Harmonic Oscillator

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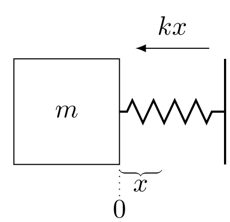
1 Simple Harmonic Oscillator

The simple harmonic oscillator comes about, most commonly, from a mass on spring. No friction.



Gives us the differential equation

$$m\ddot{x} - kx = 0$$

We solve this equation by solving the characteristic equation

$$r^2 - \frac{k}{m} = 0$$
$$r = \pm\sqrt{-\frac{k}{m}}$$

Since both k and m are positive

$$r = \pm i\sqrt{\frac{k}{m}}$$

and the general solution is

$$x = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

We let

$$\omega_0 = \sqrt{\frac{k}{m}}$$

This is both a useful simplification and a meaningful label, the *angular frequency* of the oscillation. Therefore we have

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

or

$$x = A\cos(\omega_0 t + \phi)$$

Both of these forms a useful, I will work with both.

1.1 Initial Conditions

We will solve for A and B when given initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0$$

1.1.1 Form 1

$$\begin{aligned} x &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \\ \dot{x} &= -A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t) \end{aligned}$$

the initial conditions give

$$\begin{aligned} x(0) &= A = x_0 \\ \dot{x}(0) &= B\omega_0 = \dot{x}_0 \end{aligned}$$

So the particular solution is

$$x = x_0\cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0}\sin(\omega_0 t)$$

1.1.2 Form 2

$$\begin{aligned} x &= A\cos(\omega_0 t + \phi) \\ \dot{x} &= -A\omega_0\sin(\omega_0 t + \phi) \end{aligned}$$

the initial conditions give

$$\begin{aligned} x(0) &= A\cos\phi = x_0 \\ \dot{x}(0) &= -A\omega_0\sin\phi = \dot{x}_0 \end{aligned}$$

We can get that

$$\cos\phi = \frac{x_0}{A}, \qquad \sin\phi = -\frac{\dot{x}_0}{A\omega_0}$$

So

$$\cos^2\phi + \sin^2\phi = 1 = \frac{x_0^2}{A^2} + \frac{\dot{x}_0^2}{A^2\omega_0^2}$$
$$1 = \frac{1}{A^2}\left(x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}\right)$$
$$A^2 = x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}$$
$$A = \sqrt{x_0^2 + \dot{x}_0^2/\omega_0^2}$$

and

$$\tan\phi = -\frac{\dot{x}_0}{A\omega_0}\frac{A}{x_0} = -\frac{\dot{x}_0}{\omega_0 x_0}$$

So the particular solution is

$$x = \cos\left(\omega_0 t - \frac{\dot{x}_0}{\omega_0 x_0}\right)\sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}}$$

This looks way worse, but it gives the actual amplitude of the wave, and the *actual phase*, whatever thats useful for. So it's physically more useful to look at, as a engineer or something, but it's harder to do math with.

If you let A' and B' be the constants for **Form 1** of the solution,

$$A' = x_0, \qquad B' = \frac{\dot{x}_0}{\omega_0}$$

And then you can relate the two forms as

$$A^2 = (A')^2 + (B')^2, \qquad \tan\phi = \frac{B'}{A'}$$

Which is an interesting relation.

1.1.3 Initial Conditions Sidenote

Say our initial conditions don't start at $t = 0$, say we have initial condition $x(t_0) = x_0$, we let $t = t' - t_0$ so that

$$x = A\cos(\omega_0 t + \phi) \qquad \text{becomes}$$
$$x = A\cos\big(\omega_0(t' - t_0) + \phi\big)$$

Plugging in the initial condition will still give us the same values for constants A and ϕ. The fact that the initial conditions don't start at zero just doesn't mean the solution changes, the same things happen, just at a different time. So if need to know what happens a later time, we just shift that time aswell. If we wanted to know that happens at $t = t_1$, then we just plug in $t_1 + t_0$.

1.2 Energy