
Counting Variations

Find a natural counting question whose answer is a double exponential

2 solutions

Subsets

Functions

∃ more

Power set of S - set of subsets of S

$\mathbb{P}(S)$

$\mathbb{P}(\{a, b\}) = \{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \}$

$|\ \mathbb{P}(S)\ | = 2^{|S|}$

$|\ \mathbb{P}(\{a,b\})\ | = 4 = 2^2 = 2^{|\{a,b\}|}$ ✅

$\mathbb{P}(S)$ is a set

What about power set of $\mathbb{P}(S)$?

Find a natural counting question whose answer is a double exponential

$\mathbb{P}(\ \mathbb{P}(S)\ )$ - set of subsets of $\mathbb{P}(S)$

$|\ \mathbb{P}(S)\ | = 2^{|S|}$

$|\ \mathbb{P}(\ \mathbb{P}(S)\ )\ | = 2^{|\mathbb{P}(S)|} = 2^{2^{|S|}}$

$\mathbb{P}(\{a, b\}) = \{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \}$

$\mathbb{P}(\ \mathbb{P}(\{a, b\})\ ) = \mathbb{P}(\{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \})$

$= \{\ \{\}, \{\ \{\}\ \}, \{\ \{a\}\ \}, \ldots, \{\ \{\}, \{a\}\ \}, \ldots, \{\{\},\{a\},\{b\},\{a,b\}\}\ \}$

$|\ \mathbb{P}(\ \mathbb{P}(\{a,b\})\ )\ | = 2^{|\mathbb{P}(\{a,b\})|} = 2^{2^{|\{a,b\}|}}$

$|\ \mathbb{P}(\ \mathbb{P}([n])\ )| = 2^{2^{n}}$

Double exponential

Solution 2: Binary Functions

Functions from A to B B^A $\# = |B|^{|A|}$

 # Binary functions of n binary variables

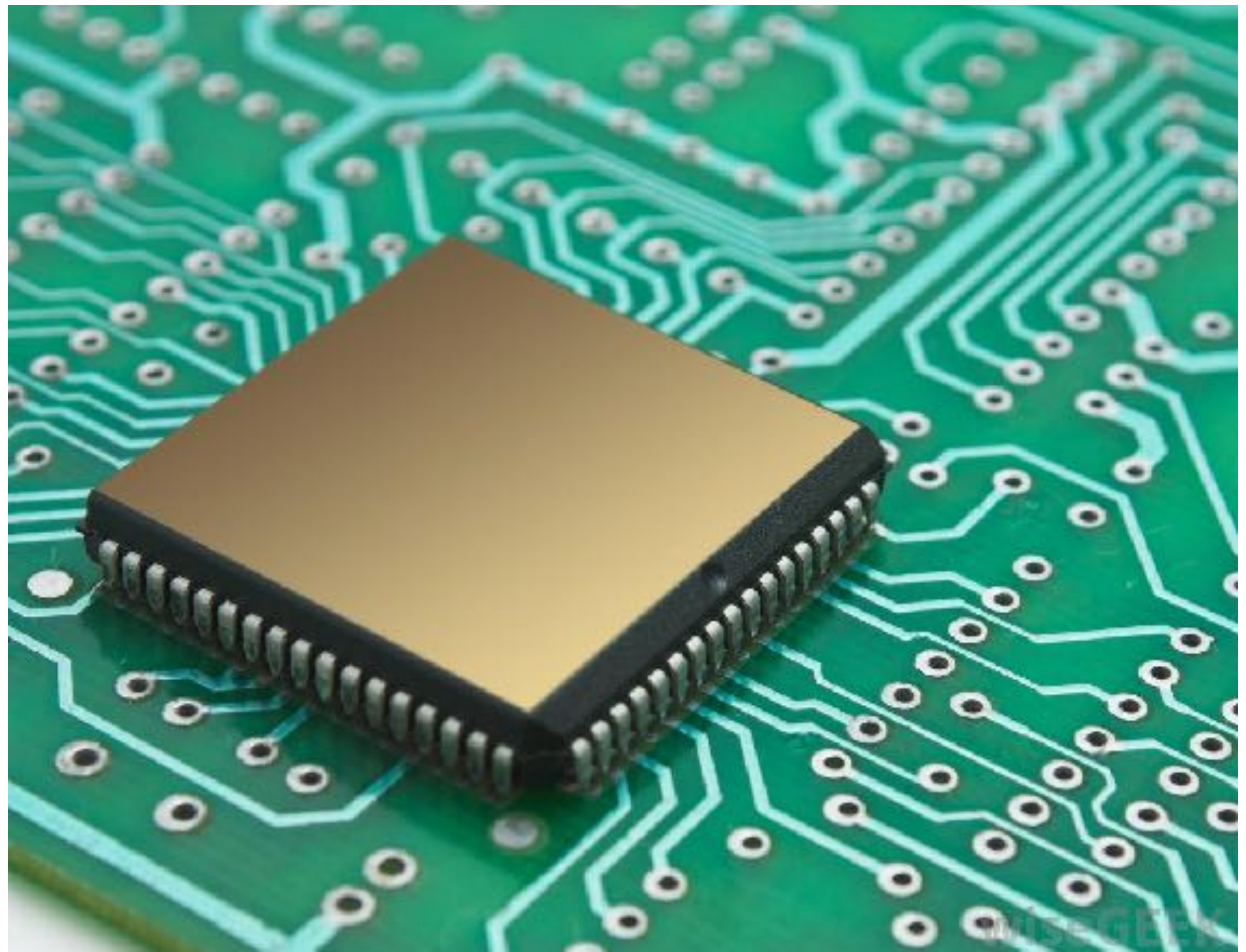
Functions from $\{0,1\}^n$ to $\{0,1\}$ $\{0,1\}^{\{0,1\}^n}$

$\# = |\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$

Double exponential

x	f(x)	
000	0	2
001	0	2
⋮		⋮
111	1	2

2^n (rows)   n (bits of x) 2^{2^n}



Circuit with n binary inputs, one binary output

Can implement 2^{2^n} functions $2^{63} = \frac{1}{2} \cdot 2^{2^6}$


Combine
addition
subtraction
multiplication
Determine size of more complex sets
PINs with various constraints
Counting Variations

PIN

Personal Identification Number   2174

\# 4-digit PIN's = ?

$D = \{0,\ldots,9\}$ Set of digits

$\{\text{4-digit PIN's}\} = D^4$ Cartesian Power

X

$|D^4| = |D|^4 = 10^4 = 10{,}000$

Variable Length

\# 3-5 digit PINs

314 2246 79380

$D = \{0,\ldots,9\}$

$\{PINs\} = D^3 \cup D^4 \cup D^5$

Disjoint Union

$\# PINs = |D^3 \cup D^4 \cup D^5| = |D^3| + |D^4| + |D^5|$

$= 10^3 + 10^4 + 10^5$

$= 1{,}000 + 10{,}000 + 100{,}000$

$= 111{,}000$

Forbidden Patterns

4-digit PIN's

Forbidden All same 3333 0000, 1111, ..., 9999 # = 10

Consecutive 3456 0123, 1234, ..., 6789 # = 7

Forbidden All same ∪ Consecutive ⊕ 10 + 7 = 17

Allowed D^4 - Forbidden ⊖ 10,000 - 17 = 9,983

4-digit PINs

PINs Containing Zero

PIN's containing 0

8093

~~2534~~

Use latex for formulas
→ slides less colorful

D = {0,1, …, 9}

$Z = \{0\}$ $\overline{Z} = \{1, 2, ..., 9\}$

$\overline{Z} = Z^c$ set of non-zero digits

$x^n \triangleq x_1, \ldots, x_n$

n-digit sequence

$\exists Z = \{x^n \in D^n : \exists i \ x_i \in Z\}$

{n-digit PINs containing 0}

$|\exists Z| = ?$

Start with 2 digits

2 ways

Inclusion exclusion

Subtraction rule

2-Digits: Inclusion-Exclusion

$\exists Z = \{x_1 x_2 \colon \exists i \; x_i = 0\}$ 00 03 50 ~~73~~

$Z_1 = \{x_1 x_2 \colon x_1 = 0\}$ 00 03 ~~50~~ ~~73~~ $|Z_1| = 10$

$Z_2 = \{x_1 x_2 \colon x_2 = 0\}$ 00 ~~03~~ 50 ~~73~~ $|Z_2| = 10$

$$\exists Z = Z_1 \cup Z_2$$

$$|\exists Z| = |Z_1| + |Z_2| - |Z_1 \cap Z_2|$$

$$= 10 + 10 - 1 = 19$$

$$Z_1 \cap Z_2 = \{00\}$$

$$|Z_1 \cap Z_2| = 1$$

2-Digits: Complement Rule

$\exists Z = \{x_1 x_2 \colon \exists i \; x_i = 0\}$ 00 03 50 ~~73~~

$\Omega = D^2$ All 2-digit PINs

$\overline{\exists Z} = \overline{\{x_1 x_2 : \exists i \; x_i = 0\}} = \{x_1 x_2 : \forall i \; x_i \neq 0\} = \overline{Z} \times \overline{Z}$ 73 44 19 ~~50~~

both digits nonzero

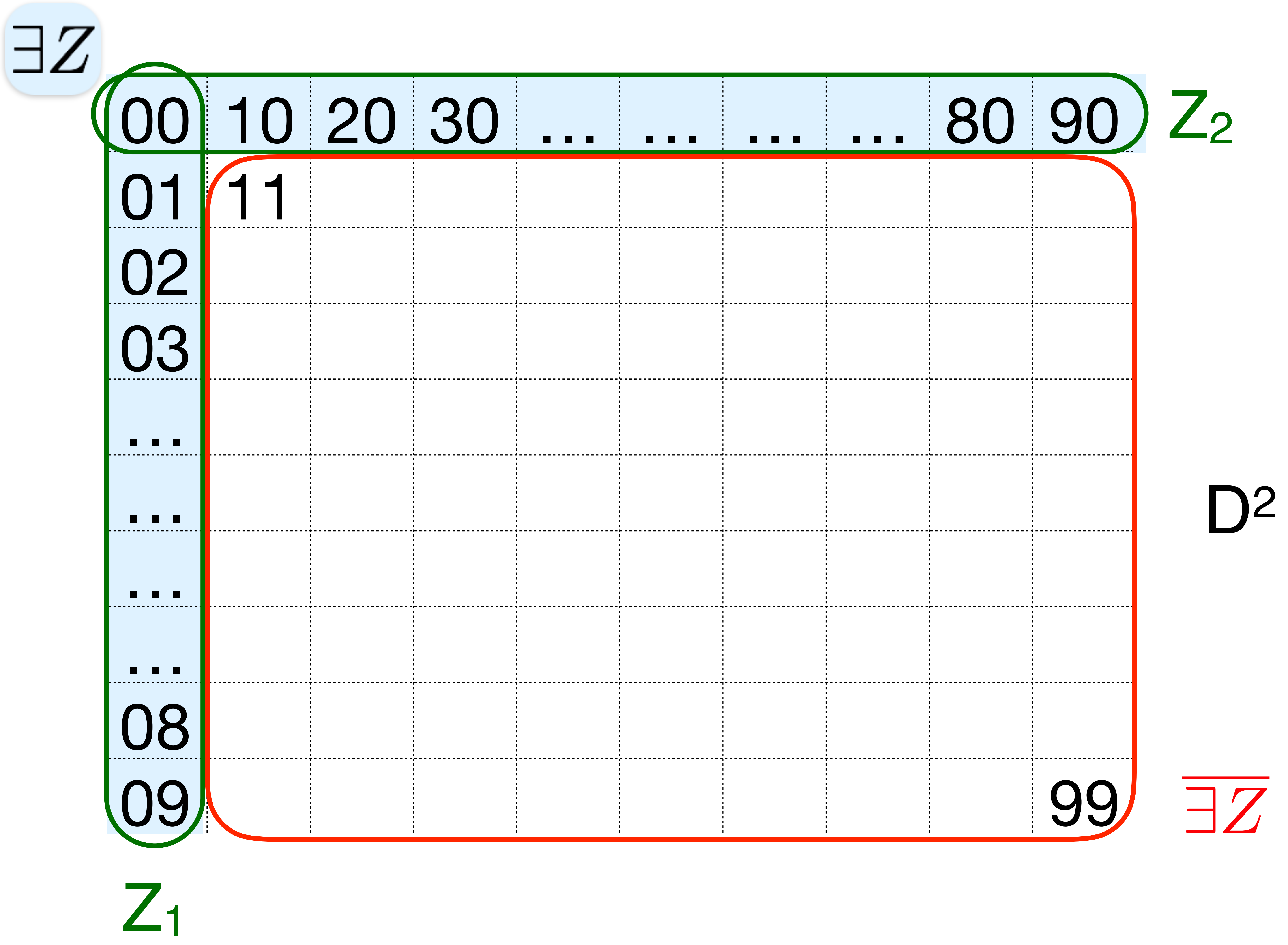
{1,..,9} x {1,..,9}

$$|\overline{\exists Z}| = |\overline{Z} \times \overline{Z}| = |\overline{Z}|^2 = 9^2 = 81$$

$$\exists Z = D^2 - \overline{\exists Z}$$

$$|\exists Z| = |D^2| - |\overline{\exists Z}| = 100 - 81 = 19$$

 or ⊖ Better?



Inclusion-Exclusion

$|Z_1| + |Z_2| - |Z_1 \cap Z_2|$

$= 10 + 10 - 1 = 19$

Complement $|\overline{\exists Z}| = 9^2 = 81$ 100 - 81 = 19

n Digit: Inclusion Exclusion

$$\exists Z = \{x^n \colon \exists i \; x_i = 0\} \qquad x^n \triangleq x_1, \ldots, x_n$$

$$Z_i = \{x^n \colon x_i = 0\}$$

n=4 $\quad Z_2 = \{x0yz\} \quad Z_4 = \{xyz0\}$

$$\exists Z = Z_1 \cup \ldots \cup Z_n$$

$$\begin{aligned} |\exists Z| \; = \; & |Z_1| + |Z_2| + \ldots + |Z_n| \\ & - |Z_1 \cap Z_2| - |Z_1 \cap Z_3| - \ldots - |Z_{n-1} \cap Z_n| \\ & + |Z_1 \cap Z_2 \cap Z_3| + \ldots + |Z_{n-2} \cap Z_{n-1} \cap Z_n| \\ & \ldots \\ & + (-1)^{n-1} |Z_1 \cap Z_2 \cap \ldots \cap Z_n| \end{aligned}$$

n Digits: Complement

$$\overline{\exists Z} = \overline{\{x^n | \exists i \; x_i \in Z\}} = \{x^n | \forall i \; x_i \notin Z\} = (\overline{Z})^n \triangleq \forall \overline{Z}$$

all digits nonzero

X

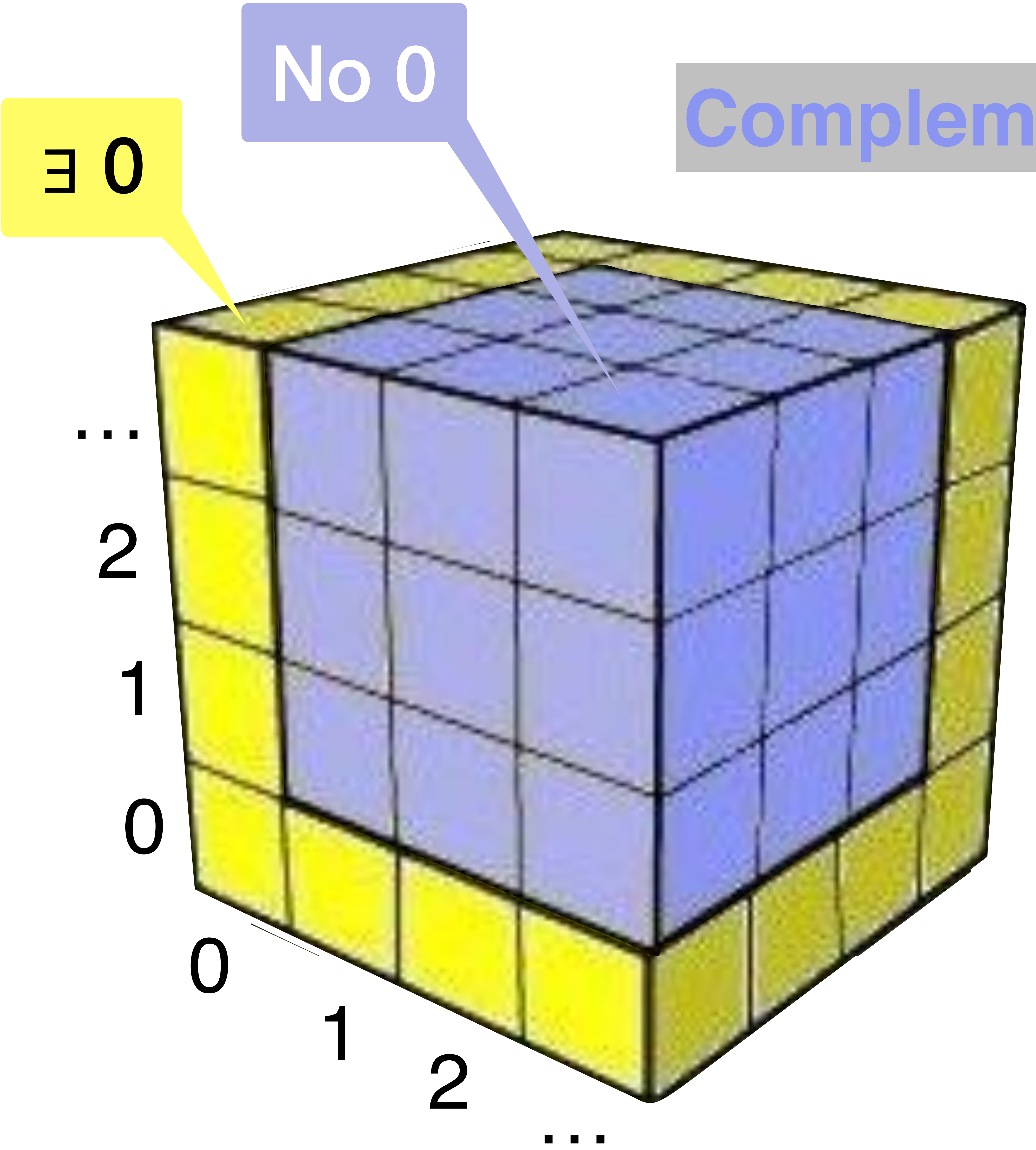
$$|\forall \overline{Z}| = |\overline{Z}|^n = 9^n$$

$$\exists Z = D^n - \forall \overline{Z}$$

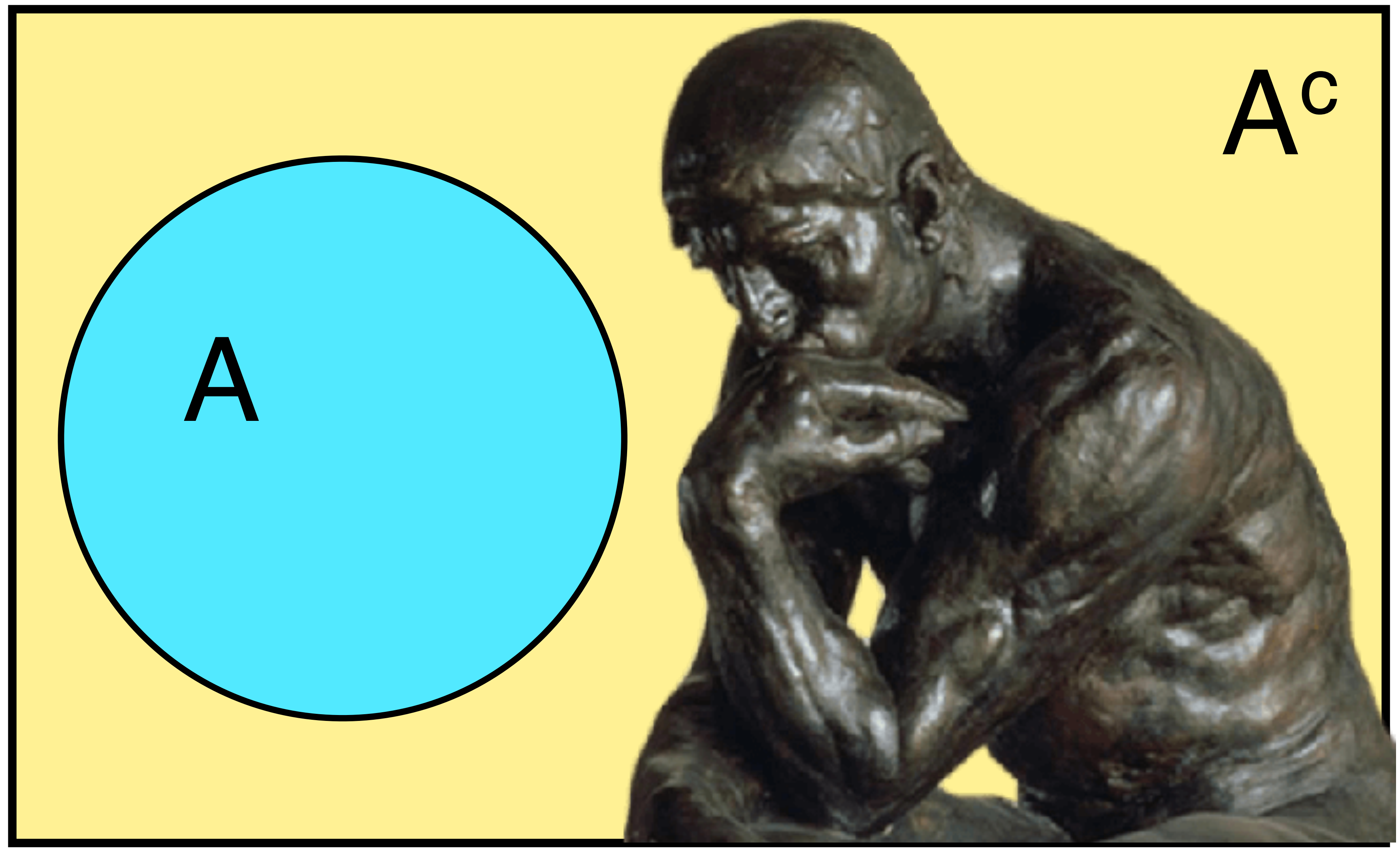
$$|\exists Z| = |D^n| - |\forall \overline{Z}| = 10^n - 9^n$$

Visualize

Inclusion exclusion



Complement



Counting Variations

Combined

addition

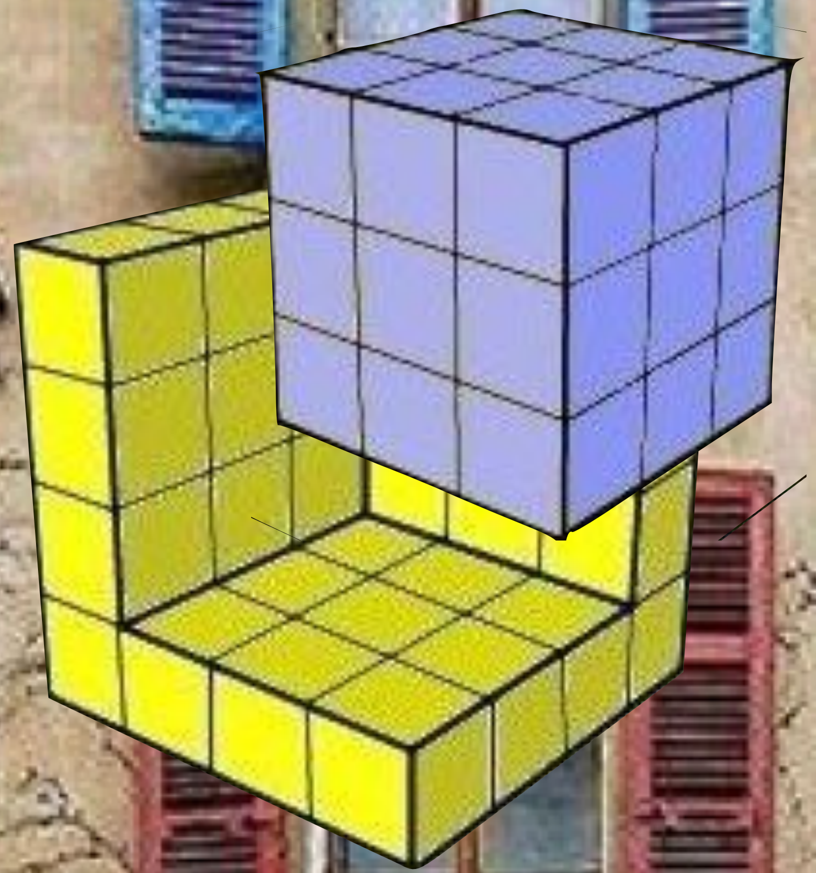
subtraction

multiplication

Determine size of more complex sets

PINs with various constraints

At least one 0



Trees