k}

<u>Note</u>: If $x(t)$ is an even signal; $x(t) = x(-t)$ then

$a_k = a_{-k}$

If $x(t)$ is an odd signal, $x(t) = -x(-t)$ then

$a_k = -a_{-k}$

6 Conjugate symmetry:

$x^*(t)$ has FSC a^*_{-K}

conjugate

Note: If $x(t)$ is a real function then $a_K = a^*_{-K}$

and a_0 must be real

i, If $x(t)$ is an real even function then a_K must

be real and even

ii, If $x(t)$ is an real odd function then a_K must be

imaginary and odd with $a_0 = 0$

*7. Multiplication:

$x(t)\,y(t)$ has FSC $c_K = \sum_{l=-\infty}^{\infty} a_l b_{K-l}$

8. Differentiation:

$\frac{d}{dt}x(t)$ has FSC $(j\omega_0 K)a_K$

9. Integration:

$\int x(t)\,dt$ has FSC $\frac{1}{(j\omega_0 K)} \cdot a_K$

10. Periodic convolution:

$\int_T x(\tau)\,y(t-\tau)\,d\tau$ has FSC $T a_K b_K$

time periodic of signal

Parseval Theorem:

The average power of a periodic signal is given by,

$$P_{avg} = \frac{1}{T}\int_T |x(t)|^2 dt$$

eg: $x(t) = \cos \omega t$

$\omega = \frac{2\pi}{T}$ $\quad P_{avg} = \frac{1}{T}\int_T \cos^2 \omega t \, dt$

$$= \frac{1}{2T}\int_0^T (1+\cos 2\omega t)\, dt$$

$$= \frac{1}{2T}\left[T + \left(\cos 2 \tfrac{2\pi}{T} . T - 1\right)\right] = \frac{1}{2}$$

FSC are a_k a_1 and a_{-1} both are $\frac{1}{2}$.

From theorem,

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$(P_{avg})_{\cos \omega t} = (a_1)^2 + (a_{-1})^2$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Q. If $x(t)$ is a real and odd function with time period $T=2$ and the FSC $a_k = 0$, $|k|>1$ and $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$. Specify 2 different signals which satisfy the given condition.

$$P_{avg} = 1 = (a_0)^2 + (a_1)^2 + (a_{-1})^2 + 0$$

$\Rightarrow$ $\cancel{a_0 = A/V}$ ($\because a_0 = 0$) $\quad a_1^2 + a_{-1}^2 = 1 \quad$ odd $(a_1 = -a_{-1})$

$\Rightarrow 2a_1^2 = 1$

$a_1 = \pm\frac{1}{j\sqrt{2}} \quad a_{-1} = \mp\frac{1}{j\sqrt{2}}$

$\therefore$ a_k should be imaginary

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_{-1} e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}$$

$$= \frac{-1}{j\sqrt{2}} e^{-j\omega_0 t} + \frac{1}{j\sqrt{2}} e^{j\omega_0 t}$$

$$= \frac{1}{j\sqrt{2}} (e^{j\omega_0 t} - e^{-j\omega_0 t}) = \frac{2j\sin\omega t}{j\sqrt{2}} = \underline{\underline{\sqrt{2}\sin\omega t}}$$

also for $a_1 = -\frac{1}{j\sqrt{2}} = -a_{-1}$ $\quad x(t) = \underline{\underline{-\sqrt{2}\sin\omega t}}$

Fourier series transform for LTI system:

The frequency response of a system is given by

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$H(j\omega_0 k) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0 k\tau}\, d\tau$$

If $x(t)$ is a periodic signal then $x(t)$ can be written as,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

Similarly the fourier series for $h(t)$,

$$b_k = \frac{1}{T}\int_0^T h(t)\, e^{-jk\omega_0 t}\, dt \qquad h(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

FSC of $h(t)$

$$b_k = \frac{1}{T} \cdot H(j\omega_0 k)$$

For FSC of $y(t)$;

$$y(t) = x(t) * h(t) = \int_T x(\tau)\, h(t-\tau)\, d\tau \Rightarrow \text{has FSC } T a_k b_k$$

$\therefore$ $y(t)$ has FSC $T a_k b_k = T\, a_k \cdot \dfrac{H(j\omega_0 k)}{T}$

$$= a_k H(j\omega_0 k)$$

$$\rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k H(jk\omega)}_{\text{FSC}} e^{jk\omega t}$$

(9) Conjugate symmetry for real signals

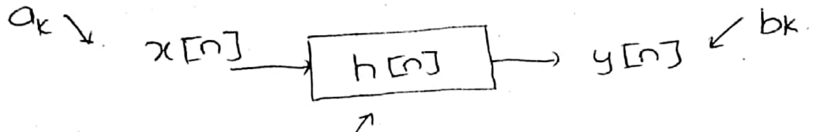
* If $x[n]$ is real, then $a_k = a_{-k}^*$
* If $x[n]$ is real and even, a_k will be real and even
* " " " " odd, " " purely imaginary and odd.

(10) Parseval's theorem.

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Discrete time Fourier series and LTI system:

The frequency response of LTI system :-



$$H(e^{j\Omega}) = \sum_{x=-\infty}^{\infty} h[n] e^{-j\Omega n}$$

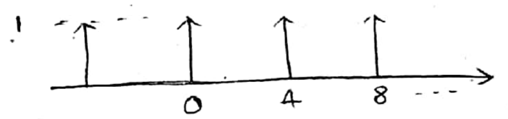
Fourier series of $x[n] = \sum_{k\langle N\rangle} a_k e^{jk\Omega n}$

$\therefore$ The output $y[n] = x[n]\, H(e^{j\Omega_0 n}) \Big|_{n = k\Omega_0}$

and FSC of output $y[n]$ is given by,

$b_k = a_k H(e^{jk\Omega_0})$ where $\Omega_0 = \frac{2\pi}{N}$

Q Find the FSC for the given periodic signal.



A $x[n] = \sum_{k\langle N\rangle} a_k e^{jk\Omega n}$ $\quad a_k = \frac{1}{N} \sum_{n\langle N\rangle} x[n] e^{-jk\Omega n}$ $\quad \Omega = \frac{2\pi}{N}$

$N = 4 \rightarrow \Omega = \frac{\pi}{2}$ $\quad a_k = \frac{1}{4} \sum_{n\langle 4} x[n] e^{-jk n\frac{\pi}{2}}$ $\quad 0, 1, 2, 3$

$$= \frac{1}{4}\left[x[0] \right] = \frac{1}{4}$$

$a_k = \frac{1}{4}$ for all k.

Q: Consider LTI system with impulse response $h[n] = \left(\frac{1}{2}\right)^{|n|}$ Find FS representation for output $y[n]$ if :

i) $x[n] = \sum_{k=-\infty}^{\infty} \delta(n-4k)$

ii) If $x[n]$ is a periodic signal with period $N=6$; $x[n] = 1$; $n = 0, \pm 1$
$= 0$; $n = \pm 2, \pm 3$

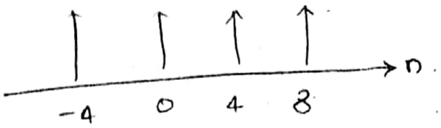
A: $$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = \sum \left(\frac{1}{2}\right)^{|n|} e^{-j\Omega n}$$

$$= \sum_{-\infty}^{0} \left(\frac{1}{2}\right)^{-n} e^{-j\Omega n} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\Omega n}$$

$$= \frac{1}{1-\frac{1}{2}e^{+j\Omega}} + \frac{\cancel{\frac{1}{2}e^{-j\Omega}}\ 1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-2e^{-j\Omega}}$$

i)

$x[n] =$ 

$N = 4$

$a_k = \frac{1}{4}$

$$b_k = \frac{1}{4}\left[\frac{1}{1-\frac{1}{2}e^{-jk\frac{\pi}{2}}} - \frac{1}{1-2e^{-jk\frac{\pi}{2}}}\right]$$

ii) $N = 6$ $\quad a_k =$

Fourier transform:

For periodic signals, the complex exponential building blocks are harmonically related but for aperiodic signals they are infinitesimally closed in frequency and its representation in terms of linear combination takes the form of integration rather than a sum.

The resulting spectrum of coefficients for representation of aperiodic signals is called a Fourier transform.

If $x(t)$ is a continuous time aperiodic signal then the fourier transform is.

$$X(j\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

Note

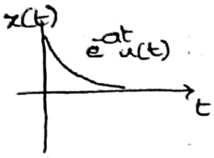
The fourier transform of a signal is possible when the signal is absolutely integrable.

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

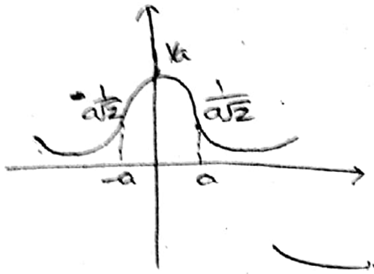
Q: $x(t) = e^{-at}u(t)$. Find fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt = \frac{-e^{-(a+j\omega)t}}{a+j\omega}\Bigg|_{0}^{\infty}$$

$$= \frac{1}{(a+j\omega)}$$



$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

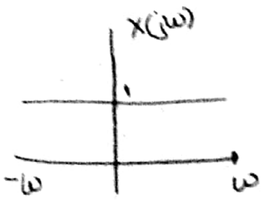


frequency spectrum
or
magnitude "
or
frequency response

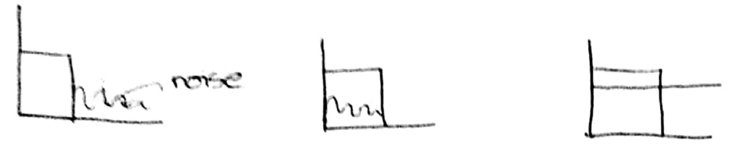
Q: $x(t) = \delta(t)$ Find $X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= 1 \qquad \left[\int \delta(t-t_0)\, \underset{\phi(t)}{e^{-j\omega t}}\, dt = \phi(t_0) = e^{0} = 1 \right]$$



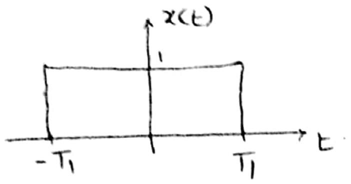
From $-\infty$ to $+\infty$, frequency is constant magnitude so filtering it is very difficult. Hence this signals impact in frequency domain is dangerous.



Acceptable Acceptable Dangerous.

Q: $x(t) = 1 \; ; \quad |t| < T_1$

$\quad\quad\; = 0 \; ; \quad |t| > T_1$



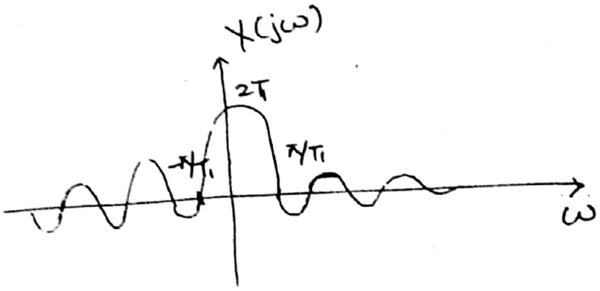
$$X(j\omega) = \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}\,dt = -\left.\frac{e^{-j\omega t}}{j\omega}\right|_{-T_1}^{T_1}$$

$$= \frac{-1}{j\omega}\left(e^{-j\omega T_1} - e^{+j\omega T_1}\right)$$

$$= \frac{2\sin\omega T_1}{\omega}$$

$\left(\operatorname{sinc}(x) = \dfrac{\sin \pi x}{\pi x}\right)$

$$= 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$



suitable frequency range = $\left[-\frac{\pi}{T_1}, \frac{\pi}{T_1}\right]$

Fourier transform for periodic signals:

Consider signal $x(t)$ with fourier transfor $X(j\omega)$ where

$$X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,e^{j\omega t}\,d\omega = \int \delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

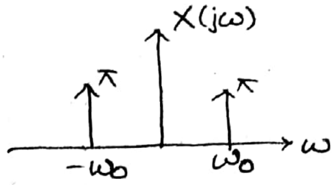
inverse fourier transform

$$\rightarrow x(t) = e^{j\omega_0 t}$$

Q: $x(t) = \cos\omega_0 t$ Find $X(j\omega)$.

$X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$ for $x(t) = e^{j\omega_0 t}$

$x(t) = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \rightarrow X(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$



Q: $x(t) = \sin\omega_0 t = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$X(j\omega) = \underbrace{\dfrac{\pi}{j}\,\delta(\omega-\omega_0) - \dfrac{\pi}{j}\,\delta(\omega+\omega_0)}$

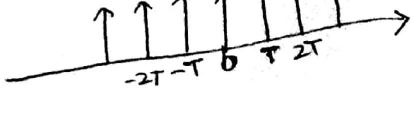
two components - same magnitude but diff. phase.

Note:

Any periodic signal $x(t)$ can be represented by a linear combination of complex exponential with the help of Fourier series coefficient.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow {}^{*}X(j\omega) = \sum_{k=-\infty}^{\infty} a_k\, 2\pi\,\delta(\omega - k\omega_0)$$

Q: If $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$. Find fourier transform.



$a_k = \dfrac{1}{T}$ for all k.

$\hookrightarrow \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi \, \delta(\omega - k\omega_0)$$

Properties of Fourier Transform

① Linearity

$$x(t) \xrightarrow{FT} X(j\omega)$$

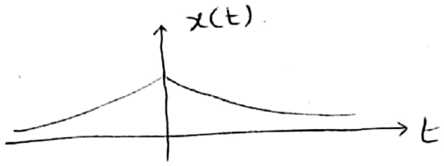
$$y(t) \xrightarrow{FT} Y(j\omega)$$

$$ax(t) + by(t) \xrightarrow{FT} aX(j\omega) + bY(j\omega)$$

Q $x(t) = e^{-a|t|}, \quad a > 0$

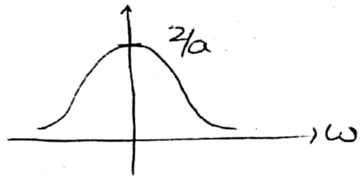
A $x(t) = e^{-at}, \quad t > 0$

$e^{at}, \quad t < 0$



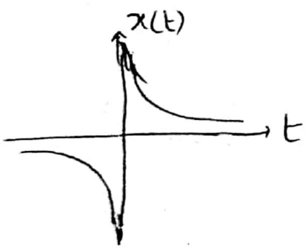
$$X(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

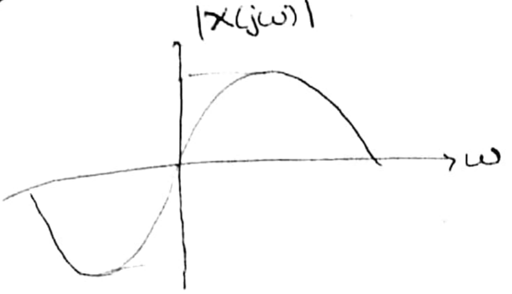


Q $x(t) = e^{-at} u(t) - e^{at} u(-t)$

$\;\;\;\;(t>0) \quad\quad (t<0)$

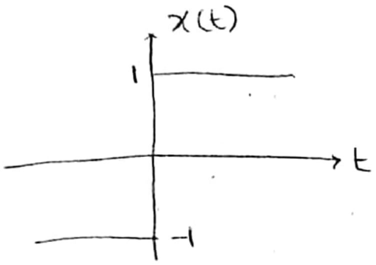


A
$$x(t) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt - \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt = \frac{1}{a + j\omega} - \frac{1}{a - j\omega} = \frac{-2j\omega}{a^2 + \omega^2}$$



t

Q $x(t) = sgn(t)$



$$x(t) = u(t) - u(-t)$$

$$sgn(t) = \lim_{a \to 0} \{ e^{-at} u(t) - e^{at} u(-t) \}$$

$$= \lim_{a \to 0} \frac{-2j\omega}{\omega^2 + a^2} = \frac{-2j}{\omega} = \underline{\underline{\frac{2}{j\omega}}} \; *$$

Note -

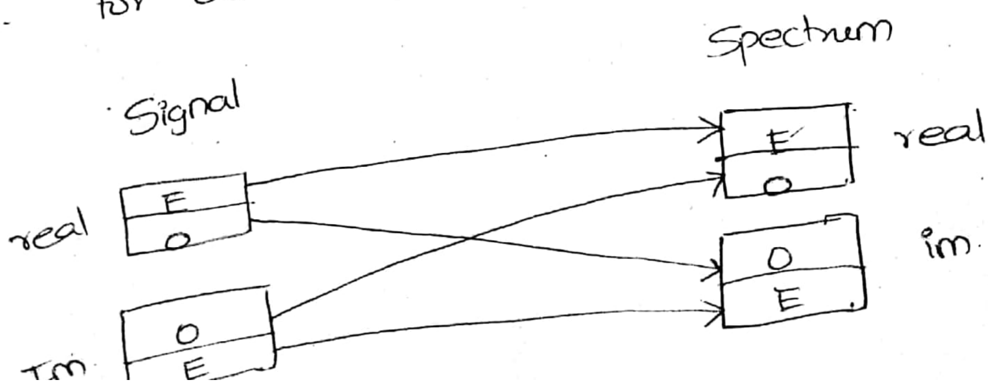
We cannot find Fourier transform of u(t) directly because we will be able to find out fourier transform of signal if and only if it is absolutely integrable.

$$\because \; sgn(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}(sgn(t) + 1)$$

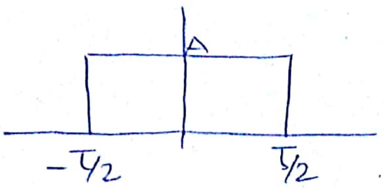
$$\mathcal{F}(u(t)) = \mathcal{F}\left[\frac{1}{2}(sgn(t) + 1)\right] = \frac{1}{j\omega} + \pi\delta(\omega)$$

Note :- For even and odd signals



Q: For a signal $x(t) = A\,rect\left(\frac{t}{\tau}\right)$

A:



$$X(j\omega) = A\int_{-\tau/2}^{\tau/2} e^{-j\omega t}\,dt = -\frac{A}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right)$$

$$= \frac{2A}{\omega}\left(\frac{e^{\frac{j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}}}{2j}\right)$$

$$= \frac{2A}{\pi\omega}\sin\frac{\omega\tau}{2}$$

$$= \frac{At}{\pi\cdot\frac{\omega t}{2\pi}}\cdot\sin\cdot\pi\frac{\omega t}{2\pi}$$

$$= At\,sinc\left(\frac{\omega\tau}{2\pi}\right) \qquad \left(sinc\,x = \frac{\sin\pi x}{\pi x}\right)$$

(2) Duality :-

$$x(t) \xrightarrow{FT} X(\omega)$$

$$X(t) \xrightarrow{FT} 2\pi x(-\omega)$$

eg: $\delta(t) \longrightarrow 1$

$1 \longrightarrow 2\pi\delta(\omega)$.

Q: If $x(t) = \dfrac{2a}{a^2+t^2}$ Find FT of signal.

A: $FT = 2\pi\, e^{-a|\omega|}$

Q: $x(t) = \frac{1}{\pi t}$

A: $sgn(t) \longrightarrow \frac{2}{j\omega}$

$\frac{2}{jt} \longrightarrow \cancel{2}\pi\, sgn(-\omega)$

$\frac{1}{\pi t} \longrightarrow j\,sgn(-\omega)$

$= -j\,sgn(\omega)$ (∵ sgn is odd fnc).

Q: $x(t) = \frac{\sin at}{\pi t}$

$A\,rect\left(\frac{t}{\tau}\right) \longrightarrow A\cdot\frac{\sin \omega\tau}{\pi\omega}$ or $A\tau\, sinc\,\frac{\omega\tau}{2\pi}$

for $A = 1$, $\tau = 2a$.

$rect\left(\frac{t}{2a}\right) \longrightarrow 1\cdot 2a\cdot sinc\,\frac{\omega\cdot 2a}{2\pi} = 2a\,sinc\,\frac{a\omega}{\pi}$.

$2a\,sinc\,\frac{at}{\pi} \longrightarrow 2\pi\,rect\left(\frac{-\omega}{2a}\right)$

$\downarrow$ $2a\cdot\frac{\sin at}{at} \longrightarrow \cancel{2}\pi\,rect\left(\frac{-\omega}{2a}\right)$.

$= \frac{\cancel{2}\sin at}{t}$

$\rightarrow$ $\frac{\sin at}{\pi t} \longrightarrow rect\left(\frac{-\omega}{2a}\right)$

③

Scaling :-

$x(t) \xrightarrow{FT} X(\omega)$

$x(at) \xrightarrow{FT} \frac{1}{|a|}X(\omega/a)$

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④ Time shifting:

$$x(t) \xrightarrow{FT} X(\omega)$$

$$x(t-t_0) \xrightarrow{FT} e^{-j\omega t_0} X(\omega)$$

Q: $x(t) = e^{-2t} u(t-2)$

$$e^{-at}u(t) \longrightarrow \frac{1}{j\omega+a}$$

$$x(t) = e^{-2(t-2+2)} u(t-2)$$

$$= e^{-2(t-2)} u(t-2) \cdot e^{-4} \cancel{u(t-2)}$$

$$= e^{-2j\omega} \frac{e^{-2t}}{j\omega+2} \cdot e^{-4}$$

⑤ Frequency shifting property:

This property is also called modulation property.

$$x(t) \xrightarrow{FT} X(\omega)$$

$$x(t)e^{j\omega_0 t} \longrightarrow X(\omega-\omega_0)$$

↳ modulation : changing freq, spectrum of original signal

Q: $x(t) = \cos\omega_0 t$

$$\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

$$\delta(t) \longrightarrow 1$$

$$1 \longrightarrow 2\pi\delta(\omega)$$

$$\frac{1}{2} \longrightarrow \pi\delta(\omega)$$

Q: $x(t) = \dfrac{4\cos 2t}{t^2+1} = 2\left(\dfrac{e^{j2t}}{t^2+1} + \dfrac{e^{-j2t}}{t^2+1}\right) = \dfrac{2}{t^2+1} e^{j2t} + \dfrac{2}{t^2+1} e^{-j2t}$

$$= 2\pi e^{-\cancel{a}|\omega-2|} + 2\pi e^{-|\omega+2|}$$

$$e^{-a|t|} \rightarrow \frac{2a}{a^2+\omega^2}$$

$a=1: \ e^{-|t|} \rightarrow \frac{2}{\omega^2+1}$

$$\frac{2}{t^2+1} \longrightarrow 2\pi e^{-a|-\omega|}$$

$$= 2\pi e^{-a|\omega|}$$

$$= 2\pi\left(e^{-|\omega-2|} + e^{-|\omega+2|}\right)$$

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(6) Differentiation in time domain:

$$x(t) \xrightarrow{FT} X(\omega)$$

$$\frac{d^n}{dx^n} x(t) \longrightarrow (j\omega)^n X(\omega)$$

Q: If $x(t) = \text{sgn}\,t$

$$\text{sgn}(t) \longrightarrow \frac{2}{j\omega}$$

$$\text{sgn}(t) = 2u(t) - 1$$

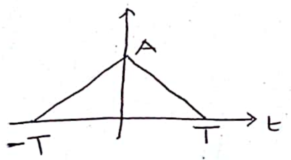
$$x(t) \longrightarrow X(\omega)$$

$$\frac{d}{dt}(x(t)) \longrightarrow j\omega \cdot X(\omega).$$

$$2\delta(t) \longrightarrow j\omega X(\omega).$$

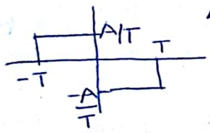
$$X(\omega) = \frac{2}{j\omega}.$$

Q:



$= x(t).$

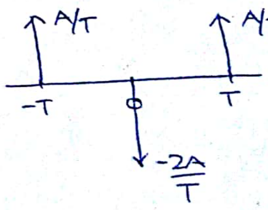
$$\frac{d}{dx} x(t) =$$



$A\{u(t)$

$$= \frac{A}{T}u(t+T) - \frac{2A}{T}u(t) + \frac{A}{T}u(t-T).$$

$$\frac{d^2}{dx^2} x(t) = \frac{A}{T}\delta(t+T) - \frac{2A}{T}\delta(t) + \frac{A}{T}\delta(t-T).$$

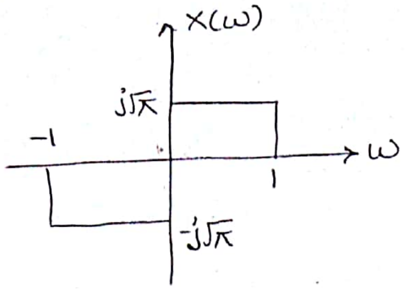


$\Big\downarrow$ F.T

$$\frac{A}{T}e^{j\omega T} - \frac{2A}{T} + \frac{A}{T}e^{-j\omega T}.$$

$$X(\omega) = \frac{1}{(j\omega)^2}\left(\frac{A}{T}e^{j\omega T} - \frac{2A}{T} + \frac{2A}{T}e^{-j\omega T}\right)$$

Q:-



find $\left.\frac{d\,x(t)}{dt}\right|_{t=0}$

$$\because \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$\cancel{\because \frac{d}{dt}x(t) \to j\omega X(\omega)}$$

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)j\omega e^{j\omega t}\,d\omega$$

$$\left.\frac{d\,x(t)}{dt}\right|_{t=0} = \frac{1}{2\pi}\int_{-1}^{1} j\omega X(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{0} -j\omega\cdot(-j\sqrt{\pi})\,d\omega + \int_{0}^{1} j\omega\cdot(j\sqrt{\pi})\,d\omega\right]$$

$$= \frac{j^2\cdot\sqrt{\pi}}{2\pi} = \frac{-1}{2\sqrt{\pi}}$$

⑥ Frequency domain differentiation:

$$x(t) \longrightarrow X(\omega)$$

$$-jt\,x(t) \longrightarrow \frac{d}{d\omega}X(\omega)$$

$$t\,x(t) \longrightarrow j\frac{d}{d\omega}X(\omega)$$

Q: $x(t) = te^{-at}u(t)$

$\because X_1'(\omega) = \dfrac{1}{a+j\omega}$ for $e^{-at}u(t)$

$$\frac{d}{d\omega} X_1'(\omega) = \frac{-j}{(a+j\omega)^2}$$

$$X_1(\omega) = \frac{-j}{(a+j\omega)^2}$$

for $x(t)$, $X(\omega) = j \cdot \dfrac{-j}{(a+j\omega)^2} = \dfrac{1}{(a+j\omega)^2}$

⑦ Convolution

$x_1(t) \longrightarrow X_1(\omega)$

$x_2(t) \longrightarrow X_2(\omega)$

$x_1(t) * x_2(t) \longrightarrow X_1(\omega)\, X_2(\omega)$

Q: $x(t) = te^{-at}u(t)$

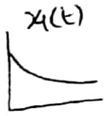
$\downarrow$

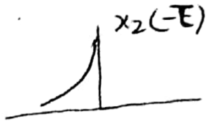
$x_1 = e^{-at}u(t)$

$x_2 = e^{-at}u(t)$

$FT = \dfrac{1}{a+j\omega} \dfrac{1}{a+j\omega} = \dfrac{1}{(a+j\omega)^2}$

$x_1(t) = e^{-at}u(t)$ $\qquad$ $x_2(t) = e^{-at}u(t)$





$x_1 * x_2 =$  $= \displaystyle\int_0^t e^{-at} e^{at}$

$= te^{-at}u(t)$

⑧ Frequency domain convolution

$$x_1(t)\cdot x_2(t) \longrightarrow \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

⑨ Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Q: $\displaystyle\int_{-\infty}^{\infty} \mathrm{sinc}^2(5t)\, dt$

$x(t) = \mathrm{sinc}(5t) \qquad X(\omega) = ?$

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \xrightarrow{FT} AT\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

$$x(t) \xrightarrow{FT} X(\omega)$$

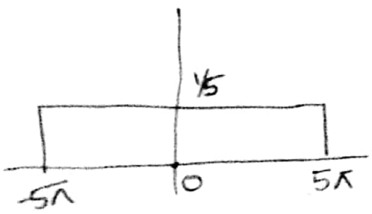
$$X(t) \xrightarrow{FT} 2\pi\, x(-\omega)$$

$$AT\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) = \mathrm{sinc}\,5\omega \Rightarrow \frac{T}{2\pi} = 5 \qquad T = 10\pi$$

$$A = \frac{1}{10\pi}$$

$$\text{FT of } \mathrm{sinc}(5\omega) = 2\pi \cdot \frac{1}{10\pi}\,\mathrm{rect}\left(\frac{-\omega}{10\pi}\right)$$

$$= \frac{1}{5}\,\mathrm{rect}\left(\frac{\omega}{10\pi}\right) \quad (\because \text{even fnc})$$



$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{5}\,\mathrm{rect}\left(\frac{\omega}{10\pi}\right)\right)^2 d\omega = \frac{1}{2\pi}\int_{-5\pi}^{5\pi}\frac{1}{25}\,d\omega = \frac{1}{5}$$

Q: It is given that A is real and non-negative and $\mathcal{F}^{-1}((1+j\omega)X(\omega)) = Ae^{-2t}u(t)$. Find x(t). $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{2\pi}{2\pi} = 1$$

$$(1+j\omega)X(\omega) = F(Ae^{-2t}u(t))$$

$$= \frac{A}{2+j\omega}$$

$$X(\omega) = \frac{A}{(1+j\omega)(2+j\omega)} = A\left[\frac{1}{1+j\omega} - \frac{1}{2+j\omega}\right]$$

$$x(t) = \mathcal{F}^{-1}(X(\omega)) = Ae^{-t}u(t) - Ae^{-2t}u(t)$$

$$\frac{1}{2\pi} \cdot 2\pi = A^2 \int_{-\infty}^{\infty} (e^{-t}u(t) - e^{-2t}u(t))^2 dt$$

$$= A^2 \int_{-\infty}^{\infty} (e^{-2t}u(t) + e^{-4t}u(t) - 2e^{-t}u(t)e^{-2t}u(t)) \cdot dt$$

$$= A^2 \int_{0}^{\infty} (e^{-2t} + e^{-4t} - 2e^{-3t}) dt = A^2\left(\frac{e^{-2t}}{-2} + \frac{e^{-4t}}{-4} + \frac{2e^{-3t}}{3}\right)\Bigg|_0^{\infty}$$

$$= A^2\left(0 - \left(-\frac{1}{2} - \frac{1}{4} + \frac{2}{3}\right)\right)$$

$$\Rightarrow \frac{A^2}{12} = \cancel{2\pi} \cdot 1$$

$$A^2 = 12 \qquad A = \sqrt{12}$$

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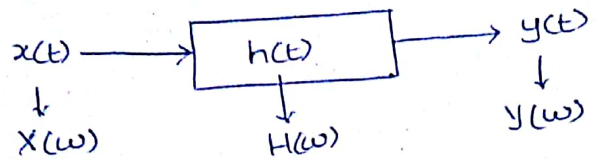
Fourier transform and LTI system:-



$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$\Rightarrow \quad H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

frequency response = response of output wrt input in freq. domain.

$$\Rightarrow \quad |Y(\omega)|^2 = |X(\omega)|^2 |H(\omega)|^2$$

Q: For an LTI system the differential equation is given by

$y'(t) + 2y(t) = x(t) + x'(t)$. Find, $h(t)$ of system

impulse response

FT

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$$

$$Y(\omega)[2 + j\omega] = X(\omega)\{1 + j\omega\}$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1 + j\omega}{2 + j\omega} = H(\omega)$$

$$H(\omega) = 1 - \frac{1}{2 + j\omega}$$

Inverse FT.

$$h(t) = \delta(t) - e^{-2t} u(t)$$

Q: Frequency response of an LTI system is given by

$$H(\omega) = -j \;;\; \omega > 0$$
$$= j \;;\; \omega < 0$$

Find the impulse response.

$$H(\omega) = -j \begin{cases} 1 \;;\; \omega > 0 \\ -1 \;;\; \omega < 0 \end{cases} = -j\,\mathrm{sgn}(\omega)$$

$$\mathrm{sgn}(t) \xrightarrow{FT} \frac{2}{j\omega}$$

$$\frac{2}{jt} \xrightarrow{FT} 2\pi\,\mathrm{sgn}(-\omega) \Rightarrow \frac{1}{jt} \longrightarrow -\pi\,\mathrm{sgn}(\omega)$$

$$\Rightarrow \frac{1}{\pi t} \longrightarrow -j\,\mathrm{sgn}(\omega)$$

$$\therefore\; h(t) = \frac{1}{\pi t}$$

Q: For a LTI system $x(t) = u(t)$ and $h(t) = e^{-2t}u(t)$. Find $y(t)$

$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega)H(\omega)$$

$$= \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \cdot \frac{1}{2+j\omega} = \frac{\pi}{2}\delta(\omega) + \frac{1}{j\omega}\cdot\frac{1}{2+j\omega}$$

$$= \frac{\pi\delta}{2} + \frac{1}{2j\omega - \omega^2} = \frac{\pi}{2}\delta(\omega) + \frac{1}{2}\left[\frac{1}{j\omega} - \frac{1}{2+j\omega}\right]$$

($\because \delta(\omega)$ valid only at $\omega = 0$)

$$= \frac{\pi}{2}\delta(\omega) + \frac{1}{2j\omega} - \frac{1}{2(2+j\omega)}$$

$$\Big\downarrow \text{IFT}$$

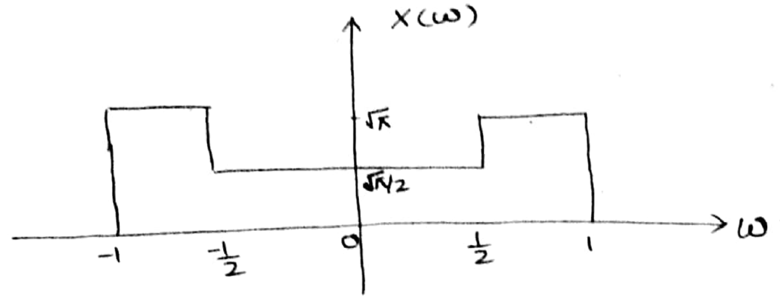
$$y(t) = \frac{1}{4} + \frac{1}{4}\mathrm{sgn}(t) - \frac{1}{2}e^{-2t}u(t)$$

$$1 \longrightarrow 2\pi\delta(\omega)$$

$$\mathrm{sgn}(t) \longrightarrow \frac{2}{j\omega}$$

$$e^{-at}u(t) \longrightarrow \frac{1}{a+j\omega}$$

Q. Find the energy of given spectrum.



$\cancel{\text{Area} = \int \omega X(\omega)}$

$\cancel{= \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{2}}$

$\cancel{X(\omega) = \frac{3\sqrt{\pi}}{2\omega}}$

Parseval's theorem.

$$\text{Energy} \Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\cdot \int_{-1}^{-1/2} (\sqrt{\pi})^2 d\omega + \int_{-1/2}^{1/2} \left(\frac{\sqrt{\pi}}{2}\right)^2 d\omega$$

$$+ \int_{1/2}^{1} (\sqrt{\pi})^2 d\omega$$

$\cancel{= \frac{1}{2\pi}\int_{-1}^{1} \frac{9\pi}{4}\cdot \frac{d\omega}{\omega^2}}$

$\cancel{+ \frac{9}{8}\, \frac{\omega^{-1}}{-1}\Big|_{-1}^{1}}$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{2}\right)$$

$$= \underline{\underline{\frac{5}{8}}}$$

Q: For an ideal low pass filter, the frequency response is given by

$$H(\omega) = 1, \quad |\omega| < \omega_c$$
$$= 0, \quad |\omega| > \omega_c$$

and if input $x(t) = e^{-2t} u(t)$ Find ω_c if Energy of $y(t)$ is equal to half of energy of $x(t)$.

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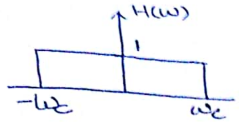
$H(\omega) \rightarrow$



$$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} e^{-4t} dt = \frac{1}{4}$$

($\because u(t) = 1$ for $t > 0$)

So, $E_{y(t)} = \frac{1}{8}$

$$|Y(\omega)|^2 = |X(\omega)|^2 |H(\omega)|^2 = \frac{1}{2+j\omega} \cdot 1 \qquad X(\omega) = \frac{1}{2+j\omega}$$

By Parsevals theorem;

$$E_{y(t)} = \int_{-\infty}^{\infty} |y(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$$

$$\frac{1}{8} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \left(\left|\frac{1}{2+j\omega}\right|\right)^2 \cdot 1 \cdot d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{1}{4+\omega^2} d\omega$$

$$\frac{\pi}{4} = \frac{1}{2} \tan^{-1} \frac{\omega}{2} \Big|_{-\omega_c}^{\omega_c} = \tan^{-1} \frac{\omega_c}{2}$$

$$\frac{\omega_c}{2} = 1 \quad \Rightarrow \quad \omega_c = 2$$

Fourier Transform of discrete time series.

$$FT \rightarrow X(e^{j\omega}) = X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$IFT \rightarrow x[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

Q: $x[n] = a^n u[n]$ Find FT.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

Properties of DTFT :-

1. Linearity :

$$x_1[n] \longrightarrow X_1(e^{j\omega})$$

$$x_2[n] \longrightarrow X_2(e^{j\omega})$$

$$a x_1[n] + b x_2[n] \longrightarrow a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

2. Time shifting:

$$x_1[n] \longrightarrow X_1(e^{j\omega})$$

$$x_1[n-n_0] \longrightarrow e^{-j\omega n_0} X_1(e^{j\omega})$$

3. Periodicity:

$$X(e^{j\omega + 2\pi}) = X(e^{j\omega})$$

4. Frequency shifting property:

$$e^{j\omega_0 n} x[n] \longrightarrow X(e^{j(\omega - \omega_0)})$$

5. Time scaling:

$$x_k[n] = x[n/k] \longrightarrow X(e^{j\omega k})$$

6 Frequency domain differentiation

$$n x[n] \longrightarrow j \frac{d}{d\omega} X(e^{j\omega})$$

7 Convolution :

$$x_1[n] * x_2[n] \longrightarrow X_1(e^{j\omega}) X_2(e^{j\omega})$$

8 Conjugate symmetry :

$$x[n] \longrightarrow X(e^{j\omega})$$

$$x^*[n] \longrightarrow X^*(e^{-j\omega})$$

Note:-

i. If $x[n]$ is real and even, FT will be real and even.

ii. If $x[n]$ is real and odd, FT will be imaginary and odd.

Q: $y[n] = \left(\frac{1}{4}\right)^n u[n-3]$ Find DTFT.

$$X'(\Omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n-3]\, e^{-j\omega n} = \sum_{n=3}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n}$$

$$y[n] = \frac{1}{4^3} \cdot \left(\frac{1}{4}\right)^{n-3} u[n-3]$$

$$= \frac{1}{64} \left(\frac{1}{4}\right)^{n-3} u[n-3] = \frac{1}{64} \cdot x[n-3]$$

let $x[n] = \left(\frac{1}{4}\right)^n u[n]$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \frac{1}{4^n} e^{-j\omega n}$$

$$= \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$X'(\Omega) = \frac{1}{64} \cdot e^{-3j\omega} \left(\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right)$$

(time shifting)

Q: $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j10\omega}}$ Find the signal.

time scaling :- $k=10$ for $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}} \rightarrow x[n] = \left(\frac{1}{2}\right)^n u[n]$

for $k=10$ $X(e^{jk\omega}) \longleftrightarrow x[n/k] = x[n/10] = \left(\frac{1}{2}\right)^{n/10} u[n/10]$

Some standard DTFT's :

$$\delta[n] \longleftrightarrow 1$$

$$1 \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$\cos\omega_0 n \longleftrightarrow \pi \sum_{k=-\infty}^{\infty} \delta(\omega-\omega_0-2\pi k) + \pi \sum_{k=-\infty}^{\infty} \delta(\omega+\omega_0-2\pi k)$$

Parseval's theorem:-

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Q: For a LTI system, $x[n]$ is input and $y[n]$ is output

System described by $y[n]$ is ; $y[n] = a\,y[n-1] + b\,x[n] + x[n-1]$

Find relation b/w a and b such that the DC gain of system is unity.

DTFT $\longrightarrow$ $Y(e^{j\omega}) = aY(e^{j\omega})e^{-j\omega} + bX(e^{j\omega}) + X(e^{j\omega})e^{-j\omega}$

$$Y(e^{j\omega})[1 - ae^{-j\omega}] = X(e^{j\omega})[b + e^{-j\omega}]$$

$$\underbrace{\frac{Y(e^{j\omega})}{X(e^{j\omega})}}_{\frac{\text{output}}{\text{input}}} = \underbrace{H(e^{j\omega})}_{\text{gain}} = \frac{b+e^{-j\omega}}{1-ae^{-j\omega}}$$

$\underbrace{\text{DC}}_{\omega=0}$ gain $= 1$ $\Rightarrow$ $\frac{b+1}{1-a} = 1$

$$\underline{a+b=0}$$

Q. The filter blocks the response at $f = 1/3$ and it provides unit gain at $f = 1/8$.

Find DC gain of filter if the impulse response of the filter is $h[n] = \{\alpha, \underset{\uparrow}{\beta}, \alpha\}$

$h[n] = \alpha\delta[n+1] + \beta\delta[n] + \alpha\delta[n-1]$

DTFT

$H(e^{j\omega}) = \alpha e^{j\omega} + \beta + \alpha e^{-j\omega}$

$f = 1/3$ response $= 0$

$\omega = \frac{2\pi}{3}$

$$0 = \alpha\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + \beta + \alpha\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$0 = -\alpha + \beta \Rightarrow \alpha = \beta.$$

$f = 1/8$ $H(e^{j\omega}) = 1.$

$\omega = \frac{\pi}{4}$

$$1 = \alpha\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) + \alpha + \alpha\left(\frac{1}{\sqrt{2}} + \frac{-j}{\sqrt{2}}\right)$$

$$= \alpha + \sqrt{2}\alpha$$

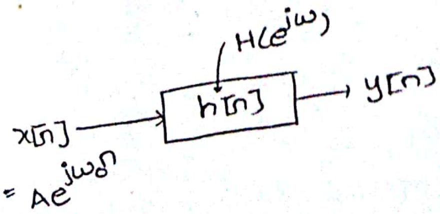
$$\Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}+1}$$

DC gain $= \alpha + \beta + \alpha = 3\alpha = \frac{3}{\sqrt{2}+1}$

$\omega = 0.$

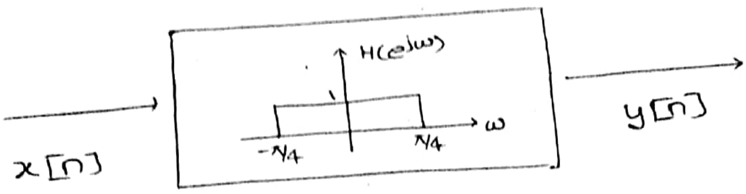
Note:

Whenever the input signal to an LTI system is complex exponential, then the output of the system can be given by



$$y[n] = x[n]\, H(e^{j\omega})\Big|_{\omega = \omega_0}.$$

Q: If $x[n] = \cancel{\sin\frac{\pi}{3}}\ 2\cos\frac{\pi}{6}n$



$x[n] = 2\cos\frac{\pi}{6}n.$

$\omega_0 = \frac{\pi}{6}, -\frac{\pi}{6}$ $\qquad H(e^{j\omega})|_{\omega_0} = 1$

So, $y[n] = x[n]$