# Expectations

# What Matters

Important random-variable properties?

**Range**

Min & max values of X

Lowest & highest temperature / salary

$x_{min} = \min \{ x \in \Omega \mid p(x) > 0 \}$

$x_{max} = \max \{ x \in \Omega \mid p(x) > 0 \}$

**Average**

Average temperature / salary

**Range average**

$$\frac{x_{\min} + x_{\max}}{2} \;?$$

**Element average**

$$\frac{1}{|\Omega|} \sum_{x \in \Omega} x \;?$$

or over x s.t. $p(x)>0$

# Sample Mean

$\Omega = \{0,\ldots,100\}$ $p(0)=.8$ $p(90)=.1$ $p(100)=.1$ all other $p(x)= 0$

**Range average** $(x_{min}+x_{max})/2$ $(0+100)/2 = 50$

**Element average** positive probabilities $(0+90+100)/3 = 63.3$

Ten samples Typical 0, 0, 0, 0, 90, 0, 0, 0, 100, 0

**Sample mean** $(8\cdot 0+1\cdot 90+1\cdot 100)/10 = 190/10 = 19$

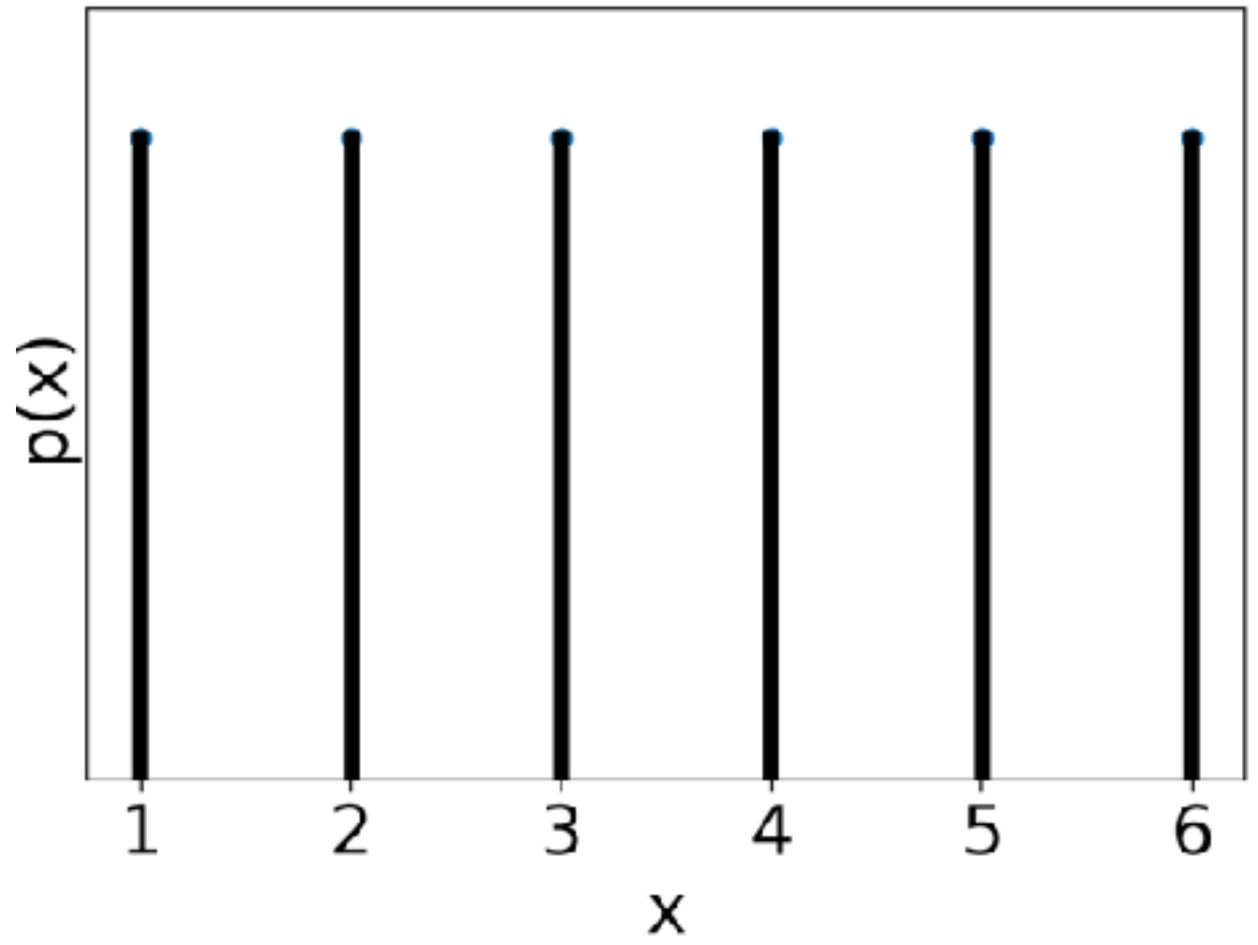
More representative of what we will observe

# Fair Die

Roll a fair die n → ∞ times

Average of the observed values = ?

Each value ~n/6 times



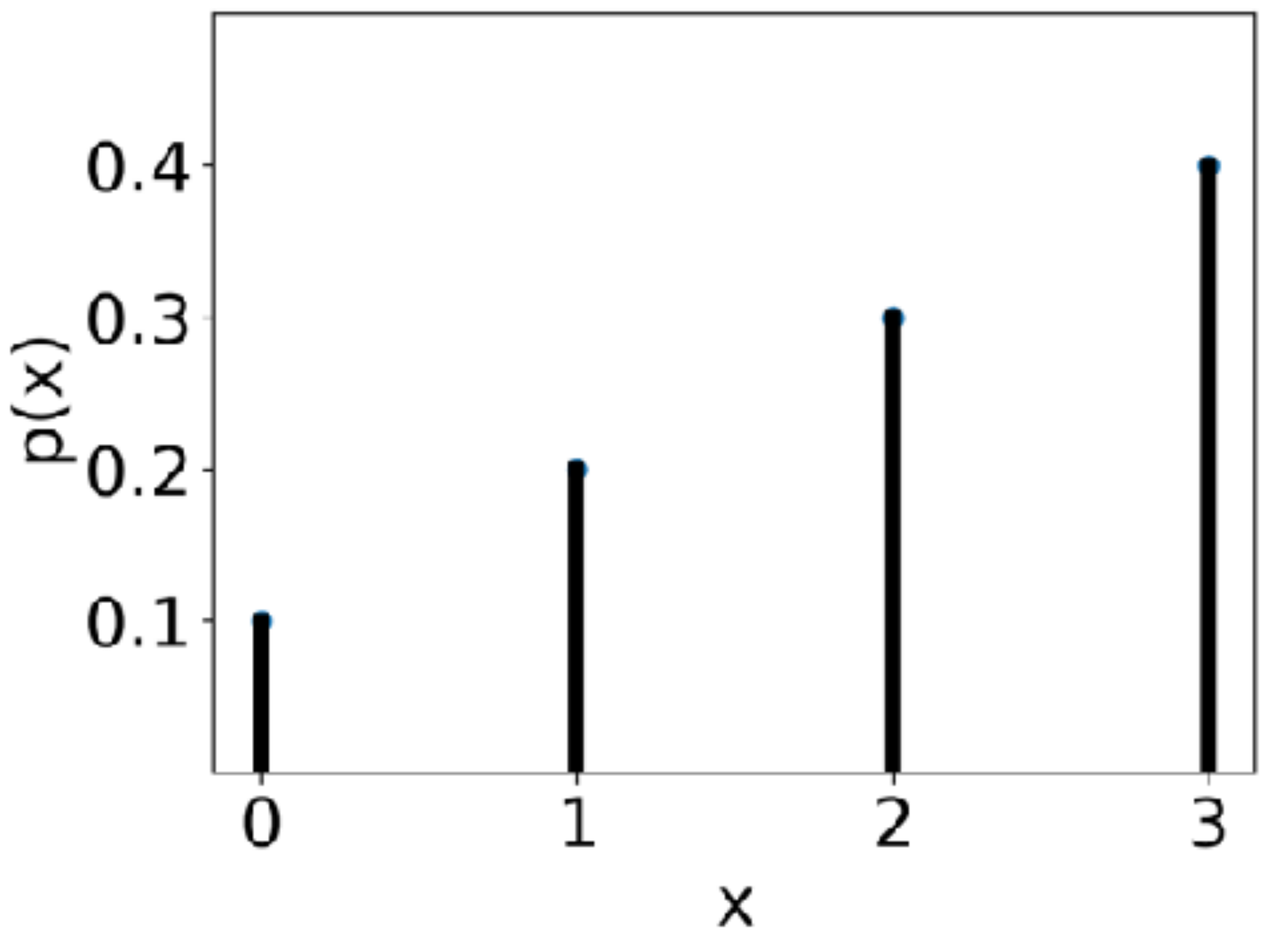

Average

$$\frac{\frac{n}{6}\cdot 1+\frac{n}{6}\cdot 2+\ldots+\frac{n}{6}\cdot 6}{n}=\frac{1+\ldots+6}{6}=\frac{1}{6}\,\frac{(1+6)\cdot 6}{2} = 3.5$$

1,…,6 → Average = 3.5

# 4-Sided Die

| Side | Prob | Appear |
|---|---|---|
| 1 | .1 | .1n |
| 2 | .2 | .2n |
| 3 | .3 | .3n |
| 4 | .4 | .4n |
| | 1 | n |





Average $= \dfrac{.1n \cdot 1 + .2n \cdot 2 + .3n \cdot 3 + .4n \cdot 4}{n}$

$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4$ $= 3$

Arithmetic average $(1+2+3+4)/4 = 2.5$

Probabilities skew to the right

# Expectation

In $n \to \infty$ samples x will appear $p(x) \cdot n$ times

$$\text{Average} = \frac{\sum_x [P(x) \cdot n] \cdot x}{n} = \boxed{\sum_x P(x) \cdot x \stackrel{\text{def}}{=} E(X)}$$

Expectation

Mean

E(X) also denoted EX $\mu_X$ $\mu$

Not random constant property of the distribution

# Fair Die

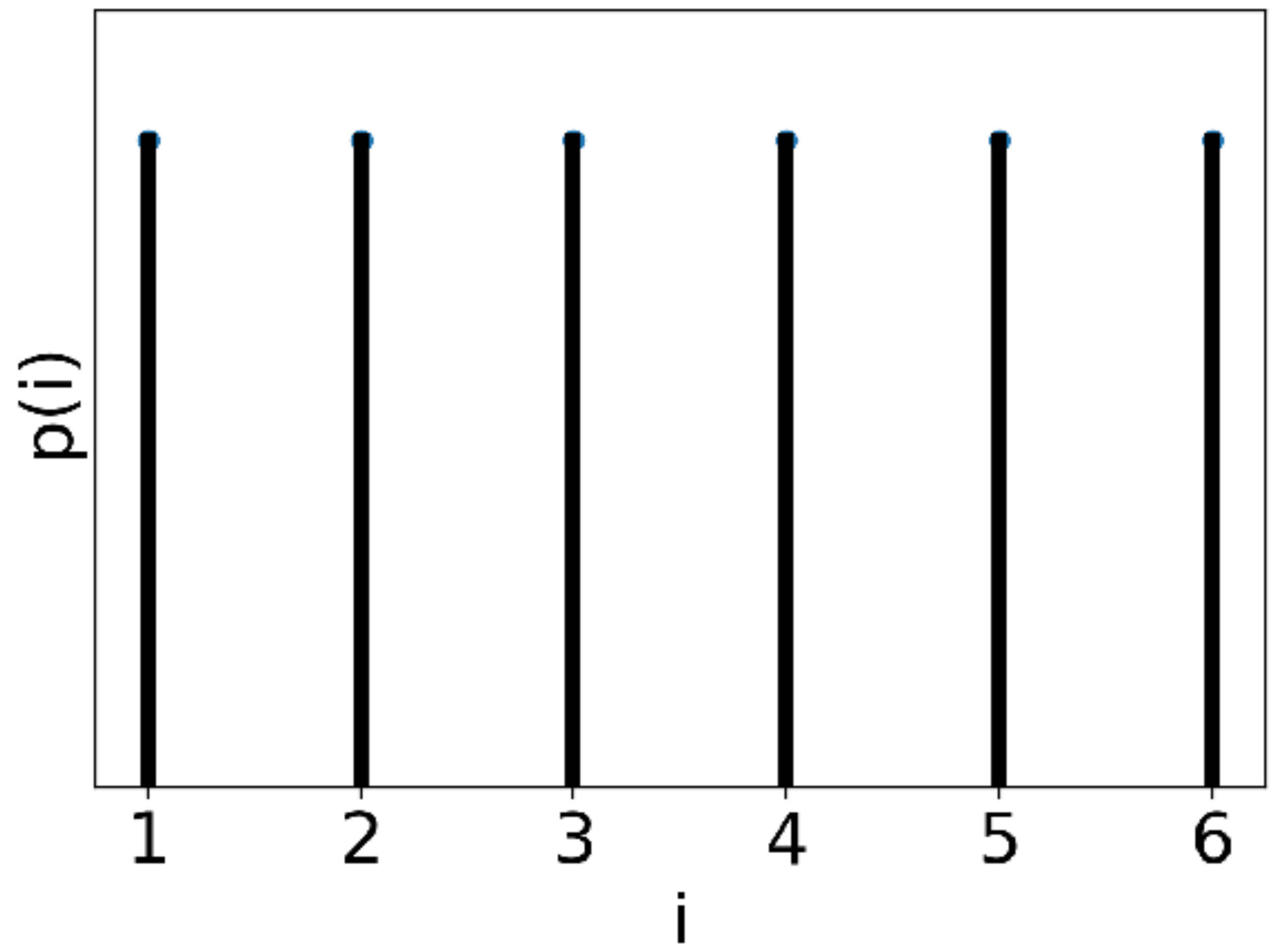
$$E(X) = \sum_{i=1}^{6} P(i) \cdot i$$

$$= \sum_{i=1}^{6} \frac{1}{6} \cdot i$$

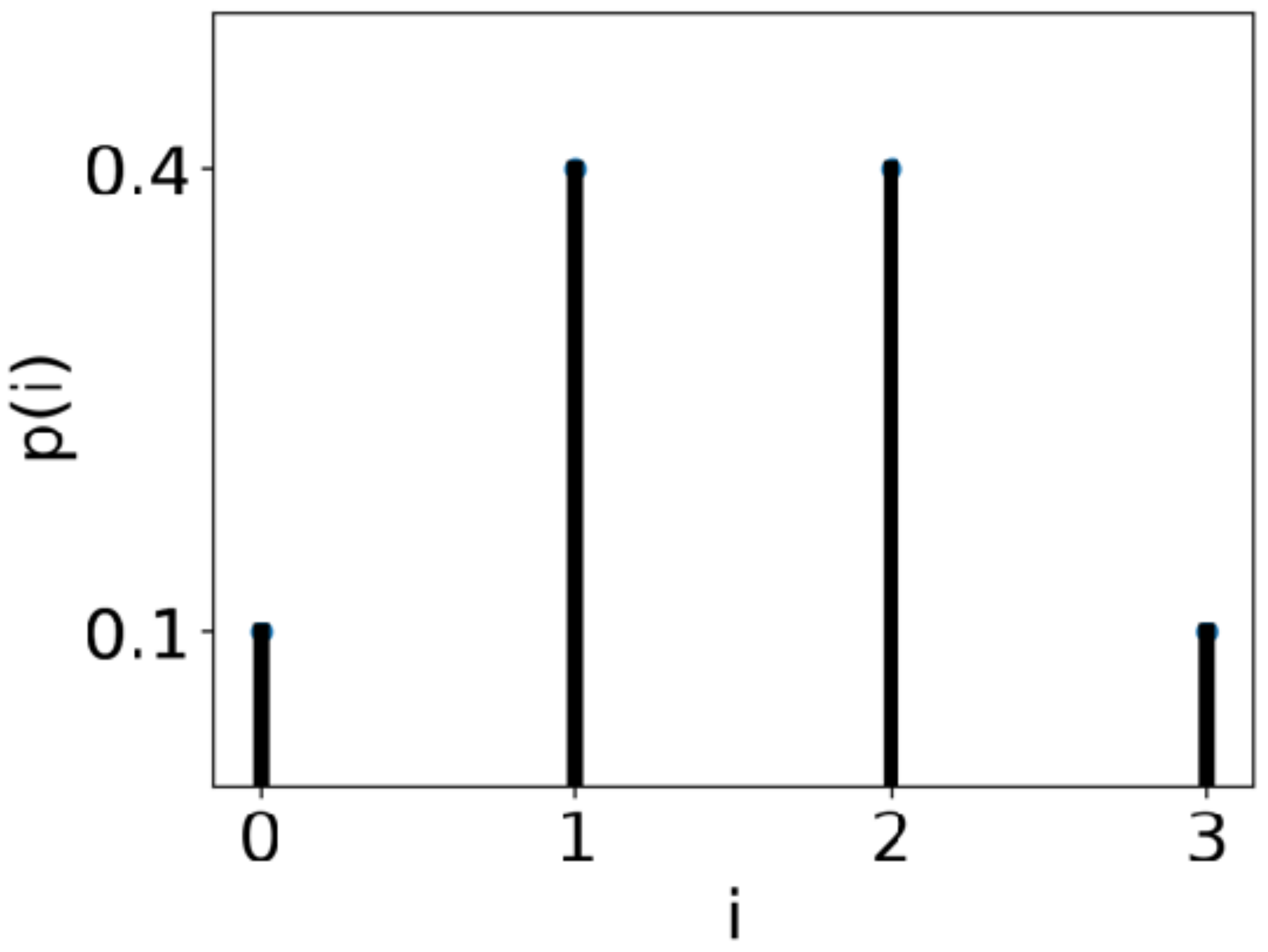
$$= \frac{1 + 2 + \ldots + 6}{6}$$

$$= \frac{1}{6} \frac{(1+6) \cdot 6}{2}$$

$$= \frac{7}{2} = 3.5$$ ✅




# 4 Sided- Die




$$E(X) = \sum_{i=1}^{4} p_i \cdot i$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4$$

$$= 3$$ ✅

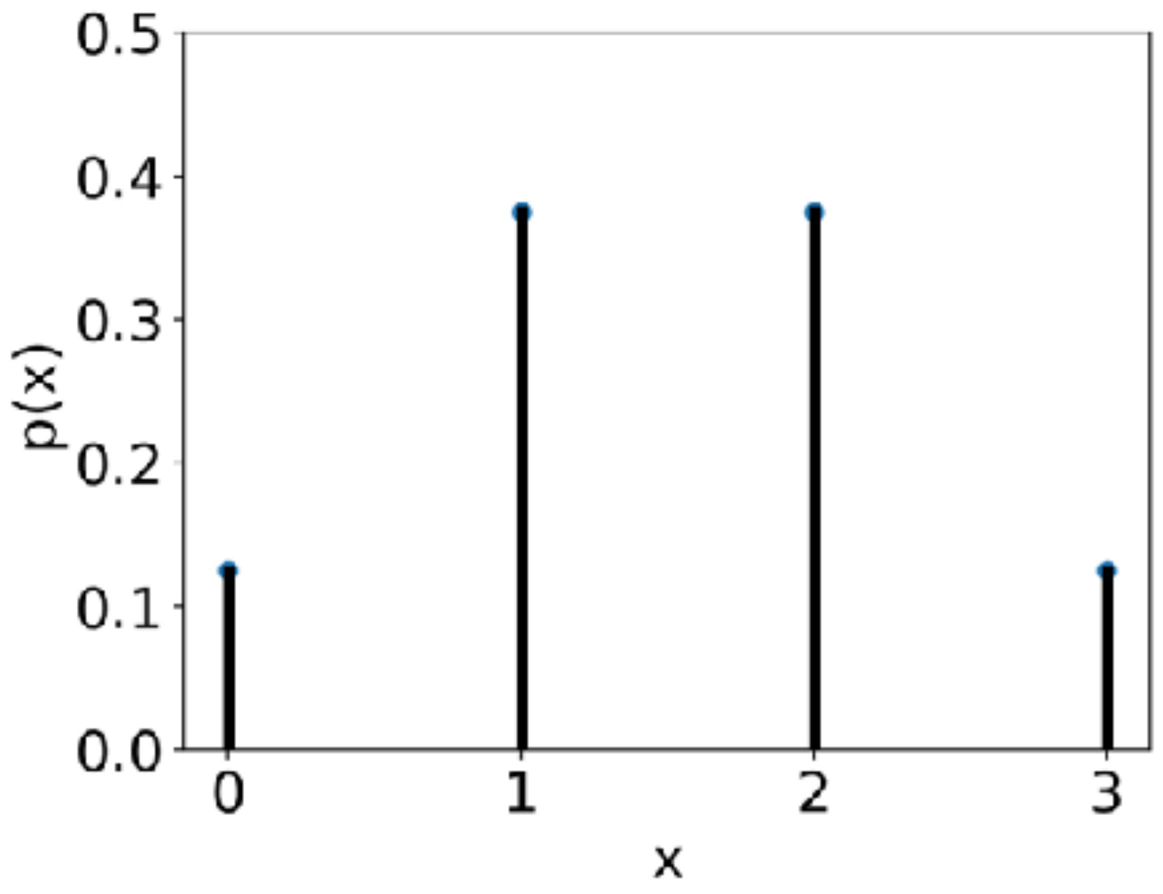
# 3 Coins

Toss a coin 3 times

X - # heads

E(X) = ?

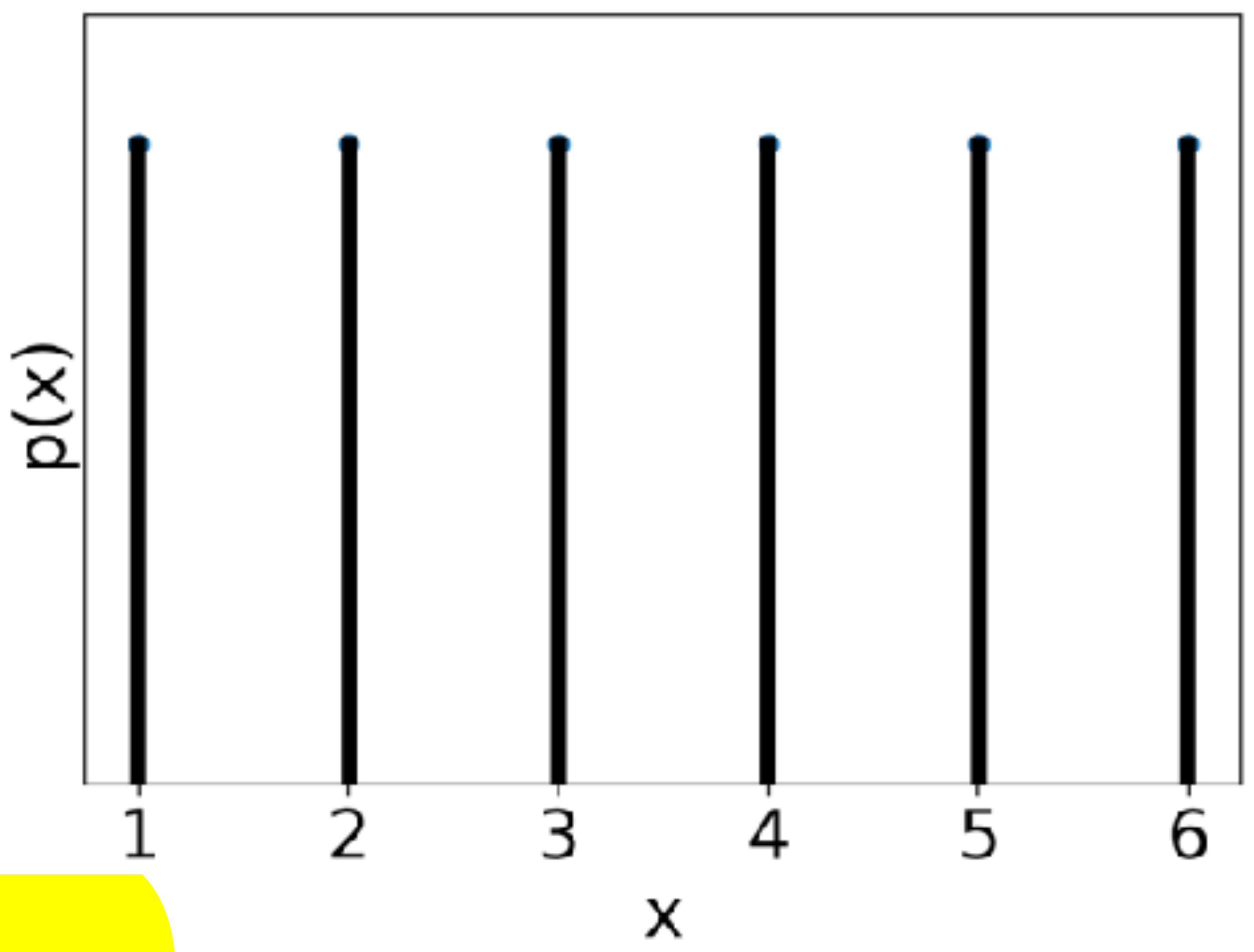
| x | outcomes | p(x) |
|---|---|---|
| 0 | ttt | ⅛ |
| 1 | tth,tht,htt | ⅜ |
| 2 | thh, hth, hht | ⅜ |
| 3 | hhh | ⅛ |




$\sum P(x) \cdot x$ $= \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3$ $= 1.5$

# heads ranges from 0 to 3, on average 1.5 ✅

# Uniform Variables




X uniform over Ω

$$p(x) = \frac{1}{|\Omega|}$$

$$E(X) = \sum_{x \in \Omega} p(x) \cdot x = \sum_{x \in \Omega} \frac{1}{|\Omega|} \cdot x = \frac{1}{|\Omega|} \sum_{x \in \Omega} x$$

E(X) is the arithmetic average of elements in Ω

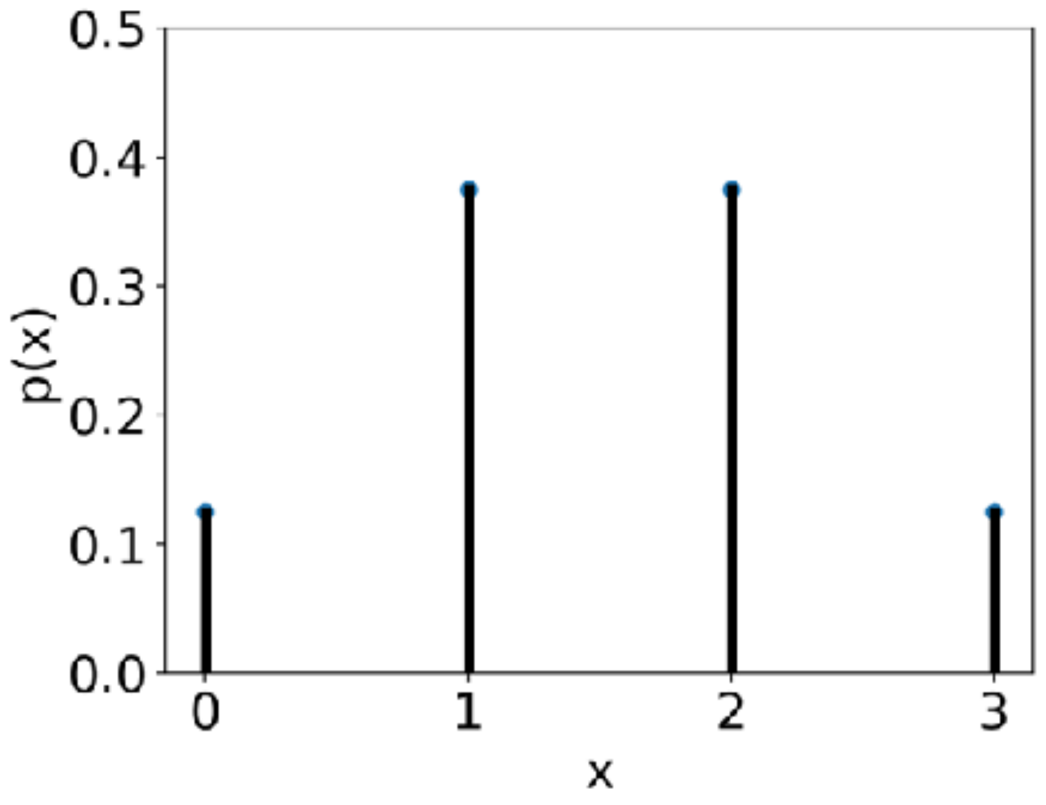
$$E(X) = \frac{1+2+\ldots+6}{6} = 3.5$$

# Symmetry

A distribution p is symmetric around a if for all x > 0, p(a+x) = p(a-x)

If p is symmetric around a, then E(X) = a

| x | outcomes | P(x) |
|---|---|---|
| 0 | ttt | ⅛ |
| 1 | tth,tht,htt | ⅜ |
| 2 | thh, hth, hht | ⅜ |
| 3 | hhh | ⅛ |




Symmetric around 1.5

E(X) = 1.5

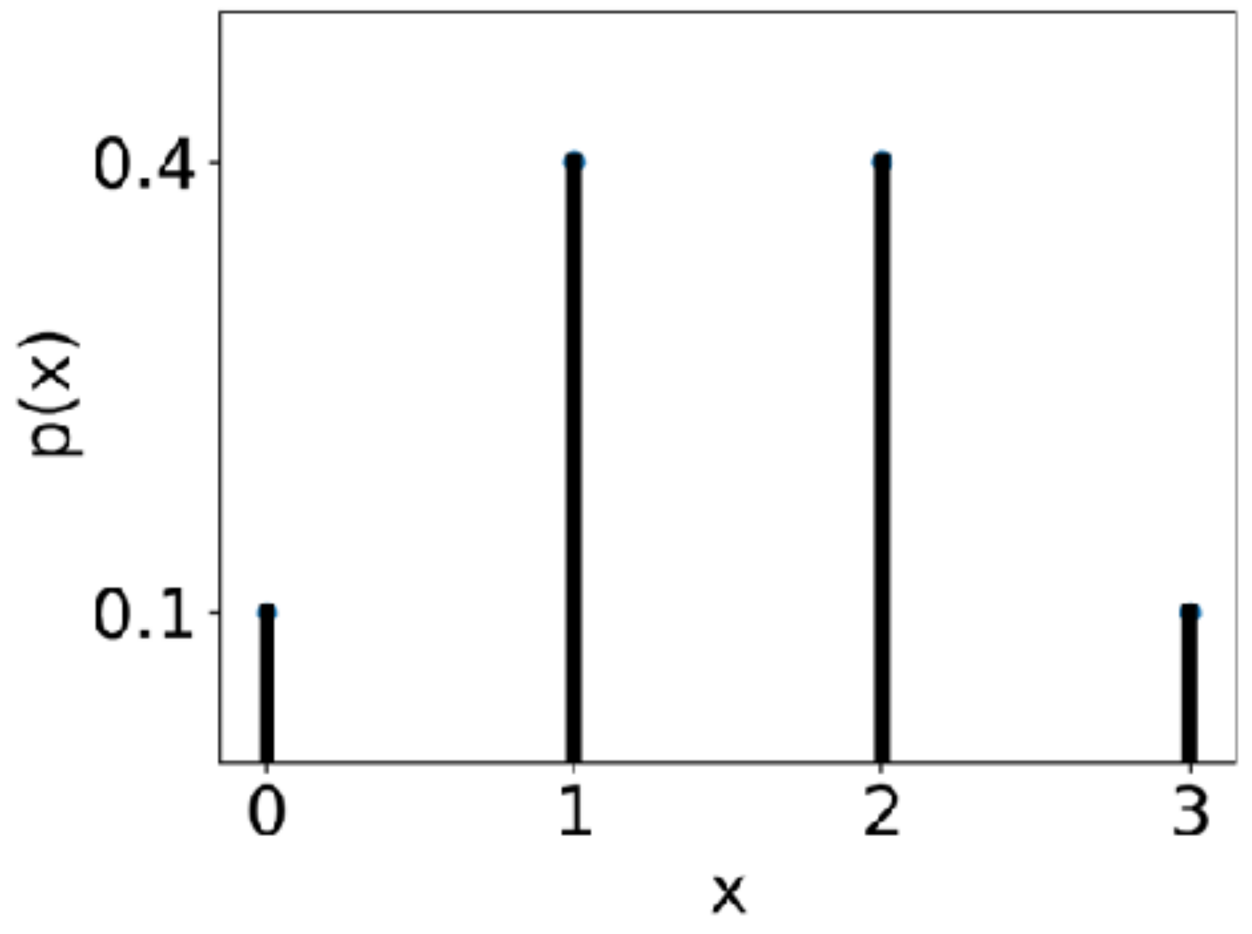
# Properties

$E(X)$ Despite notation Not random Number Property of distribution

$E(X) = 1.5$

$x_{min} \leq E(X) \leq x_{max}$ = iff $X = c$

$0 \leq E(X) \leq 3$




X is a constant, namely $X=c$ → $E(X)=c$

$E(E(X)) = E(X)$

# Is Expectation Expected?

$\mu = EX$ - expectation of X

Do we expect to see it?

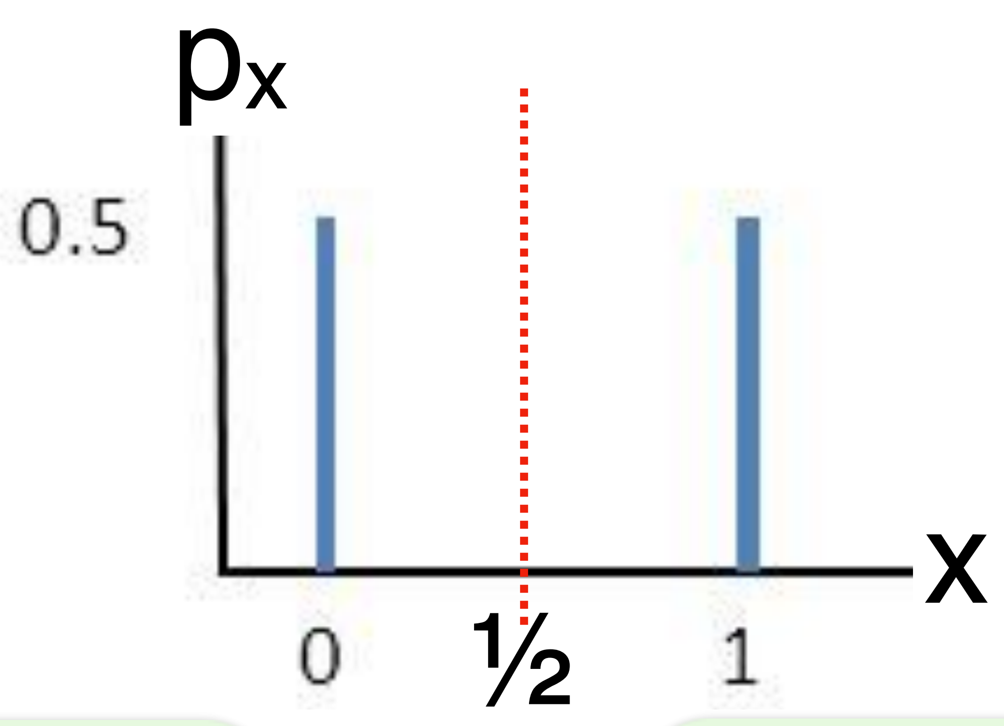
Is $p_\mu$ high?

Not necessarily

We may never see it!

$X \in \{0,1\}$

$p_0 = p_1 = 0.5$




$EX = 0 \cdot p_0 + 1 \cdot p_1 = 0 \cdot ½ + 1 \cdot ½ = ½$

Symmetric around ½

½ will never happen!

Many samples → average = ½

EX - average of large sample

Not necessarily likely

May not be observed at all

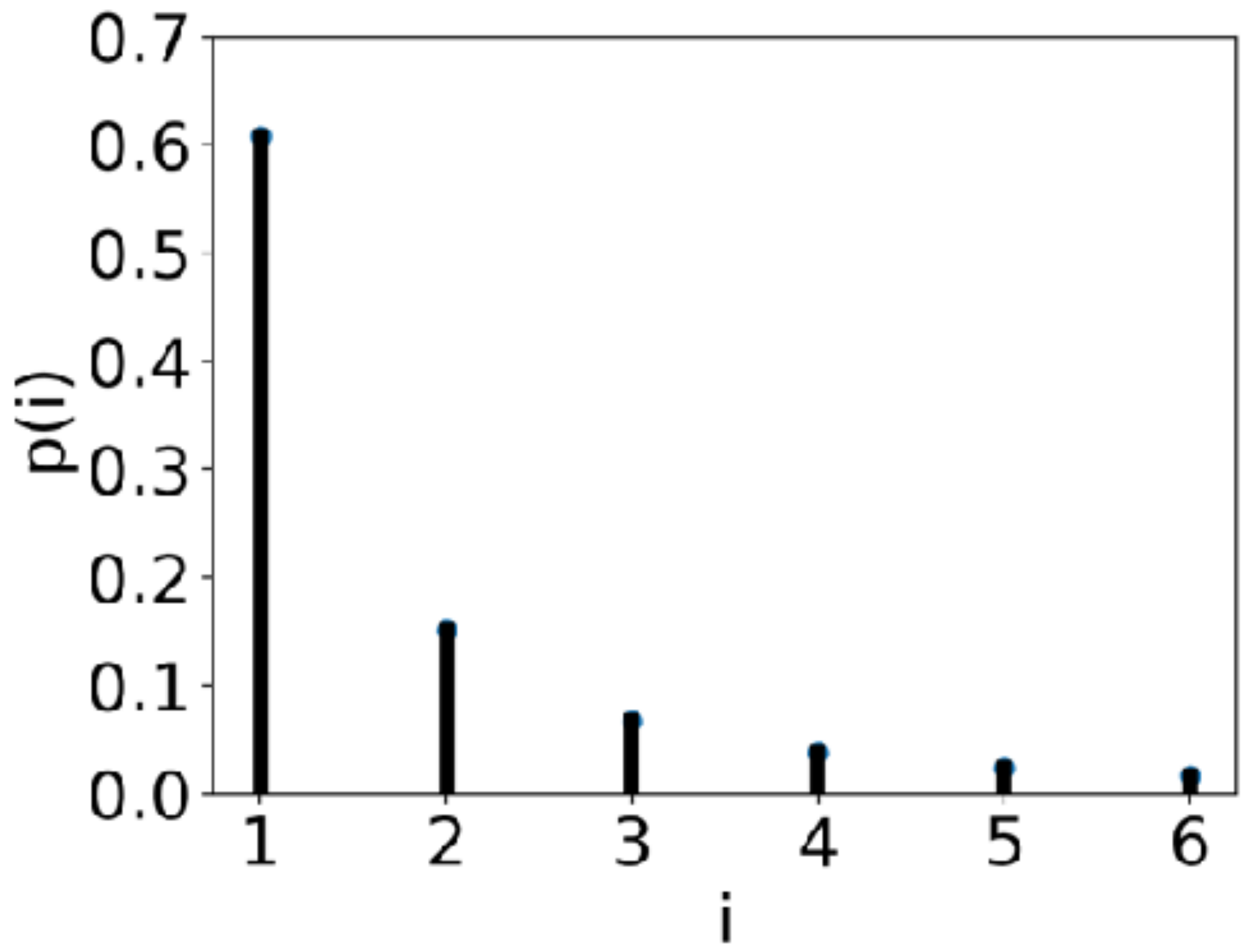
# Infinite Expectation

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Basel problem

Euler → famous

$$\frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = 1$$




$$p_i = \frac{6}{\pi^2} \cdot \frac{1}{i^2}$$

probability distribution over $\mathbb{P}$

$$E(X) = \sum_{i=1}^{\infty} i \cdot p_i = \frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

Many samples

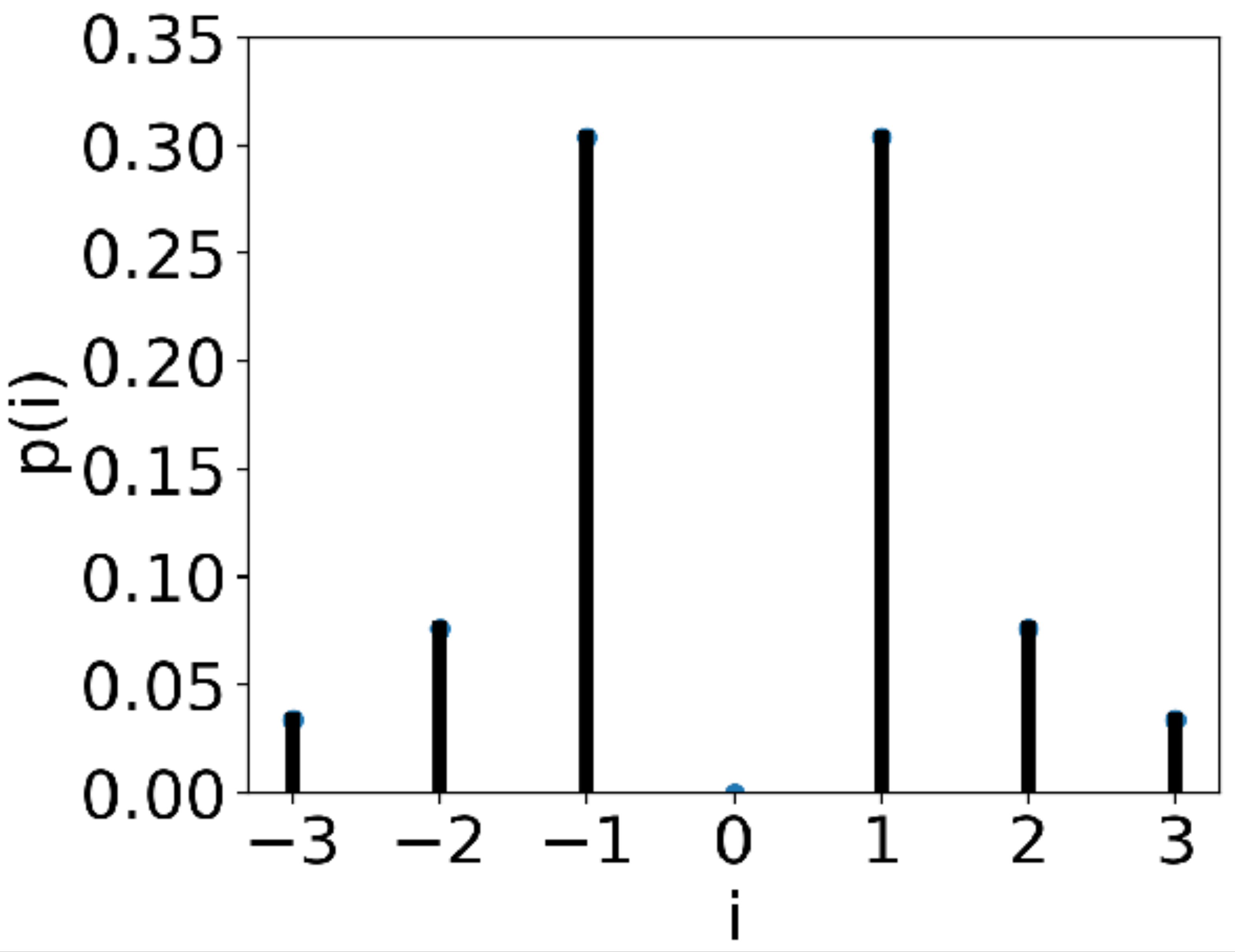
Average will go to ∞

# Undefined Expectation

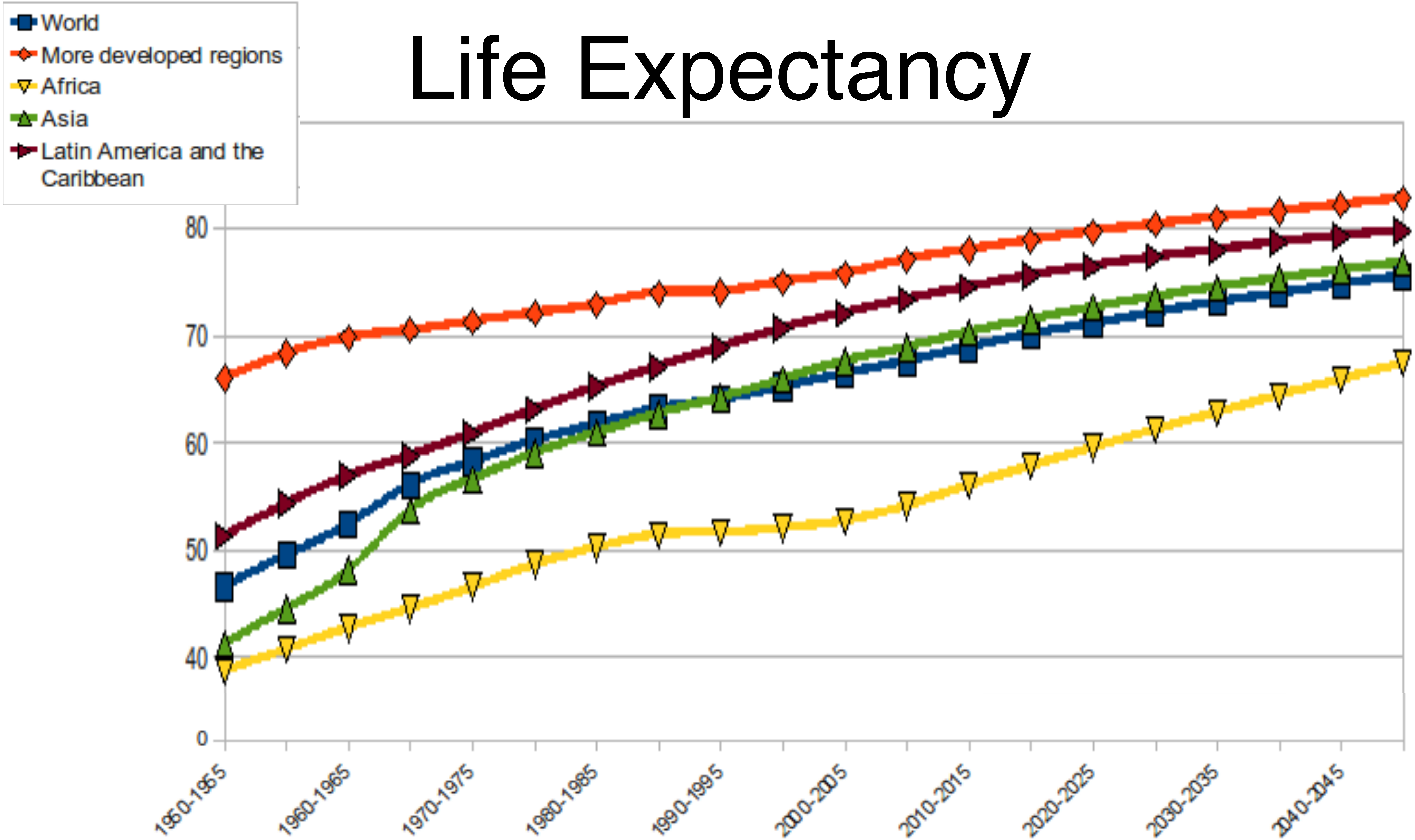
$$p_i = \frac{3}{\pi^2} \cdot \frac{1}{i^2} \quad \text{for} \quad i \neq 0$$

E(X) = ∞ - ∞

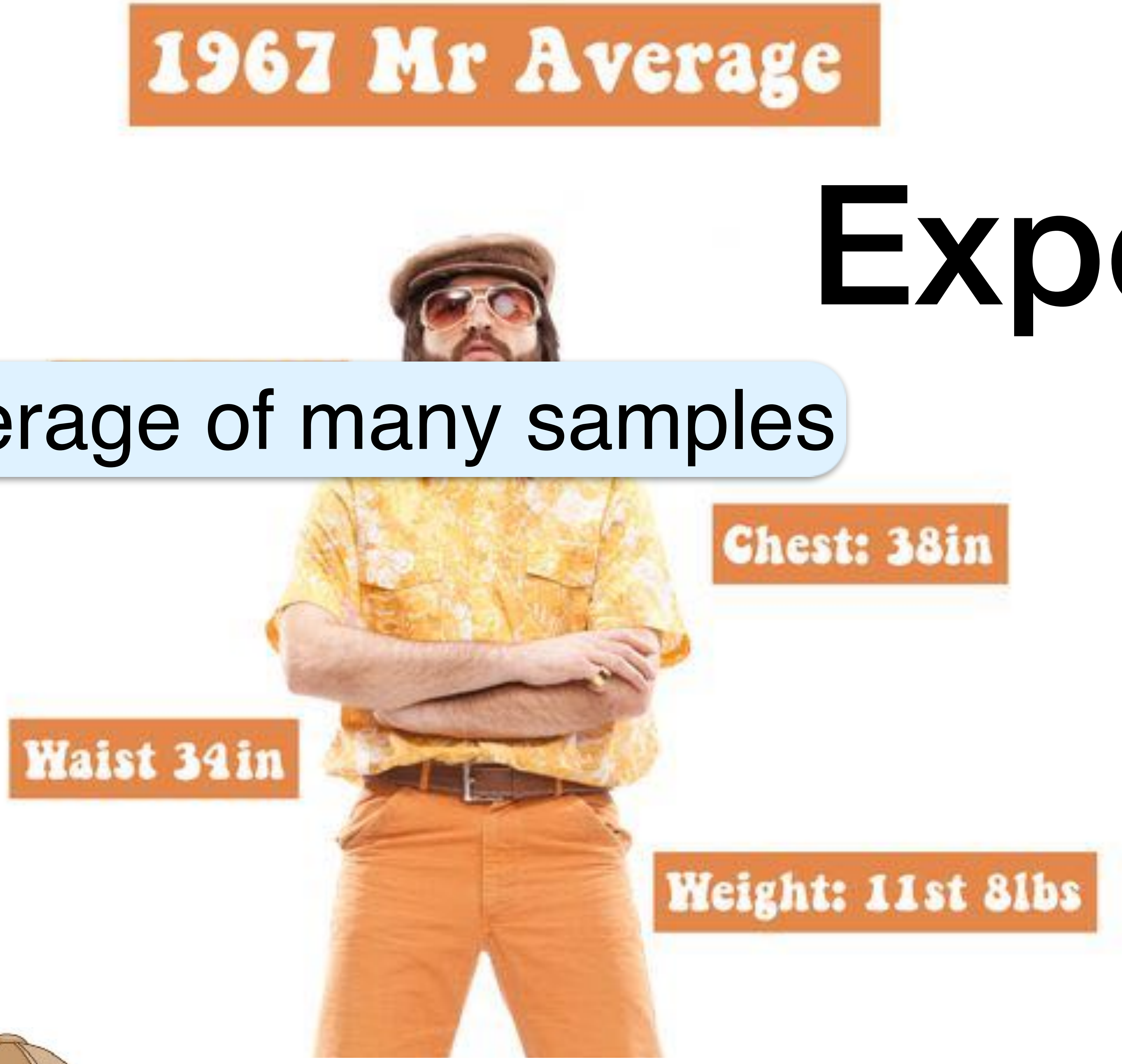
Undefined



# Life Expectancy



# Expectation




Average of many samples


Expectations of Functions
of Random Variables

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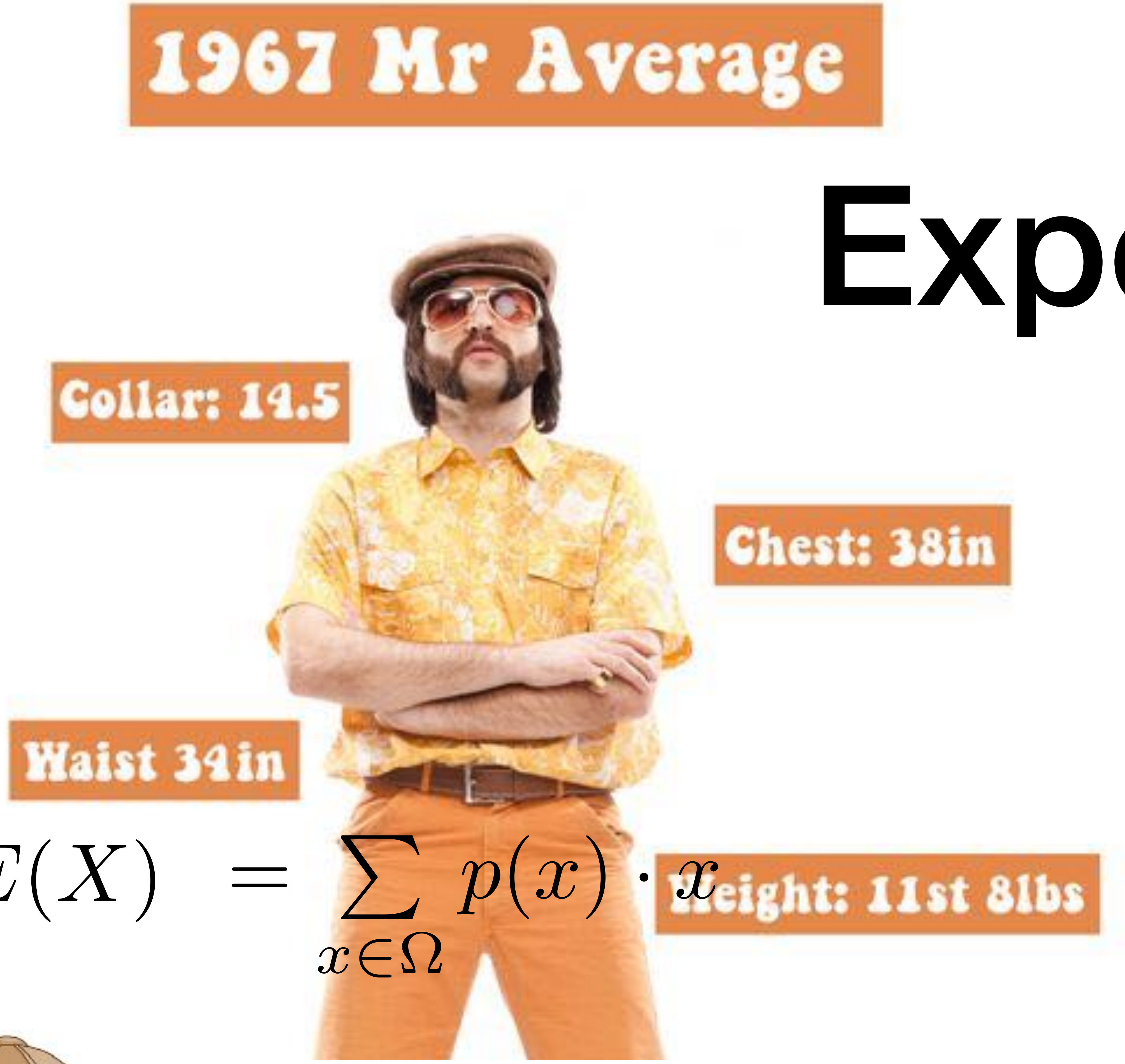


# Expectation


$$E(X) = \sum_{x \in \Omega} p(x) \cdot x$$

Expectations of Functions
of Random Variables