

# Covariance

# Do Expectations Multiply?

$$E(XY) = \sum_{x,y} xy \cdot p(x, y) \qquad E(XY) \stackrel{?}{=} EX \cdot EY$$

$$X = Y = \begin{matrix} & \begin{matrix} -1 & 1 \end{matrix} & y \\ \begin{matrix} -1 & 1 \end{matrix} & \begin{array}{|c|c|} \hline \frac{1}{2} & 0 \\ \hline 0 & \frac{1}{2} \\ \hline \end{array} & \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \\ x & \frac{1}{2} & \frac{1}{2} \end{matrix}$$

$$EX = EY = 0 \qquad EX \cdot EY = 0$$

$$E(XY) = EX^2 = E(1) = 1 \qquad E(XY) \neq EX \cdot EY$$

Expectations do not always multiply!      Satisfy any relation?

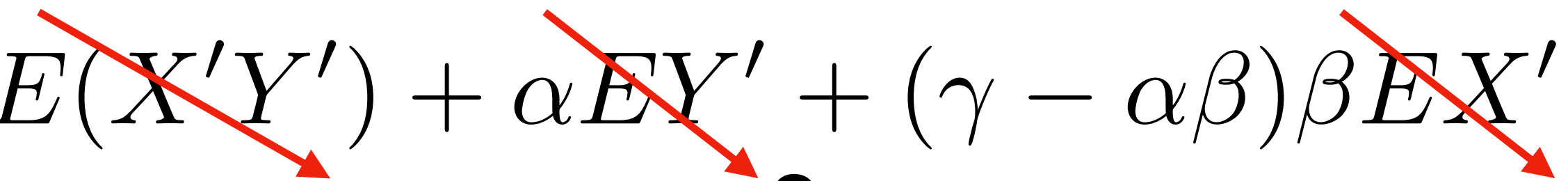
# Wild World of Product Expectations

For any  $\alpha, \beta, \gamma \ni X, Y$  with:  $EX = \alpha$   $EY = \beta$   $E(XY) = \gamma$

$$Y' = X' = \begin{cases} -1 & \frac{1}{2} \\ +1 & \frac{1}{2} \end{cases} \quad EX' = EY' = 0 \quad E(X'Y') = E[(X')^2] = 1$$

$$X = (\gamma - \alpha\beta)X' + \alpha \quad Y = Y' + \beta$$

$$EX = \alpha \quad EY = \beta$$

$$\begin{aligned} E(XY) &= E((\gamma - \alpha\beta)X' + \alpha)(Y' + \beta) \\ &= (\gamma - \alpha\beta)E(X'Y') + \alpha EY' + (\gamma - \alpha\beta)\beta EX' + \alpha\beta \\ &= \gamma \quad \quad \quad \underset{1}{\quad} \quad \underset{0}{\quad} \quad \underset{0}{\quad} \end{aligned}$$


Can we still say something about  $E(XY)$ ?

# Covariance

Sufficient, and easier, to understand 0-mean variables

“Centralize”  $X, Y$ , consider expectation of centralized product

$$\begin{aligned}\sigma_{X,Y} &\triangleq \text{Cov}(X, Y) \triangleq E[(X - \mu_X) \cdot (Y - \mu_Y)] \\ &= E(XY) - E(X\mu_Y) - E(\mu_X Y) + E(\mu_X \mu_Y) \\ &= E(XY) - E(X)\mu_Y - \mu_X E(Y) + \mu_X \mu_Y \\ &= E(XY) - \mu_X \mu_Y\end{aligned}$$

If seems complex, think of  $E(XY)$  for 0-mean variables

Amount  $X$  and  $Y$  vary together

# Properties

$$\text{Cov}(X, X) = EX^2 - \mu_X^2 = V(X)$$

$$\text{Cov}(X, Y) = E((X - \mu_X)(Y - \mu_Y)) = \text{Cov}(Y, X)$$

$$\text{Cov}(aX, Y) = E(aXY) - \mu_{aX}\mu_Y = aE(XY) - a\mu_X\mu_Y = a\text{Cov}(X, Y)$$

$$\begin{aligned}\text{Cov}(X + a, Y) &= E(((X + a) - \mu_{X+a})(Y - \mu_Y)) \\ &= E(X - \mu_X)(Y - \mu_Y) = \text{Cov}(X, Y)\end{aligned}$$

Intuitively if  $X$  changes by  $\sigma_X$ ,  $Y$  grows by  $\sigma_{X,Y} \cdot \sigma_X \cdot \sigma_Y$

# Correlation Coefficient

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Pearson's correlation coefficient

Properties:

$$\rho_{X,X} = 1 \quad \rho_{X,-X} = -1$$

$$\rho_{X,Y} = \rho_{Y,X}$$

$$\rho_{aX+b, cY+d} = \text{sign}(ac) \cdot \rho_{X,Y}$$

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

If  $X \nearrow$  by  $\sigma_X$ , by how many  $\sigma_Y$  do we expect  $Y$  to  $\nearrow$

Bounds on  $\rho_{X,Y}$ ?

# Cauchy-Schwarz Inequality

$E(X \cdot Y)$  can't take all possible values

$$|E(XY)| \leq \sqrt{EX^2} \cdot \sqrt{EY^2}$$

For any  $\alpha$

$$0 \leq E(\alpha X + Y)^2 = \alpha^2 EX^2 + 2\alpha E(XY) + EY^2$$

True for all  $\alpha$ , so discriminant must be negative

$$4(EXY)^2 - 4EX^2 \cdot EY^2 \leq 0$$

$$(EXY)^2 \leq EX^2 \cdot EY^2$$

# Correlation Coefficient

$$|E(X - \mu_X)(Y - \mu_Y)| \leq \sqrt{E(X - \mu_X)^2 \cdot E(Y - \mu_Y)^2}$$

Namely

$$|\sigma_{X,Y}| \leq \sigma_X \cdot \sigma_Y$$

$$\rho_{X,Y} \triangleq \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$|\rho_{X,Y}| \leq 1$$



# Examples

Uncorrelated:  $EXY = 0$

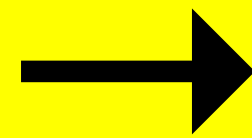
$$X, Y \sim B\left(\frac{1}{2}\right)$$

Correlation			
Positive	$X, X + Y$	$X, 2X + Y$	$\min(X, Y), \max(X, Y)$
Uncorrelated	$X, Y$	$3X, 4Y$	
Negative	$X, -Y$	$Y, -X$	$ X - Y , \min(X, Y)$

$$X = 3Y$$

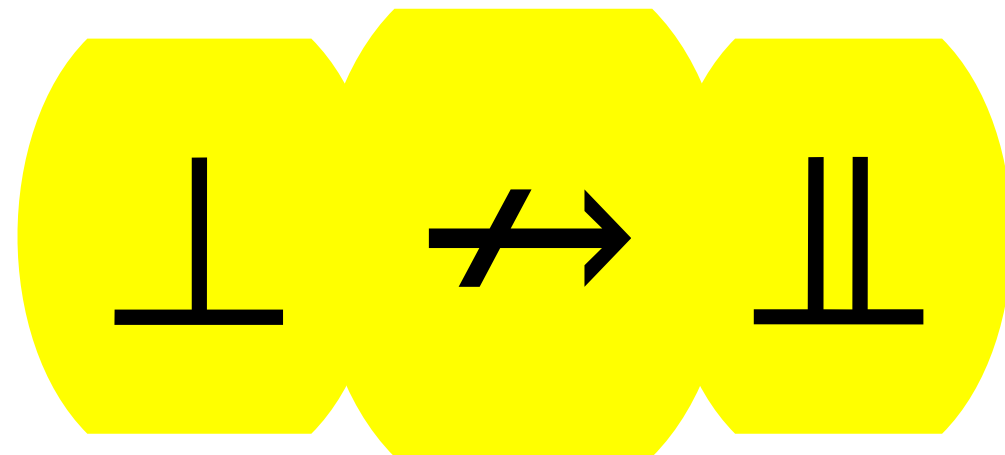
$$\text{Cov}(X, Y) = 3\text{Var}(X)$$

$$P = 1$$

$\perp\!\!\!\perp$  $\perp$ 

Independent implies uncorrelated

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy \cdot p(x, y) \\ &= \sum_x \sum_y xy \cdot p(x)p(y) \\ &= \sum_x x \cdot p(x) \sum_y y \cdot p(y) \\ &= E(X) \cdot E(Y) \end{aligned}$$



Independent  $\rightarrow$  uncorrelated      **Uncorrelated**  $\overset{?}{\rightarrow}$  independent

$$X = \begin{cases} -1 & \frac{1}{2} \\ +1 & \frac{1}{2} \end{cases}$$

$$X = -1 \rightarrow Y = 0$$

$$X = +1 \rightarrow Y = \begin{cases} +1 \\ -1 \end{cases}$$

	-1	0	1	Y
-1		1/2		
1	1/4		1/4	
X				

Uncorrelated

$$EX = 0 \quad EY = 0$$

$$E(XY) = \frac{1}{4} \cdot -1 + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 1 = 0 = EX \cdot EY$$

**Clearly dependent**

Note: Uncorrelated binary random pairs are independent

# Variance

$$V(X + Y) \stackrel{?}{=} V(X) + V(Y)$$

$$\begin{aligned} V(X + Y) &= E(X + Y)^2 - (E(X + Y))^2 \\ &= E(X^2 + 2XY + Y^2) - (EX + EY)^2 \\ &= EX^2 + 2E(XY) + EY^2 - (E^2X + 2EX \cdot EY + E^2Y) \\ &= EX^2 - E^2X + EY^2 - E^2Y + 2(E(XY) - EX \cdot EY) \\ &= V(X) + V(Y) + 2(E(XY) - EX \cdot EY) \\ &= V(X) + V(Y) + 2\text{Cov}(X, Y) \\ &= \text{iff } \text{Cov}(X, Y) = 0 \quad \text{Uncorrelated} \end{aligned}$$

$$V(X+Y) \text{ may be } \geq \quad = \quad \text{or} \quad \leq \quad V(X)+V(Y)$$

$$X \perp Y \overset{\rightarrow}{\iff} \sigma_{X,Y} = 0 \quad \overset{\rightarrow}{\iff} V(X+Y) = V(X)+V(Y)$$

$$X \perp\!\!\!\perp Y \rightarrow X \perp Y \rightarrow V(X+Y) = V(X)+V(Y)$$

$$X,Y \sim B(\tfrac{1}{2})$$

$$Y=X \qquad \qquad \qquad \sigma_{X,Y}=V(X)$$

$$V(X+Y) = V(2X) = 4V(X) = V(X)+V(Y)+2V(X)$$

$$Y=-X$$

$$V(X+Y) = V(0) - 0 = V(X)+V(X)-2V(X)$$