(1) DPV Problem 8.4. Note that for part (d), while you will receive partial credit for an algorithm with running time $O(|V|^4)$, to receive full credit you must give an algorithm with time $O(|V|)$.

Consider the CLIQUE problem restricted to graphs in which every vertex has degree at most 3. Call this problem CLIQUE-3.

(a) Prove that CLIQUE-3 is in NP.

If given a solution S to the problem consisting of vertices in the graph, we can prove that S forms a clique by looping through each pair of vertices in $O(|V|^2)$ time so it is NP.

(b) What is wrong with the following proof of NP-completeness for CLIQUE-3?
We know that the CLIQUE problem in general graphs is NP-complete, so it is enough to present a reduction from CLIQUE-3 to CLIQUE. Given a graph G with vertices of degree 3, and a parameter g, the reduction leaves the graph and the parameter unchanged: clearly the output of the reduction is a possible input for the CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, the NP-completeness of CLIQUE-3.

The proof has CLIQUE-3 reducing to CLIQUE. This implies that CLIQUE-3 is not harder than CLIQUE. However, this is wrong as the reduction is backwards. The proof should instead reduce CLIQUE to CLIQUE-3.

(c) It is true that the VERTEX COVER problem remains NP-complete even when restricted to graphs in which every vertex has degree at most 3. Call this problem VC-3. What is wrong with the following proof of NP-completeness for CLIQUE-3?
We present a reduction from VC-3 to CLIQUE-3. Given a graph $G = (V, E)$ with node degrees bounded by 3, and a parameter b, we create an instance of CLIQUE-3 by leaving the graph unchanged and switching the parameter to $|V| - b$. Now, a subset $C \subseteq V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G. Therefore G has a vertex cover of size $\leq b$ if and only if it has a clique of size $\geq |V| - b$. This proves the correctness of the reduction and, consequently, the NP-completeness of CLIQUE-3.

The proof states, "a subset $C \subseteq V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G". This statement is false and must be changed to show that C is a vertex cover if and only if the complimntary set is an independentt set in G.

(d) Describe an $O(|V|^4)$ algorithm for CLIQUE-3.

The algorithm should take in a graph G with a number k representin the number of vertices.
Use a series of if statements to evaluate whether the graph satisfies CLIQUE-3.
If $k > 4$ then it cannot satisfy CLIQUE-3.
If $k == 4$ then iterate through all the vertices and check if they all have a degree of three, if not, then it does not satisfy CLIQUE-3.
If $k == 3$ then iterate through all the vertices and check if they all have a degree of two, if not, then it does not satisfy CLIQUE-3.
If $k == 2$ then iterate through all the vertices and check if they all have a degree of one, if not, then it does not satisfy CLIQUE-3.
If $k == 3$ then it satisfies CLIQUE-3.
This algorithm will be $O(|V|)$ because it uses a conditional and the maximum it can ever be on any of the paths is $O(|V|)$.

(2) DPV Problem 8.10 (a) and (b). Give a clear reduction (i.e., an algorithm solving one problem using the other as a black box) and show that the problem is NP-Complete; it is not enough to just say that A is a generalization of B, you must provide a reduction.

Proving NP-completeness by generalization. For each of the problems below, prove that it is NP-complete by showing that it is a generalization of some NP-complete problem we have seen in this chapter.

(a) SUBGRAPH ISOMORPHISM: Given as input two undirected graphs G and H, determine whether G is a subgraph of H (that is, whether by deleting certain vertices and edges of H we obtain a graph that is, up to renaming of vertices, identical to G), and if so, return the corresponding mapping of $V(G)$ into $V(H)$.

This problem can be solved using a generalization of clique which takes an input $CLIQUE(G, k)$ where G is a graph and k is the number of strongly connected components in graph H. We can say that G is a subset of H if and only if H has a subset of size k with which you can create a subset of G. Therefore, using $CLIQUE$, we obtain a complete graph of size k. This graph will be a subset of G if there is a clique of size k in G. If this clique exists, it will be proven as NP-complete.

(b) LONGEST PATH: Given a graph G and an integer g, find in G a simple path of length g.

This problem is a generalization of $RUDRATA - PATH$ problem with an input $RUDRATA - PATH(G, g)$ where G has g vertices. Additionally, we would want to make sure G is undirected. The $RUDRATA - PATH$ with these inputs will find the longest path by itself so, as $RUDRATA - PATH$ is NP-complete, longest path will be NP-complete.

(3) DPV Problem 8.10 (d) and (e). Give a clear reduction (i.e., an algorithm solving one problem using the other as a black box) and show that the problem is NP-Complete; it is not enough to just say that A is a generalization of B, you must provide a reduction.

(d) DENSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at least b edges between them.

This problem can be solved using a generalization of clique. If we assume n represents the number of edges, take $a = n$ and $b = \frac{n(n-1)}{2}$. Then take the input as $CLIQUE(G, n)$. This will leave a subgraph with the original number of vertices and $n(n-1)2$ edges. This will imply that there must be an edge between each pair of vertices and it forms a clique so the graph will be NP-complete.

(e) SPARSE SUBGRAPH: Given a graph and two integers a and b, find a set of a vertices of G such that there are at most b edges between them.

This is a generalization of the vertex cover problem with an input taken as (G, a) where a represents the maximum number of vertices in the cover. Such a vertex cover will create a sparse subgraph with b reoresenting the number of edges. The problem would be phrased like sparseSubGraph($V = (G, a), b = |E|$). Thus, since vertex cover is NP-complete, sparse subgraph will be NP-complete.

(3) DPV Problem 8.14.

Prove that the following problem is NP-complete: given an undirected graph $G = (V, E)$ and an integer k, return a clique of size k as well as an independent set of size k, provided both exist.

To show the problem is NP, if there is a solution of size k which is a clique inside of G as well as an independent set of size k, the problem will be shown to be NP if both of these statements are true.
Next, show we can solve independent set if this is true. If given a graph G and number k, we must determine if an independent set of size k exists in G.
We will implement a new algorithm that evaluates the clique problem and the independent set problem. First take G and k as inputs and first adds k new vertices to G with all these k vertices are connected to each other in a clique and not connected to other vertices in G. This new graph trivially has a a clique of size k. If we run the algorithm to evaluate the previous problems on the modified graph, it being true represents G has an independent set of size k while false represents that it does not. This will show the problem is NP-complete as it is formed through generalizations of clique and independant set.