Rosen 5.2, Exercise 6:

a) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

3,6,9,12,13,15,16,18,19,20,21,22,23,...
All numbers greater than or equal to 18 are added to the list.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

$P(n) =$ "Postage of n cents formed using just 3-cent and 10-cent stamps"

Base Step: $n = 18$
$6 \cdot 3 = 18$
Base step is satisfied because 18 can be made using six 3-cent stamps.

Induction Step: If we assume $P(k)$ is true, then to prove P$(k + 1)$ is true:
If k is made up of three or more 3-cent stamps, then $k + 1$ can be made by replacing three of the 3-cent stamps with one 10-cent stamp.
If there are less than three 3-cent stamps, according to the base step saying the lowest value is 18, this situation will only occur when there are two or more 10-cent stamps in k. It is possible to make $k + 1$ by replacing two 10-cent stamps with seven 3-cent stamps.
Thus, it is always possible to make $P(k + 1)$ when k is greater than or equal to 18.

By principle of mathematical induction, $P(n)$ is true for all positive integers greater than or equal to 18.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

$P(n) =$ "Postage of n cents formed using just 3-cent and 10-cent stamps"

Base Step: $n = 18$, $n = 19$, $n = 20$
$6 \cdot 3 = 18$
$P(18)$ is satisfied because 18 can be made using six 3-cent stamps.
$1 \cdot 10 + 3 \cdot 3 = 19$
$P(19)$ is satisfied because 11 can be made using one 10-cent stamp and three 3-cent stamps.
$2 \cdot 10 = 20$
$P(20)$ is satisfied because 20 can be made using two 10-cent stamps.
Thus, the base step is satisfied.

Induction Step: If we assume $P(18), P(19), ..., P(k)$ are true, then to prove P$(k + 1)$ is true:
Then, to prove $P(k + 1)$ is true, a 3-cent coin can be added to $P(k - 2)$, which was already assumed to be true.

By principle of strong induction, $P(n)$ is true for all positive integers greater than or equal to 18.

Rosen 5.2, Exercise 18:

Use strong induction to show that when a simple polygon P with consecutive vertices $v_1, v_2, ..., v_n$ is triangulated into $n-2$ triangles, the $n-2$ triangles can be numbered $1, 2, ..., n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, ..., n-2$.

Base Step: $n = 3$
Base step is satisfied because when $n = 3$, that means there is 1 triangle and therefore the triangle numbered 1 must have 1, 2 and 3 as the vertices. This satisfies the condition since v_1 is a vertex of triangle 1.

Induction Step:
If a simple polygon with n sides has vertices $v_m, v_{m+1}, ..., v_{m+n-3}$ after being triangulated and the triangles are labeled m to $m+n-3$, then it satisfies the condition of v_i being a vertice of i.

Vertices v_{m+n-1} or v_{m+n-2} will be enpoints of a diagonal after an n shaped polygon has been triangluated leaving two cases:

Case 1: v_{m+n-1} is an endpoint
When v_{m+n-1} is connected to a vertex v_k to form a diagonal, it will split the polygon into 2.
If we use v_{k-1} as v_{m+n-1}, there will be two polygons, one having vertices v_m to v_{k-1} and the other having vertices v_k to v_{m+n-3}.
This satisfies the required number of vertices.

Case 2: v_{m+n-2} is an endpoint
Similarly, there will be two polygons formed if we use v_k.
If we use v_{k+1} as v_{m+n-2} and v_{k+2} as v_{m+n-1}, there will be two polygons, one having vertices v_m to v_k and the other having vertices v_{k+1} to v_{m+n-3}.
This satisfies the required number of vertices.

Hence, the induction step evaluates to true and the statement is true by the principle of strong induction

Rosen 5.2, Exercise 30:

Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.
Basis Step: $a^0 = 1$ is true by the definition of $a0$.
Inductive Step: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1$$

The inductive step showed that a^j is equal to one for all nonnegative integers but when evaluating to try to find a^1, we would need $k = 0$, so the inductive step would show:

$$a^{0+1} = \frac{a^0 \cdot a^0}{a^{0-1}}$$

This results in the presence of a^{-1} which we dont know the value of due to j being a negative number in this scenario.

Rosen 5.2, Exercise 40:

Use the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [$Hint$: Use the Archimedean property, given in Appendix 1, to find a positive integer A with $A > 1/(y - x)$. Then show that there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

1. According to the Archimedian property, every real number x has an integer n such that $n > x$.

2. By the principle of real numbers, if x and y are both real numbers, then $1/(y - x)$ is also a real number.

3. Combining the two above lines, it is possible to show that there exists an integer A such that $A > 1/(y-x)$, which can be simplified to show $Ax + 1 < Ay$.

4. If there is a number x, then there will exist an integer n such that $n - 1 \leq x < n$

5. By combining lines (3) and (4), there exists $n - 1 \leq Ax < n$, which is equivalent to $n \leq Ax + 1 < Ay$.

6. Line (5) will further give $Ax < n < Ay$ or $x < \frac{n}{A} < y$.

7. If $\frac{n}{A}$ is assumed to be r, then because n and A are both integers, it proves that there is a rational number r such that $x < r < y$.