Linearity of Expectation

Expectation

$$
\begin{aligned}
Eg(X) &= \sum_z z \cdot P(g(x) = z) \\
&= \sum_z z \sum_{x \in g^{-1}(z)} p(x) \\
&= \sum_z \sum_{x \in g^{-1}(z)} z \cdot p(x) \\
&= \sum_z \sum_{x \in g^{-1}(z)} g(x)p(x) \\
&= \sum_x g(x)p(x)
\end{aligned}
$$

$$p(x) \to p(x, y)$$

$$g(x) \to g(x, y)$$

$$\sum_x \to \sum_{x,y}$$

Linearity of Expectation

$$
\begin{aligned}
E(X+Y) &= \sum_x \sum_y (x+y) \cdot p(x,y) \\
&= \sum_x \sum_y x \cdot p(x,y) + \sum_x \sum_y y \cdot p(x,y) \\
&= \sum_x x \sum_y p(x,y) + \sum_y y \sum_x p(x,y) \\
&= \sum_x x \cdot p(x) + \sum_y y \cdot p(y) \\
&= EX + EY
\end{aligned}
$$

Expectation of sum =
sum of expectations

The Hat Problem

	H_1	H_2	H_3	H
1 2 3	1	1	1	3
1 3 2	1	0	0	1
2 1 3	0	0	1	1
2 3 1	0	0	0	0
3 1 2	0	0	0	0
3 2 1	0	1	0	1

$\mathbb{1}_i$ - indicator function i^{th} student caught their own hat

H - # students who caught their own hat

$$H = \sum_{i=1}^{n} \mathbb{1}_i$$

$\mathbb{1}_i$ - Bernoulli

$$P(\mathbb{1}_i = 1) = \frac{\#\text{ permutations of } (\sigma_1, \ldots, \sigma_n) \text{ when } \sigma_i = i}{\#\text{ permutations of } (\sigma_1, \ldots, \sigma_n)} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$E(\mathbb{1}_i) = P(\mathbb{1}_i = 1) = \frac{1}{n}$$

$$E(H) = E\left(\sum_{i=1}^{n} \mathbb{1}_i\right) = \sum_{i=1}^{n} E(\mathbb{1}_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Variance

Expectations add $E(X+Y) = EX + EY$

Do variances? $V(X+Y) \stackrel{?}{=} V(X) + V(Y)$

$$\begin{aligned}
V(X+Y) &= E(X+Y)^2 - (E(X+Y))^2 \\
&= E(X^2 + 2XY + Y^2) - (EX + EY)^2 \\
&= EX^2 + 2E(XY) + EY^2 - (E^2X + 2EX \cdot EY + E^2Y) \\
&= EX^2 - E^2X + EY^2 - E^2Y + 2(E(XY) - EX \cdot EY) \\
&= V(X) + V(Y) + 2(E(XY) - EX \cdot EY)
\end{aligned}$$

$E(XY) = EX \cdot EY$?

Do expectations multiply?

Linearity of Expectation