Data modeling: CSCI E-106

Applied Linear Statistical Models

Chapter 4 - Simultaneous Inferences and Other Topics in Regression Analysis

Joint Estimation of β_0 and β_1

- To draw inferences with confidence coefficient 0.95 about both the intercept β_0 and slope β_1.
  - We could use the methods of Chapter 2 to construct separate 95 percent confidence intervals for β_0 and β_1.
  - If the inferences are independent, the probability of both being correct would be $(.95)^2 = 0.9025$.
  - These would not provide 95 percent confidence that the conclusions for both β_0 and β_1, are correct.
- A family confidence coefficient of 0.95 would indicate that if repeated samples are selected and interval estimates for both β_0 and β_1 are calculated for each sample by specified procedures would lead to a family of estimates where both confidence intervals are correct.

Bonferroni Joint Confidence Intervals

- Developing joint confidence intervals for β_0 and β_1 with a specified family confidence coefficient
  - each statement confidence coefficient is adjusted to be higher than 1- α so that the family confidence coefficient is at least 1 – α
- The ordinary confidence limits for β_0 and β_1 with 1 – α

$$b_0 \pm t(1-\alpha/2; n-2)s\{b_0\}$$
$$b_1 \pm t(1-\alpha/2; n-2)s\{b_1\}$$

Bonferroni Joint Confidence Intervals, cont'd

- What is the probability that one or both of these intervals are incorrect?
  - A_1 : the event that the first C.I. does not cover β_0
  - A_2 : the event that the second C.I. does not cover β_1

$\Longrightarrow$ $P(A_1) = \alpha$ and $P(A_2) = \alpha$

- Bonferroni inequality:
  - $P(\bar{A}_1 \cap \bar{A}_2) \geq 1 - \alpha - \alpha = 1 - 2\alpha$

Bonferroni Joint Confidence Intervals, cont'd

- β_0 and β_1 are separately estimated with 0.95% C.I. $\Longrightarrow$ the Bonferroni inequality guarantees us a family confidence coefficient of at least 90 percent that both intervals based on the same sample are correct.
- The 1- α family confidence limits for β_0 and β_1, for regression model (2.1) by the Bonferroni procedure are:

$$b_0 \pm Bs\{b_0\} \quad b_1 \pm Bs\{b_1\}$$
$$B = t(1 - \alpha/4; n - 2)$$

Bonferroni Joint Confidence Intervals, cont'd

- For the Toluca Company example, 90 percent family confidence intervals for β_0 and β_1 require B = t(1 - .10/4; 23) = t(0.975; 23) = 2.069.
- We have from Chapter 2:

$$b_0 = 62.37 \qquad s\{b_0\} = 26.18$$
$$b_1 = 3.5702 \qquad s\{b_1\} = .3470$$

- Hence, the respective confidence limits for β_0 and β_1, are

$$8.20 \leq \beta_0 \leq 116.5$$
$$2.85 \leq \beta_1 \leq 4.29$$

Bonferroni Joint Confidence Intervals, cont’d

- toluca<-read.table("toluca.txt",header=T)
- attach(toluca)
- fitreg<-lm(Hrs~Size)
- confint(fitreg)
- confint(fitreg,level=1-alpha/2) # Bonferroni C.I.

Bonferroni Joint Confidence Intervals, cont'd

- The Bonferroni 1 - α family confidence coefficient is actually a lower bound on the true family confidence coefficient.
- If g interval estimates are desired with family confidence coefficient 1 - α, constructing each interval estimate with statement confidence coefficient 1 – α /g will suffice.

$$P\left(\bigcap_{i=1}^{g} \bar{A}_i\right) \geq 1 - g\alpha$$

- The Bonferroni technique is ordinarily most useful when the number of simultaneous estimates is not too large.
- It is not necessary with the Bonferroni procedure that the C.I. have the same statement confidence coefficient. ($P(A_1) + P(A_2) = \alpha$)
- Joint confidence intervals can be used directly for testing.

Bonferroni Joint Confidence Intervals, cont'd

- The estimators b_0 and b_1 are usually correlated, but the Bonferroni simultaneous confidence limits recognize this correlation by means of the bound on the family confidence coefficient.

$$\sigma\{b_0, b_1\} = -\bar{X}\sigma^2\{b_1\}$$

- Note that if $\bar{X}$ is positive, b_0 and b_1 are negatively correlated, implying that if the estimate b_1 is too high, the estimate b_0 is likely to be too low, and vice versa.

- When the independent variable is $X_i - \bar{X}$, as in the alternative model (1.6), b_0^* and b_1 are uncorrelated according to because the mean of the $X_i - \bar{X}$ observations is zero.

Simultaneous Estimation of Mean Responses

- The mean responses at a number of X levels need to be estimated.
  - Example: Toluca Company: the mean number of work hours for X = 30, 65, 100 units
- Two procedures for simultaneous estimation of a number of different mean responses:
  - Working-Hotelling
  - Bonferroni
- A family confidence coefficient is needed for estimating several mean responses:
  - even though all estimates are based on the same fitted regression line, the separate interval estimates of $E\{Y_h\}$ at the different X_h levels need not all be correct or all be incorrect.
  - The combination of sampling errors in b_0 and b_1 may be such that the interval estimates of $E\{Y_h\}$ will be correct over some range of X levels and incorrect elsewhere.

Working-Hotelling Procedure

- Working-Hotelling procedure: based on the confidence band for the regression line (Chap. 2.6)

$$\hat{Y}_h \pm Ws\{\hat{Y}_h\}, \quad W^2 = 2F(1-\alpha; 2, n-2)$$

- The confidence band contains the entire regression line $\Rightarrow$ contains the mean responses at all X levels
- The simultaneous confidence limits for g mean responses $E\{Y_h\}$ for model (2.1) withe the Working-Hotelling procedure:

$$\hat{Y}_h \pm Ws\{\hat{Y}_h\}, \quad W^2 = 2F(1-\alpha; 2, n-2)$$

  - $\hat{Y}_h = b_0 + b_1 X_h$
  - $s\{\hat{Y}_h\} = MSE\left[\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right]$

Working-Hotelling Procedure, cont'd

- Example: Toluca Company:
  - Y_h at X = 30, 65, 100 units
  - $1 - \alpha = 0.90$; $F(0.90; 2, 23) = 2.549$
  - $W^2 = 5.098 \Rightarrow W = 2.258$

X_h	$\hat{Y}_h$	$s\{\hat{Y}_h\}$
30	169.5	16.97
65	294.4	9.918
100	419.4	14.27

$\Rightarrow$

$$131.2 = 169.5 - 2.258(16.97) \le E\{Y_h\} \le 169.5 + 2.258(16.97) = 207.8$$
$$272.0 = 294.4 - 2.258(9.918) \le E\{Y_h\} \le 294.4 + 2.258(9.918) = 316.8$$
$$387.2 = 419.4 - 2.258(14.27) \le E\{Y_h\} \le 419.4 + 2.258(14.27) = 451.6$$

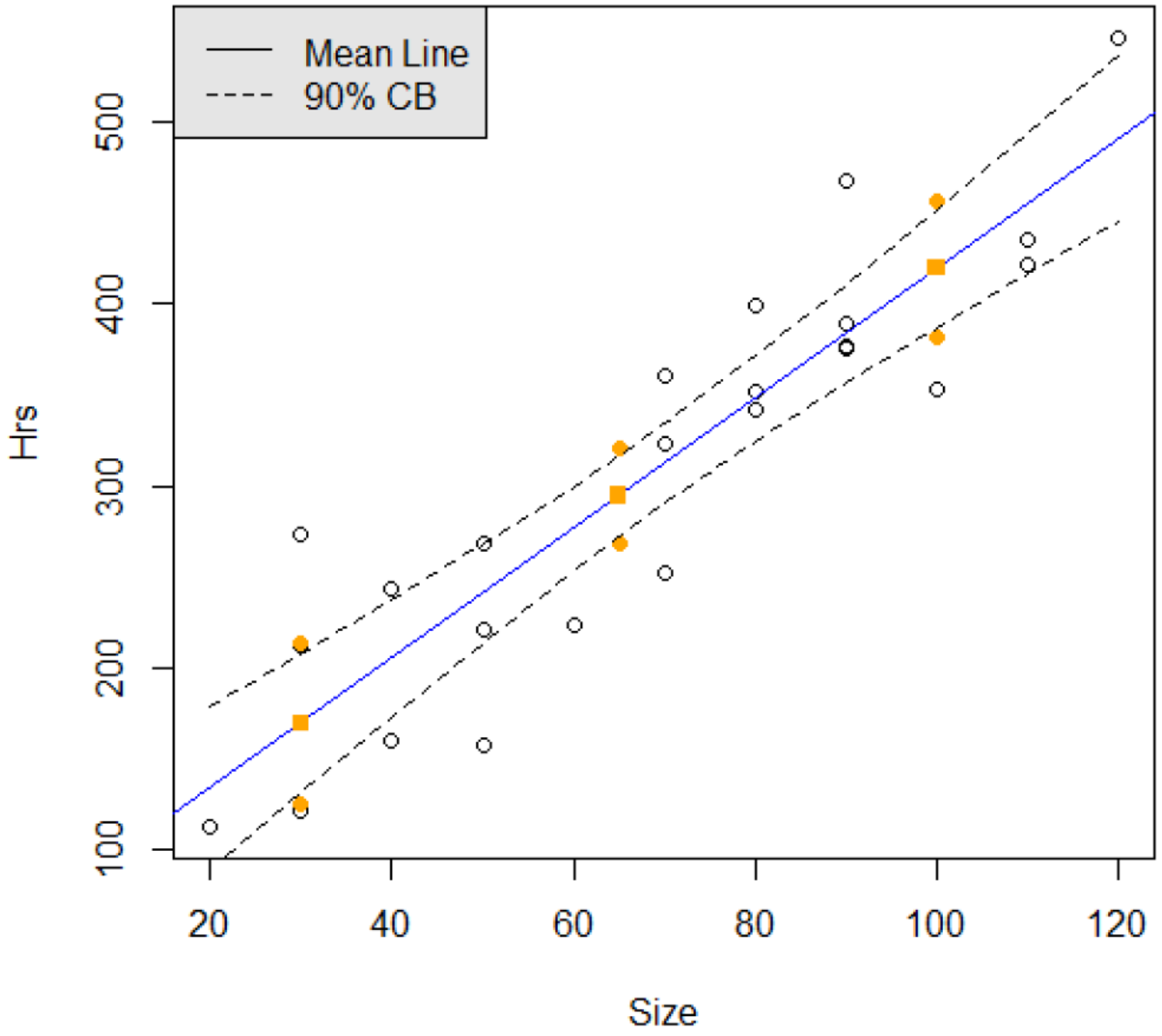
- concluded that the mean number of work hours required is between 131.2 and 207.8 for lots of 30 parts, between 272.0 and 316.8 for lots of 65 parts, and between 387.2 and 451.6 for lots of 100 parts with family confidence coefficient .90.

Working-Hotelling Procedure, cont'd

- `fitreg<-lm(Hrs~Size)`
- `Xh<-c(30,65,100)`
- `pred<-predict.lm(fitreg,data.frame(Size = c(Xh)),`
- `se.fit = T, level = 0.9)`
- `W <-rep(sqrt( 2 * qf(0.90, 2, n-2) ),length(Xh))`
- `rbind(pred$fit-W * pred$se.fit,pred$fit+W * pred$se.fit)`

```
         1        2        3
[1,] 131.1542 272.0351 387.1591
[2,] 207.7897 316.8229 451.6130
```

Working-Hotelling Procedure, cont'd



Bonferroni procedure

- When $E\{Y_h\}$ is to be estimated for g levels X_h with family confidence coefficient 1-α, the Bonferroni confidence limits for regression model (2.1) are:

$$\hat{Y}_h \pm Bs\{\hat{Y}_h\}$$

- Where B=t(1-α/2g;n-2), and *g* is the number of confidence intervals in the family

Bonferroni procedure, cont'd

- Example: Toluca Company:
  - Y_h at X = 30, 65, 100 units
  - 1 − α = 0.90; F(0.90; 2, 23) = 2.549
  - B = t(1 − 0.1/(2 x 3); 23) = 2.263

X_h	$\hat{Y}_h$	$s\{\hat{Y}_h\}$
30	169.5	16.97
65	294.4	9.918
100	419.4	14.27

$$\Rightarrow \begin{aligned} 131.1 = 169.5 - 2.263(16.97) \le E\{Y_h\} \le 169.5 + 2.263(16.97) = 207.9 \\ 272.0 = 294.4 - 2.263(9.918) \le E\{Y_h\} \le 294.4 + 2.263(9.918) = 316.8 \\ 387.1 = 419.4 - 2.263(14.27) \le E\{Y_h\} \le 419.4 + 2.263(14.27) = 451.7 \end{aligned}$$

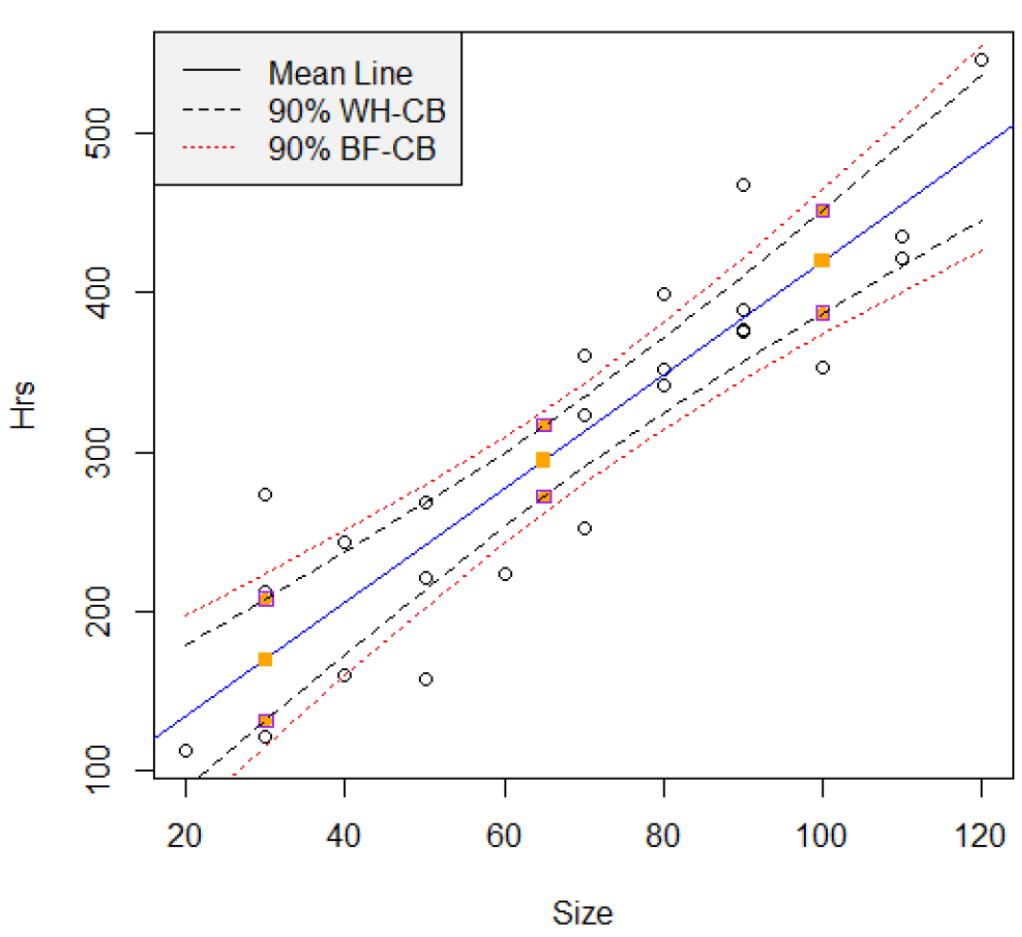
- concluded that the mean number of work hours required is between 131.1 and 207.9 for lots of 30 parts, between 272.0 and 316.8 for lots of 65 parts, and between 387.1 and 451.7 for lots of 100 parts with family confidence coefficient .90.

Bonferroni procedure, cont'd

- `fitreg<-lm(Hrs~Size)`
- `Xh<-c(30,65,100)`
- `pred<-predict.lm(fitreg,data.frame(Size = c(Xh)),`
- `se.fit = T, level = 0.9)`
- `B=rep(qt(1-alpha/(2*length(Xh)),n-2),length(Xh))`
- `rbind(pred$fit-B * pred$se.fit, pred$fit+B * pred$se.fit)`

```
         1        2        3
[1,] 131.0570 271.9783 387.0774
[2,] 207.8868 316.8797 451.6947
```

Bonferroni procedure, cont'd



- The Working-Hotelling confidence limits are slightly tighter than, or the same as, the Bonferroni limits.
- If the number of statements is small, the Bonferroni limits may be tighter. For larger families, the Working-Hotelling confidence limits will always be the tighter, since W stays the same for any number of statements in the family whereas B becomes larger as the number of statements increases.
- Both the Working-Hotelling and Bonferroni procedures provide lower bounds to the actual family confidence coefficient.

Simultaneous Prediction Intervals for New Observations

- The simultaneous predictions of g new observations on Y in g independent trials at g different levels of X.
- Two procedures:
  - Scheffé using F distribution

$$\hat{Y}_h \pm Ss\{pred\}, \quad S^2 = gF(1-\alpha;\, g,\, n-2)$$

  - Bonferroni using t distribution

$$\hat{Y}_h \pm Bs\{pred\}, \quad B = t(1-\alpha/2g;\, n-2)$$

where,

$$s^2\{pred\} = MSE\left[1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum (X_i - \bar{X})^2}\right]$$

Simultaneous Prediction Intervals for New Observations, cont'd

- Example: Toluca Company:
  - Predit $Y_{h(new)}$ at X_h = 80, 100 units
  - $1 - \alpha = 0.95$; $S^2 = 2*F(0.95; 2, 23) = 6.844 \Rightarrow S = 2.616$
  - $B = t(1 - 0.05/(2 \times 2); 23) = 2.398$
  - Bonferroni procedure will yield somewhat tighter prediction limits as ($B \leq S$)

X_h	$\hat{Y}_h$	$s\{pred\}$	$Bs\{pred\}$
80	348.0	49.91	119.7
100	419.4	50.87	122.0

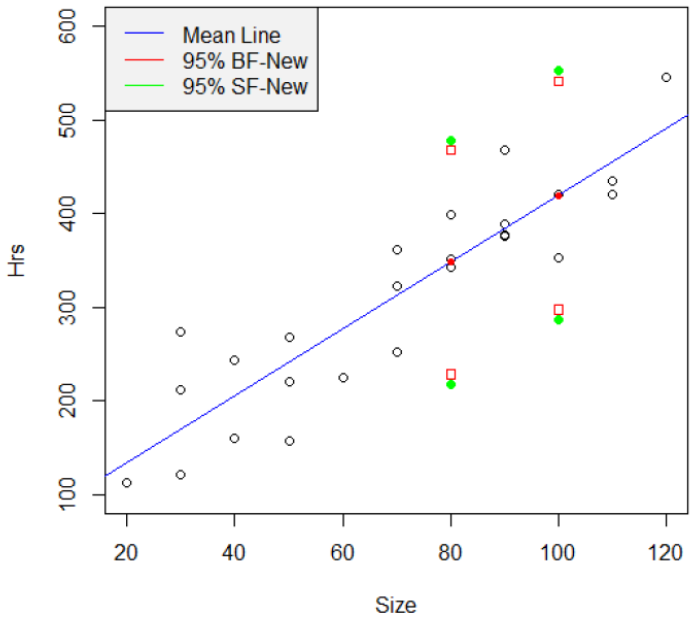
$\Rightarrow$

$$228.3 = 348.0 - 119.7 \leq Y_{h(new)} \leq 348.0 + 119.7 = 467.7$$
$$297.4 = 419.4 - 122.0 \leq Y_{h(new)} \leq 419.4 + 122.0 = 541.4$$

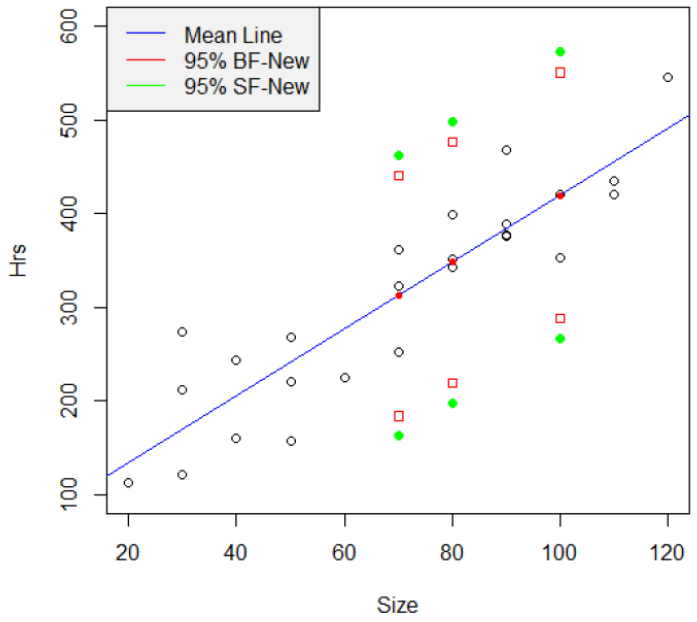
Simultaneous Prediction Intervals for New Observations, cont'd

```
Xh<-c(80,100)
g<-length(Xh)
alpha<-0.05
CI.New <- predict.lm(fitreg,data.frame(Size = c(Xh)),
se.fit = T, level = 1-alpha)
M <-rbind(rep(qt(1 - alpha / (2*g), fitreg$df),g),
rep(sqrt( g * qf( 1 - alpha, g, fitreg$df)),g))
spred <- sqrt( CI.New$residual.scale^2 + (CI.New$se.fit)^2 ) # (2.38)
pred.new <- t(rbind(
"Yh" = Xh,
"s.pred" = spred,
"fit" = CI.New$fit,
"lower.B" = CI.New$fit - M[1,] * spred,
"upper.B" = CI.New$fit + M[1,] * spred,
"lower.S" = CI.New$fit - M[2,] * spred,
"upper.S" = CI.New$fit + M[2,] * spred))
```

Simultaneous Prediction Intervals for New Observations, cont'd



(a) $g = 2$



(b) $g = 3$

- Simultaneous prediction intervals for g new observations on Y at g different levels of X with a 1 – α family confidence coefficient are wider than the corresponding single prediction intervals.
- Note that both the B and S multiples for simultaneous predictions become larger as g increases. This contrasts with simultaneous estimation of mean responses where the B multiple becomes larger but not the W multiple. When g is large, both the B and S multiples for simultaneous predictions may become so large that the prediction intervals will be too wide to be useful.

Regression through Origin

- Sometimes the regression function is known to be linear and to go through the origin at (0,0).

$$Y_i = \beta_i X_i + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2)$$

- The regression function for model is E{Y}=β_1X , straight line through origin with slope β_1

- The point estimator (LSE and MLE):

$$b_1 = \frac{\sum X_i Y_i}{\sum X_i^2}$$

Regression through Origin, cont'd

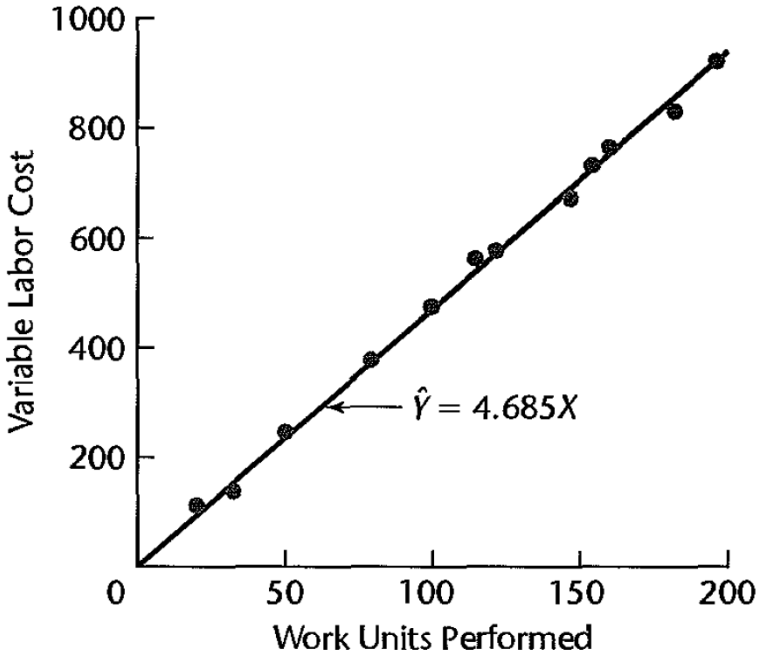
Estimate of	Estimated Variance	Confidence Limits	
β_1	$s^2\{b_1\} = \frac{MSE}{\sum X_i^2}$	$b_1 \pm ts\{b_1\}$	(4.18)
$E\{Y_h\}$	$s^2\{\hat{Y}_h\} = \frac{X_h^2 MSE}{\sum X_i^2}$	$\hat{Y}_h \pm ts\{\hat{Y}_h\}$	(4.19)
$Y_{h(\text{new})}$	$s^2\{\text{pred}\} = MSE\left(1 + \frac{X_h^2}{\sum X_i^2}\right)$	$\hat{Y}_h \pm ts\{\text{pred}\}$	(4.20)

where: $t = t(1 - \alpha/2; n-1)$

Regression through Origin, cont'd

Example: Warehouse Data

Warehouse i	(1) Work Units Performed X_i	(2) Variable Labor Cost (dollars) Y_i	(3) X_iY_i	(4) X_i^2	(5) $\hat{Y}_i$	(6) e_i
1	20	114	2,280	400	93.71	20.29
2	196	921	180,516	38,416	918.31	2.69
3	115	560	64,400	13,225	538.81	21.19
...	...	...	...	...	...	...
10	147	670	98,490	21,609	688.74	−18.74
11	182	828	150,696	33,124	852.72	−24.72
12	160	762	121,920	25,600	749.64	12.36
Total	1,359	6,390	894,714	190,963	6,367.28	22.72



$$b_1 = \frac{\sum X_i Y_i}{\sum X_i^2} = \frac{894{,}714}{190{,}963} = 4.68527 \quad \Rightarrow \quad \hat{Y} = 4.68527X$$

Regression through Origin, cont'd

- Important Caution for Using Regression through Origin:

$$Y_i = \beta_i X_i + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2)$$

- The residuals do not sum to zero usually
- The only constraint

$$\sum X_i e_i = 0$$

- The residuals plot will usually not be balanced around the zero line.

Regression through Origin, cont'd

- SSE = $\sum e_i^2$ may exceed SSTO = $\sum (Y_i - \bar{Y})^2$ when
  - data form a curvilinear pattern or a
  - Linear pattern with an intercept away from the origin.

- If SSE ≥ SSTO $\Rightarrow R^2 \leq 0$ as $1 - \frac{SSE}{SSTO} \leq 0$

- regression-through-the-origin model (4.10) needs to be evaluated for aptness.
  - it is generally a safe practice not to use regression-through-the-origin model and instead use the intercept regression model.

Regression through Origin, cont'd

- SSE = $\sum e_i^2$ may exceed SSTO =$\sum(Y_i - \bar{Y})^2$ when
- Some statistical packages calculate R^2 for regression through the origin and hence will sometimes show a negative value for R^2. Other statistical packages calculate R^2 using the total uncorrected sum of squares SSTOU (uncorrected SS).
- This procedure avoids obtaining a negative coefficient but lacks any meaningful interpretation.
- The ANOVA tables for regression through the origin shown in the output for many statistical packages are based on SSTOU = $\sum Y_i^2$, SSRU=$\sum \hat{Y}_i^2 = b_1^2 \sum X_i^2$, and SSE=$\sum(Y_i - b_1 X_i)^2$. It can be shown that these sums of squares are additive: SSTOU = SSRU + SSE.

Effects of Measurement Errors

- Measurement Errors in the response variable (Y):
  - no new problems are created when these errors are uncorrelated and not biased(positive and negative measurement errors tend to cancel out).
  - absorbed in the model error term ε

- Measurement Errors in the predictor variable (X):
  - pressure in a tank
  - temperature in an oven
  - speed of a production line

Effects of Measurement Errors, cont'd

Illustration:

- X_i: the true age of the ith employee
- X_i^*: the age reported by the employee of the employment record
- Measurement Error δ_i:

$$\delta_i = X_i^* - X_i$$

- The regression model:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

Effects of Measurement Errors, cont'd

- Observe only X_i^*:

$$Y_i = \beta_0 + \beta_1(X_i^* - \delta_i) + \varepsilon_i \tag{4.23}$$

$$\Rightarrow Y_i = \beta_0 + \beta_1 X_i^* + (\varepsilon_i - \beta_1\delta_i) \tag{4.24}$$

- X_i^* is a random variable is correlated with $\varepsilon_i - \beta_1\delta_i$
- $\varepsilon_i - \beta_1\delta_i$ is not independent of X_i^* $(\because \delta_i = X_i^* - X_i)$
- To determine the dependence: Conditions

$$E\{\delta_i\} = 0$$

$$E\{\varepsilon_i\} = 0$$

$$E\{\delta_i\varepsilon_i\} = 0$$

$$\Rightarrow E\{X_i^*\} = X_i$$

Effects of Measurement Errors, cont'd

- $E\{\delta_i\} = 0$: the reported ages would be unbiased estimates of the true ages
- $E\{\varepsilon_i\} = 0$: the model error term have expectation 0
- $E\{\delta_i \varepsilon_i\} = 0$: the measurement error δ_i not be correlated with ε_i

$$\Rightarrow \sigma\{\delta_i, \varepsilon_i\} = 0$$

- The covariance between X_i^* and $\varepsilon_i - \beta_1 \delta_i$: not zero

$$E\{\varepsilon_i - \beta_1 \delta_i\} = 0;$$
$$\sigma\{X_i^*, \varepsilon_i - \beta_1 \delta_i\} = E\{\delta_i \varepsilon_i - \beta_1 \delta_i^2\} = -\beta_1 \sigma^2\{\delta_i\}$$

Effects of Measurement Errors, cont'd

Assume $(Y, X^*) \sim$ bivariate normal distribution

- The conditional distribution of Y_i given X_i^*: normal and independent

$$\text{Conditional mean: } E\{Y_i|X_i^*\} = \beta_0^* + \beta_1^* X_i^*$$

$$\text{Conditional variance: } \sigma^2_{Y|X^*}$$

$$\Rightarrow \beta_1^* = \beta_1[\sigma_X^2/(\sigma_X^2 + \sigma_Y^2)] < \beta_1$$

If σ_Y^2 is small relative to σ_X^2, the bias would be small; otherwise the bias may be substantial

Berkson Model

- There is one situation where measurement errors in X are no problem. (Berkson Model)
  - The observation X_i^* is fixed quantity
  - The unobserved true value X is a random variable
  - The measurement error: $\delta_i = X_i^* - X_i$ (no constraint on the relation between X_i^* and δ_i)
  - $E\{\delta_i\} = 0$
  - Still applicable for the Berkson case:

$$Y_i = \beta_0 + \beta_1 X_i^* + (\varepsilon_i - \beta_1 \delta_i) \qquad (4.28)$$

Berkson Model, cont'd

- $E\{\varepsilon_i - \beta_1\delta_i\} = 0;\ E\{\varepsilon_i\} = 0;\ E\{\delta_i\} = 0$

$$\Rightarrow \sigma\{X_i^*, \varepsilon_i - \beta_1\delta_i\} = 0 \quad (\because X_i^* \text{ is a constant})$$

- Least squares procedures can be applied for the Berkson case: unbiased estimators b_0, b_1
  - The error terms have expectation zero
  - The predictor variable is a constant, and the error terms are not correlated with it.

Inverse Prediction

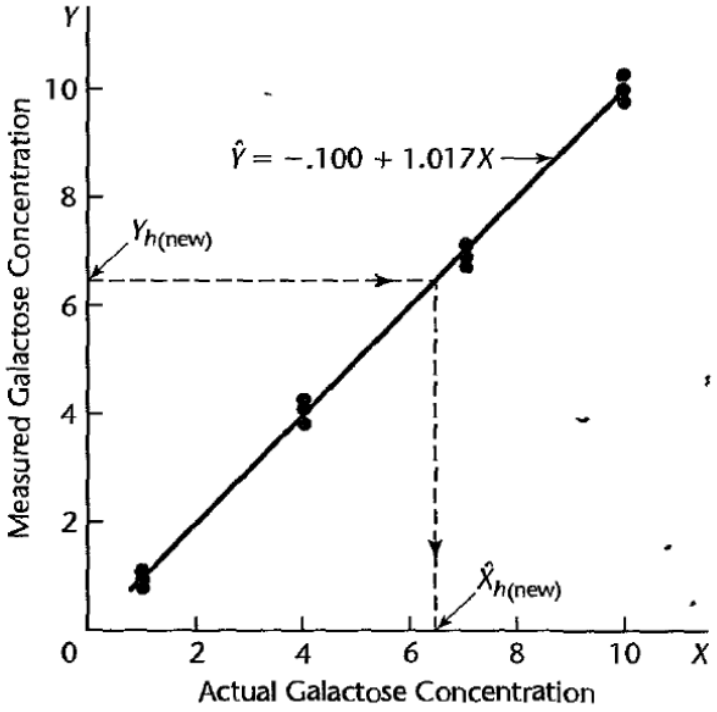


Figure : Scatter Plot and Fitted Regression Line-calibration Example.

Inverse Prediction,cont'd

- A trade association analyst has regressed the selling price of a product (Y) on its cost (X). The selling price $Y_{h(new)}$ for another firm not belonging to the trade association is known, and it is desired to estimate the cost $X_{h(new)}$ for this firm.
- A regression analysis of the amount of decrease in cholesterol level (Y) achieved with a given dosage of a new drug (X) has been conducted. A physician is treating a new patient for whom the cholesterol level should decrease by the amount $Y_{h(new)}$. It is desired to estimate the appropriate dosage level $X_{h(new)}$ to be administered to bring about the needed cholesterol decrease $Y_{h(new)}$

Inverse Prediction, cont'd

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

$$\hat{Y} = b_0 + b_1 X$$

$$\Rightarrow \hat{X}_{h(new)} = \frac{Y_{h(new)} - b_0}{b_1} \quad b_1 \neq 0$$

- $\hat{X}_{h(new)}$ is the MLE of $X_{h(new)}$ for normal error regression model (2.1)
- Approximate $1 - \alpha$ confidence limits for $X_{h(new)}$:

$$\hat{X}_{h(new)} \pm t(1 - \alpha/2; n - 2)s\{predX\}$$

$$s^2\{predX\} = \frac{MSE}{b_1^2}[1 + 1/n + \frac{(\hat{X}_{h(new)} - \overline{X})^2}{\sum(X_i - \overline{X})^2}]X$$

Inverse Prediction, cont'd

- A medical researcher studied a new, quick method for measuring low concentration of galactose (sugar) in the blood. Linear regression model (2.1) was fitted with the following results:

$$n = 12 \qquad b_0 = -.100 \qquad b_1 = 1.017 \qquad MSE = .0272$$

$$s\{b_1\} = .0142 \qquad \bar{X} = 5.500 \qquad \bar{Y} = 5.492 \qquad \sum(X_i - \bar{X})^2 = 135$$

$$\hat{Y} = -.100 + 1.017X$$

- H_o:β_1 =0 vs H_a: $\beta_1 \neq 0$ $\Rightarrow$ conclude H_a: $\beta_1 \neq 0$
- $Y_{h(new)} = 6.52 \Rightarrow \hat{X}_{h(new)} = 6.509 \Rightarrow$ $6.13 \leq \hat{X}_{h(new)} \leq 6.89$

Inverse Prediction, cont'd

- The inverse prediction problem is also known as a calibration problem since it is applicable when inexpensive, quick, and approximate measurements (Y) are related to precise, often expensive, and time-consuming measurements (X) based on n observations. The resulting regression model is then used to estimate the precise measurement $X_{h(new)}$ for a new approximate measurement $Y_{h(new)}$.
- The approximate confidence interval (4.32) is appropriate if the quantity:

$$\frac{[t(1-\alpha/2;n-2)]^2 MSE}{b_1^2 \sum (X_i - \overline{X})^2} \quad \text{is small; less than } 0.1$$

Choice of X Levels

- By experiment: X's are under control of the experimenter
- Among other things: consider
  1. How many levels of X should be investigated?
  2. What shall the two extreme levels be?
  3. How shall the other levels of X be spaced?
  4. How many observations should be taken at each level of X?
- No single answer: different purposes lead to different answers
  - estimate the intercept
  - predict new observations
  - estimate mean responses
  - other purposes

Choice of X Levels, cont'd

The variances developed for the regression model(2.1):

$$\sigma^2\{b_0\} = \sigma^2[1/n + \frac{\overline{X}^2}{\sum(X_i - \overline{X})^2}]$$

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum(X_i - \overline{X})^2}$$

$$\sigma^2\{\widehat{Y}_h\} = \sigma^2[\frac{1}{n} + \frac{(X_h - \overline{X})^2}{\sum(X_i - \overline{X})^2}]$$

$$\sigma^2\{pred\} = \sigma^2[1 + \frac{1}{n} + \frac{(X_h - \overline{X})^2}{\sum(X_i - \overline{X})^2}]$$

Choice of X Levels, cont'd

- Estimate β_1: $\min s^2\{b_1\} \Leftrightarrow \max \sum(X_i - \overline{X})^2 \Rightarrow$ using two levels of X at the two extremes and placing half of the observations at each of the two levels
- Estimate Y_h: $\bar{X} = X_h$

Choose: (Cox advices)

- 2 levels: only interested in whether there is an effect and its direction
- 3 levels: goal is describing relation and any possible curvature
- 4 or more levels:further description of response curve and any potential non-linearity such as an asymptotic value