\documentclass[10pt]{article}

% math fonts

\usepackage{amsmath,amsfonts,amsthm,amssymb}

% to insert graphics

\usepackage{graphicx}

% to change margins of the pages

\usepackage[margin=0.9in]{geometry}

% Makes equations flush left

\usepackage{fleqn}

% This generates a page header with your name in it.

\usepackage{fancyhdr}

\pagestyle{fancy}

\fancyhf{}

\lhead{Algorithms Spring 2019}

\rhead{HW07 solution by Sriyuth Sagi}

\rfoot{Page \thepage}

% This package makes it easy to have boxes around large text.

\usepackage{framed}

\begin{document}

{\bf (1) DPV Problem 8.4. Note that for part (d), while you will receive partial credit for an algorithm with running time $O(|V |^4)$, to receive full credit you must give an algorithm with time $O(|V |)$.} \\

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Consider the CLIQUE problem restricted to graphs in which every vertex has degree at most 3. Call this problem CLIQUE-3.\\

(a) Prove that CLIQUE-3 is in NP.\\

\\

If given a solution $S$ to the problem consisting of vertices in the graph, we can prove that S forms a clique by looping through each pair of vertices in $O(|V|^2)$ time so it is NP.\\

\\

(b) What is wrong with the following proof of NP-completeness for CLIQUE-3?\\

We know that the CLIQUE problem in general graphs is NP-complete, so it is enough to present a reduction from CLIQUE-3 to CLIQUE. Given a graph G with vertices of degree ≤ 3, and a parameter g, the reduction leaves the graph and the parameter unchanged: clearly the output of the reduction is a possible input for the CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, the NP-completeness of CLIQUE-3.\\

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The proof has CLIQUE-3 reducing to CLIQUE. This implies that CLIQUE-3 is not harder than CLIQUE. However, this is wrong as the reduction is backwards. The proof should instead reduce CLIQUE to CLIQUE-3.\\

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(c) It is true that the VERTEX COVER problem remains NP-complete even when restricted to graphs in which every vertex has degree at most 3. Call this problem VC-3. What is wrong with the following proof of NP-completeness for CLIQUE-3?\\

We present a reduction from VC-3 to CLIQUE-3. Given a graph $G = (V, E)$ with node degrees bounded by 3, and a parameter $b$, we create an instance of CLIQUE-3 by leaving the graph unchanged and switching the parameter to $|V | - b$. Now, a subset $C \subseteq V$ is a vertex cover in $G$ if and only if the complementary set $V - C$ is a clique in $G$. Therefore $G$ has a vertex cover of size $\leq b$ if and only if it has a clique of size $\geq |V | - b$. This proves the correctness of the reduction and, consequently, the NP-completeness of CLIQUE-3.\\

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The proof states, "a subset $C \subseteq V$ is a vertex cover in $G$ if and only if the complementary set $V - C$ is a clique in $G$". This statement is false and must be changed to show that $C$ is a vertex cover if and only if the complimntary set is an independentt set in $G$.\\

\\

(d) Describe an $O(|V |^4)$ algorithm for CLIQUE-3.\\

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The algorithm should take in a graph $G$ with a number $k$ representin the number of vertices.\\

Use a series of if statements to evaluate whether the graph satisfies CLIQUE-3.\\

If $k > 4$ then it cannot satisfy CLIQUE-3.\\

If $k == 4$ then iterate through all the vertices and check if they all have a degree of three, if not, then it does not satisfy CLIQUE-3.\\

If $k == 3$ then iterate through all the vertices and check if they all have a degree of two, if not, then it does not satisfy CLIQUE-3.\\

If $k == 2$ then iterate through all the vertices and check if they all have a degree of one, if not, then it does not satisfy CLIQUE-3.\\

If $k == 3$ then it satisfies CLIQUE-3.\\

This algorithm will be $O(|V |)$ because it uses a conditional and the maximum it can ever be on any of the paths is $O(|V |)$.

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{\bf (2) DPV Problem 8.10 (a) and (b). Give a clear reduction (i.e., an algorithm solving one problem using the other as a black box) and show that the problem is NP-Complete; it is not enough to just say that A is a generalization of B, you must provide a reduction.} \\

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Proving NP-completeness by generalization. For each of the problems below, prove that it is NP-complete by showing that it is a generalization of some NP-complete problem we have seen in this chapter.\\

(a) SUBGRAPH ISOMORPHISM: Given as input two undirected graphs $G$ and $H$, determine whether $G$ is a subgraph of $H$ (that is, whether by deleting certain vertices and edges of $H$ we obtain a graph that is, up to renaming of vertices, identical to $G$), and if so, return the corresponding mapping of $V (G)$ into $V (H)$.\\

\\

This problem can be solved using a generalization of clique which takes an input $CLIQUE(G,k)$ where $G$ is a graph and $k$ is the number of strongly connected components in graph $H$. We can say that $G$ is a subset of $H$ if and only if $H$ has a subset of size $k$ with which you can create a subset of $G$. Therefore, using $CLIQUE$, we obtain a complete graph of size $k$. This graph will be a subset of $G$ if there is a clique of size $k$ in $G$. If this clique exists, it will be proven as NP-complete.\\

\\

(b) LONGEST PATH: Given a graph $G$ and an integer $g$, find in $G$ a simple path of length $g$.\\

\\

This problem is a generalization of $RUDRATA-PATH$ problem with an input $RUDRATA-PATH(G,g)$ where $G$ has $g$ vertices. Additionally, we would want to make sure $G$ is undirected. The $RUDRATA-PATH$ with these inputs will find the longest path by itself so, as $RUDRATA-PATH$ is NP-complete, longest path will be NP-complete.

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{\bf (3) DPV Problem 8.10 (d) and (e). Give a clear reduction (i.e., an algorithm solving one problem using the other as a black box) and show that the problem is NP-Complete; it is not enough to just say that A is a generalization of B, you must provide a reduction.} \\

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(d) DENSE SUBGRAPH: Given a graph and two integers $a$ and $b$, find a set of a vertices of $G$ such that there are at least $b$ edges between them.\\

\\

This problem can be solved using a generalization of clique. If we assume $n$ represents the number of edges, take $a = n$ and $b = \dfrac{n(n-1)}{2}$. Then take the input as $CLIQUE(G,n)$. This will leave a subgraph with the original number of vertices and ${n(n-1)}{2}$ edges. This will imply that there must be an edge between each pair of vertices and it forms a clique so the graph will be NP-complete.\\

\\

(e) SPARSE SUBGRAPH: Given a graph and two integers $a$ and $b$, find a set of a vertices of $G$ such that there are at most $b$ edges between them.\\

\\

This is a generalization of the vertex cover problem with an input taken as $(G,a)$ where $a$ represents the maximum number of vertices in the cover. Such a vertex cover will create a sparse subgraph with $b$ reoresenting the number of edges. The problem would be phrased like sparseSubGraph($V=(G,a),b=|E|$). Thus, since vertex cover is NP-complete, sparse subgraph will be NP-complete.

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{\bf (3) DPV Problem 8.14.} \\

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Prove that the following problem is NP-complete: given an undirected graph $G = (V, E)$ and an integer $k$, return a clique of size $k$ as well as an independent set of size $k$, provided both exist.\\

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To show the problem is NP, if there is a solution of size $k$ which is a clique inside of G as well as an independent set of size $k$, the problem will be shown to be NP if both of these statements are true. \\

Next, show we can solve independent set if this is true. If given a graph $G$ and number $k$, we must determine if an independent set of size $k$ exists in $G$. \\

We will implement a new algorithm that evaluates the clique problem and the independent set problem. First take $G$ and $k$ as inputs and first adds $k$ new vertices to $G$ with all these $k$ vertices are connected to each other in a clique and not connected to other vertices in $G$. This new graph trivially has a a clique of size $k$. If we run the algorithm to evaluate the previous problems on the modified graph, it being true represents G has an independent set of size $k$ while false represents that it does not. This will show the problem is NP-complete as it is formed through generalizations of clique and independant set.

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