

**Rosen 6.1, Exercise 22(g):**

How many positive integers less than 1000

g) have distinct digits?

Done by counting the amount of numbers for 1, 2 or three digits seperately. For:

1 digit numbers: there are 9 integers.

2 digit numbers: from 10 to 99 there are 90 numbers however, excluding numbers like 11, 22, ..., 99, there are 81 integers.

3 digit numbers: from 100 to 999, there are 900 numbers, but in order to exclude repeating digits, we can use a different method. The first digit can be selected from among 9 integers ranging from 1 to 9, the second is another 9 integers ranging from 0 to 9 but excluding the one used in the first digit and the third is selected from among 8 integers ranging from 0 to 9 excluding those used in the first and second. Therefore, the number of integers with distinct digits is  $9 \cdot 9 \cdot 8 = 648$ .

In conclusion, the number of distinct positive integers less than 1000 is  $9 + 81 + 648 = 738$ .

**Rosen 6.1, Exercise 50:**

How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

To have five consecutive 0s or 1s, the starting position of the consecutive numbers can be any position up till the 6th. The number of bit strings can be found by grouping them according to the starting positions of the consecutive sequence.

If it is the first position, there are  $2^5$  possible combinations.

In the remaining five positions, each will have  $2^4$  positions.

Therefore, there are  $2^5 \cdot 5(2^4) = 112$  possible combinations for five consecutive digits of one of the two.

Therefore, to find the bit strings with either five consecutive 0s or five consecutive 1s, we must multiply this 112 by two and further subtract two because of the presence of 0000011111 and 1111100000 in both creating an overlap.

$$2(112) - 2 = 222$$

Therefore there are 222 bit strings of length 10 that contain either five consecutive 0s or five consecutive 1s.

**Rosen 6.1, Exercise 72:**

Use mathematical induction to prove the product rule for  $m$  tasks from the product rule for two tasks.

$P(m)$  = "Suppose that a procedure can be broken down into a sequence of two tasks. If there are  $n_1$  ways to do the first task and for each of these ways of doing the first task, there are  $n_2$  ways to do the second task and  $n_m$  ways to do the  $m$  task, then there are  $n_1 \cdot n_2 \cdot \dots \cdot n_m$  ways to do the procedure."

Base Step: for  $n = 2$

$P(2)$  is true because  $P(2)$  results in  $n_1 \cdot n_2$  according to the statement which matches the product rule for 2 tasks.

Induction Step: If we assume  $P(k)$  is true, then we must prove  $P(k + 1)$  to also be true.

If, following the statement, the first event occurs  $n_1$  ways, the second occurs  $n_2$  and the  $k$ th occurs  $n_k$ , then the  $(k + 1)$ th occurs  $n_{k+1}$  ways.

If  $P(k)$  is true and occurs  $n_1 \cdot n_2 \cdot \dots \cdot n_k$  ways, then by using the product rule for two tasks, we can show:  
 $(n_1 \cdot n_2 \cdot \dots \cdot n_k) \cdot n_{k+1} = n_1 \cdot n_2 \cdot \dots \cdot n_k \cdot n_{k+1}$

This shows that  $P(k + 1)$  is satisfied and the induction step is true.

Thus, by the principle of mathematical induction, the statement is true for integers greater than or equal to 2.

**Rosen 6.2, Exercise 10:**

Let  $(x_i, y_i), i = 1, 2, 3, 4, 5$ , be a set of five distinct points with integer coordinates in the  $xy$  plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

To find the midpoint of a line segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$ , the formula is  $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ .

The equation for midpoint indicates that for the midpoint to have integer coordinates, the combined  $x$  values and combined  $y$  values must both be divisible by two. This will only result if  $x_1$  and  $x_2$  are both equal or both odd and same for  $y_1$  and  $y_2$ .

There are four possible combinations of even and odd, being (odd, odd), (odd, even), (even, odd), and (even, even).

So, given five distinct points, the pigeonhole principle will show that at least two will have matching combinations of even or odd. Therefore, the midpoint of the line joining these two will have integer coordinates.

**Rosen 6.2, Exercise 14(a):**

a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.

Take the first ten integers and divide them into five pairs that add up to eleven:

(1, 10)

(2, 9)

(3, 8)

(4, 7)

(5, 6)

If seven integers are chosen, then there can only be three unchosen integers. If each of these unchosen integers denies a pair, they will only be able to deny the presence of up to three pairs. This would mean that there must be at least two pairs of these integers with the sum 11.

**Rosen 6.2, Exercise 40:**

Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

If it is assumed that there are  $n$  number of party attendees, a person may know 0 to  $n - 1$  other people.

Using a contradiction, if we say that  $n$  people know a different number of others, then for every number from 0 to  $n - 1$  there must be a person who knows that many others.

However, if a person has  $n - 1$  friends, it is not possible for another to have 0 friends. So, rather than from 0 to  $n - 1$ , it must actually be 1 to  $n - 1$  possibilities.

By the pigeonhole principle, there must be at least two people with the same number of friends, so the contradiction is false and the initial statement is true.

**Rosen 6.3, Exercise 32(c,d):**

How many strings of six lowercase letters from the English alphabet contain

c) the letters  $a$  and  $b$  in consecutive positions with  $a$  preceding  $b$ , with all the letters distinct?

Because all letters are distinct and  $a$  and  $b$  are taken, there are 24 possible letters remaining. At the same time, because there will be two positions taken up by  $a$  and  $b$ , there will be 4 positions remaining.

$$P(24, 4)$$

The position of  $ab$  can also be in five different possible locations in the string. This results in the equation:

$$5 \cdot P(24, 4) = 1275120$$

d) the letters  $a$  and  $b$ , where  $a$  is somewhere to the left of  $b$  in the string, with all the letters distinct?

Same as above, because all letters are distinct and  $a$  and  $b$  are taken, there are 24 possible letters remaining. At the same time, because there will be two positions taken up by  $a$  and  $b$ , there will be 4 positions remaining.

$$P(24, 4)$$

The two letters can be arranged can be arranged in 6 possible positions. however, the number of times  $a$  will be before  $b$  will amount to half of these.

$$\frac{P(6,2)}{2}$$

Combining the two it is possible to derive:

$$P(24, 4) \cdot \frac{P(6,2)}{2} = 3825360$$

**Rosen 6.3, Exercise 42:**

Find a formula for the number of ways to seat  $r$  of  $n$  people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

The formula for permutations of  $n$  choices  $r$  times is  $\frac{n!}{(n-r)!}$ . The formula functions to number each of the seats and allows us to assign  $r$  number of people to specific seats.

However, it is necessary to divide the formula by  $r$  because there are  $r$  number of positions that can start listing other, equivalent permutations.

However, it is also necessary to multiply the divisor by 2 because the permutations can hold the same seating in the opposite direction. However, this additional divisor only applies if  $r > 2$  because if it is less than or equal, then direction won't matter.

Therefore, the formula would be:

$$\frac{n!}{r \cdot (n-r)!} \text{ if } r > 2$$
$$\frac{n!}{2r \cdot (n-r)!} \text{ if } r \leq 2$$



**Rosen 6.3, Exercise 44:**

How many ways are there for a horse race with four horses to finish if ties are possible? [*Note* : Any number of the four horses may tie.]

The number of methods can be evaluated by finding the number of possible results as test cases:

4 distinct placements -  $4! = 24$

1st, 2nd and tie for 3rd -  $\frac{4!}{2!} = 12$

1st, tie for 2nd and 3rd -  $\frac{4!}{2!} = 12$

tie for 1st, 2nd and 3rd -  $\frac{4!}{2!} = 12$

tie for 1st and tie for 2nd -  $\frac{4!}{2! \cdot 2!} = 6$

1st and triple tie for 2nd -  $\frac{4!}{3!} = 4$

triple tie for 1st and 2nd -  $\frac{4!}{3!} = 4$

quadruple tie - 1

$$24 + 12 + 12 + 12 + 6 + 4 + 4 + 1 = 75$$

There are 75 ways for a horse race with four horses to finish if ties are possible.