Rosen 5.3, Exercise 14:

Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

$P(n) = f_{n+1}f_{n-1} - f_n^2 = (-1)^n$

Base Step: when $n = 1$

$$\begin{aligned} f_{(1)+1}f_{(1)-1} - f_{(1)}^2 &= 1 \cdot 0 - (1^2) \\ &= -1 \\ &= (-1)^{(1)} \end{aligned}$$

Which shows that the base step is satisfied.

Induction Step: If we assume the satement holds for $P(k)$, then we must prove that it holds for $P(k+1)$

$$\begin{aligned} f_{(k+1)+1}f_{(k+1)-1} - f_{(k+1)}^2 &= f_{k+2}f_k - f_{k+1}^2 \\ &= f_k(f_k - f_{k+1}) - f_{k+1}^2 \\ &= f_k^2 - f_k f_{k+1} - f_{k+1}^2 \\ &= -f_{k+1}(f_k + f_{k+1}) + f_k^2 \\ &= -f_{k+1}f_{k-1} + f_k^2 \\ &= (-1)^1 \cdot (f_{k+1}f_{k-1} - f_k^2) \\ &= (-1)^1 \cdot (-1)^k \\ &= (-1)^{k+1} \end{aligned}$$

Thus, the induction step is shown to hold true for $P(k+1)$.

By the principle of mathematical induction, $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

Rosen 5.3, Exercise 26:

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step : $(0, 0) \in S$.

Recursive step : If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

a) List the elements of S produced by the first five applications of the recursive definition.

first application: (2, 3), (3, 2)
second application: (4, 6), (5, 5), (6, 4)
third application: (6, 9), (7, 8), (8, 7), (9, 6)
fourth application: (8, 12), (9, 11), (10, 10), (11, 9), (12, 8)
fifth application: (10, 15), (11, 14), (12, 13), (13, 12), (14, 11), (15, 10)

b) Use strong induction on the number of applications of the recursive step of the definition to show that $5|a + b$ when $(a, b) \in S$.

Let $P(n) =$ "$5|a + b$ when $(a, b) \in S$"

Base Step: let $n = 0$
$(0, 0) \in S$
$5|0 + 0$
This satisfies the base step because 0 is divisible by any number.

Induction Step: If $P(1), P(2), ..., P(k)$ are true, we must prove $P(k + 1)$ is also true
If $P(k)$ is $(a, b) \in S$, then $P(k + 1)$ must be either $(a + 2, b + 3) \in S$ or $(a + 3, b + 2) \in S$, therefore, we must prove both satisfy the condition.
For $(a + 2, b + 3) \in S$ or $(a + 3, b + 2) \in S$, adding both terms from either will result in $a + b + 5$ and since we know $a + b$ is divisible by five and 5 is divisible by five, it is possible to conclude $5|(a + 2) + (b + 3)$ and $5|(a + 3) + (b + 2)$.
Thus the induction step is satisfied because $P(k + 1)$ is satisfied.

By the principle of strong induction, the statement is true.

c) Use structural induction to show that $5|a + b$ when $(a, b) \in S$.

Let $P(n) =$ "$5|a + b$ when $(a, b) \in S$"

Base Step: let $n = 0$
$(0, 0) \in S$
$5|0 + 0$
This satisfies the base step because 0 is divisible by any number.

Recursive Step: Assume $(a, b) \in S$ with $5|a + b$. The two possible next steps are $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$
Since $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$ both give $a + b + 5$ and since we know $a + b$ is divisible by five and 5 is divisible by five, it is possible to conclude $5|(a + 2) + (b + 3)$ and $5|(a + 3) + (b + 2)$.
Thus, the recursive step is satisfied.

By the principle of structural induction, the statement is true.

Rosen 5.3, Exercise 44:

Use structural induction to show that $l(T)$, the number of leaves of a full binary tree T , is 1 more than $i(T)$, the number of internal vertices of T .

For full binary tree T, $l(T) = i(T) + 1$

Base Step: A binary tree made from just the vertex
$T = 0$
$leaves = (0) + 1 = 1$
$internal\ vertices = (0) = 0$
A binary tree with a single vertice has 1 leaf and 0 internal vertices which is shown in the base step, thus, the base step is satisfied.

Recursive Step: If there are two full binary trees, T_1 and T_2, must show the statement holds true for the conjunction of both shown by $T_{1,2}$

Because, when combining two trees, the number of internal vertices will be the combination of both along with one new intenal vertice from the connection between both trees. Thus:
$i(T_{1,2}) = i(T_1) + i(T_2) + 1$

On the other hand, even after both trees are combined, this will not result in the formation of new leaves. Thus:
$T_{1,2} = l(T_1) + l(T_2)$

To show the statement holds trye for $T_{1,2}$:

$$
\begin{aligned}
T_{1,2} &= l(T_1) + l(T_2) \\
&= (i(T_1) + 1) + (i(T_2) + 1) \\
&= (i(T_1) + i(T_2) + 1) + 1 \\
&= i(T_{1,2}) + 1
\end{aligned}
$$

Thus the induction step is shown to hold true when both trees are combined.

By the principle of structural induction, the statement holds true.

Rosen 5.4, Exercise 8:

Give a recursive algorithm for finding the sum of the first n positive integers.

procedure $sum(n :$ positive integer$)$
if $n = 1$ **then**
  return 1
else
  return $sum(n - 1) + n$

Rosen 5.4, Exercise 16:

Prove that the recursive algorithm for finding the sum of the first n positive integers you found in Exercise 8 is correct.

Proof By Induction:

Base Step: when n = 1

The algorithm returns 1 due to the if statement. The sum of numbers till 1 also evaluates to 1 so the base step can be found to be true.

Induction Step: assume that the algorithm is correct for a number $sum(k)$, to prove the inductive step, the algorithm must evaluate for $sum(k+1)$ so:

$sum(k) = \sum_{n=1}^{k} n$

$$\begin{aligned} sum(k+1) &= sum(k) + k + 1 \\ &= \sum_{n=1}^{k} n + k + 1 \\ &= \sum_{n=1}^{k+1} n + k + 1 \\ &= \sum_{n=1}^{(k+1)} n \end{aligned}$$

The equation now gives the sum for the first $k+1$ integers thus the induction step is proven to evaluate to true.

By the principle of mathematical induction, the algorithm found in Exercise 8 is correct.