

The background of the entire slide is a repeating pattern of US pennies. The pennies are arranged in a grid-like fashion, with some showing the profile of Abraham Lincoln and others showing the Union Shield. The text is overlaid on this background.

Number of successes in n Bernoulli trials

Useful in many applications

μ , V , σ

Binomial distribution

21 heads

19 tails

The Binomial Distribution

n independent Bernoulli experiments

\bar{p}, q

Each “success” with same probability p

“failure” with probability $1 - p$

$B_{p,n}$ - distribution of # successes

n independent coin flips

$P(\text{heads}) = p$

$B_{p,n}$ - distribution of # heads

$B_{n,p}$ more common

Use $B_{p,n}$

Generalizes B_p ,

Natural for Poisson Binomial

No confusion: $n \in \mathbb{N}, 0 \leq p \leq 1$

Applications

Positive responses to a treatment

Faulty components

Rainy days in a month

Delayed flights

Small n

n independent experiments

Success probability p

Failure probability $q = 1 - p$

$b_{p,n}(k)$ - probability of k successes

$0 \leq k \leq n$

n = 0

k	$b_{p,0}(k)$
0	1

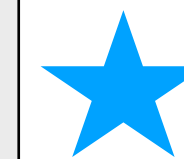
n = 1

k	$b_{p,1}(k)$
0	q
1	p

$$p+q=1$$

n = 2

k	seq's	$b_{p,2}(k)$
0	00	q^2
1	01,10	$2pq$
2	11	p^2



$$p^2+2pq+q^2=(p+q)^2=1^2=1$$

General n and k

$n \perp B_p$ experiments

successes $0 \leq k \leq n$

$b_{p,n}(k) = p(k \text{ successes})$

Every k-success sequence: n-k failures, probability $p^k \cdot q^{n-k}$

$\binom{n}{k}$ such sequences

$$= \binom{n}{k} p^k q^{n-k}$$

Distribution over $n+1$ values

Σ Will It ADD?

$$0 \leq k \leq n$$

$$p(X = k) = b_{p,n}(k) = \binom{n}{k} p^k q^{n-k}$$

$$\sum_{k=0}^n b_{p,n}(k) = \sum_{k=0}^n \binom{n}{k} p^k q^{n-k}$$

$$= (p + q)^n$$

$$= 1^n = 1$$

**YES IT
ADDS!**

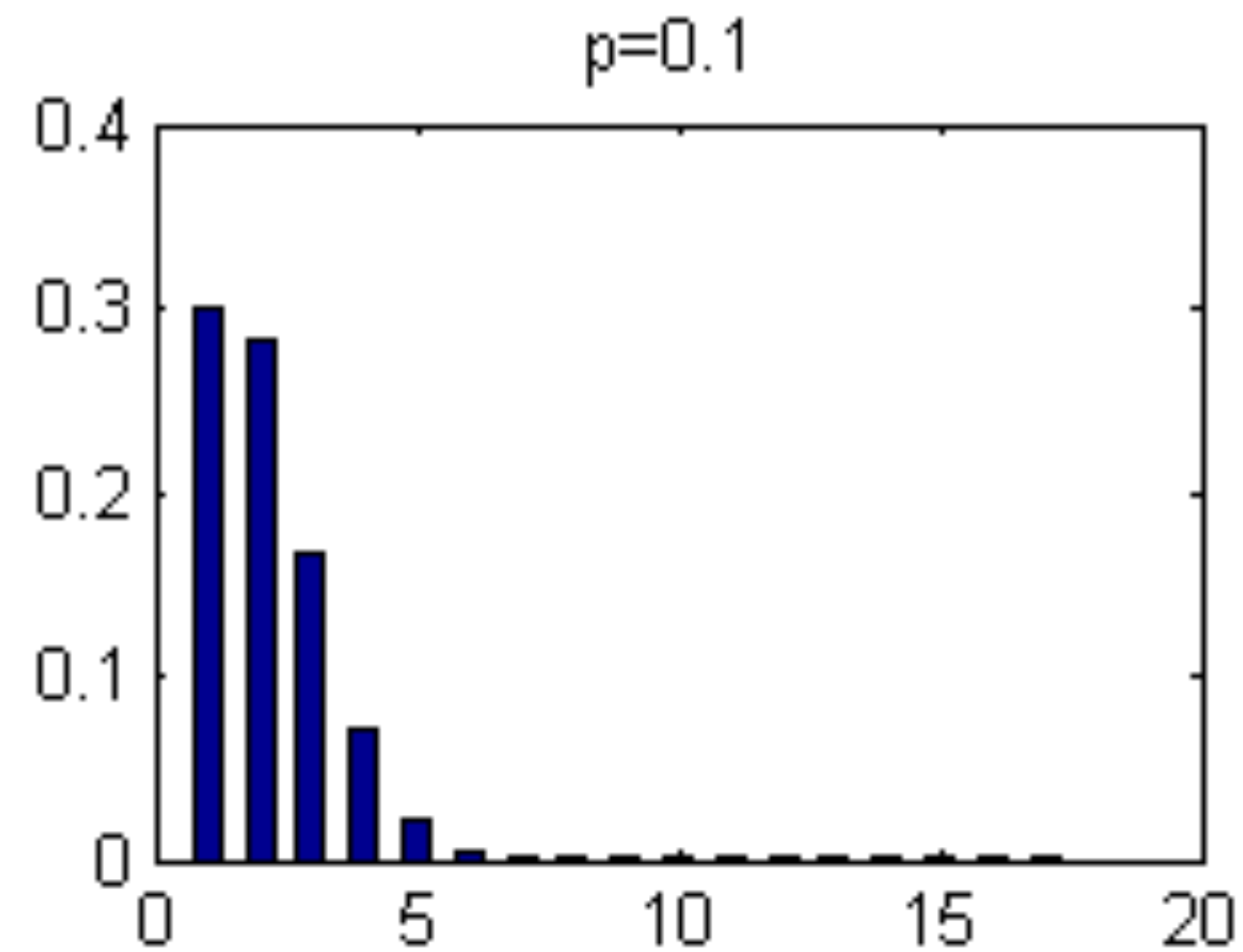
Binomial Theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Typical Distributions

$n=18$

$b_{p,18}(k)$



Notebook: experiment with different p and n

Multiple Choice

Exam has 6 multiple-choice questions, each with 4 possible answers

Each question, student selects one of the 4 answers randomly

$X = \# \text{ correct answers} \sim B_{1/4, 6}$

Passing: ≥ 4 correct answers $P(\text{passing}) = ?$

$$P(4) = \binom{6}{4} \cdot \left(\frac{1}{4}\right)^4 \cdot \left(\frac{3}{4}\right)^2 \approx 0.0329$$

$$P(5) = \binom{6}{5} \cdot \left(\frac{1}{4}\right)^5 \cdot \left(\frac{3}{4}\right)^1 \approx 0.00439$$

$$P(6) = \binom{6}{6} \cdot \left(\frac{1}{4}\right)^6 \cdot \left(\frac{3}{4}\right)^0 \approx 0.000244$$

$$P(\geq 4) = P(4) + P(5) + P(6) \approx 0.03759$$

Binomial as a Sum

$B_{p,n}$ a sum of n B_p

$X_1, \dots, X_n \sim B_p \perp\!\!\!\perp$

$$X \stackrel{\text{def}}{=} \sum_{i=1}^n X_i$$

$$P(X = k) = P(\text{exactly } k \text{ of } X_1, \dots, X_n \text{ are } 1) = \binom{n}{k} p^k q^{n-k} = b_{p,n}(k)$$

$$X \sim B_{p,n}$$

Apply to mean and variance

Mean and Variance

$$X \sim B_{p,n}$$

$$X = \sum_{i=1}^n X_i \quad X_1, \dots, X_n \sim B_p \quad \perp$$

$$E(X) = E\left(\sum_{i=1}^n X_i\right) \stackrel{\text{LE}}{=} \sum E X_i \stackrel{B_p}{=} \sum p = np$$

$$V(X) = V\left(\sum_{i=1}^n X_i\right) \stackrel{\perp}{=} \sum V(X_i) \stackrel{B_p}{=} \sum pq = npq$$

$$\sigma = \sqrt{npq}$$

Multiple Choice

Exam has 6 multiple-choice questions, each with 4 possible answers

For each question, student selects one of the 4 answers randomly

$X = \# \text{ correct answers} \sim B_{1/4, 6}$

Mean $EX = np = 6 \cdot \frac{1}{4} = 1.5$

Standard deviation $\sigma = \sqrt{npq} = \sqrt{6 \cdot \frac{1}{4} \cdot \frac{3}{4}} = \frac{\sqrt{18}}{4}$

Why Vote

For simplicity odd # voters: $2n + 1$

Each equally likely D or R

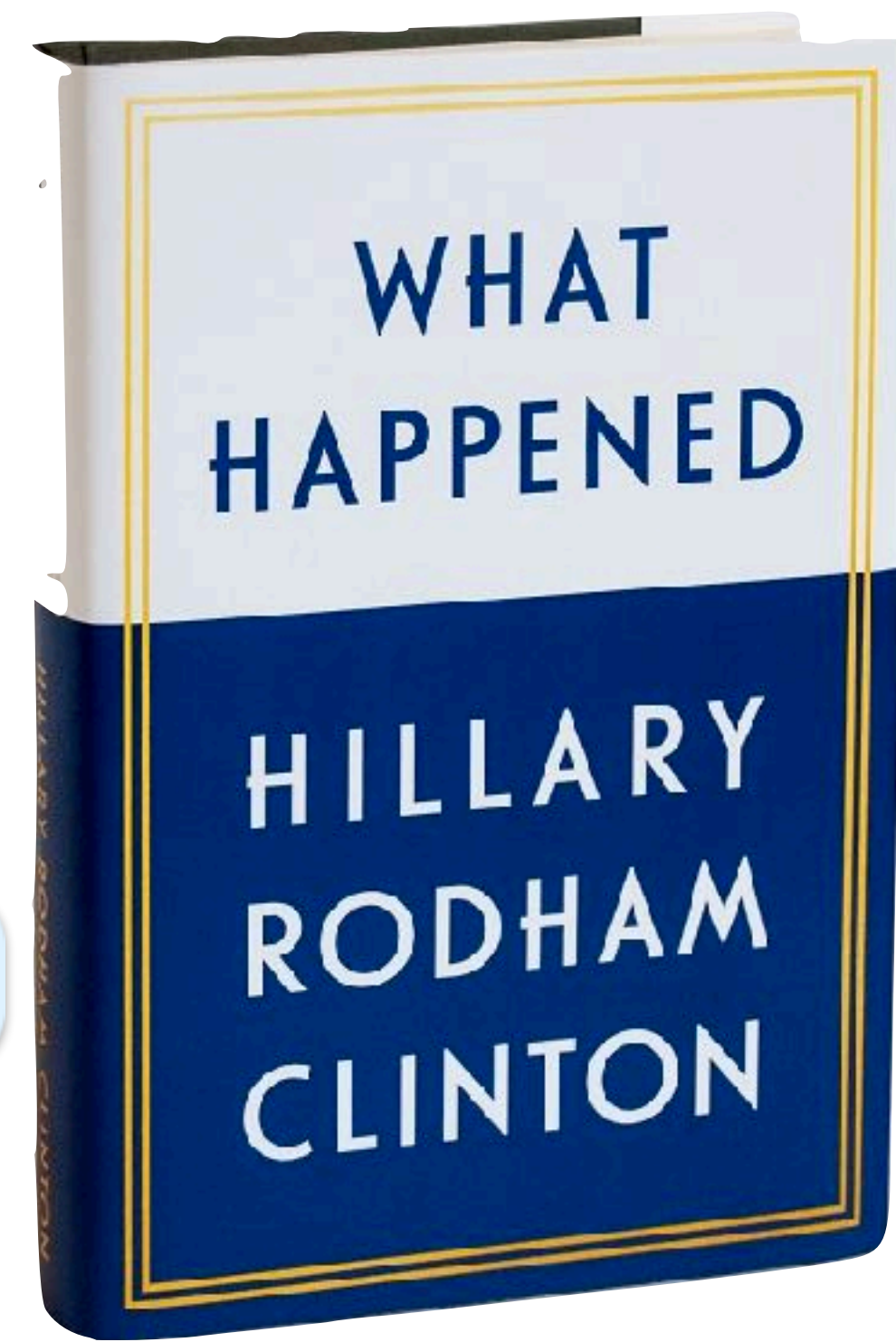
$P(\text{voter makes a difference}) = P(\text{other } 2n \text{ voters equally split})$

$$b_{p,n}(k) = \binom{n}{k} p^k q^{n-k} = \binom{2n}{n} \frac{1}{2^n} \cdot \frac{1}{2^n}$$

Stirling's Approximation

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

$$\begin{aligned} &= \frac{(2n)!}{n! \cdot n! \cdot 2^n \cdot 2^n} \\ &\approx \frac{\sqrt{2\pi \cdot 2n} \left(\frac{2n}{e}\right)^{2n}}{(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n)^2 2^{2n}} \\ &= \frac{1}{\sqrt{\pi n}} \end{aligned}$$



Poisson Binomial

Generalizes the binomial distribution

$n \geq 1$	Binomial	$B_{p,n}$	For $1 \leq i \leq n$	$X_i \sim B_p$	\perp	$X = \sum_{i=1}^n X_i$
	Poisson Binomial	PB_{p_1, \dots, p_n}		$X_i \sim B_{p_i}$		

$PB_{1/4, 2/3}$

$X_1 \sim B_{1/4}$

$X_2 \sim B_{2/3}$

\perp

X_1	X_2	P	X
0	0	$3/4 \cdot 1/3 = 1/4$	0
0	1	$3/4 \cdot 2/3 = 1/2$	1
1	0	$1/4 \cdot 1/3 = 1/12$	1
1	1	$1/4 \cdot 2/3 = 1/6$	2

X	P(x)
0	$1/4$
1	$7/12$
2	$1/6$

Expectation and Variance

$$X \sim PB_{p_1, p_2, \dots, p_n} \quad X = \sum_{i=1}^n X_i \quad X_i \sim B_{p_i}$$

$$E(X) = E\left(\sum_{i=1}^n X_i\right) \stackrel{\text{LE}}{=} \sum_{i=1}^n EX_i \stackrel{B_{p_i}}{=} \sum_{i=1}^n p_i$$

$$V(X) = V\left(\sum_{i=1}^n X_i\right) \stackrel{\text{II}}{=} \sum_{i=1}^n V(X_i) \stackrel{B_{p_i}}{=} \sum_{i=1}^n p_i(1 - p_i)$$

p(k)

No closed form

Computationally

Homework

$B_{p,n}$ Number of successes in n B_p trials

$$b_{p,n}(k) = \binom{n}{k} p^k \bar{p}^{n-k}$$

$$\mu = np$$

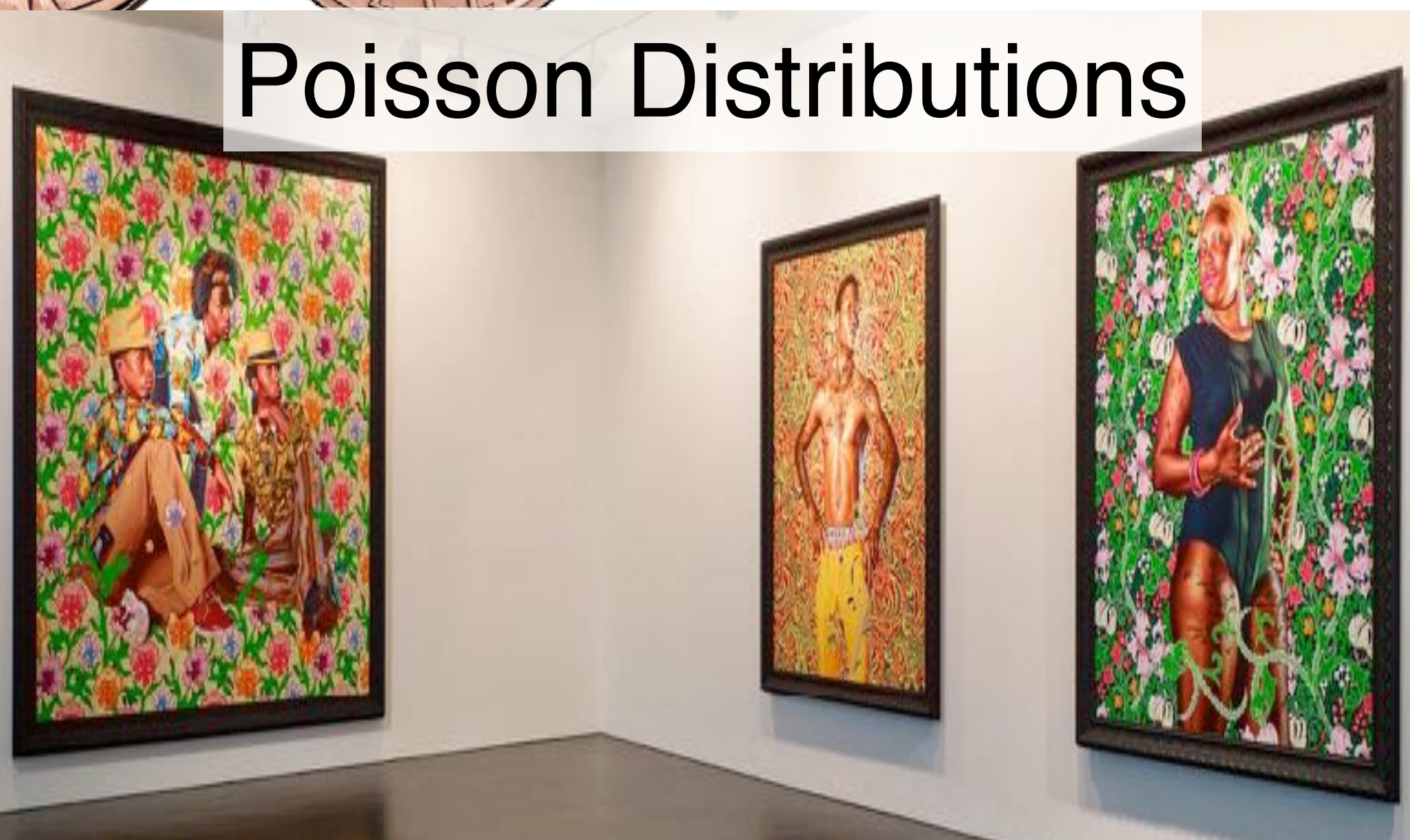
$$V = np\bar{p}$$

$$\sigma = \sqrt{np\bar{p}}$$

Why vote

Binomial distribution

Poisson Distributions



Coin Flips

Most basic convergence to average is $B(p)$

Flip n $B(p)$ coins, average # 1's will approach np

Probability of a sequence with k 1's and $n-k$ 0's is $p^k q^{n-k}$

Wolog assume $p > 0.5$, then most likely is 1^n

Yet by WLLN with probability $\rightarrow 1$ we see roughly pn 1's and qn 0's

Why do we observe these sequences and not the most likely ones?

Strength in #s. # sequences of a given composition increases near $1/2$

pn balances # x probability.