Two Variables

Why 2

Outcomes often result from multiple factors

Rain temperature and humidity

Economy unemployment and inflation

Hiring experience and salary

Student # classes GPA

Human condition profession age happiness salary location dinner plans

Two Fair Coins

$$U, V \sim B(1/2)$$



Several ways to indicate distribution

Explicit
$$P(u,v) \stackrel{\text{def}}{=} P(U=u, V=v) = \frac{1}{4} \quad \forall \{u,v\} \in \{0,1\}$$

1-d table

u	V	P(u,v)
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

2-d table

Use U, V, for several examples

Min - Max

U,V ~ B(1/2)



$$X = min(U,V)$$

$$X = min(U,V)$$
 $Y = max(U,V)$

u	V	min	max	
0	0	0	0	} 1/4
0	1	0	1	1/2
1	0	0	1	\frac{1}{2}
1	1	1		<pre>} 1/4</pre>

$$y = max$$

$$x = \min \quad 0 \quad \frac{1}{4}$$

	0	1
0	1/4	1/2
1	0	1/4

Product - Sum

$$X = U \cdot V$$

$$Y = U+V$$

y

0 1 2

1 1/2 0

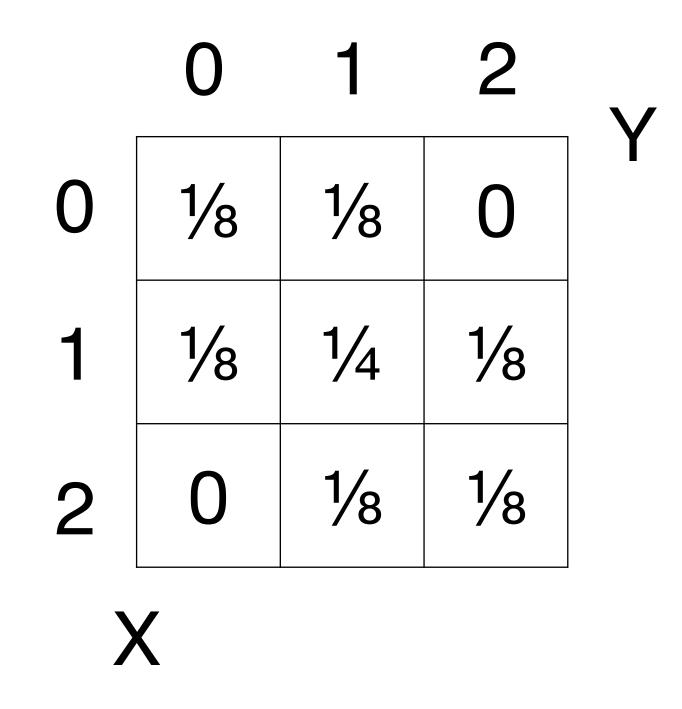
1 0 0 1/4

3 Coins

$$U_1, U_2, U_3 \sim B(1/2)$$

$$X = U_1 + U_2$$
 # heads among first 2

U ₁	U ₂	U ₃	X	Y
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	2
1	0	0	1	0
1	0	1	1	1
1	1	0	2	1
1	1	1	2	2



General B(p)

$$U \sim B(p), V \sim B(q)$$

$$X = min(U,V)$$
 $Y = max(U,V)$

$$Y = max(U,V)$$

$$y = max$$

$$0 \quad 1$$

$$x = min \quad 0 \quad \overline{p}\overline{q} \quad p\overline{q} + \overline{p}q$$

$$1 \quad 0 \quad pq$$

General?

Joint Distribution

X, Y - random variables

Joint distribution: P: probability of every possible (x,y) pair

$$p(x,y) \stackrel{\text{def}}{=} P(X = x, Y = y)$$

$$\forall x,y \ p(x,y) \geq 0$$

$$\sum_{x,y} p(x,y) = 1$$

Joint Distribution Tells All

Joint distribution determines probabilities of all events

$$0 1 y$$

$$0 0.1 0.2$$

$$X 1 0.3 0.4$$

$$P(X \le Y) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 1)$$

$$= P(0, 0) + P(0, 1) + P(1, 1)$$

$$= 0.1 + 0.2 + 0.4$$

$$= 0.7$$

Marginals

Marginal of X
$$P(x) = P_X(x) = P(X = x) = \sum_y p(x,y)$$
Rule of total probability

Marginal of Y $P(y) = P_Y(y) = P(Y = y) = \sum_x p(x,y)$
0 1 y
0 0.1 0.2 $\leftarrow P(X = 0) = .3$
X 1 0.3 0.4 $\leftarrow P(X = 1) = .7$

$$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1)$$

= $P(0,0)+P(0,1) = .1 + .2 = .3$

$$P(x | y) = \frac{p(x,y)}{p(y)}$$

Conditionals of 1 y

$$P(y \mid x) = \frac{p(x,y)}{p(x)}$$

0 0.1 0.2
$$\leftarrow P(X = 0) = 0.3$$

x 1 0.3 0.4

$$P(Y = 0 | X = 0) = \frac{P(X = 0, Y = 0)}{P(X = 0)} = \frac{0.1}{0.3} = \frac{1}{3}$$
 $P(Y = 0) = 0.4$

$$P(Y = 0) = 0.4$$

$$P(Y = 1 | X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{0.2}{0.3} = \frac{2}{3}$$

$$P(X = 0 | Y = 0) = \frac{P(X = 0, Y = 0)}{P(Y = 0)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$P(X = 1 | Y = 0) = 1 - P(X = 0, Y = 0) = 1 - \frac{1}{4} = \frac{3}{4}$$

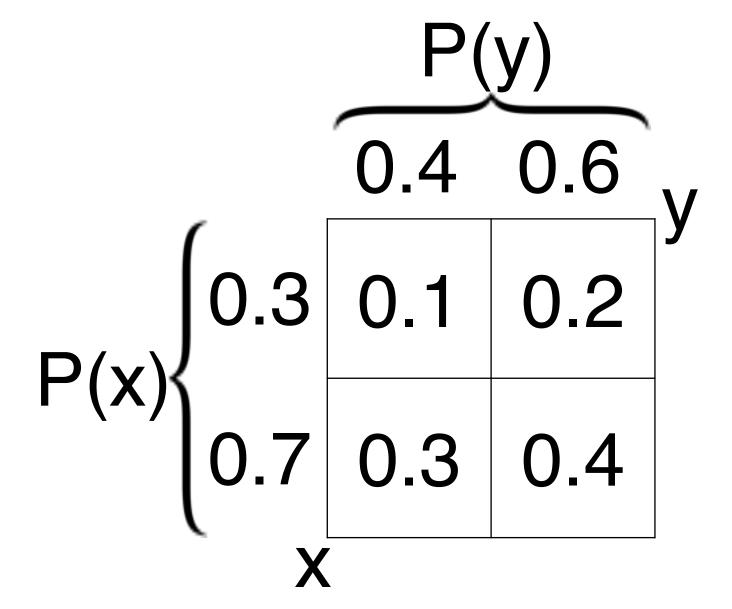
Independence

X, Y independent

$$\forall x,y$$
 $p(y \mid x) = p(y)$
 $p(x \mid y) = p(x)$
 $p(x,y) = p(x) \cdot p(y) \leftarrow \text{more robust}$

$$P(y) = 0.2 \quad 0.8 \quad y$$

$$P(x) \begin{cases} 0.6 & 0.12 \quad 0.48 \\ 0.4 & 0.08 \quad 0.32 \\ x \end{cases}$$



Independence Checks

Independent → rows proportional to each other

-- columns proportional to each other