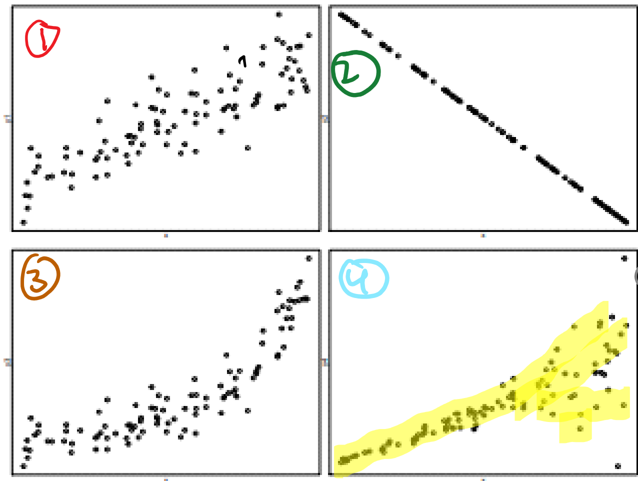
Preliminary activity II



note: trend in difference does not matter which x you pick, has different noise, while others have the same noise.

1 Description: - increasing / positive relationship
- noise
- linear

3 Description: - increasing / positive
- has noise
- curved

2 Description: - decreasing / negative
- no noise

4 Description: - increasing / positive
- has noise

Heteroscedasticity: the circumstance in which the variability of a variable is unequal across the range of values of a second variable that predicts it.

Regression analysis

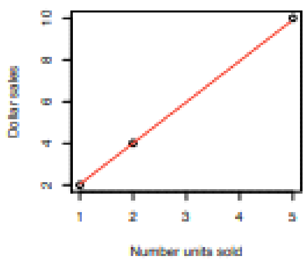
- Statistical methodology that utilizes the relation between variables.
- Predicts a response variable (or outcome) from the relation between the response and other variables.
- Regression analysis is used in many disciplines such as:

  - Business:
    - i) Forecasting: predicting future demand for a product.
    - ii) Optimization: Fine tune manufacturing and delivery processes

Functional relation

- Relation of the form
$$Y = f(X),$$
where X, Y are variables, and f is a function.
- Example: Relation between dollar sales (Y) of a product sales sold \$2 per unit and number of units sold (X):

$Y = 2x$

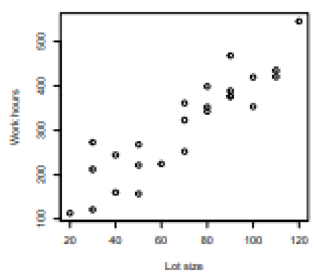


This is an example of a perfect relationship.

All observations fall on the line of functional relationship.

Statistical relation

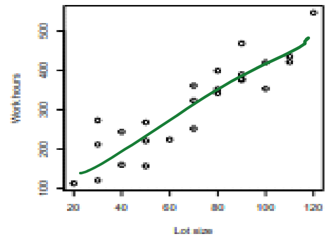
- Not a perfect relation.
- Example: A company produces replacement parts. It produces lots of varying size. The relation between the lot size and work hours is a statistical relation.



Note: There is still a trend but a greater amount of noise

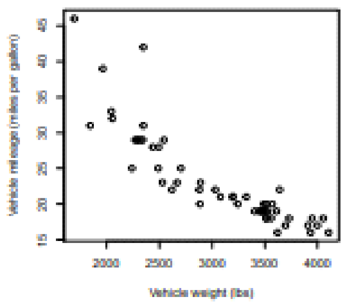
Statistical relation

- Example (contd): There is a relation between X and Y: the higher the lot size, the higher the work hours tend to be.
- Perfect relation?
  No! since there is noise and data points are scattered around the trend.
- Two lots with $X = 40$ have different Y.
- Linear or non-linear statistical relation?
  Linear statistical relation



Statistical relation

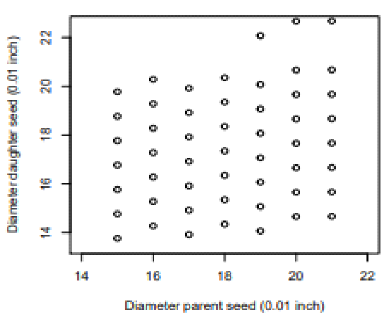
- Example: Weight and mileage for 54 cars.
- Functional or statistical relation?
- Linear statistical relation?



Galton's early considerations of regression

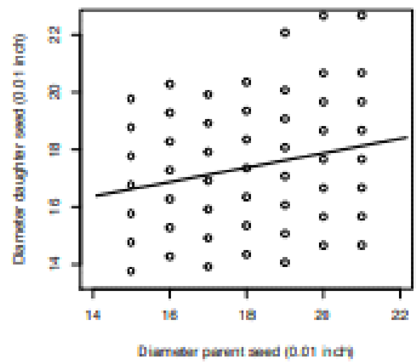
- Sir Francis Galton, English Victorian statistician, sociologist, psychologist, anthropologist, etc.
- Work on inherited characteristics of sweet peas ⇒ initial conceptualization of linear regression.
- In 1975, Galton distributed packets of sweet pea seeds to seven friends who harvested seeds from the new generations of plants and returned them to Galton.
- Galton plotted the diameter of the daughter seeds against the diameter of the mother seeds [Galton, 1894].

Galton's early considerations of regression



Galton's early considerations of regression

- Mean diameter of daughter seeds from a particular diameter of mother seed approximately a straight line with positive slope
  Tendency of diameter of daughter seeds to vary with diameter of mother seeds
- Constant variability for diameter of daughter seeds from a particular diameter of mother seed
  Random scatter around this tendency



Notation and general concepts

- **Model**: mathematical expression to describe the behavior of a random variable of interest
- **Response variable** or **outcome** Y: variable of interest
- **Predictor** or **independent variables** X: know constant variables thought to provide information on the behavior of Y
- Subscript on Y and X identifies the particular unit from which the observation was taken (X_5 for unit 5)
- **Parameters**: control behavior of the model; usually represented by Greek letters (β, σ); unknown constants to be estimated from the sample
- **Linear model**: model linear in the parameters

Note: A model is a representation of reality.
No model is a 100% accurate but can be close to matching reality.

Examples

- Dollar sales of a product sales sold \$2 per unit and number of units sold

$$Y = \beta X$$

- Diameter of daughter seeds and diameter of mother seeds

$$Y = \beta X + \varepsilon$$

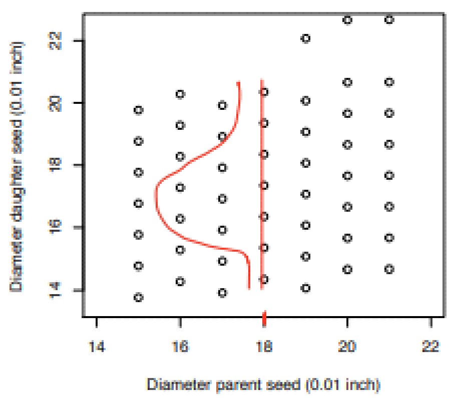
Basic concepts

Two characteristics of a statistical relation:

1. Tendency of Y to vary with X
2. Random scatter around this tendency

In a regression model:

1. The mean of Y vary in a systematic fashion with X
2. Probability distribution of Y for any given value of X



Data collection for regression analysis

Note: observational studies can't conclude cause and effect, only corrolation.

- **Observational study**
  - Investigator has no control over the explanatory variables (X)
  - Limitation: not adequate for cause-and-effect
    A strong association does not necessarily means a cause-and-effect relationship
- **Experiment**
  - Investigator exercises control over the explanatory variables (X) through random assignment
  - Random assignment balances out effect of other variables that might affect Y
  - Gold standard for cause-and-effect conclusions

Example of observational study

Study the relationship between age of employees (X) and number of days of illness last year (Y)

- Observational data because we can't control age or # of sick days-
- An observed association between X and Y does not necessarily imply that X explains Y

- Note: There maybe other factors that we have not looked at.

Example of experiment

Study the relationship between productivity and length of training of analysts working in a bank:

1. 30 analysts considered
2. randomly select 10 analysts that will be trained for 2 week; randomly select 10 other analysts that will be trained for 5 weeks; the 10 remaining will be trained for 8 weeks
3. productivity of the 30 analysts observed for a fixed time after the training

- Experiment because investigators can manipulate the value of x
  - e.g. 8 weeks
  - This could have cause and effect.

Cause-and-effect / Causation

- We observe an association between Y and X
- Does changing one of the variables imply the other to change?
- Mechanisms that can result in an observed association between Y and X:

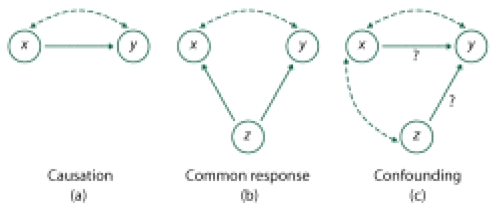


Figure 1: The dashed arrows represent association and the solid ones cause and effect link. The variable x is explanatory, y is response, and z is a lurking variable.

Regression analysis by itself provides no information about causation. Be careful in drawing causal conclusions

Overview of the steps in regression analysis

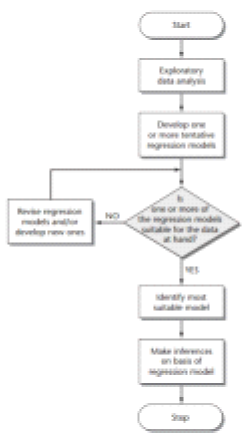


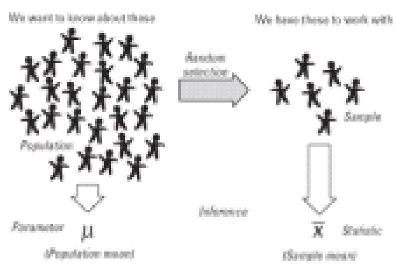
Figure 2: The steps in regression analysis [Kutner et al., 2004, p.14]

Three main purposes of regression analysis

1. **Describe**: describe the relation between diameter of daughter seeds and diameter of mother seeds.
2. **Control**: control the length of training to maximize productivity constrained by costs.
3. **Predict**: predict future demand for a product.

Parameters, estimators, and estimates

- **Parameter**: quantity of interest, quantity describing a population (or model).
  A parameter is a constant ________ (constant/random) quantity.
- **Estimator**: rule for calculating an estimate of parameter.
  An estimator is a random ________ (constant/random) quantity.
- **Estimate**: result of the estimator (for a given sample).
  An estimate is a constant ________ (constant/random) quantity.



Recall

$$\bar{x} = \frac{1}{N} \sum x_i$$

Toluca company example[1]

- Toluca Company produces replacement parts for refrigeration equipment
- Produces lots of varying size
- Cost improvement: find optimal lot size
- Key input: relationship between lot size and labor hours
- Data: lot size X and work hours Y for 25 production runs

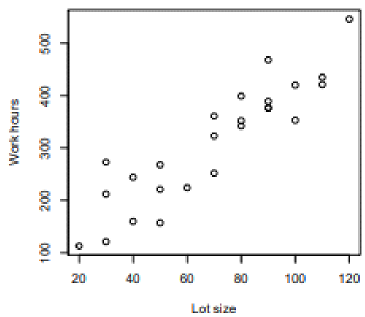
Run i	Lot size X_i	Work hours Y_i
1	80	399
2	30	121
...	...	...
24	80	342
25	70	323

← how much time it takes to produce the items

ex/ 399 hours to create 80 fridges

[1] From [Kutner et al., 2004], page 19

Toluca company example



From the scatter plot:
- looks like a linear model

Simple linear model

Suppose we have n observed pairs (X_i, Y_i), $i = 1, \ldots, n$. The simple linear model is:

$$\boxed{Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i,}$$

where

4 assumptions

- Y_i is the observed value of Y on unit i,
- β_0 and β_1 are parameters,
- X_i is the observed value of X on unit i, and
- ε_i are random errors that have zero mean $\mathrm{E}(\varepsilon_i) = 0$, with common variance $\mathrm{Var}(\varepsilon_i) = \sigma^2$, and pairwise independent.

$\varepsilon_i \perp \varepsilon_j,\ i \neq j$

Simple linear model

Exercise 1

Show that the random errors satisfy

$$E(\varepsilon_i \varepsilon_j) = \begin{cases} 0 & \text{if } i \neq j \\ \sigma^2 & \text{if } i = j \end{cases}$$

Recall Assumptions about random errors

1) $E(\varepsilon_i) = 0$
2) $Var(\varepsilon_i) = \sigma^2$
3) pairwise independent; thus $Cov(\varepsilon_i, \varepsilon_j) = 0 \quad i \neq j$

For this proof there are 2 cases $i = j$ & $i \neq j$

- $i = j$:

wts $E(\varepsilon_i \cdot \varepsilon_i) = E(\varepsilon_i^2) = \sigma^2$

we know: $Var(\varepsilon_i) = \sigma^2$

so $Var(\varepsilon_i) = E(\varepsilon_i^2) - (\underbrace{E(\varepsilon_i)}_{0})^2$, first assumption says $E(\varepsilon_i) = 0$, second assumption says $Var(\varepsilon_i) = \sigma^2$

$\sigma^2 = \sigma^2 - 0$

$\sigma^2 = \sigma^2$

Recall
$Var(x) = E(x^2) - (E(x))^2$

- $i \neq j$: we want to show $E(\varepsilon_i \varepsilon_j) = 0$

$0 = E(\varepsilon_i \varepsilon_j) - \underbrace{E(\varepsilon_i)}_{0} E(\varepsilon_j)$, first assumption

$0 = E(\varepsilon_i \varepsilon_j)$

Recall
$Cov(\varepsilon_i, \varepsilon_j) = 0$

Important features

Simple linear model

constant $\Rightarrow$ understand constant as not random.

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

Parameters are always constant. We don't know them but they are constant

1. The response Y_i is a sum of two terms:
   - A constant term
   - A random term

   The outcome Y_i is random (constant/random)

   y_i is constant + random = random

2. $E(Y_i) = \beta_0 + \beta_1 X_i$, where $E(Y_i)$ is a shortcut for $E(Y_i|X_i)$ the mean of Y when $X = X_i$.

   $E(y_i) = E(\beta_0 + \beta_1 x_i + \varepsilon_i) = E(\beta_0) + E(\beta_1 x_i) + E(\varepsilon_i)$, linearity

   $= \beta_0 + \beta_1 x_i$

   Thus, the functional relationship between the true mean of Y_i and X_i is a straight line with intercept β_0 and slope β_1

Important features

Simple linear model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

3. $\mathrm{Var}(Y_i) = \sigma^2$, where $\mathrm{Var}(Y_i)$ is a shortcut for $\mathrm{Var}(Y_i|X_i)$ the ~~mean~~ variance of Y when $X = X_i$.

$\mathrm{Var}(y_i) = \mathrm{Var}(\underbrace{\beta_0 + \beta_1 X_i}_{\text{constant}} + \varepsilon_i) = \mathrm{Var}(\varepsilon_i) = \sigma^2$

Recall

- Variance is not linear.
- you could use expectation but too hard.
- $\beta_0 + \beta_1 x_i$ is a constant

$\mathrm{Var}(\text{constant} + \text{r.v}) = \mathrm{Var}(\text{r.v})$

4. The outcomes Y_i are pairwise independent because the errors ε_i are pairwise independent.

i.e $y_i \perp$ from y_j when $i \neq j$

Reminder: normal distribution

A random variable X is normal if its probability density function is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\},$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ are the parameters of the distribution. We say that X is normally distributed with mean $\mathrm{E}(X) = \mu$ and variance $\mathrm{Var}(X) = \sigma^2$ and we write

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

Additional Assumption

Sometimes we make an additional assumption that random error are normally distributed.

- This assumption is only used when explicitly stated.

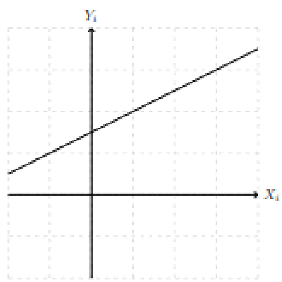
Simple linear model with normal errors

- The random errors are sometimes assumed to be normally distributed.
- Simple linear model with normal errors:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i,$$

where
  - β_0 and β_1 are parameters,
  - ε_i are independently and identically distributed (i.i.d.) with normal distribution with mean 0 and variance σ^2.
- In what follows, we suppose a simple linear model (errors not necessarily normal) unless otherwise specified.

Interpretation of the regression parameters



- If the scope of the model includes $X = 0$, the intercept β_0 is the mean of Y when $X = 0$ (no meaning otherwise)
- The slope β_1 is the change in the mean of Y per unit increase of X

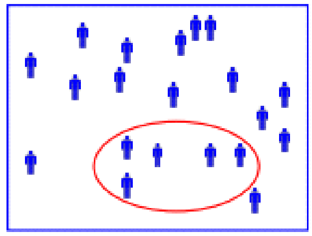
Estimation of the parameters

- Postulated model:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

- Observed values (X_i, Y_i)
- Parameters β_0 and β_1 unknown and to be estimated from the sample.
- Two estimation methods:
  1. Least squares
  2. Maximum likelihood

  $\Rightarrow$ Estimates $\widehat{\beta}_0$ and $\widehat{\beta}_1$



- See the model as describing the population
- select sample at random
- observe x & y values
- estimate β_0 and β_1 from the sample

$\wedge$ ← hat means estimate, not true value.

Method of least squares

Simple linear model:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

- Parameters β_0 and β_1 to be estimated from the data.
- **Goal**: find the best estimates $\widehat{\beta}_0$ and $\widehat{\beta}_1$ given the data.
- What does *best* mean?
- **Least square**: best by criterion

$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n} (\underbrace{Y_i - \beta_0 - \beta_1 X_i}_{y_i - E(y_i)})^2$$

$\underline{y_i - \beta_0 - \beta_1 x_i}$ is the deviation of Y_i from its expected value.

- Least square estimators of β_0 and β_1: $\widehat{\beta}_0$ and $\widehat{\beta}_1$ that minimize criterion Q.

We want to minimize the sum of the errors to get the best one

Least square estimators

Find $\widehat{\beta}_0$ and $\widehat{\beta}_1$ that minimize criterion

$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)^2$$

given the data.

1. Write the normal equations (derivatives of Q set to 0).
2. Find the critical points (solution of the normal equations).
3. Determine whether the critical point is a maximum or a minimum (we will skip this step).

1) $\frac{\partial Q}{\partial \beta_0} = 2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) \quad (-1)$

$0 = -2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)$

$\frac{\partial Q}{\partial \beta_1} = 2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)\cdot(-x_i)$

$0 = 2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)\cdot(-x_i)$

2) β_0 : Critical points

$0 = -2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)$

$0 = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n}\beta_0 - \beta_1\sum_{i=1}^{n} x_i$

$\sum_{i=1}^{n}\beta_0 = \sum_{i=1}^{n} y_i - \beta_1\sum_{i=1}^{n} x_i$

$n\beta_0 = \sum_{i=1}^{n} y_i - \beta_1\sum_{i=1}^{n} x_i$

$\beta_0 = \frac{\sum_{i=1}^{n} y_i}{N} - \beta_1\frac{\sum_{i=1}^{n} x_i}{N}$

$\hat{\beta}_0 = \bar{y} - \beta_1\bar{x}$

β_1: Critical point

$0 = 2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)\cdot(-x_i)$

$0 = \sum_{i=1}^{n}(y_i x_i - \beta_0 x_i - \beta_1 x_i^2)$

$0 = \sum_{i=1}^{n} y_i x_i - \beta_0\sum_{i=1}^{n}(x_i) - \beta_1\sum_{i=1}^{n} x_i^2$

$\beta_1\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i x_i - \beta_0\sum_{i=1}^{n} x_i$

$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - \beta_0\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2}$

$\hat{\beta}_1, \hat{\beta}_0$ are r.v / estimators.

Least square estimators

Least square estimators of β_1 and β_0:

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n} X_i Y_i - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2},$$

$$\widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1\overline{X},$$

where

- $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and
- $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

are the sample mean of Y and X, respectively.

January 11, 2019 8:48 AM

Least square estimators

Exercise 2

Show that

$$\boxed{\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}.}$$

Start from: $\dfrac{\hat{\sum_{i=1}^{n}}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$ and go to $\hat{\beta}_1$.

Regression equation

Model: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ $E(y_i) = \beta_0 + \beta_1 x_i$; True mean of y when $X = x$

to understand as $E(y_i \mid x_i)$

$$\boxed{\begin{array}{l}\text{Regression equation or fitted regression line} \\ \widehat{Y} = \widehat{\beta}_0 + \widehat{\beta}_1 X\end{array}}$$

where $\widehat{Y}$ is the estimated mean of the response variable at level X of the explanatory.

Gauss-Markov theorem

Theorem 1
Consider the simple linear model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i.$$

Suppose that the following assumptions concerning the random errors (called Gauss-Markov assumptions) are satisfied:

- *They have mean zero:* $E(\varepsilon_i) = 0$,
- *They are homoscedastic:* $\mathrm{Var}(\varepsilon_i) = \sigma^2 < \infty$, *and*
- *There are uncorrelated* $Cov(\varepsilon_i, \varepsilon_j) = 0, \forall i \neq j$.

Then the least squares estimators $\widehat{\beta}_0$ *and* $\widehat{\beta}_1$ *are unbiased and have minimum variance among all unbiased linear estimators.*

Proof of the Gauss-Markov theorem

Step 1 Exercise: Prove that the least squares estimators are unbiased, i.e. prove that

$$\mathrm{E}(\widehat{\beta}_1) = \beta_1 \quad \text{and} \quad \mathrm{E}(\widehat{\beta}_0) = \beta_0$$

Step 2 To be proven later: The least squares estimators have minimum variance among all unbiased linear estimators.

CHECK NEXT

PAGE !

January 16, 2019 8:40 AM

Parameter θ

Estimator $\hat{\theta}$

$\hat{\theta}$ is an <u>unbiased</u> estimator of θ

if $E(\hat{\theta}) = \theta$

Otherwise, $\hat{\theta}$ is biased and its bias is $E(\hat{\theta}) - \theta$

Prof asks to prove stuff if shit is biased

$$\hat{\beta}_1 = \frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}$$

$$E(\hat{\beta}_1) = E\left(\frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}\right)$$, # use linearity dawg.

Note

$E\left(\frac{x}{y}\right) \neq \frac{E(x)}{E(y)}$

usually!

$$= \frac{1}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}\left\{E(\sum x_i y_i) - E\left(\frac{1}{n}\sum x_i \sum y_i\right)\right\}$$

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

$E(y_i) = \beta_0 + \beta_1 x_i$

$$= \frac{1}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}\left\{\sum x_i E(y_i) - \frac{1}{n}\sum x_i \sum E(y_i)\right\}$$

$$= \frac{1}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}\left\{\underbrace{\sum x_i(\beta_0 + \beta_1 x_i)}_{\beta_0 \sum x_i + \beta_1 \sum x_i^2} - \frac{1}{n}\sum x_i \underbrace{\sum(\beta_0 + \beta_1 x_i)}_{n\beta_0 + \beta_1 \sum x_i}\right\}$$

$$= \frac{\beta_1 \sum x_i^2 - \frac{1}{n}\beta_1(\sum x_i)^2}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}$$

$$= \beta_1$$

Conclusion

$E(\hat{\beta}_1) = \beta_1$, i.e

$\hat{\beta}_1$ is an unbiased estimator of β_1.

January 16, 2019 8:48 AM

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

$E(\hat{\beta}_0) = E(\bar{y}) - E(\hat{\beta}_1)\bar{x}$, linearity of expectation daug

Recall $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}$$

$E(\bar{y}) = E(\frac{1}{n}\sum y_i) = \frac{1}{n}\sum(y_i) = \frac{1}{n}\sum(\beta_0 + \beta_1 x_i)$

linearity of E

$= \frac{1}{n} n\beta_0 + \frac{1}{n}\beta_1 \sum x_i$

$= \beta_0 + \beta_1 \bar{x}$

$E(\hat{\beta}_1) = \beta_1 \rightarrow$ because $\hat{\beta}_1$ is an unbiased estimator.

Conclusion: $E(\hat{\beta}_1) = \beta_0$, i.e

$\hat{\beta}_0$ is an unbiased estimator of β_0

Toluca company example

Using R, we find:

$$\sum_{i=1}^{n} X_i = 1750 \quad \sum_{i=1}^{n} Y_i = 7807 \quad \sum_{i=1}^{n} X_i Y_i = 617180$$

$$\sum_{i=1}^{n} X_i^2 = 142300 \quad n = 25$$

Exercise 3

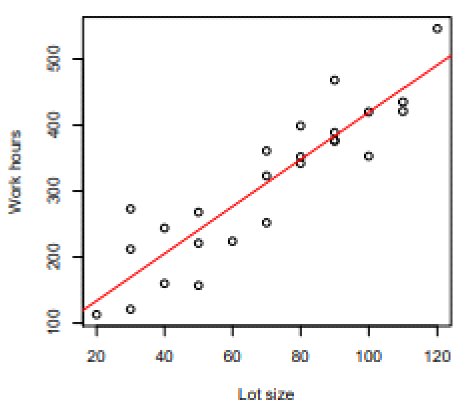
1. *Compute the least squares estimates of β_1 and β_0.*
2. *What is the regression equation?*
3. *Interpret the parameters.*

Take formula in plug in

Test is more proof based

Toluca company example

check if 0 is in the range otherwise don't interpret the intercept because x = 0 is not in the range of the observed x values.



$\hat{\beta}_1 = 3.5702$

$\hat{\beta}_0 = 62.37$

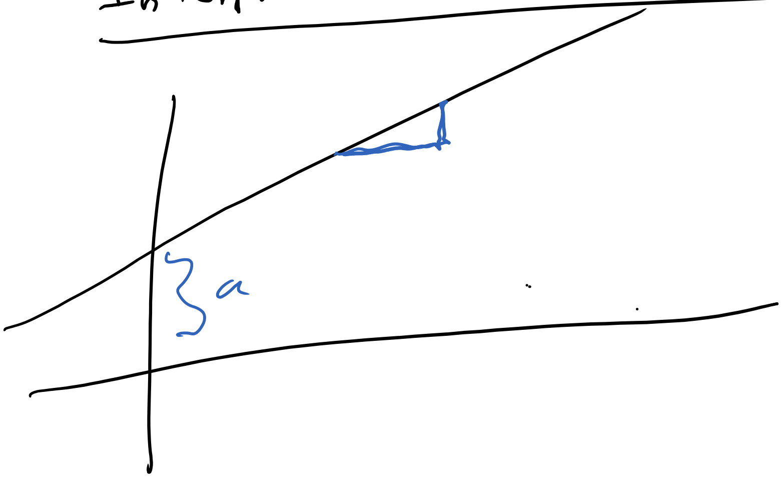
$\hat{y} = 62.37 + 3.5702\, x$

estimation mean of y.

The estimated work time increases by 3.57 hours when the lot size increases by 1 unit one additional unit is produced.

January 16, 2019 9:00 AM

Interpretation of the regression parameters



$y = a + bx$

a: obtained y when $x = 0$

$b = \frac{\Delta y}{\Delta x}$

if $\Delta x = 1$

$b = \Delta y$

b is the difference in Y when X increases by 1 unit.

Copy down the slides, later k! :)

Toluca company example: R output

y X Hours $= \beta_0 + \beta_1 \text{Size} + \varepsilon_i$

```
Call:
lm(formula = Hours ~ Size, data = toluca)
```

name of the data set

linear model

```
Residuals:
    Min      1Q  Median      3Q     Max
-83.876 -34.088  -5.982  38.826 103.528

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   62.366     26.177   2.382   0.0259 *
Size           3.570      0.347  10.290 4.45e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 48.82 on 23 degrees of freedom
Multiple R-squared:  0.8215,Adjusted R-squared:  0.8138
F-statistic: 105.9 on 1 and 23 DF,  p-value: 4.449e-10
```

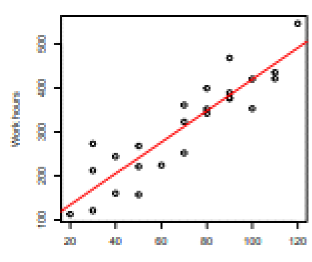
$\hat\beta_0$ β_1

Preliminary exercise: Toluca company example

- Regression equation (or fitted regression line)

$$\hat{Y} = 62.37 + 3.5702X,$$

  where
  - X is the lot size, and
  - Y is the work hours.



What is:

- The predicted work hours for a new production run for a lot size of 60?

  $\hat{y} = 62.37 + 3.5402 \cdot 60 = 276.582$

- The estimated population mean work hours for a lot size of 60?

  $\hat{y} = 62.37 + 3.5402 \cdot 60 = 276.582$

Predicted values and residuals

- Fitted regression line:

$$\widehat{Y} = \widehat{\beta}_0 + \widehat{\beta}_1 X$$

- Fitted value: value of Y computed from the regression line:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i.$$

  Fitted value $\widehat{Y}_i$ used as:
  - **Prediction** of the value of Y for particular value X_i of X. Sometimes written $\widehat{Y}_{pred_i}$.
  - **Estimate** of the population mean of Y for particular value X_i of X.

- A residual is the deviation of the observed value Y_i from the fitted value $\widehat{Y}_i$

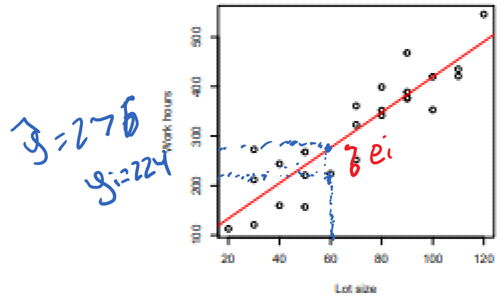
$$e_i = Y_i - \widehat{Y}_i$$

Toluca company example

For production run 6, the lot size was 60 and 224 work hours were required.

Exercise 4

1. *What is the fitted value for this observation?*
2. *What is the residual?*
3. *Where can we read the observed work hours (Y), the fitted work hours, and the residual in the scatterplot?*



Exercises

- From the textbook[2]
  - 1.20
  - 1.21
- From the slides[3]
  - Slide 26
  - Slide 37
  - Slide 40
  - Slide 41
  - Slide 47

Go do this or slap yourself!

Karan style slide questions dawg.

[2] Solutions in the student manual (CD) provided with the textbook
[3] Partial solutions posted on Quercus

January 11, 2019 12:34 PM

Proofs

1) $\sum e_i = \sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n}\hat{\beta}_0 - \sum_{i=1}^{n}\hat{\beta}_1 x_i = \sum_{i=1}^{n} y_i - n\hat{\beta}_0 - \hat{\beta}_1 \sum x_i = \sum y_i - \sum y_i + \hat{\beta}_1 \sum x_i - \hat{\beta}_1 \sum x_i = 0$

$e_i = y_i - \hat{y}_i = y_i - (\hat{\beta}_0 - \hat{\beta}_1 x_i)$

sub $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ $= \sum y_i - n\bar{y} + n\hat{\beta}_1 \bar{x} - \hat{\beta}_1 \sum x_i$

3) To prove: $\sum y_i = \sum \hat{y}_i$

From ex 1/ we know $\sum e_i = \sum(y_i - \hat{y}_i) = 0 \iff \sum y_i - \sum \hat{y}_i = 0$

$\Rightarrow \sum y_i = \sum \hat{y}_i$