Functions of Random Variables

X - income, $Y = 0.3X$ - income tax

X - driving speed, $Y = 2^x$ - speeding ticket

$X \sim f_X$ $\quad$ f_X known distribution

$Y = g(X)$ $\quad$ g known deterministic function

f_Y ? $\quad$ What is the distribution of Y?

Power-law pdf's

$a > -1$

$\int_0^1 \underbrace{(a+1)x^a}_{\geq 0} dx = x^{a+1}\Big|_0^1 = 1$

$a < -1$ later

$$f(x) = \begin{cases} (a+1)x^a & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

pdf!

$F(x) = \int_0^x (a+1)u^a du = u^{a+1}\Big|_0^x = x^{a+1}$

$a \leq x \leq 1$

Use $f(x) = 3x^2$

$F(x) = x^3$

$0 \leq x \leq 1$

$$X \sim f_X(x) = 3x^2$$

$$F_X(x) = x^3$$

$$0 \le x \le 1$$

g ↗

$$Y = g(X) \triangleq X^{\frac{3}{2}}$$

$$F_Y(y) = P(Y \le y) \qquad 0 \le y \le 1$$

$$= P(X^{\frac{3}{2}} \le y)$$

$$= P(X \le y^{\frac{2}{3}})$$

$$= F_X(y^{\frac{2}{3}})$$

$$= y^2$$

$$F_Y(0) = 0$$

$$F_Y(1) = 1$$ ✅

$$f_Y(y) = F'_Y(y) = 2y$$

g ↘

$$Y = g(X) \triangleq X^{-3}$$

$$F_Y(y) = P(Y \le y) \qquad y \ge 1$$

$$= P(X^{-3} \le y)$$

$$= P(X \ge y^{-\frac{1}{3}})$$

$$= 1 - P(X \le y^{-\frac{1}{3}})$$

$$= 1 - F_X(y^{-\frac{1}{3}})$$

$$= 1 - y^{-1}$$

$$F_Y(1) = 0$$

✅ $$F_Y(\infty) = 1$$

$$f_Y(y) = F'_Y(y) = y^{-2}$$

$g \nearrow$

Generalized form for increasing Y=g(X)

$$\boxed{f_X(x) = 3x^2 \quad F_X(x) = x^3 \quad g(x) = x^{\frac{3}{2}} \quad h(y) = y^{\frac{2}{3}} \quad f_Y(y) = 2y}$$

$$F_Y(y) \triangleq P(Y \leq y)$$

$$f_Y(y) = F'_Y(y)$$

$$= P(g(X) \leq y)$$

$$P(X^{\frac{3}{2}} \leq y)$$

$$= [F_X(h(y))]'$$

$$[(y^{\frac{2}{3}})^3]'$$

$$= P(X \leq g^{-1}(y))$$

$$P(X \leq y^{\frac{2}{3}})$$

$$= F'_X(h(y)) \cdot h'(y)$$

$$3(y^{\frac{2}{3}})^2 \cdot \frac{2}{3} y^{-\frac{1}{3}}$$

$$= F_X(g^{-1}(y))$$

$$F_X(y^{\frac{2}{3}})$$

$$= f_X(h(y)) \cdot h'(y)$$

$$2y$$

$$= F_X(h(y))$$

$$h(y) \stackrel{\text{def}}{=} g^{-1}(y)$$

g $\searrow$

Generalized form for decreasing Y=g(X)

$$F_Y(y) \triangleq P(Y \leq y)$$

$$= P(g(X) \leq y)$$

$$= P(X \geq g^{-1}(y))$$

$$= 1 - P(X \leq g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(h(y))$$

$$f_Y(y) = F'_Y(y)$$

$$= [1 - [F_X(h(y))]'$$

$$= -F'_X(h(y)) \cdot h'(y)$$

$$= -f_X(h(y)) \cdot h'(y)$$

Combining

$g \nearrow$ $$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$g \searrow$ $$f_Y(y) = -f_X(h(y)) \cdot h'(y)$$

For both $$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

Alternative formulation $$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \Big|_{y=g(x)}$$

Functions of Random Variables

Uniform Distributions