Harmonic Oscillator

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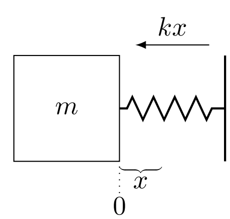
1 Simple Harmonic Oscillator

The simple harmonic oscillator comes about, most commonly, from a mass on spring. No friction.



Gives us the differential equation

$$m\ddot{x} - kx = 0$$

We solve this equation by solving the characteristic equation

$$\begin{aligned} r^2 - \frac{k}{m} &= 0 \\ r &= \pm\sqrt{-\frac{k}{m}} \end{aligned}$$

Since both k and m are positive

$$r = \pm i\sqrt{\frac{k}{m}}$$

and the general solution is

$$x = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

We let

$$\omega_0 = \sqrt{\frac{k}{m}}$$

This is both a useful simplification and a meaningful label, the *angular frequency* of the oscillation. we should then rewrite the differential equation

$$\ddot{x} - \omega_0^2 x = 0$$

And it's solution

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

or

$$x = A\cos(\omega_0 t + \phi)$$

Both of these forms are useful, I will work with both. First one is **Form 1** and second one is **Form 2**.

1.1 Initial Conditions

We will solve for A and B when given initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0$$

1.1.1 Form 1

$$\begin{aligned} x &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \\ \dot{x} &= -A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t) \end{aligned}$$

the initial conditions give

$$\begin{aligned} x(0) &= A = x_0 \\ \dot{x}(0) &= B\omega_0 = \dot{x}_0 \end{aligned}$$

So the particular solution is

$$\boxed{x = x_0\cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0}\sin(\omega_0 t)}$$

1.1.2 Form 2

$$\begin{aligned} x &= A\cos(\omega_0 t + \phi) \\ \dot{x} &= -A\omega_0\sin(\omega_0 t + \phi) \end{aligned}$$

the initial conditions give

$$\begin{aligned} x(0) &= A\cos\phi = x_0 \\ \dot{x}(0) &= -A\omega_0\sin\phi = \dot{x}_0 \end{aligned}$$

We can get that

$$\cos\phi = \frac{x_0}{A}, \qquad \sin\phi = -\frac{\dot{x}_0}{A\omega_0}$$

So

$$
\begin{aligned}
\cos^2\phi + \sin^2\phi = 1 &= \frac{x_0^2}{A^2} + \frac{\dot{x}_0^2}{A^2\omega_0^2} \\
1 &= \frac{1}{A^2}\left(x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}\right) \\
A^2 &= x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2} \\
A &= \sqrt{x_0^2 + \dot{x}_0^2/\omega_0^2}
\end{aligned}
$$

and

$$\tan\phi = -\frac{\dot{x}_0}{A\omega_0}\frac{A}{x_0} = -\frac{\dot{x}_0}{\omega_0 x_0}$$

So the particular solution is

$$\boxed{x = \cos\left(\omega_0 t - \frac{\dot{x}_0}{\omega_0 x_0}\right)\sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}}}$$

This looks way worse, but it gives the actual amplitude of the wave, and the *actual phase*, whatever thats useful for. So it's physically more useful to look at, as a engineer or something, but it's harder to do math with.

If you let A' and B' be the constants for **Form 1** of the solution,

$$A' = x_0, \qquad B' = \frac{\dot{x}_0}{\omega_0}$$

And then you can relate the two forms as

$$A^2 = (A')^2 + (B')^2, \qquad \tan\phi = -\frac{B'}{A'}$$

Which is an interesting relation.

1.1.3 Initial Conditions Sidenote

Say our initial conditions don't start at $t = 0$, say we have initial condition $x(t_0) = x_0$, we let $t = t' - t_0$ so that

$$
\begin{aligned}
x &= A\cos(\omega_0 t + \phi) \qquad \text{becomes} \\
x &= A\cos\big(\omega_0(t' - t_0) + \phi\big)
\end{aligned}
$$

Plugging in the initial condition will still give us the same values for constants A and ϕ. The fact that the initial conditions don't start at zero just doesn't mean the solution changes, the same things happen, just at a different time. So if need to know what happens a later time, we just shift that time aswell. If we wanted to know that happens at $t = t_1$, then we just plug in $t_1 + t_0$.

1.2 Energy

We find the kinetic energy of our system

$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

recall that $\omega_0^2 = k/m$

$$K = \frac{1}{2}kA^2\sin^2(\omega_0 t + \phi)$$

and then to potential

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$$

And then we can get the total energy

$$\begin{aligned} E &= K + U = \frac{1}{2}kA^2\sin^2(\omega_0 t + \phi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) \\ E &= \frac{1}{2}kA^2 \end{aligned}$$

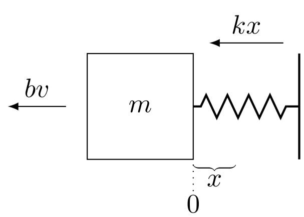
The energy is constant, as it should be.

2 Damped Harmonic Oscillator

The damped harmonic oscillator comes about when we simple add friction to the mass on a spring. A specific type of friction, *viscous friction.* A type of friction I know nothing about, only that

$$f_{\text{visc}} = -bv$$

Yeah, cool, whatever. I don't really care. We don't care about viscous friction. We care about a term that is *against* motion, and proportional to velocity. Take this and use it.



Gives us the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solve it, again, using the characteristic equation

$$\begin{aligned} mr^2 + br + k &= 0 \\ r^2 + \frac{b}{m}r + \frac{k}{m} &= 0 \end{aligned}$$

Yes, the quadratic equation

$$r = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

what? this isn't what you expected? *Look Closer*. Anyway, this should be cleaned up, it would be nice if we could keep $\omega_0^2 = k/m$.

$$r = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}$$

If there's anway to simplfy this, it's to pull out an ω_0^2, to get a one, we like ones.

$$r = -\frac{b}{2m} \pm \omega_0 \sqrt{\left(\frac{b}{2m\omega_0}\right)^2 - 1}$$

Now it's the perfect time to let

$$\zeta = \frac{b}{2m\omega_0} = \frac{b}{2\sqrt{km}}$$

$$r = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

An finally it's still worth to let

$$\omega_1 = \omega_0\sqrt{\zeta^2 - 1}$$

so

$$r = -\zeta\omega_0 \pm \omega_1$$

We apply these substitutions to the differential equation

$$\ddot{x} - 2\zeta\omega_0\dot{x} - \omega_0^2 x = 0$$

And our solution

$$x = Ae^{-(\zeta\omega_0 + \omega_1)t} + Be^{-(\zeta\omega_0 - \omega_1)t}$$

How nice, except we need to do more work. we know ζ is always positive, but ω_1 may or may not be complex. If $\zeta > 1$ then ω_1 is real and the solution remains. But if $\zeta < 1$, then ω_1 is complex and we would like to write our solution as

$$x = e^{-\zeta\omega_0 t}\big(A\cos(\omega_1 t) + B\sin(\omega_1 t)\big)$$

or

$$x = Ae^{-\zeta\omega_0 t}\cos(\omega_1 t + \phi)$$

If $\zeta = 1$ than $\omega_1 = 0$ and the solution is

$$x = (A + Bt)e^{-\omega_0 t}$$

Will now analyze all 3 equations

2.1 Overdamped