\documentclass[10pt]{article}

% math fonts

\usepackage{amsmath,amsfonts,amsthm,amssymb}

% to insert graphics

\usepackage{graphicx}

% to change margins of the pages

\usepackage[margin=0.9in]{geometry}

% Makes equations flush left

\usepackage{fleqn}

% This generates a page header with your name in it.

\usepackage{fancyhdr}

\pagestyle{fancy}

\fancyhf{}

\lhead{FOCS Fall 2018}

\rhead{HW08 solution by Sriyuth Sagi}

\rfoot{Page \thepage}

% This package makes it easy to have boxes around large text.

\usepackage{framed}

\begin{document}

{\bf Rosen 5.3, Exercise 14:} \\

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Show that $f\_{n+1} f\_{n-1} - f\_n^2 = (-1)^n$ when $n$ is a positive integer.\\

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$P(n) = f\_{n+1} f\_{n-1} - f\_n^2 = (-1)^n$\\\\

Base Step: when $n = 1$\\

$f\_{(1)+1} f\_{(1)-1} - f\_{(1)}^2 = 1 \cdot 0 - (1^2)$

\hspace{2.4cm}$= -1$

\hspace{2.4cm}$= (-1)^{(1)}$\\

Which shows that the base step is satisfied.\\\\

Induction Step: If we assume the satement holds for $P(k)$, then we must prove that it holds for $P(k+1)$\\

$f\_{(k+1)+1} f\_{(k+1)-1} - f\_{(k+1)}^2 = f\_{k+2} f\_{k} - f\_{k+1}^2$

\hspace{3.5cm}$= f\_k(f\_{k} - f\_{k+1}) - f\_{k+1}^2$

\hspace{3.5cm}$= f\_k^2 - f\_{k} f\_{k+1} - f\_{k+1}^2$

\hspace{3.5cm}$= -f\_{k+1}(f\_{k} + f\_{k+1}) + f\_{k}^2$

\hspace{3.5cm}$= -f\_{k+1}f\_{k-1} + f\_{k}^2$

\hspace{3.5cm}$= (-1)^1 \cdot (f\_{k+1}f\_{k-1} - f\_{k}^2)$

\hspace{3.5cm}$= (-1)^1 \cdot (-1)^k$

\hspace{3.5cm}$= (-1)^{k+1}$\\

Thus, the induction step is shown to hold true for $P(k+1)$.\\\\

By the principle of mathematical induction, $f\_{n+1} f\_{n-1} - f\_n^2 = (-1)^n$ when $n$ is a positive integer.

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{\bf Rosen 5.3, Exercise 26:} \\

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Let $S$ be the subset of the set of ordered pairs of integers defined recursively by\\\\

$Basis$ $step:$ $(0, 0) \in S$.\\\\

$Recursive$ $step:$ If $(a,b) \in S$, then $(a + 2,b + 3) \in S$ and $(a+3,b+2) \in S$.\\\\

a) List the elements of $S$ produced by the first five applications of the recursive definition.\\\\

first application: (2, 3), (3, 2)\\

second application: (4, 6), (5, 5), (6, 4)\\

third application: (6, 9), (7, 8), (8, 7), (9, 6)\\

fourth application: (8, 12), (9, 11), (10, 10), (11, 9), (12, 8)\\

fifth application: (10, 15), (11, 14), (12, 13), (13, 12), (14, 11), (15, 10)\\\\

b) Use strong induction on the number of applications of the recursive step of the definition to show that $5|a+b$ when $(a,b) \in S$.\\

Let $P(n) = $ "$5|a+b$ when $(a,b) \in S$"\\

Base Step: let $n = 0$\\

$(0,0) \in S$\\

$5|0+0$\\

This satisfies the base step because 0 is divisible by any number.\\

Induction Step: If $P(1),P(2),...,P(k)$ are true, we must prove $P(k+1)$ is also true\\

If $P(k)$ is $(a,b) \in S$, then $P(k+1)$ must be either $(a+2,b+3) \in S$ or $(a+3,b+2) \in S$, therefore, we must prove both satisfy the condition.\\

For $(a+2,b+3) \in S$ or $(a+3,b+2) \in S$, adding both terms from either will result in $a + b + 5$ and since we know $a+b$ is divisible by five and 5 is divisible by five, it is possible to conclude $5|(a+2)+(b+3)$ and $5|(a+3)+(b+2)$.\\

Thus the induction step is satisfied because $P(k+1)$ is satisfied.\\

By the principle of strong induction, the statement is true.\\\\

c) Use structural induction to show that $5 | a + b$ when $(a,b) \in S$.\\

Let $P(n) = $ "$5|a+b$ when $(a,b) \in S$"\\

Base Step: let $n = 0$\\

$(0,0) \in S$\\

$5|0+0$\\

This satisfies the base step because 0 is divisible by any number.\\

Recursive Step: Assume $(a,b) \in S$ with $5 | a + b$. The two possible next steps are $(a + 2,b + 3) \in S$ and $(a+3,b+2) \in S$\\

Since $(a+2,b+3) \in S$ and $(a+3,b+2) \in S$ both give $a + b + 5$ and since we know $a+b$ is divisible by five and 5 is divisible by five, it is possible to conclude $5|(a+2)+(b+3)$ and $5|(a+3)+(b+2)$.\\

Thus, the recursive step is satisfied.\\

By the principle of structural induction, the statement is true.

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{\bf Rosen 5.3, Exercise 44:} \\

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Use structural induction to show that $l(T )$, the number of leaves of a full binary tree $T$ , is 1 more than $i(T )$, the number of internal vertices of $T$ .\\

For full binary tree $T$, $l(T) = i(T) + 1$\\\\

Base Step: A binary tree made from just the vertex\\

$T = 0$\\

$leaves = (0) + 1 = 1$\\

$internal$ $vertices = (0) = 0$\\

A binary tree with a single vertice has 1 leaf and 0 internal vertices which is shown in the base step, thus, the base step is satisfied.\\\\

Recursive Step: If there are two full binary trees, $T\_1$ and $T\_2$, must show the statement holds true for the conjunction of both shown by $T\_{1,2}$\\\\

Because, when combining two trees, the number of internal vertices will be the combination of both along with one new intenal vertice from the connection between both trees. Thus:\\

$i(T\_{1,2}) = i(T\_1) + i(T\_2) + 1$\\\\

On the other hand, even after both trees are combined, this will not result in the formation of new leaves. Thus:\\

$T\_{1,2} = l(T\_1) + l(T\_2)$\\\\

To show the statement holds trye for $T\_{1,2}$:\\

$T\_{1,2} = l(T\_1) + l(T\_2)$

\hspace{0.2cm}$= (i(T\_1) + 1) + (i(T\_2) + 1)$

\hspace{0.2cm}$= (i(T\_1) + i(T\_2) + 1) + 1$

\hspace{0.2cm}$= i(T\_{1,2}) + 1$\\

Thus the induction step is shown to hold true when both trees are combined.\\\\

By the principle of structural induction, the statement holds true.

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{\bf Rosen 5.4, Exercise 8:} \\

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Give a recursive algorithm for finding the sum of the first $n$ positive integers.\\

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\textbf{procedure} $sum(n:$ positive integer$)$\\

\textbf{if} $n = 1$ \textbf{then}

\textbf{return} 1\\

\textbf{else}

\textbf{return} $sum(n - 1) + n$\\

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{\bf Rosen 5.4, Exercise 16:} \\

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Prove that the recursive algorithm for finding the sum of the first $n$ positive integers you found in Exercise 8 is correct.\\

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Proof By Induction:\\\\

Base Step: when n = 1\\\\

The algorithm returns 1 due to the if statement. The sum of numbers till 1 also evaluates to 1 so the base step can be found to be true.\\\\

Induction Step: assume that the algorithm is correct for a number $sum(k)$, to prove the inductive step, the algorithm must evaluate for $sum(k+1)$ so:\\\\

$sum(k) = \sum\_{n=1}^{k} n$\\\\\\

$sum(k+1) = sum(k) + k + 1$\\

\hspace{1.3cm}$= \sum\_{n=1}^{k} n + k + 1$\\

\hspace{1.3cm}$= \sum\_{n=1}^{k+1} n + k + 1$\\

\hspace{1.3cm}$= \sum\_{n=1}^{(k+1)} n$\\\\

The equation now gives the sum for the first $k+1$ integers thus the induction step is proven to evaluate to true.\\\\

By the principle of mathematical induction, the algorithm found in Exercise 8 is correct.

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