Two Variables

Why 2

Outcomes often result from multiple factors

Rain temperature and humidity

Economy unemployment and inflation

Hiring experience and salary

Student # classes GPA

Human condition profession age cholesterol salary happiness location dinner plans …

Two Fair Coins

$U,V \sim B(1/2)$ $\perp\!\!\!\perp$

Several ways to indicate distribution

Explicit $P(u,v) \stackrel{\text{def}}{=} P(U{=}u, V{=}v) = ¼ \quad \forall \{u,v\} \in \{0,1\}$

1-d table

u	v	P(u,v)
0	0	¼
0	1	¼
1	0	¼
1	1	¼

2-d table

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{v} \\ & & 0 & 1 \\ \hline u & 0 & ¼ & ¼ \\ & 1 & ¼ & ¼ \end{array}$$

Use U, V, for several examples

Min - Max

$U,V \sim B(1/2)$ $\perp\!\!\!\perp$

$X = \min(U,V)$ $Y = \max(U,V)$

u	v	min	max	
0	0	0	0	¼
0	1	0	1	½
1	0	0	1	
1	1	1	1	¼

y = max

x = min

	0	1
0	¼	½
1	0	¼

Product - Sum

$X = U \cdot V$

$Y = U + V$

		y		
		0	1	2
x	0	¼	½	0
	1	0	0	¼

3 Coins

$U_1, U_2, U_3 \sim B(1/2)$ $\perp\!\!\!\perp$

$X = U_1 + U_2$ # heads among first 2

$Y = U_2 + U_3$ # heads among last 2

U_1	U_2	U_3	X	Y
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	2
1	0	0	1	0
1	0	1	1	1
1	1	0	2	1
1	1	1	2	2

X \ Y	0	1	2
0	⅛	⅛	0
1	⅛	¼	⅛
2	0	⅛	⅛

General B(p)

$U \sim B(p)$, $V \sim B(q)$ $\quad \perp\!\!\!\perp$

$X = \min(U,V)$ $\qquad$ $Y = \max(U,V)$

u \ v		
	$\bar{p}\bar{q}$	$\bar{p}q$
	$p\bar{q}$	pq

y = max

x = min	0	1
0	$\bar{p}\bar{q}$	$p\bar{q}+\bar{p}q$
1	0	pq

General?

Joint Distribution

X, Y - random variables

Joint distribution: P: probability of every possible (x,y) pair

$$p(x,y) \stackrel{\text{def}}{=} P(X = x, Y = y)$$

$$\forall x,y \; p(x,y) \geq 0$$

$$\sum_{x,y} p(x,y) = 1$$

Joint Distribution Tells All

Joint distribution determines probabilities of all events

x \ y	0	1
0	0.1	0.2
1	0.3	0.4

$$P(X \leq Y) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 1)$$

$$= P(0, 0) + P(0, 1) + P(1, 1)$$

$$= 0.1 + 0.2 + 0.4$$

$$= 0.7$$

Marginals

Marginal of X $P(x) \stackrel{\text{def}}{=} P_X(x) \stackrel{\text{def}}{=} P(X = x) = \sum_y p(x,y)$

Rule of total probability

Marginal of Y $P(y) \stackrel{\text{def}}{=} P_Y(y) \stackrel{\text{def}}{=} P(Y = y) = \sum_x p(x,y)$

x \ y	0	1	
0	0.1	0.2	← $P(X = 0) = .3$
1	0.3	0.4	← $P(X = 1) = .7$

$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1)$

$= P(0,0)+P(0,1) \;\; = .1 + .2 = .3$

Conditionals

$$P(x \mid y) = \frac{p(x,y)}{p(y)}$$

$$P(y \mid x) = \frac{p(x,y)}{p(x)}$$

x \ y	0	1
0	0.1	0.2
1	0.3	0.4

$\leftarrow P(X = 0) = 0.3$

$\uparrow$ $P(Y = 0) = 0.4$

$$P(Y = 0 \mid X = 0) = \frac{P(X = 0 \;,\; Y = 0)}{P(X = 0)} = \frac{0.1}{0.3} = \frac{1}{3}$$

$$P(Y = 1 \mid X = 0) = \frac{P(X = 0 \;,\; Y = 1)}{P(X = 0)} = \frac{0.2}{0.3} = \frac{2}{3}$$

$$P(X = 0 \mid Y = 0) = \frac{P(X = 0 \;,\; Y = 0)}{P(Y = 0)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$P(X = 1 \mid Y = 0) = 1 - P(X = 0 \;,\; Y = 0) = 1 - \frac{1}{4} = \frac{3}{4}$$

Independence

X, Y independent $\qquad X \perp\!\!\!\perp Y$

$\forall x,y \qquad p(y \mid x) = p(y)$

$p(x \mid y) = p(x)$

$p(x,y) = p(x) \cdot p(y)$ ← more robust

P(y), y / P(x), x

	0.2	0.8
0.6	0.12	0.48
0.4	0.08	0.32

$\perp\!\!\!\perp$

P(y), y / P(x), x

	0.4	0.6
0.3	0.1	0.2
0.7	0.3	0.4

$\not\perp\!\!\!\perp$

Independence Checks

Independent → rows proportional to each other

→ columns proportional to each other

$X \sim B(½)$

$Y = X$

$x \backslash y$	0	1
0	½	0
1	0	½

$\not\perp\!\!\!\perp$

$Y = 1 - X$

$x \backslash y$	0	1
0	0	½
1	½	0

$\not\perp\!\!\!\perp$