Data modeling: CSCI E-106

Applied Linear Statistical Models

Chapter 7 – Multiple Regression II

Illustration for the extra sum of squares

Example 1: A study of the relation of amount of body fat: a sample of 20 healthy females: 25-34 years old

- Y : body fat
- X_1: triceps skinfold thickness
- X_2: thigh circumference
- X_3: midarm circumference

It would be very helpful if a regression model with some or all these predictor variables could provide reliable estimates of amount of body fat.

Illustration for the extra sum of squares, cont'd

Table : Basic Data-Body Fat Example.

Subject i	Triceps Skinfold Thickness X_{i1}	Thigh Circumference X_{i2}	Midarm Circumference X_{i3}	Body Fat Y_i
1	19.50	43.10	29.10	11.90
2	24.70	49.80	28.20	22.80
3	30.70	51.90	37.00	18.70
4	29.80	54.30	31.10	20.10
5	19.10	42.20	30.90	12.90
6	25.60	53.90	23.70	21.70
7	31.40	58.50	27.60	27.10
8	27.90	52.10	30.60	25.40
9	22.10	49.90	23.20	21.30
10	25.50	53.50	24.80	19.30
11	31.10	56.60	30.00	25.40
12	30.40	56.70	28.30	27.20
13	18.70	46.50	23.00	11.70
14	19.70	44.20	28.60	17.80
15	14.60	42.70	21.30	12.80
16	29.50	54.40	30.10	23.90
17	27.70	55.30	25.70	22.60
18	30.20	58.60	24.60	25.40
19	22.70	48.20	27.10	14.80
20	25.20	51.00	27.50	21.10

Illustration for the extra sum of squares, cont'd

(a) Regression of Y on X_1 (b) Regression of Y on X_2

TABLE 7.2 Regression Results for Several Fitted Models—Body Fat Example.

(a) Regression of Y on X_1
$\hat{Y} = -1.496 + .8572 X_1$

Source of Variation	SS	df	MS
Regression	352.27	1	352.27
Error	143.12	18	7.95
Total	495.39	19	

Variable	Estimated Regression Coefficient	Estimated Standard Deviation	t^*
X_1	$b_1 = .8572$	$s\{b_1\} = .1288$	6.66

(b) Regression of Y on X_2
$\hat{Y} = -23.634 + .8565 X_2$

Source of Variation	SS	df	MS
Regression	381.97	1	381.97
Error	113.42	18	6.30
Total	495.39	19	

Variable	Estimated Regression Coefficient	Estimated Standard Deviation	t^*
X_2	$b_2 = .8565$	$s\{b_2\} = .1100$	7.79

Illustration for the extra sum of squares, cont'd

(c) Regression of Y on X_1 and X_2 (d) Regression of Y on X_1, X_2 and X_3

TABLE 7.2
(Continued).

(c) Regression of Y on X_1 and X_2
$\hat{Y} = -19.174 + .2224X_1 + .6594X_2$

Source of Variation	SS	df	MS
Regression	385.44	2	192.72
Error	109.95	17	6.47
Total	495.39	19	

Variable	Estimated Regression Coefficient	Estimated Standard Deviation	t^*
X_1	$b_1 = .2224$	$s\{b_1\} = .3034$	.73
X_2	$b_2 = .6594$	$s\{b_2\} = .2912$	2.26

(d) Regression of Y on X_1, X_2, and X_3
$\hat{Y} = 117.08 + 4.334X_1 - 2.857X_2 - 2.186X_3$

Source of Variation	SS	df	MS
Regression	396.98	3	132.33
Error	98.41	16	6.15
Total	495.39	19	

Variable	Estimated Regression Coefficient	Estimated Standard Deviation	t^*
X_1	$b_1 = 4.334$	$s\{b_1\} = 3.016$	1.44
X_2	$b_2 = -2.857$	$s\{b_2\} = 2.582$	−1.11
X_3	$b_3 = -2.186$	$s\{b_3\} = 1.596$	−1.37

Illustration for the extra sum of squares, cont'd

Notations:

- Assume X_1 is in the model
  - $SSR(X_1)$: The regression sum of squares
  - $SSE(X_1)$: The error sum of squares

  - measure the marginal effect of adding X_2 (another variable) to the regression model when X_1 is already in the model
    - $SSR(X_2|X_1)$: *The extra sum of squares gained by adding* X_2

- Assume X_1 and X_2 are in the model
  - $SSR(X_1, X_2)$: The regression sum of squares
  - $SSE(X_1, X_2)$: The error sum of squares
  - measure the marginal effect of adding X_3 (another variable) to the regression model when X_1 and X_2 are already in the model
    - $SSR(X_3|X_1, X_2)$: *The extra sum of squares gained by adding* X_3

Illustration for the extra sum of squares, cont'd

(a) Regression of Y on X_1
$\hat{Y} = -1.496 + .8572X_1$

Source of Variation	SS	df	MS
Regression	352.27	1	352.27
Error	143.12	18	7.95
Total	495.39	19	

Variable	Estimated Regression Coefficient	Estimated Standard Deviation	t^*
X_1	$b_1 = .8572$	$s\{b_1\} = .1288$	6.66

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$\hat{Y} = -19.174 + .2224X_1 + .6594X_2$

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X_2	$b_2 = .6594$	$s\{b_2\} = .2912$	2.26

An extra sum of squares:

$SSR(X_2|X_1) = SSE(X_1) - SSE(X_1, X_2) = 143.12 - 109.95 = 33.17$

$= SSR(X_1, X_2) - SSR(X_1) = 385.44 - 352.27 = 33.17$

$SSR(X2|X1)$

- the marginal increase in the regression sum of squares (SSR)
- reflects the additional or extra reduction in the error sum of squares (SSE) associated with $X2$, given that $X1$ is already included in the model

Illustration for the extra sum of squares, cont'd

- The marginal reduction in the *SSE* = The marginal increase in *SSR*

- SSTO = SSR + SSE:
  - measure the variability of Y_i and does not depend on the regression model fitted
  - Any reduction in SSE implies an identical increase in SSR

Illustration for the extra sum of squares, cont'd

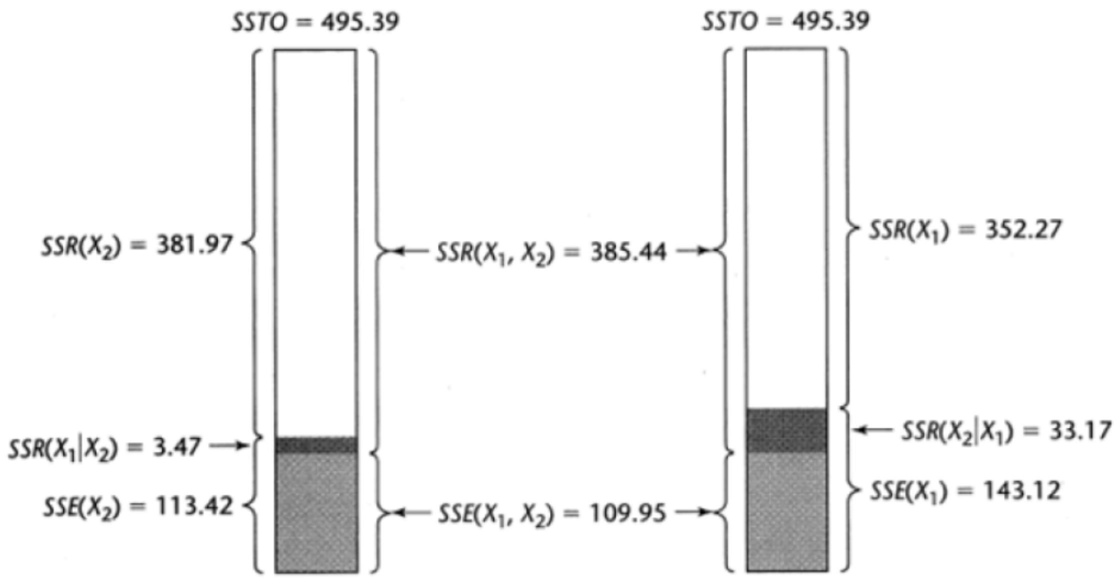

Figure : Schematic Representation of Extra Sums of Squares-Body Fat Example.

$$
\begin{aligned}
SSR(X_2|X_1) &= SSE(X_1) - SSE(X_1, X_2) = 143.12 - 109.95 = 33.17 \\
&= SSR(X_1, X_2) - SSR(X_1) = 385.44 - 352.27 = 33.17
\end{aligned}
$$

Illustration for the extra sum of squares, cont'd

(c) Regression of Y on X_1 and X_2

$\hat{Y} = -19.174 + .2224X_1 + .6594X_2$

Source of Variation	SS	df	MS
Regression	385.44	2	192.72
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(d) Regression of Y on X_1, X_2, and X_3

$\hat{Y} = 117.08 + 4.334X_1 - 2.857X_2 - 2.186X_3$

Source of Variation	SS	df	MS
Regression	396.98	3	132.33
Error	98.41	16	6.15
Total	495.39	19	

Variable	Estimated Regression Coefficient	Estimated Standard Deviation	t^*
X_1	$b_1 = 4.334$	$s\{b_1\} = 3.016$	1.44
X_2	$b_2 = -2.857$	$s\{b_2\} = 2.582$	−1.11
X_3	$b_3 = -2.186$	$s\{b_3\} = 1.596$	−1.37

An extra sum of squares: adding X_3

$$SSR(X_3|X_1,X_2) = SSE(X_1,X_2) - SSE(X_1,X_2,X_3) = 109.95 - 98.41 = 11.54$$
$$= SSR(X_1,X_2,X_3) - SSR(X_1,X_2) = 396.98 - 385.44 = 11.54$$

An extra sum of squares: adding X_2, X_3

$$SSR(X_2,X_3|X_1) = SSE(X_1) - SSE(X_1,X_2,X_3) = 143.12 - 98.41 = 44.71$$
$$= SSR(X_1,X_2,X_3) - SSR(X_1) = 396.98 - 352.27 = 44.71$$

Illustration for the extra sum of squares, cont'd

Extra Sums of Squares

- An extra sum of squares measures the marginal decrease in the error sum of squares when one or several predictor variables are added to the regression model, given that other variables are already in the model.

- Equivalently, one can view the extra sum of squares as measuring the marginal increase in the regression sum of squares

- Extra: $SSE \downarrow$; $SSR \uparrow$

Definitions

Extra Sums of Squares for two variables:

If X_1 is the extra variable:

$$\begin{aligned} SSR(X_1|X_2) &= SSE(X_2) - SSE(X_1, X_2) \\ &= SSR(X_1, X_2) - SSR(X_2) \end{aligned}$$

If X_2 is the extra variable:

$$\begin{aligned} SSR(X_2|X_1) &= SSE(X_1) - SSE(X_1, X_2) \\ &= SSR(X_1, X_2) - SSR(X_1) \end{aligned}$$

Definitions, cont'd

Extra Sums of Squares for three variables:

If X_3 is the extra variable:

$$SSR(X_3 | X_1, X_2) = SSE(X_1, X_2) - SSE(X_1, X_2, X_3)$$
$$= SSR(X_1, X_2, X_3) - SSR(X_1, X_2)$$

If X_2 , X_3 are the extra variables:

$$SSR(X_3, X_2 | X_1) = SSE(X_1) - SSE(X_1, X_2, X_3)$$
$$= SSR(X_1, X_2, X_3) - SSR(X_1)$$

Extensions for more variables are straightforward and easily follow as above.

Decomposition of SSR into Extra Sums of Squares

Consider the cade of two X variables. We begin with the SSTO identity (2.50) for variable X_1:

$$SSTO = SSE(X_1) + SSR(X_1)$$

From slide 12:

$$SSR(X_2|X_1) = SSE(X_1) - SSE(X_1, X_2) \Rightarrow SSE(X_1) = SSR(X_2|X_1) + SSE(X_1, X_2)$$

Then,

$$\begin{aligned} SSTO &= SSR(X_1) + SSE(X_1, X_2) \\ &= SSR(X_1, X_2) - SSR(X_1) + SSR(X_1) + SSE(X_1, X_2) \\ &= SSR(X_1, X_2) + SSE(X_1, X_2) \end{aligned}$$

Also From slide 12; $SSR(X_2, X_1) = SSR(X_1) + SSR(X_2|X_1)$

Decomposition of SSR into Extra Sums of Squares, cont'd

Decomposition $SSR(X_2, X_1) = SSR(X_1) + SSR(X_2|X_1)$

- $SSR(X_1)$: measuring the contribution by including X_1 alone in the model
- $SSR(X_2|X_1)$: measuring the addition contribution when X_2 is included, given that X_1 is already in the model
- The order of the X variables is arbitrary

$$SSR(X_2, X_1) = SSR(X_2) + SSR(X_1|X_2)$$

Decomposition of SSR into Extra Sums of Squares, cont'd

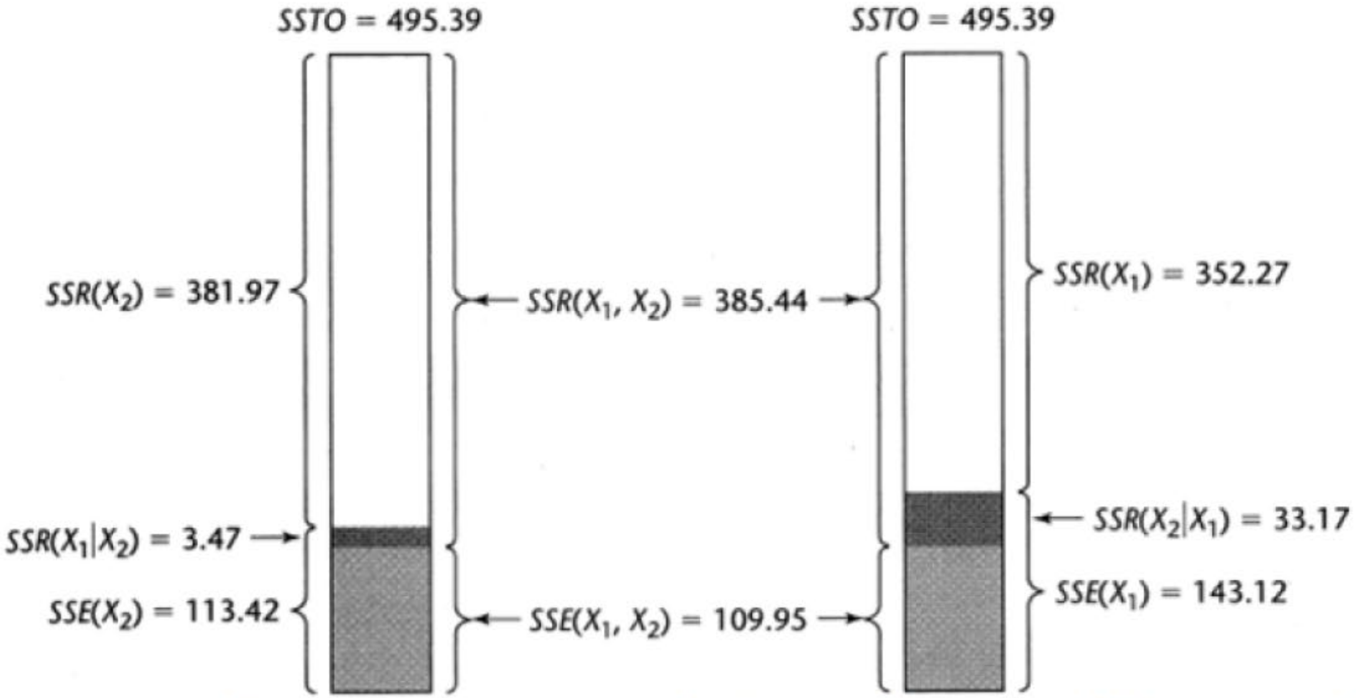


Figure : Schematic Representation of Extra Sums of Squares-Body Fat Example.

Decomposition of SSR into Extra Sums of Squares, cont'd

When the regression model contains three X variables (X_1, X_2, X_3):

$$\begin{aligned} SSR(X_1, X_2, X_3) &= SSR(X_1) + SSR(X_2|X_1) + SSR(X_3|X_1, X_2) \\ &= SSR(X_2) + SSR(X_3|X_2) + SSR(X_1|X_2, X_3) \\ &= SSR(X_3) + SSR(X_1|X_3) + SSR(X_2|X_1, X_3) \\ &= SSR(X_1) + SSR(X_2, X_3|X_1) \end{aligned}$$

The number of possible decompositions becomes vast as the number of X variables in the regression model increases.

ANOVA Table Containing Decomposition of SSR

Example of ANOVA Table With Decomposition Three X Variables.

Source of Variation	SS	df	MS
Regression	$SSR(X_1, X_2, X_3)$	3	$MSR(X_1, X_2, X_3)$
X_1	$SSR(X_1)$	1	$MSR(X_1)$
$X_2 \mid X_1$	$SSR(X_2 \mid X_1)$	1	$MSR(X_2 \mid X_1)$
$X_3 \mid X_1, X_2$	$SSR(X_3 \mid X_1, X_2)$	1	$MSR(X_3 \mid X_1, X_2)$
Error	$SSE(X_1, X_2, X_3)$	$n-4$	$MSE(X_1, X_2, X_3)$
Total	$SSTO$	$n-1$	

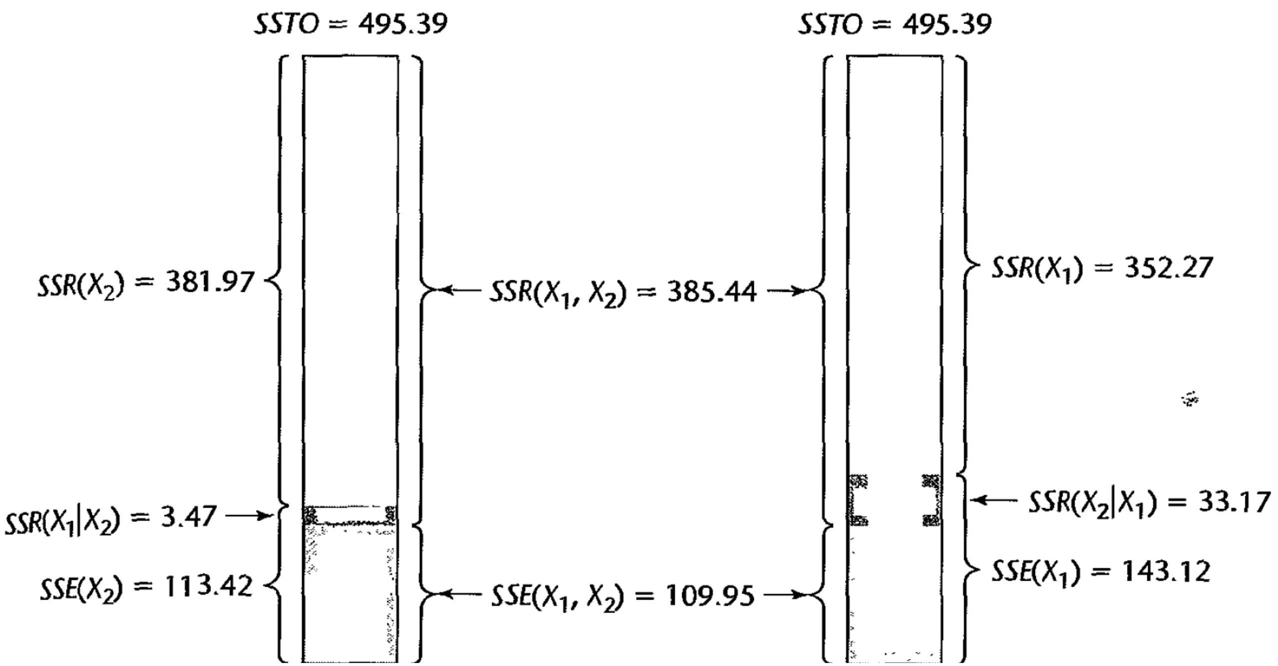
ANOVA Table Containing Decomposition of SSR, cont'd

- Each extra sum of squares involving
  - a single extra X variable has associated with it one degree of freedom
  - two extra X variables have two degrees of freedom
- Mean squares:

$$MSR(X_2|X_1) = \frac{SSR(X_2|X_1)}{1}$$

$$MSR(X_2, X_3|X_1) = \frac{SSR(X_2, X_3|X_1)}{2}$$

ANOVA Table Containing Decomposition of SSR, cont’d



Source of Variation	*SS*	*df*	*MS*
Regression	396.98	3	132.33
X_1	352.27	1	352.27
$X_2 \mid X_1$	33.17	1	33.17
$X_3 \mid X_1, X_2$	11.54	1	11.54
Error	98.41	16	6.15
Total	495.39	19	

Extra sums of squares are of interest because they occur in a variety of tests about regression coefficients where the question of concern is whether certain *X* variables can be dropped from the regression model.

Test whether a Single $\beta_k = 0$

- Test whether $\beta_k X_k$ can be dropped from a multiple regression model

$$H_0 : \beta_k = 0$$
$$H_a : \beta_k \neq 0$$

- Test statistics in (6.51b): $t^* = \frac{b_k}{s\{b_k\}}$

- The general linear test approach (Sec. 2.8): Full model vs. Reduced model

$$\boxed{F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F}}$$

Test whether a Single $\beta_k = 0$, cont'd

- The general linear test approach (Sec. 2.8) involves an extra sum of squares:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i \qquad \text{Full Model}$$

- To test the alternatives:

  H_o: $\beta_3 = 0$ vs. H_a: $\beta_3 \neq 0$

- When H_o holds:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i \qquad \text{Reduced Model}$$

- The test whether or not $\beta_3 = 0$ is a marginal test, given X_1, X_2 are already in the model

Test whether a Single $\beta_k = 0$, cont'd

- Steps:
  1. $SSE(F) = SSE(X_1, X_2, X_3)$, $df_F = n - 4$
  2. $SSE(R) = SSE(X_1, X_2)$, $df_R = n - 3$
  3. The general linear test statistic (2.70):

$$
\begin{aligned}
F^* &= \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \\
&= \frac{SSE(X_1, X_2) - SSE(X_1, X_2, X_3)}{(n-3) - (n-4)} \div \frac{SSE(X_1, X_2, X_3)}{n-4} \\
&= \frac{SSR(X_3 | X_1, X_2)}{1} \div \frac{SSE(X_1, X_2, X_3)}{n-4} \\
&= \boxed{\frac{MSR(X_3 | X_1, X_2)}{MSE(X_1, X_2, X_3)}}
\end{aligned}
$$

Test whether a Single $\beta_k = 0$, cont'd

TABLE 7.4 ANOVA Table with Decomposition of *SSR*—Body Fat Example with Three Predictor Variables.

Source of Variation	*SS*	*df*	*MS*
Regression	396.98	3	132.33
X_1	352.27	1	352.27
$X_2 \mid X_1$	33.17	1	33.17
$X_3 \mid X_1, X_2$	11.54	1	11.54
Error	98.41	16	6.15
Total	495.39	19	

Body Fat Example: Testing, H_o: $\beta_3 = 0$ vs. H_a: $\beta_3 \neq 0$

$$F^* = \frac{SSR(X_3|X_1, X_2)}{1} \div \frac{SSE(X_1, X_2, X_3)}{n-4} = \frac{11.54}{1} \div \frac{98.41}{16} = 1.88$$

$F^* = 1.88 \leq 8.53 = F(0.99; 1, 16) \Rightarrow$ conclude H_0 ($\alpha = 0.01$)

- X_3 can be dropped from the regression model that already contains X_1, X_2

Test whether a Single $\beta_k = 0$, cont'd

- R Codes For Extra Sum Of Squares With The Body Fat Example:

```
ex <- read.table("CH07TA01.txt",header=F)
n<-length(ex$V1)
frm1 <- lm(V4~V1+V2+V3,data=ex)
frm2 <- lm(V4~V1+V2,data=ex)
SSE1 <-deviance(frm1)
SSE2 <-deviance(frm2)
F<-((SSE2-SSE1)/1)/(SSE1/(n-4))
```

Test whether Several $\beta_k = 0$

- Test whether $\beta_2 X_2$ and $\beta_3 X_3$ can be dropped from the full model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i \qquad \text{Full Model}$$

- Alternative

$$H_0: \beta_2 = \beta_3 = 0$$
$$H_a: \text{not both } \beta_2 \text{ and } \beta_3 \text{ equal } 0$$

- When H_o holds:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \varepsilon_i \qquad \text{Reduced Model}$$

Test whether Several $\beta_k = 0$

- Test statistics
  1. $SSE(F) = SSE(X_1, X_2, X_3)$, $df_F = n - 4$
  2. $SSE(R) = SSE(X_1)$, $df_R = n - 2$
  3. The general linear test statistic (2.70):

$$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F}$$

$$= \frac{SSE(X_1) - SSE(X_1, X_2, X_3)}{(n-2) - (n-4)} \div \frac{SSE(X_1, X_2, X_3)}{n-4}$$

$$= \frac{SSR(X_2, X_3 | X_1)}{2} \div \frac{SSE(X_1, X_2, X_3)}{n-4}$$

$$= \frac{MSR(X_2, X_3 | X_2)}{MSE(X_1, X_2, X_3)}$$

Test whether a Single $\beta_k = 0$, cont'd

TABLE 7.4 ANOVA Table with Decomposition of *SSR*—Body Fat Example with Three Predictor Variables.

Source of Variation	*SS*	*df*	*MS*
Regression	396.98	3	132.33
X_1	352.27	1	352.27
$X_2 \| X_1$	33.17	1	33.17
$X_3 \| X_1, X_2$	11.54	1	11.54
Error	98.41	16	6.15
Total	495.39	19	

Can both X_2 and X_3 be dropped from the full model?

$$F^* = \frac{SSR(X_2, X_3|X_1)}{2} \div MSE(X_1, X_2, X_3) = \frac{33.17 + 11.54}{2} \div 6.15 = 3.63$$

$F^* = 3.63 \sim 3.63 = F(0.99; 2, 16)$ $\Rightarrow$ at the boundary of the decision rule

We may wish to make further analyses before deciding whether X_2 and X_3 should be dropped from the regression model that already contains X_1.

Comments

- Testing whether a single β_k equals zero:
  1. the t^* test statistic
  2. the F^* general linear test statistic
- Testing whether several β_k equal zero:
  1. the F^* general linear test statistic
- General linear test statistic can be expressed in term of the coefficients of multiple determination R^2

$$
\begin{aligned}
F^* &= \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \\
&= \frac{R_F^2 - R_R^2}{df_R - df_F} \div \frac{1 - R_F^2}{df_F}
\end{aligned}
$$

Comments, cont'd

Can both X_2 and X_3 be dropped from the full model?

$$F^* = \frac{0.80135-0.71110}{(20-2)-(20-4)} \div \frac{1-0.80135}{16} = 3.63$$

$$F^* = \frac{R^2_{Y|123} - R^2_{Y|1}}{(n-2)-(n-4)} \div \frac{1-R^2_{Y|123}}{n-4} = 3.63$$

Test Statistics:

$$F^* = \frac{R_F^2 - R_R^2}{df_R - df_F} \div \frac{1 - R_F^2}{df_F}$$

is not appropriate when the full and reduced regression models do not contain β_0

Summary of Tests Concerning Regression Coefficients

- Test whether all $\beta_k = 0$

$$\text{overall } F \text{ test: } F^* = \frac{MSR}{MSE} \sim F(p-1, n-p)$$

- Test whether a single $\beta_k = 0$

$$\text{partial } F \text{ test: } F^* = \frac{MSR(X_k | X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p-1})}{MSE}$$

$$\sim F(1, n-p)$$

$$\Leftrightarrow t^* = \frac{b_k}{s\{b_k\}}$$

Summary of Tests Concerning Regression Coefficients, cont'd

- Test whether some $\beta_k = 0$

$$H_0 : \beta_q = \beta_{q+1} = \cdots = \beta_{p-1} = 0$$

$$\text{patial } F \text{ test: } F^* = \frac{MSR(X_q, \ldots, X_{p-1} | X_1, \ldots, X_{q-1})}{MSE} \sim F(p-q, n-p)$$

Summary of Tests Concerning Regression Coefficients, cont'd

- When tests about regression coefficients are desired that do not involve testing whether one or several β_k equal zero, extra sums of squares cannot be used and the general linear test approach requires separate fittings of the full and reduced models.

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i \qquad \text{Full Model}$$

- Wish to test: $H_o\colon \beta_1 = \beta_2$ vs. $H_a\colon \beta_1 \neq \beta_2$

$$Y_i = \beta_0 + \beta_c (X_{i1} + X_{i2}) + \beta_3 X_{i3} + \varepsilon_i \qquad \text{Reduced Model}$$

- Wish to test: $H_o\colon \beta_1 = 3\,,\ \beta_3 = 5$ vs. $H_a\colon$ not both equalities in H_o holds

- Under H_o, $\beta_1 X_1$ and $\beta_3 X_3$ are known constants

$$Y_i - 3X_{i1} - 5X_{i2} = \beta_0 + \beta_2 X_{i2} + \varepsilon_i \qquad \text{Reduced Model}$$

Coefficients of Partial Determination

- R^2: measures the proportionate reduction in the variation of Y achieved by the introduction of the entire set of X considered in the model

- Coefficient of partial determination: measures the marginal contribution on one X variable when all others are already included in the model

Coefficients of Partial Determination, cont'd

Illustration: two predictor variables

$$\text{Model } Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$$

- $SSE(X_2)$: measures the variation in Y when X_2 is included in the model
- $SSE(X_1, X_2)$: measures the variation in Y when X_1, X_2 are included in the model
- $R^2_{Y1|2}$: the coefficient of partial determination between Y and X_{i1}, given that X_2 is in the model

$$\boxed{R^2_{Y1|2} = \frac{SSE(X_2) - SSE(X_1, X_2)}{SSE(X_2)} = \frac{SSR(X_1|X_2)}{SSE(X_2)}}$$

Coefficients of Partial Determination, cont'd

General Case: coefficients of partial determination to three or more X variables in the model

$$R^2_{Y1|23} = \frac{SSR(X_1|X_2, X_3)}{SSE(X_2, X_3)}$$

$$R^2_{Y2|13} = \frac{SSR(X_2|X_1, X_3)}{SSE(X_1, X_3)}$$

$$R^2_{Y3|12} = \frac{SSR(X_3|X_1, X_2)}{SSE(X_1, X_2)}$$

$$R^2_{Y4|123} = \frac{SSR(X_4|X_1, X_2, X_3)}{SSE(X_1, X_2, X_3)}$$

Coefficients of Partial Determination, cont'd

- Body Fat Example:

$$R^2_{Y2|1} = \frac{SSR(X_2|X_1)}{SSE(X_1)} = \frac{33.17}{143.12} = 0.232$$

$$R^2_{Y3|12} = \frac{SSR(X_3|X_1, X_2\ )}{SSE(X_1, X_2\ )} = \frac{11.54}{109.95} = 0.105$$

$$R^2_{Y1|2} = \frac{SSR(X_1|X_2)}{SSE(X_2)} = \frac{3.47}{113.42} = 0.031$$

Comments

- The coefficients of pallial determination can take on values between 0 and 1.
- Other interpretation: with a coefficient of simple determination

  - Residuals: regress Y on $X_2 \Rightarrow$ $e_i(Y|X_2) = Y_i - \hat{Y}_i(X_2)$
  - Residuals: regress X_1 on $X_2 \Rightarrow$ $e_i(X_1|X_2) = X_{i1} - \hat{X}_{i1}(X_2)$
  - R^2 between $e_i(Y|X_2)$ and $e_i(X_1|X_2)$ will be the same as $R^2_{Y1|2}$

- added variable plots or partial regression plots (Chapter 10): the strength of the relationship between Y and X_1 adjusted for X_2

$$e_i(Y|X_2) \text{ vs. } e_i(X_1|X_2)$$

Comments, cont'd

- Body Fat Example:

$$R^2_{Y1|2} = \frac{SSR(X_1|X_2)}{SSE(X_2)} = \frac{3.47}{113.42} = 0.031$$

```
ex<-read.table("CH07TA01.txt",header=F)
res1<-lm(V4~V2,data=ex)$residuals
res2<-lm(V1~V2,data=ex)$residuals
fitres<-summary(lm(res1~res2))
fitres$ r.squared
[1] 0.03061875
```

Coefficients of Partial Correlation

- Coefficient of partial correlation: (Chapter 9)

$$r_{Y2|1} = \sqrt{R^2_{Y2|1}}$$

- the same sign with the regression coefficient
- Expressed in terms of simple or other partial correlation coefficients:

$$R^2_{Y2|1} = [r_{Y2|1}]^2 = \frac{(r_{Y2} - r_{12}r_{Y1})^2}{(1 - r^2_{12})(1 - r^2_{Y1})}$$

$$R^2_{Y2|13} = [r_{Y2|13}]^2 = \frac{(r_{Y2|3} - r_{12|3}r_{Y1|3})^2}{(1 - r^2_{12|3})(1 - r^2_{Y1|3})}$$

r_{Y1}: correlation of Y and X_1
r_{12}: correlation of X_1 and X_2

Standardized Multiple Regression Model

- Roundoff errors tend to enter normal equations calculations primarily when the inverse of X'X is taken.

  - determinant that is close to zero: some variables are highly intercorrelated
  - the element of X'X substantially different: the entries in X'X cover a wide range magnitudes

Roundoff errors $\Rightarrow$ $standartized\ regression$

- Transformation: correlation transformation
  - Transformed variables fall between -1 and 1
  - becomes much less subject to roundoff errors

Lack of Comparability in Regression Coefficients

- differences in the units:

$$\hat{Y}_i = 200 + 20000X_1 + 0.2X_2$$

- Y: dollars; X_1: thousand dollars; X_2:cents
- Is X_1 the only important predicted variable ?

Correlation Transformation

- help with controlling roundoff errors
- expressing the regression coefficients in the same units

- Y is a normal random variable $\Rightarrow$ $Z = \frac{Y - \mu}{\sigma}$

- Standardizing: involving centering and scaling the variable

Correlation Transformation, cont'd

- The usual standardizations of the variables:

$$\frac{Y_i - \bar{Y}}{s_Y}; \qquad s_Y = \sqrt{\frac{\sum_i (Y_i - \bar{Y})^2}{n-1}}$$

$$\frac{X_{ik} - \bar{X}_k}{s_k}; \qquad s_k = \sqrt{\frac{\sum_i (X_{ik} - \bar{X}_k)^2}{n-1}} (k = 1, \ldots, p-1)$$

- The correlation transformation:

$$Y_i^* = \frac{1}{\sqrt{n-1}} \left(\frac{Y_i - \bar{Y}}{s_Y} \right)$$

$$X_{ik}^* = \frac{1}{\sqrt{n-1}} \left(\frac{X_{ik} - \bar{X}_k}{s_k} \right) \ (k = 1, \ldots, p-1)$$

Standardized Regression Model

A standardized regression model:

$$Y_i^* = \beta_1^* X_{i1}^* + \cdots + \beta_{p-1}^* X_{i,p-1}^* + \varepsilon_i^*$$

- no need for intercept

$$\beta_k = \left(\frac{s_Y}{s_k}\right)\beta_k^* \qquad (k = 1, \ldots, p-1)$$

$$\beta_0 = \bar{Y} - \beta_1 \bar{X}_1 - \cdots - \beta_{p-1}\bar{X}_{p-1}$$

X'X Matrix for Transformed Variables

- $\boldsymbol{r}_{XX}$: correlation matrix of the X variables

$$\underset{(p-1)\times(p-1)}{\boldsymbol{r}_{XX}} = \begin{bmatrix} 1 & r_{12} & \cdots & r_{1,p-1} \\ r_{21} & 1 & \cdots & r_{2,p-1} \\ & \vdots & & \vdots \\ r_{p-1,1} & r_{p-1,2} & \cdots & 1 \end{bmatrix}$$

- $\boldsymbol{r}_{YX}$: correlation between Y and each of X variables:

$$\underset{(p-1)\times 1}{\boldsymbol{r}_{YX}} = \begin{bmatrix} r_{Y1} \\ r_{Y2} \\ \vdots \\ r_{Y,p-1} \end{bmatrix}$$

X'X Matrix for Transformed Variables, cont'd

- The transformed variables: (no column of 1 in $\boldsymbol{X}$)

$$\underset{n\times(p-1)}{\boldsymbol{X}} = \begin{bmatrix} X_{11}^* & \cdots & X_{1,p-1}^* \\ X_{21}^* & \cdots & X_{2,p-1}^* \\ \vdots & & \vdots \\ X_{n1}^* & \cdots & X_{n,p-1}^* \end{bmatrix}$$

$$\Rightarrow \underset{(p-1)\times(p-1)}{\boldsymbol{X}'\boldsymbol{X}} = \boldsymbol{r}_{XX}$$

- All of the elements of $\boldsymbol{X}'\boldsymbol{X}$ are between -1 and 1
- $\sum (X_{i1}^*)^2 = 1$
- $\sum X_{i1}^* X_{i2}^* = \dfrac{\sum (X_{i1} - \bar{X}_1)(X_{i2} - \bar{X}_2)}{\left[\sum (X_{i1} - \bar{X}_1)^2 \sum (X_{i2} - \bar{X}_2)^2\right]^2}$

Estimated Standard Regression Coefficients

- the least squares estimator:

$$\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}\boldsymbol{Y}$$

- The least squares normal equations and estimators of the regression coefficients of the standardized regression model:

$$\boldsymbol{r}_{XX}\boldsymbol{b} = \boldsymbol{r}_{YX} \Rightarrow \boldsymbol{b} = \boldsymbol{r}_{XX}^{-1}\boldsymbol{r}_{YX}$$

$$\underset{(p-1)\times 1}{\boldsymbol{b}} = \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_{p-1}^* \end{bmatrix}$$

- $b_1^*, \ldots, b_{p-1}^*$: standardized regression coefficients

Estimated Standard Regression Coefficients, cont'd

- The standardized parameters vs. the original parameters
  - $b_k = \left(\frac{s_Y}{s_k}\right) b_k^* \qquad k = 1, \dots, p-1$
  - $b_o = \bar{Y} - b_1\bar{X}_1 - \cdots b_{p-1}\bar{X}_{p-1}$
- Illustration for p – 1 = 2:

$$\mathbf{r}_{XX} = \begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix}$$

$$\mathbf{r}_{YX} = \begin{bmatrix} r_{Y1} \\ r_{Y2} \end{bmatrix}$$

$$\mathbf{r}_{XX}^{-1} = \frac{1}{1 - r_{12}^2} \begin{bmatrix} 1 & -r_{12} \\ -r_{12} & 1 \end{bmatrix}$$

$$\Rightarrow$$

$$\mathbf{b} = \frac{1}{1 - r_{12}^2} \begin{bmatrix} 1 & -r_{12} \\ -r_{12} & 1 \end{bmatrix} \begin{bmatrix} r_{Y1} \\ r_{Y2} \end{bmatrix} = \frac{1}{1 - r_{12}^2} \begin{bmatrix} r_{Y1} - r_{12}r_{Y2} \\ r_{Y2} - r_{12}r_{Y1} \end{bmatrix}$$

$$b_1^* = \frac{r_{Y1} - r_{12}r_{Y2}}{1 - r_{12}^2}$$

$$b_2^* = \frac{r_{Y2} - r_{12}r_{Y1}}{1 - r_{12}^2}$$

Estimated Standard Regression Coefficients, cont'd

- Dwane Studios Example:

(a) Original Data

Case i	Sales Y_i	Target Population X_{i1}	Per Capita Disposable Income X_{i2}
1	174.4	68.5	16.7
2	164.4	45.2	16.8
...	...	...	...
20	224.1	82.7	19.1
21	166.5	52.3	16.0
	$\bar{Y} = 181.90$	$\bar{X}_1 = 62.019$	$\bar{X}_2 = 17.143$
	$s_Y = 36.191$	$s_1 = 18.620$	$s_2 = .97035$

Estimated Standard Regression Coefficients, cont'd

- Dwane Studios Example:

(b) Transformed Data

i	Y_i^*	X_{i1}^*	X_{i2}^*
1	−.04637	07783	−.10205
2	−.10815	−.20198	−.07901
···	···	···	···
20	.26070	.24835	.45100
21	−.09518	−.11671	−.26336

(c) Fitted Standardized Model

$$\hat{Y}^* = .7484X_1^* + .2511X_2^*$$

$$\hat{Y}^* = 0.7484X_1^* + 0.2511X_2^*$$

$$\hat{Y} = -68.860 + 1.455X_1 + 9.365X_2$$

Estimated Standard Regression Coefficients, cont'd

```
## Ex p277
library(QuantPsyc)
ex7.5<-read.table("CH07TA05.txt")
fit<-lm(V1~V2+V3,data=ex7.5)
fit
Call:
lm(formula = V1 ~ V2 + V3, data = ex7.5)

Coefficients:
(Intercept)     V2       V3
-68.857      1.455     9.366

lm.beta(fit)
V2                     V3
0.7483670      0.2511039
```

Estimated Standard Regression Coefficients, cont'd

$$\hat{Y}^* = 0.7484X_1^* + 0.2511X_2^*$$

- Does X_1 have much greater impact on sales than X_2? ($\because b_1^* > b_2^*$)

- One must be cautious about interpreting any regression coefficient whether standardized or not.

  - caution if the predictor variables
  - $r_{12} = 0.781$ in the Dwaine Studios data

Estimated Standard Regression Coefficients, cont'd

To shift from the standardized regression coefficients b_1^* and b_2^* back to the regression coefficients for the model with the original variables:

$$b_1 = \left(\frac{s_Y}{s_1}\right) b_1^* = \frac{36.191}{18.620} \times 0.7484 = 1.4546$$

$$b_1 = \left(\frac{s_Y}{s_2}\right) b_2^* = \frac{36.191}{0.97035} \times 0.2511 = 9.3652$$

$$b_o = \bar{Y} - b_1\bar{X}_1 - b_2\bar{X}_2 = 181.90 - 1.45 \times 62.02 - 9.36 \times 17.14 = -68.86$$

$$\hat{Y} = -68.86 + 1.455X_1 + 9.365X_2$$

Multicollinearity and Its Effects

Questions:

- What is the relative importance of the effects of the different predictor variables?
- What is the magnitude of the effect of a given predictor variable on the response variable?
- Can any predictor variable be dropped from the model because it has little or no effect on the response variable?
- Should any predictor variables not yet included in the model be considered for possible inclusion?

- intercorrelation or multicollinearity: the predictor variables are correlated among themselves

Uncorrelated Predicted Variables

Example: Y - crew productivity; X_1-the effect of work crew size; X_2-level of bonus pay

- $r_{12}^2 = 0 \Rightarrow$ the predictor variables are uncorrelated
- $SSR(X_1|X_2) = 231.125 = SSR(X_1)$
- $SSR(X_2|X_1) = 171.125 = SSR(X_2)$

Case i	Crew Size X_{i1}	Bonus Pay (dollars) X_{i2}	Crew Productivity Y_i
1	4	2	42
2	4	2	39
3	4	3	48
4	4	3	51
5	6	2	49
6	6	2	53
7	6	3	61
8	6	3	60

Uncorrelated Predicted Variables, cont'd

Example: Y - crew productivity; X_1-the effect of work crew size; X_2-level of bonus pay

TABLE 7.7 Regression Results when Predictor Variables Are Uncorrelated—Work Crew Productivity Example.

(a) Regression of Y on X_1 and X_2

$$\hat{Y} = .375 + \boxed{5.375 X_1} + \boxed{9.250 X_2}$$

Source of Variation	SS	df	MS
Regression	402.250	2	201.125
Error	17.625	5	3.525
Total	419.875	7	

(b) Regression of Y on X_1

$$\hat{Y} = 23.500 + \boxed{5.375 X_1}$$

Source of Variation	SS	df	MS
Regression	231.125	1	231.125
Error	188.750	6	31.458
Total	419.875	7	

(c) Regression of Y on X_2

$$\hat{Y} = 27.250 + \boxed{9.250 X_2}$$

Source of Variation	SS	df	MS
Regression	171.125	1	171.125
Error	248.750	6	41.458
Total	419.875	7	

Uncorrelated Predicted Variables, cont'd

- When two or more predictor variables are uncorrelated, the when marginal contribution of one predictor variable in reducing the error sum of squares the other predictor variables are in the model is exactly the same as when this predictor variable is in the model alone.

$$b_1 = \frac{\dfrac{\sum(X_{i1}-\bar{X}_1)(Y_i-\bar{Y})}{\sum(X_{i1}-\bar{X}_1)^2} - \left[\dfrac{\sum(Y_i-\bar{Y})^2}{\sum(X_{i1}-\bar{X}_1)^2}\right]^{1/2} r_{Y2}r_{12}}{1-r_{12}^2} \Rightarrow b_1 = \frac{\sum(X_{i1}-\bar{X}_1)(Y_i-\bar{Y})}{\sum(X_{i1}-\bar{X}_1)^2} \qquad \text{when } r_{12}=0$$

Multicollinearity and Its Effects

Predictor variables are perfectly correlated:

$$E\{Y\} = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

Figure : Example of Perfectly Correlated Predictor Variables.

TABLE 7.8 Example of Perfectly Correlated Predictor Variables.

Case i	X_{i1}	X_{i2}	Y_i	Fitted Values for Regression Function (7.58)	Fitted Values for Regression Function (7.59)
1	2	6	23	23	23
2	8	9	83	83	83
3	6	8	63	63	63
4	10	10	103	103	103

Response Functions:

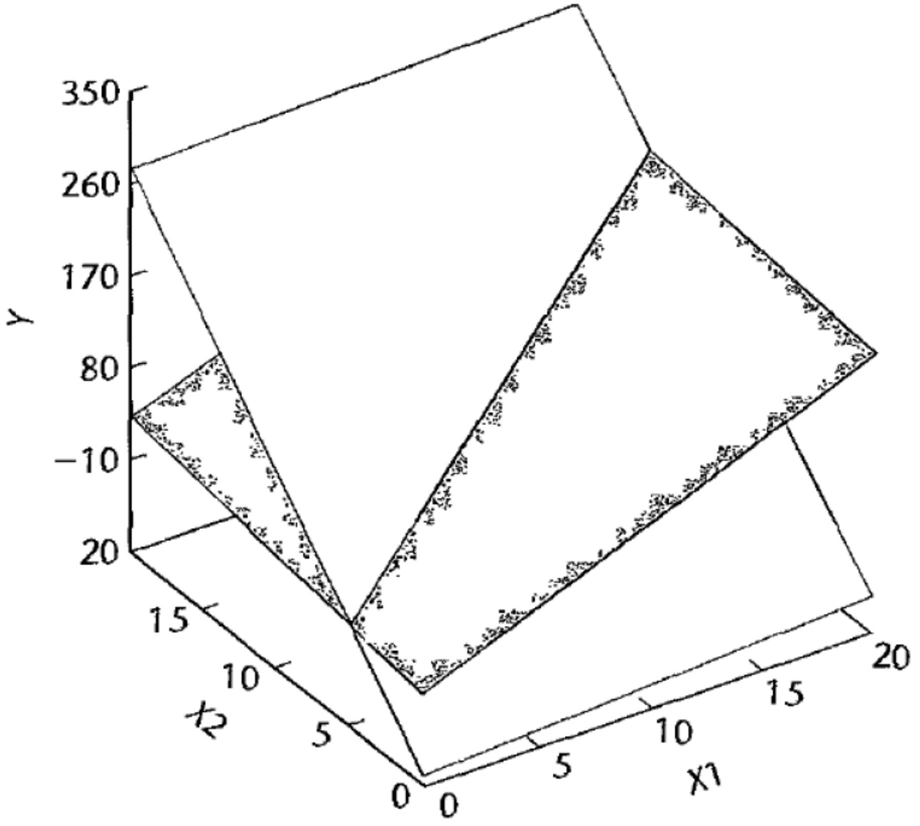
$\hat{Y} = -87 + X_1 + 18X_2$ (7.58)

$\hat{Y} = -7 + 9X_1 + 2X_2$ (7.59)

Multicollinearity and Its Effects, cont’d

Figure : Two Response Planes That Intersect when $X_2 = 5 + 0.5X_1$.

FIGURE 7.2
Two Response Planes That Intersect when $X_2 = 5 + .5X_1$.



Multicollinearity and Its Effects, cont'd

- When X_1 and X_2 are perfectly correlated, many different response functions will lead to the same perfectly fitted values for the observations.

- The perfect relation between X_1 and X_2 did not inhibit the ability to obtain a good fit to the data.

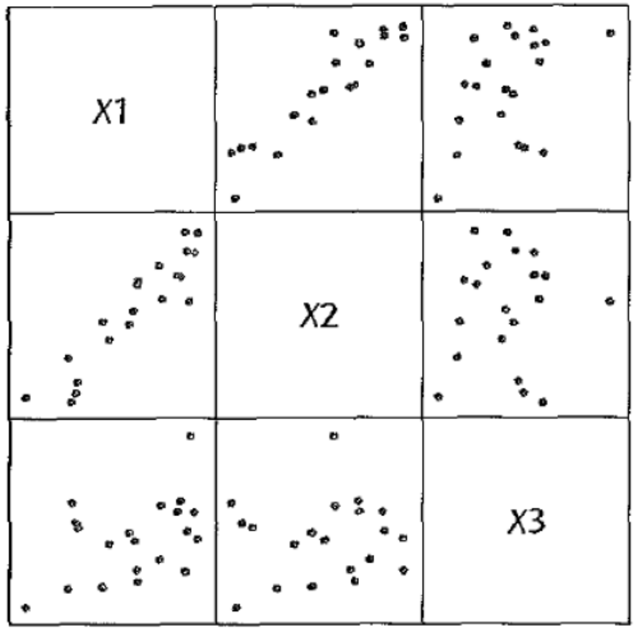
- Since many different response functions provide the same good fit, we cannot interpret any one set of regression coefficients as reflecting the effects of the different predictor variables.

Multicollinearity and Its Effects, cont'd

Figure : Scatter Plot Matrix and Correlation Matrix of the Predictor Variables-Body Fat Example.

FIGURE 7.3 Scatter Plot Matrix and Correlation Matrix of the Predictor Variables—Body Fat Example.

(a) Scatter Plot Matrix of *X* Variables



(b) Correlation Matrix of *X* Variables

$$\mathbf{r}_{XX} = \begin{bmatrix} 1.0 & .924 & .458 \\ .924 & 1.0 & .085 \\ .458 & .085 & 1.0 \end{bmatrix}$$

Multicollinearity and Its Effects, cont'd

Effects on Regression Coefficients: X_1, triceps skinfold thickness, varies markedly depending on which other variables are included in the model:

Variables in Model	b_1	b_2
X_1	.8572	—
X_2	—	.8565
X_1, X_2	.2224	.6594
X_1, X_2, X_3	4.334	−2.857

- The story is the same for the regression coefficient for X2 : the regression coefficient b_2 changes sign when X_3 is added to the model that includes X_1 and X_2

Multicollinearity and Its Effects, cont'd

Effects on $s\{b_k\}$

Variables in Model	$s\{b_1\}$	$s\{b_2\}$
X_1	.1288	—
X_2	—	.1100
X_1, X_2	.3034	.2912
X_1, X_2, X_3	3.016	2.582

- The high degree of multicollinearity among the predictor variables is responsible for the inflated variability of the estimated regression coefficients.

Multicollinearity and Its Effects, cont'd

Effects on fitted values and predictions

Variables in Model	*MSE*
X_1	7.95
X_1, X_2	6.47
X_1, X_2, X_3	6.15

- Estimated means and Predicted values are not affected

Multicollinearity and Its Effects, cont’d

Effects on the test statistics

- It is possible that when individual t tests are performed, neither β_1 or β_2 is significant.
- However, when the F test is performed for both β_1 and β_2, the results may still be significant.
- Need for more powerful diagnostics for multicollinearity