Rosen 7.1, Exercise 18:

What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?

The number of different hands of 5 cards with a deck of cards is $\binom{52}{5}$. Additionally, this combination can be any of 4 suits and for each suit, there are 10 combinations of cards that result in a straight flush. Therefore, the total number of straight flushes is given by $4 \cdot 10 = 40$. Therefore, the probability is:

$$\frac{40}{\binom{52}{5}} = \frac{1}{64974}$$

Rosen 7.1, Exercise 40:

Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

If the host does not open the losing door, to win you must select the correct door on the first try therefore, the probability will be $\frac{1}{4}$.

On the other hand, to win by changing the selected door, the first guess must be wrong, with a probability of $\frac{3}{4}$ while the second must be correct with a probability of $\frac{1}{2}$. Together, this will result in $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$.

Rosen 7.2, Exercise 16:

Show that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.

Indepedndance is shown by $p(E \cap F) = p(E) \cdot p(F)$.

$p(\overline{E}) = 1 - p(E)$
$p(\overline{F}) = 1 - p(F)$

Thus we must prove $p(\overline{E} \cap \overline{F}) = p(\overline{E}) \cdot p(\overline{F})$:

$$
\begin{aligned}
p(\overline{E}) \cdot p(\overline{F}) &= (1 - p(E))(1 - p(F)) \\
&= 1 - p(E) - p(F) + p(E) \cdot p(F) \\
&= 1 - (p(E) + p(F) - p(E \cap F)) \\
&= 1 - p(E \cup F) \\
&= p(\overline{E \cup F}) \\
&= p(\overline{E} \cap \overline{F})
\end{aligned}
$$

Therefore, $\overline{E}$ and $\overline{F}$ are also independent events.

Rosen 7.2, Exercise 38(a):

A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

a) What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

There are 11 possible situations where at least one of the die could roll a six from 36 possible total solutions when both die are rolled.

There are 2 possible situations where one of the die could roll a six while the other rolls a one to add up to seven from 36 possible total solutions when both die are rolled.

This creates the situation where, if the observer is honest that there is a six, the probability that the sum is seven is $\frac{2}{11}$

Rosen 7.3, Exercise 16(b):

Ramesh can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will be late. The probability that he is late when he rides his bicycle is only 5%. Ramesh arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

b) Suppose the boss knows that Ramesh drives 30% of the time, takes the bus only 10% of the time, and takes his bicycle 60% of the time. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes theorem using this information?

$p(F_1) = 0.3$
$p(F_2) = 0.1$
$p(F_3) = 0.6$

Using Bayes theorem:

$$p(F_1|E) = \frac{p(E|F_1) \cdot p(F_1)}{(p(E|F_1) \cdot p(F_1)) + (p(E|F_2) \cdot p(F_2)) + (p(E|F_2) \cdot p(F_2))}$$
$$= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3) + (0.2 \cdot 0.1) + (0.05 \cdot 0.6)}$$
$$= 0.75$$

Rosen 7.3, Exercise 22:

Suppose that we have prior information concerning whether a random incoming message is spam. In particular, suppose that over a time period, we find that s spam messages arrive and h messages arrive that are not spam.

a) Use this information to estimate $p(S)$, the probability that an incoming message is spam, and $p(\overline{S})$, the probability an incoming message is not spam.

$p(S) = \frac{s}{s+h}$
$p(\overline{S}) = \frac{h}{s+h}$

b) Use Bayes theorem and part(a) to estimate the probability that an incoming message containing the word w is spam, where $p(w)$ is the probability that w occurs in a spam message and $q(w)$ is the probability that w occurs in a message that is not spam.

$$\frac{p(E|S) \cdot p(S)}{(p(E|S) \cdot p(S)) + (p(E|\overline{S}) \cdot p(\overline{S}))} = \frac{p(w) \cdot s}{(p(w) \cdot s) + (q(w) \cdot h)}$$

Rosen 7.4, Exercise 10:

Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?

The coin will be flipped 2 to 6 times, with 2 to 5 flips having to result in two tails or six being every other occasion. if n is the number of flips, then the probability of stopping after n flips is $\frac{n-1}{2^n}$. The probabilities for stopping for $2 \le n \le 5$:
n = 2: $\frac{1}{4}$
n = 3: $\frac{1}{4}$
n = 4: $\frac{3}{16}$
n = 5: $\frac{2}{16}$

Therefore, the remaining probability will result in six flips. This means, using one minus the sum when $2 \le n \le 5$. The probability for n = 6:
n = 6: $\frac{3}{16}$

The expected number of times the coin is flipped is given by:
$(2 \cdot \frac{1}{4}) + (3 \cdot \frac{1}{4}) + (4 \cdot \frac{3}{16}) + (5 \cdot \frac{2}{16}) + (6 \cdot \frac{3}{16}) = \frac{15}{4} = 3.75$

Rosen 7.4, Exercise 16:

Let X and Y be the random variables that count the number of heads and the number of tails that come up when two fair coins are flipped. Show that X and Y are not independent.

X has a probability of $\frac{1}{2}$ per trial over the course of 2 trials which results in:
$P(X = 0) = \frac{1}{4}$
$P(X = 1) = \frac{1}{2}$
$P(X = 2) = \frac{1}{4}$

Y has a probability of $\frac{1}{2}$ per trial over the course of 2 trials which results in:
$P(Y = 0) = \frac{1}{4}$
$P(Y = 1) = \frac{1}{2}$
$P(Y = 2) = \frac{1}{4}$

If zero heads are tossed $(X = 0)$ and the two tails are $(Y = 0)$:
$P(X = 0 \text{ and } Y = 2) = (\frac{1}{2})^0 \cdot (\frac{1}{2})^2 = \frac{1}{4}$
On the other hand:
$P(X = 0) \cdot P(Y = 2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$P(X = 0 \text{ and } Y = 2) \neq P(X = 0) \cdot P(Y = 2)$
Thus, by the multiplication rule of independence, X and Y are not independent.

Rosen 7.4, Exercise 28:

What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

The variance in n Bernoulli trials is npq where n is the number of trials equal to 10, p is the success probability equal to $\frac{1}{6}$ and q is the failure probability equal to $\frac{5}{6}$. Thus:

$npq = 10 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{25}{18}$

Rosen 7.4, Exercise 30:

Show that if X and Y are independent random variables, then $V(XY) = E(X)^2V(Y) + E(Y)^2V(X) + V(X)V(Y)$

$$
\begin{aligned}
V(XY) &= E(X^2Y^2) - E(XY)^2 \\
&= E(X^2)E(Y^2) - V(X)V(Y) - E(X)^2E(Y)^2 + V(X)V(Y) \\
&= E(X^2)E(Y^2) - (E(X^2) - E(X)^2)(E(Y^2) - E(Y)^2) - E(X)^2E(Y)^2 + V(X)V(Y) \\
&= E(X)^2E(Y^2) + E(X^2)E(Y)^2 - 2E(X)^2E(Y)^2 + V(X)V(Y) \\
&= E(X)^2E(Y^2) - E(X)^2E(Y)^2 + E(X^2)E(Y)^2 - E(X)^2E(Y)^2 + V(X)V(Y) \\
&= E(X)^2V(Y) + E(Y)^2V(X) + V(X)V(Y)
\end{aligned}
$$