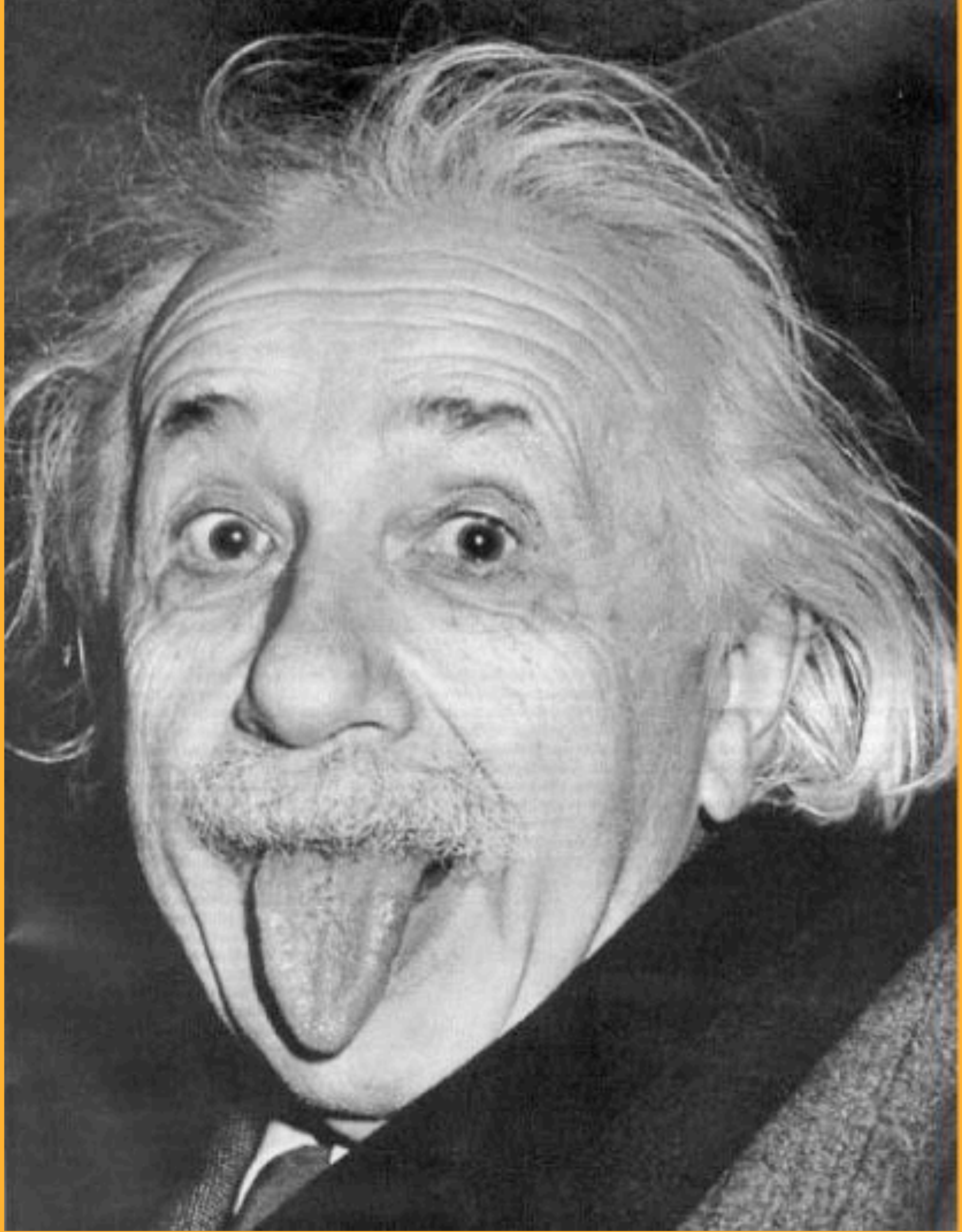
Continuous

Distributions



"Not everything that can be counted, counts; not everything that counts can be counted."

- Albert Einstein

Discrete to Continuous

Discrete distributions: Countable # values (finite or countably-infinite)

Continuous distributions: Uncountable # values, intervals

Why Continuous

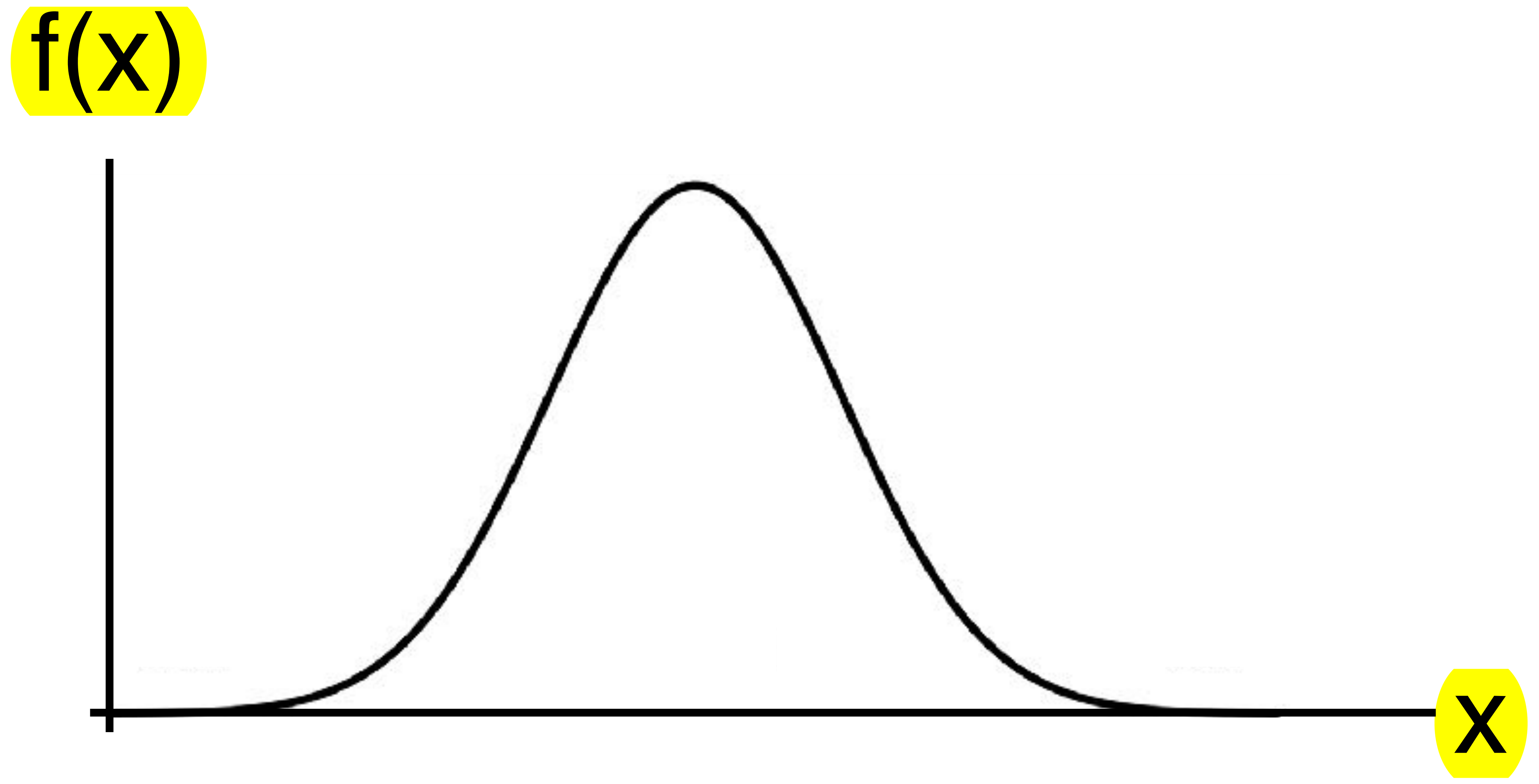
- Anything physics
  - Time: flight, delivery, disease, life
  - Space: height, storm area
  - Mass: pet, cookie
  - Temperature: air, body
- Nearly continuous variables
  - Cost: stock, house, pork bellies
  - Rates: interest, exchange, unemployment

Probability Density Function

Replaces the discrete pmf

$f(x) \geq 0$ relative likelihood of x

$\int_{-\infty}^{\infty} f(x)dx = 1$ area under curve (area)

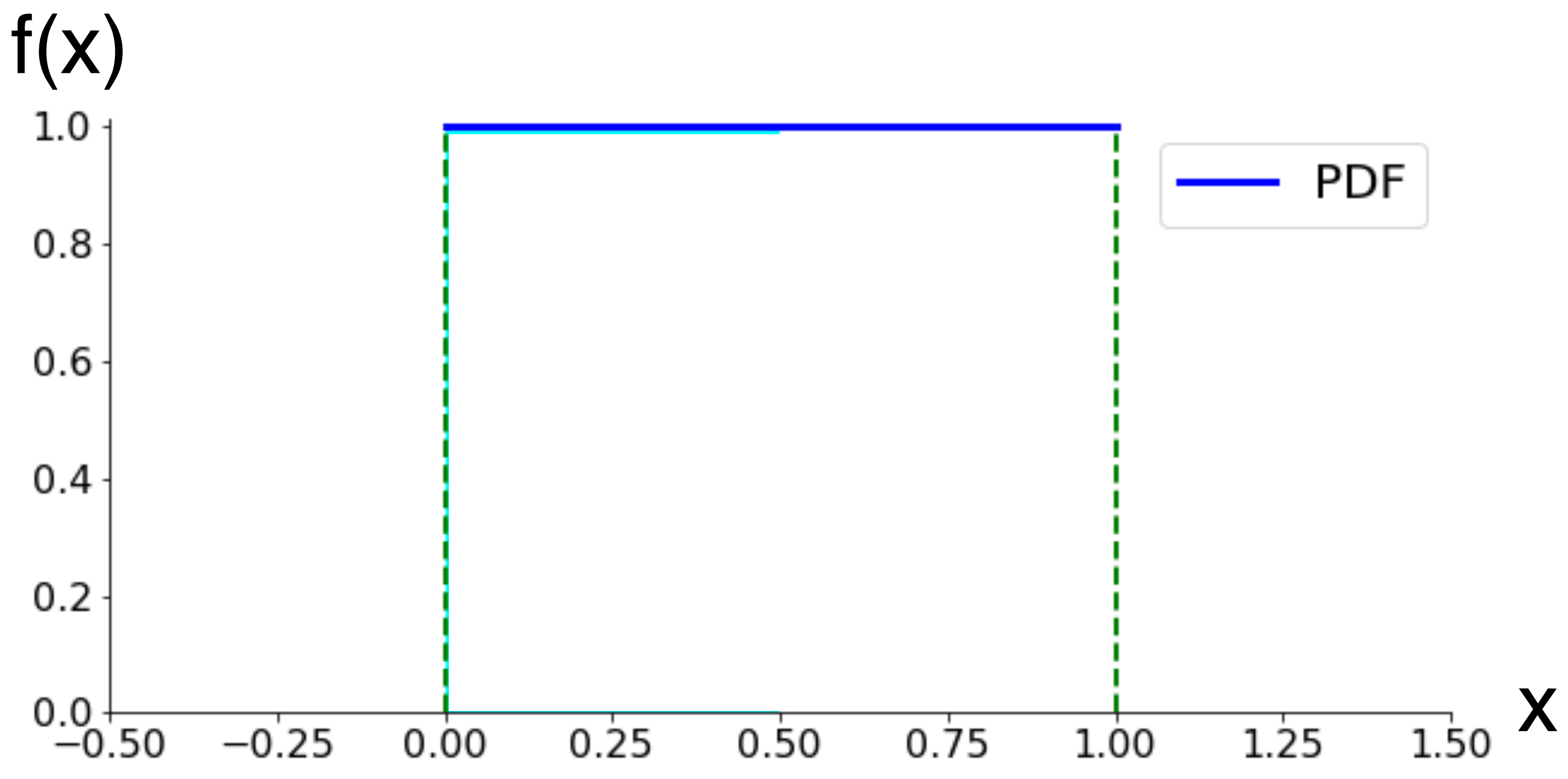


Comparison to Discrete

	Discrete	Continuous
Probability function	mass (pmf)	density (pdf)
≥ 0	$p(x) \geq 0$	$f(x) \geq 0$
∑ = 1	$\sum_x p(x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$

Uniform

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

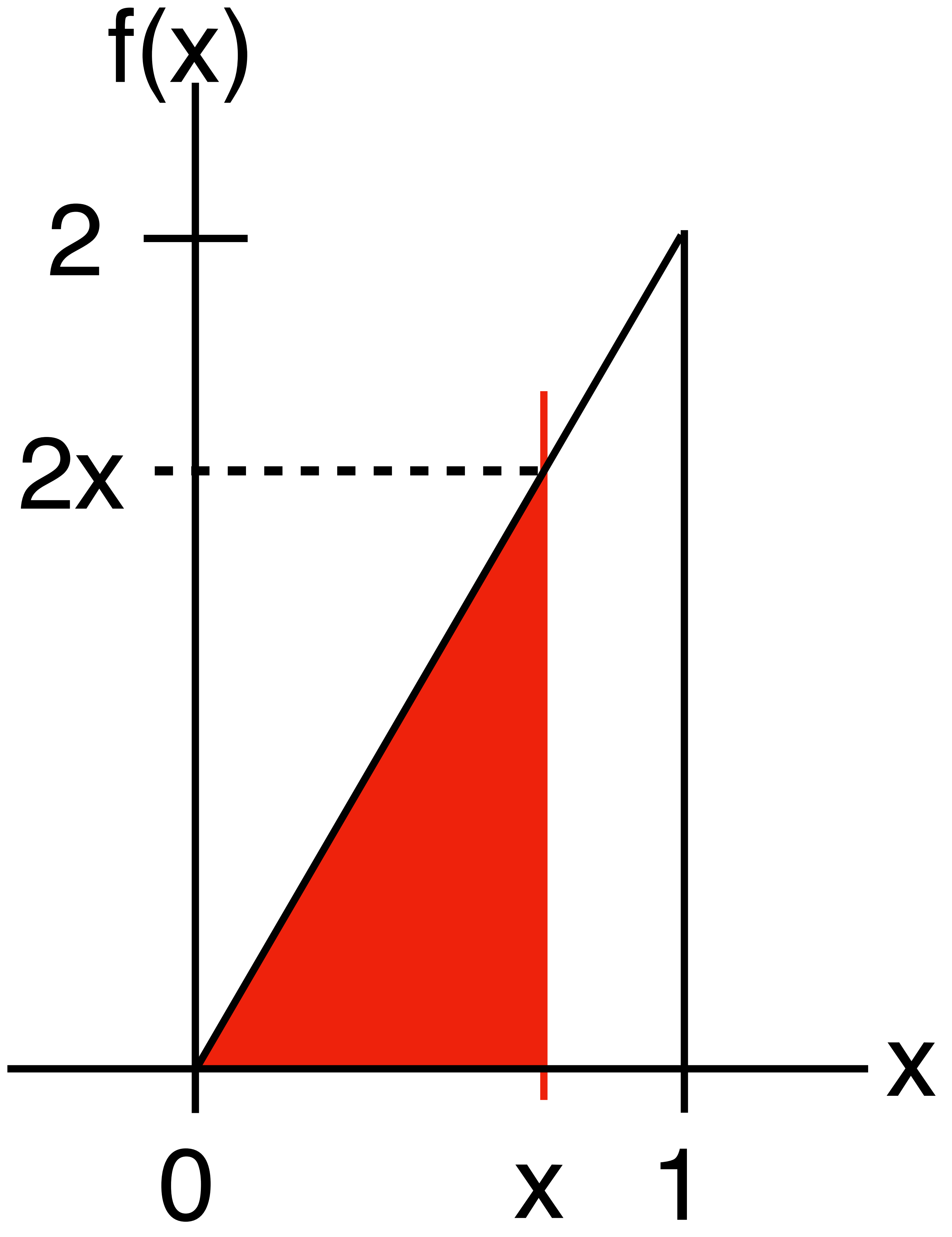


Will it $\int$? Area = 1 · 1 = 1

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = x\Big|_0^1 = 1$$

Triangle

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Will it $\int$?

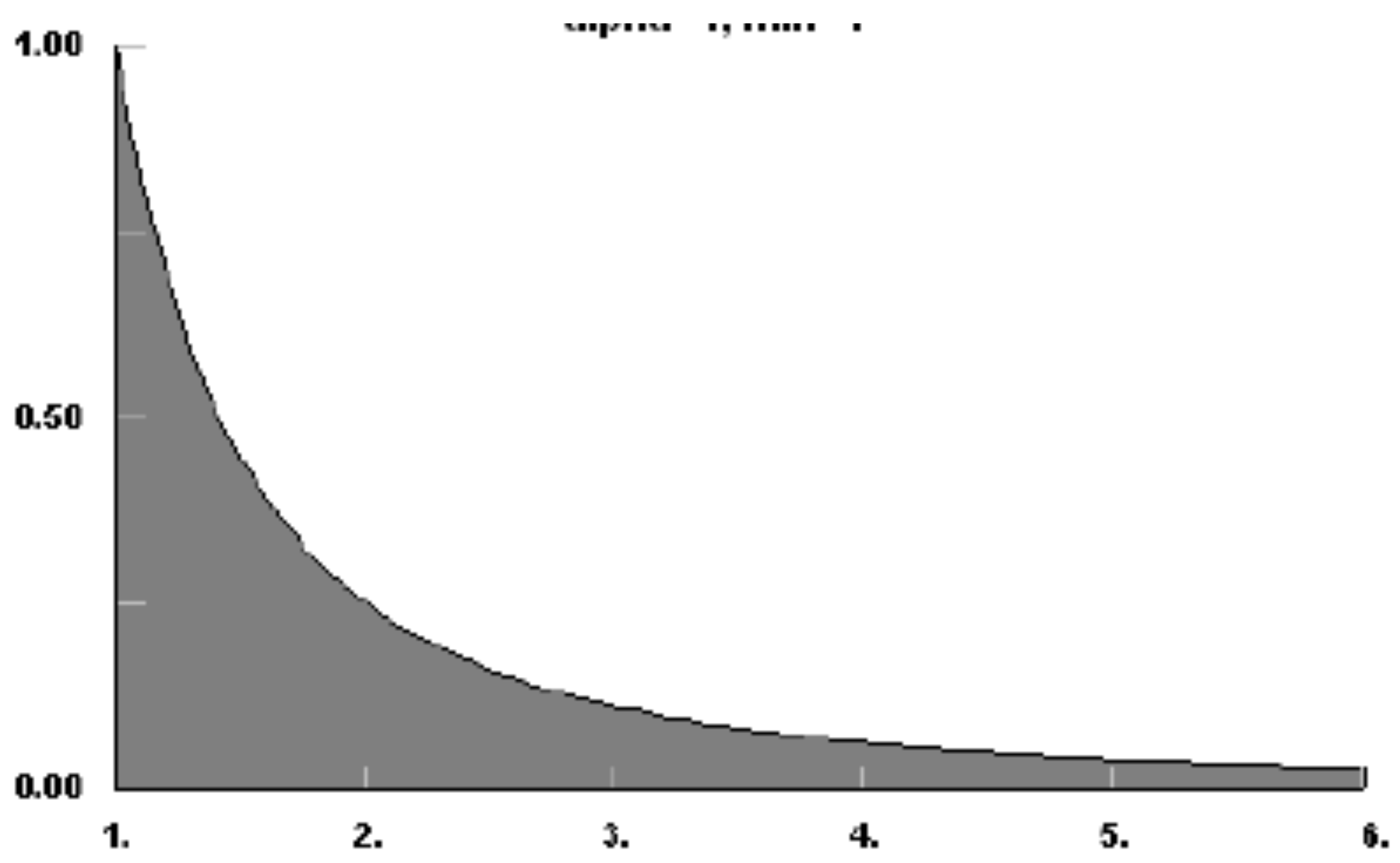
Area $= 2 \cdot 1 \cdot \frac{1}{2} = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 2xdx = x^2\Big|_0^1 = 1 - 0 = 1$$

Infinite Support

Power law

$$f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$



Will it ∑?

$$\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{u^2} du = \frac{-1}{u}\Big|_1^{\infty} = 1$$

Event Probability

	Discrete	Continuous
P(A)	$\sum_{x \in A} p(x)$	$\int_{x \in A} f(x)dx$

Typically interested in interval probability $P(a \leq X \leq b)$

Area between a and b

$P(X \leq b) - P(X \leq a)$

Cumulative distribution function

Cumulative Distribution Function (CDF)

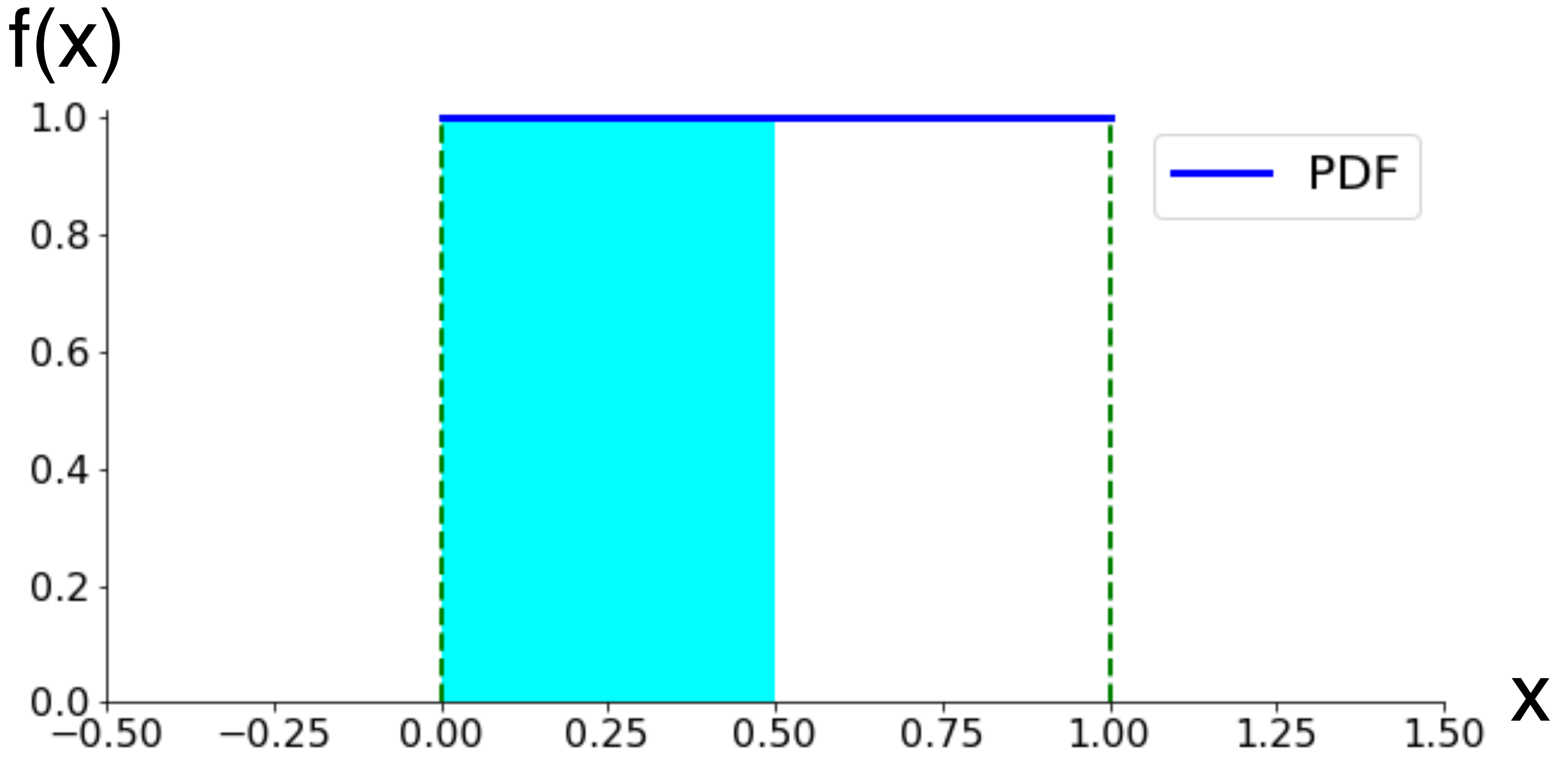
$$F(x) \triangleq P(X \leq x)$$

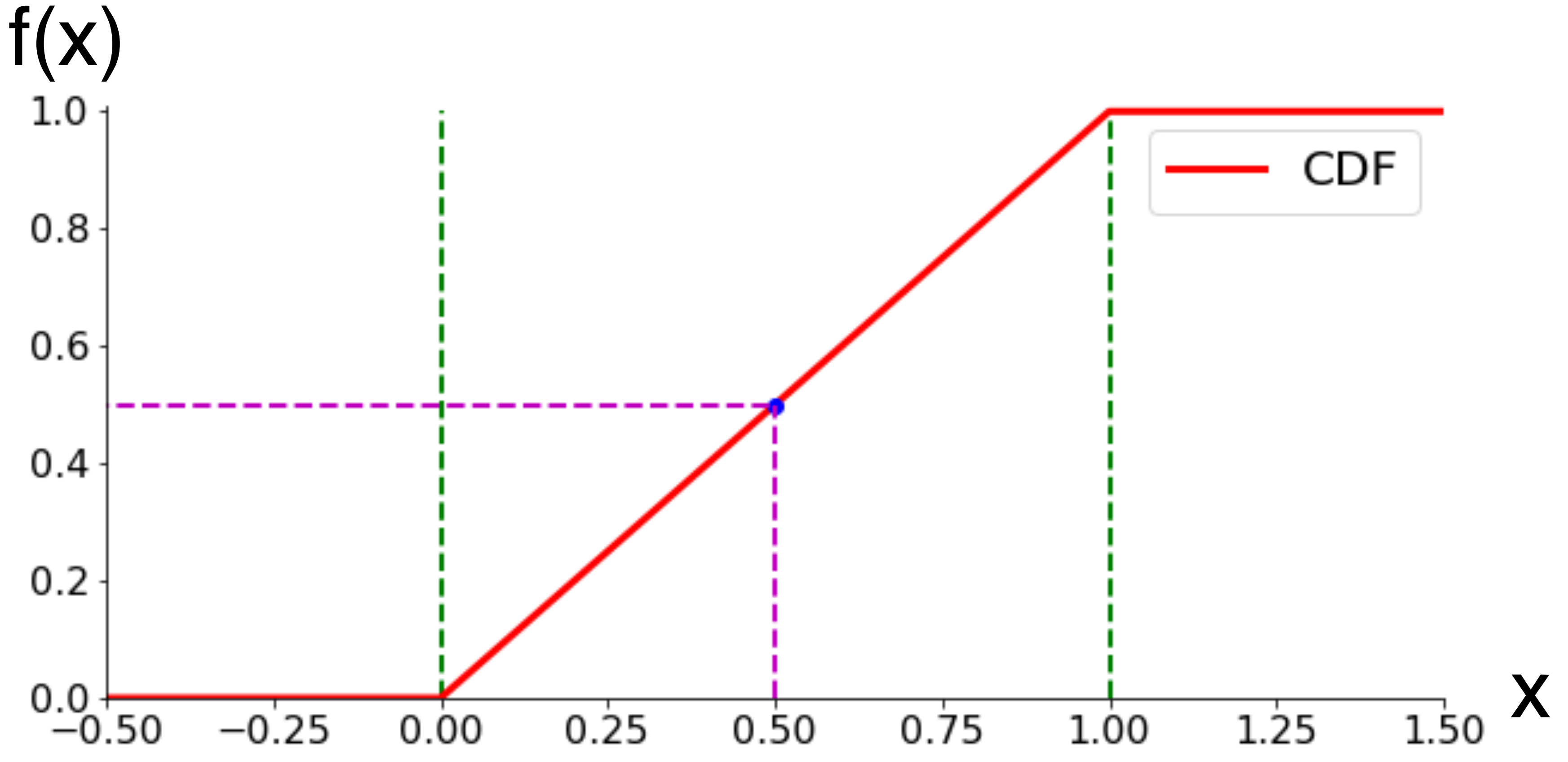
	Discrete	Continuous
PF → CDF	$\sum_{u \leq x} p(u)$	$\int_{-\infty}^{x} f(u)du$
CDF → PF	$p(x) = F(x) - F(x^*)$	$f(x) = F'(x)$

x^* - element preceding x

Uniform

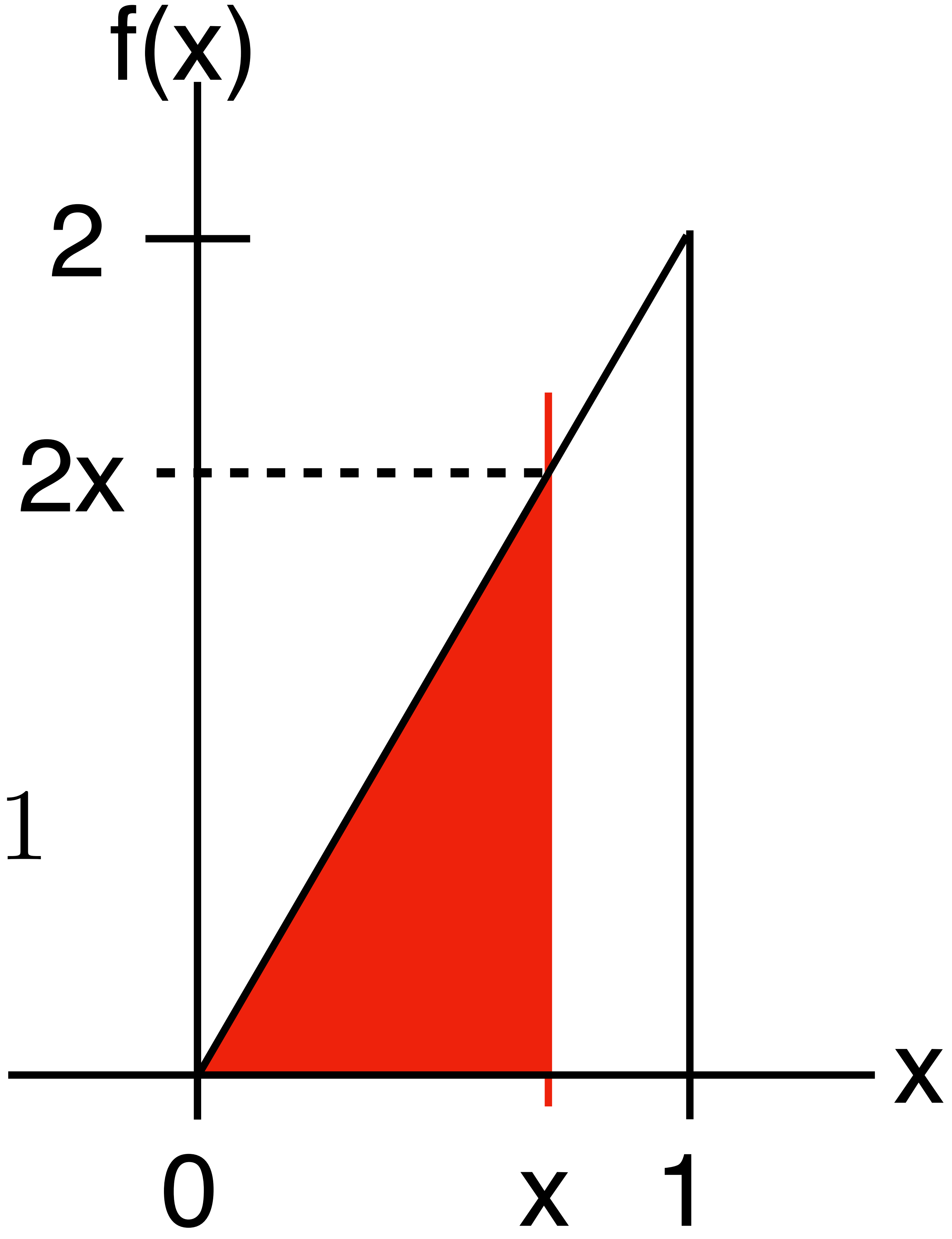
$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 0 & x \le 0 \\ \int_0^x 1du = u\Big|_0^x = x & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$





$$F'(x) = \begin{cases} (0)' = 0 & x \le 0 \\ (x)' = 1 & 0 < x < 1 \\ (1)' = 0 & 1 < x \end{cases}$$

Triangle

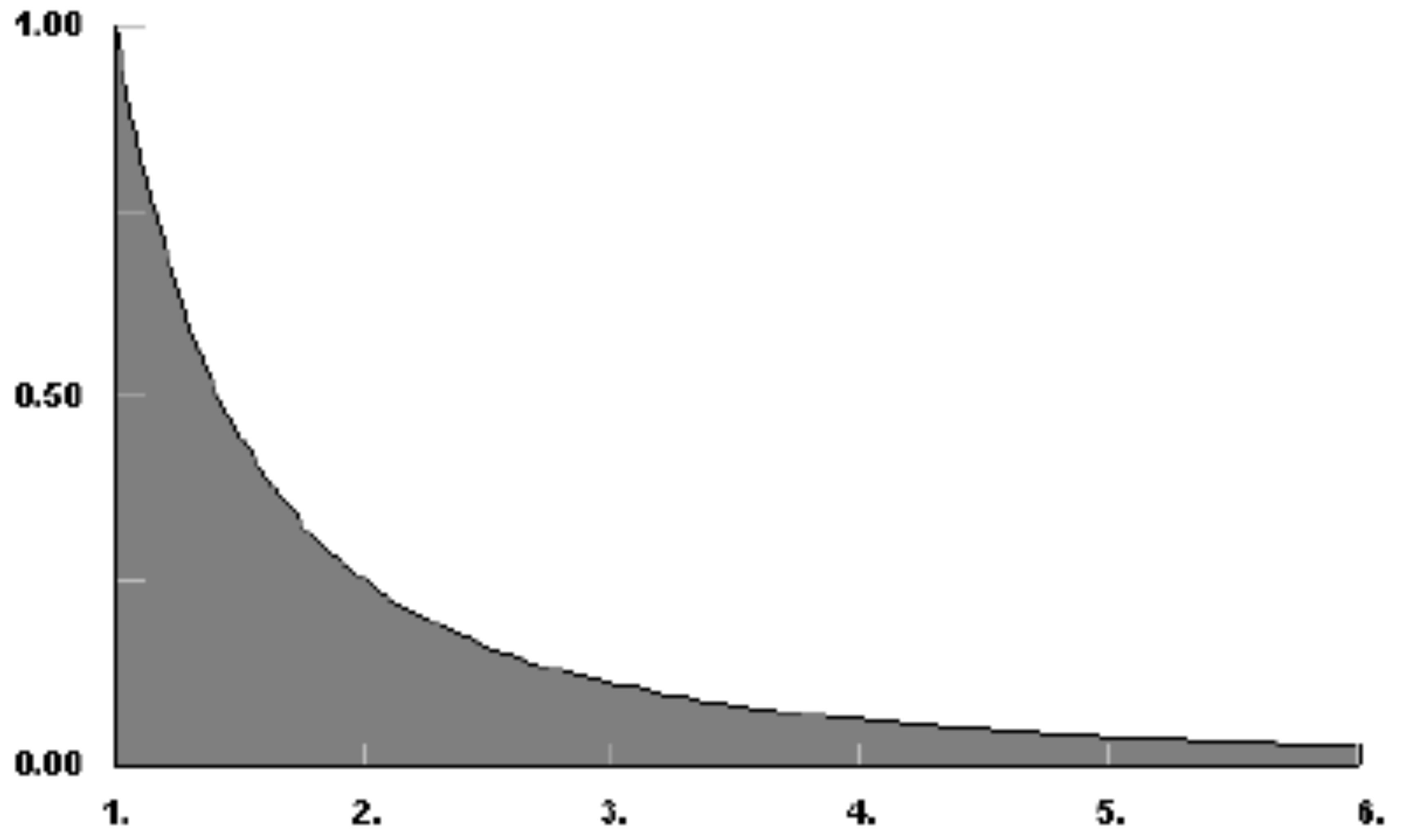


$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 0 & x \leq 0 \\ \int_0^x 2udu = u^2\big|_0^x = x^2 & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

$$F'(x) = \begin{cases} (0)' = 0 & x < 0 \\ (x^2)' = 2x & 0 < x < 1 \\ (1)' = 0 & 1 < x \end{cases}$$

Infinite Support

$$F(x) = \begin{cases} 0 & x \leq 1 \\ \int_1^x \frac{1}{u^2} du = \frac{-1}{u}\Big|_1^x = 1 - \frac{1}{x} & x \geq 1 \end{cases}$$



$$F'(x) = \begin{cases} (0)' = 0 & x < 1 \\ (1 - \frac{1}{x})' = \frac{1}{x^2} & x > 1 \end{cases}$$

Properties of the CDF

F(x) = integral

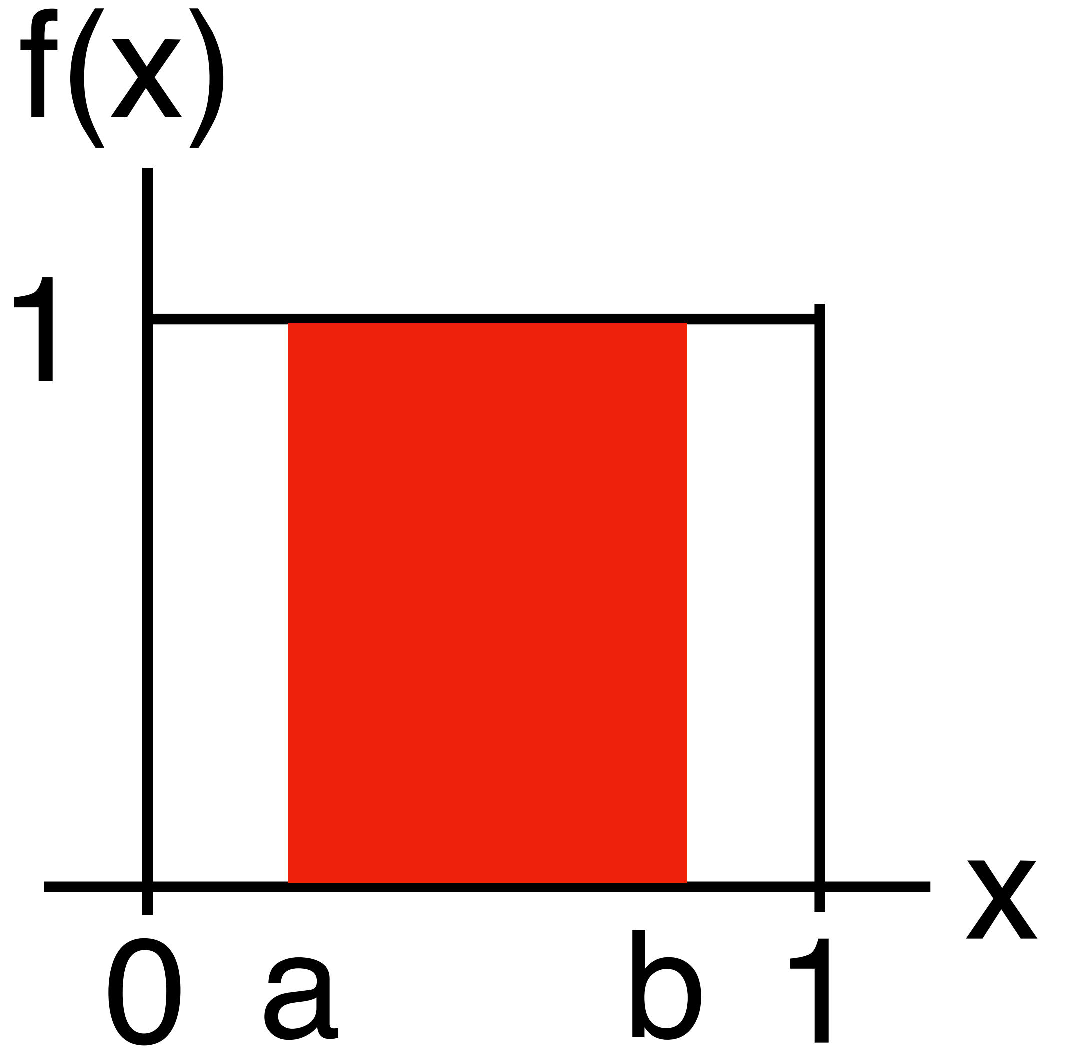
Nondecreasing

$F(-\infty)=0$

$F(\infty)=1$

Continuous

Examples



Uniform

$0 \le a \le b \le 1$

$$P(a \le X \le b) = \begin{cases} \text{Area} = (b - a) \cdot 1 = b - a \\ \int_a^b f(x)dx = \int_a^b 1 dx = x\Big|_a^b = b - a \\ F(b) - F(a) = b - a \end{cases}$$

$$P(0.6 \le X \le 1.3) = P(0.6 \le X \le 1) = 0.4$$
$$= F(1.3) - F(0.6) = 1 - 0.6 = 0.4$$

Power law

$1 \le a \le b \quad P(a \le X \le b) = F(b) - F(a) = (1 - \frac{1}{b}) - (1 - \frac{1}{a}) = \frac{1}{a} - \frac{1}{b}$

Differences

Discrete	Continuous
$p(x) \leq 1$	$f(x)$ can be >1
Generally $p(x) \neq 0$	$p(x) = 0$
Generally $P(X \leq a) \neq P(X < a)$	$P(X \leq a) = P(X < a) = F(a)$
	$P(X \geq a) = P(X > a) = 1 - F(a)$
	$P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a)$

Expectation

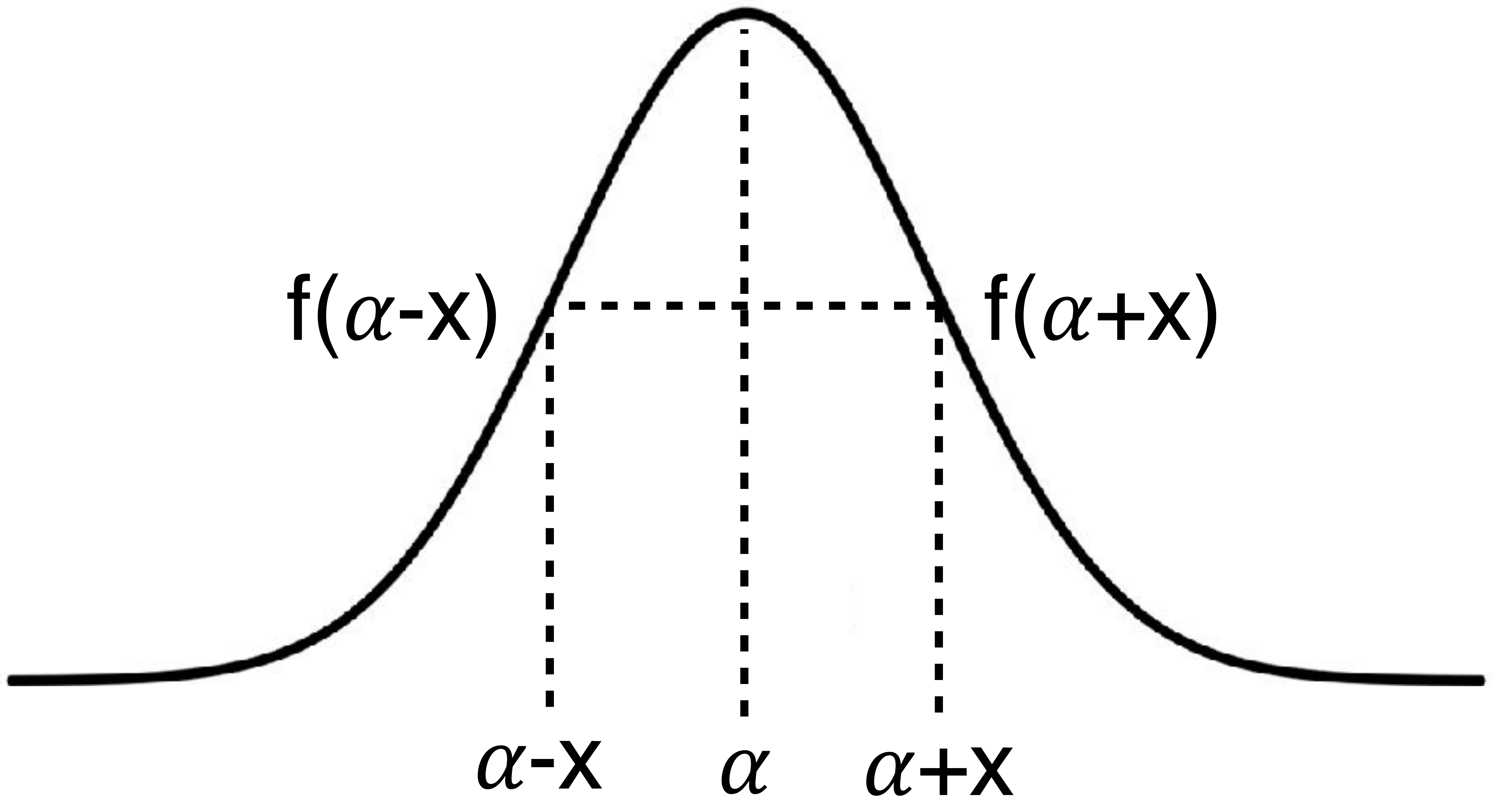
	Discrete	Continuous
EX	$\sum x \cdot p(x)$	$\int_{-\infty}^{\infty} x f(x) dx$

As discrete: Average of many samples

Properties

Support set = [a,b] $a \leq EX \leq b$

Symmetry If for some α, $f(\alpha+x)=f(\alpha-x)$ for all x then $EX = \alpha$



Examples

Uniform $EX = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x \; 1 \; dx \; = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

Triangle $EX = \int_0^1 x \;\; 2x \;\; dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

Power law $EX = \int_1^{\infty} x \; \frac{1}{x^2} \; dx = \int_1^{\infty} \frac{1}{x} \; dx = \ln x\Big|_1^{\infty} = \infty$

Later: power laws with finite expectation

Variance

	Discrete	Continuous
$V(X) \triangleq E(X-\mu)^2$	$\sum_x p(x)(x-\mu)^2$	$\int_{-\infty}^{\infty} f(x)(x-\mu)^2 dx$

$$E(X^2) - \mu^2$$

As for discrete

$$\begin{aligned}
E(X-\mu)^2 &= \int (x-\mu)^2 f(x) dx \\
&= \int (x^2 - 2x\mu + \mu^2) f(x) dx \\
&= \int x^2 f(x)\, dx - 2\mu \int x f(x)\, dx + \mu^2 \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

$$\sigma = \sqrt{V(X)}$$

Examples

Uniform

$$EX = \tfrac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 \ 1 \ dx = \tfrac{x^3}{3}\Big|_0^1 = \tfrac{1}{3}$$

$$V(X) = E(X^2) - (EX)^2 = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}$$

$$\sigma = \tfrac{1}{\sqrt{12}} = \tfrac{1}{2\sqrt{3}}$$

Triangle

$$EX = \tfrac{2}{3}$$

$$EX^2 = \int_0^1 x^2 \ 2x \ dx = \tfrac{2}{4} x^4\Big|_0^1 = \tfrac{1}{2}$$

$$V(X) = E(X^2) - (EX^2) = \tfrac{1}{2} - (\tfrac{2}{3})^2 = \tfrac{9-8}{18} = \tfrac{1}{18}$$

$$\sigma = \tfrac{1}{\sqrt{18}} = \tfrac{1}{3\sqrt{2}}$$

Discrete vs. Continuous

	Discrete	Continuous
Prob. Fun.	pmf - p	pdf - f
≥ 0	$p(x) \geq 0$	$f(x) \geq 0$
∑ = 1	$\sum p(x) = 1$	$\int f(x)dx = 1$
P(A)	$\sum_{x \in A} p(x)$	$\int_{x \in A} f(x)dx$
F(X)	$\sum_{u \leq x} p(u)$	$\int_{-\infty}^{x} f(u)du$
μ = E(X)	$\sum xp(x)$	$\int xf(x)dx$
V(X)	$\sum (x-\mu)^2 p(x)$	$\int (x-\mu)^2 f(x)dx$

p(x) ≤ 1

f can be larger

f(x) = F'(x)

P(a ≤ X ≤ b) = P(a < X < b) = F(b) - F(a)

Functions of Continuous Random Variables