Look at syllabus for details Page!

Five Quizzes 25%
Midterm Exam 30%
Final exam 45%

Office hours: 4:00 - 6:00
IC340

1. Introduction to Linear Programming

1.1 Introduction of Linear programming (LP)

To understand best how to apply a mathematical theory to solution for some practical problem:

- ***Recognition of the problem***
- ***Formulation of a mathematical model***
- ***Solution of the mathematical problem***
- ***Translation of the results back into the context of the original problem***

Optimization via (Freshman) Calculus

Example: how to enclose the largest area: Suppose we have 100 meters of fencing, and want to enclose a rectangular area up against a long straight wall. How big an area can we enclose?

Recall Area $= L \times W = A(x)$, $0 \le x \le 50$

$= x(100-2x)$

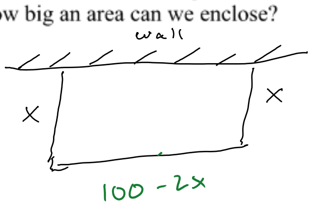
$= 100x - 2x^2$

$A'(x) = 100 - 4x \triangleq 0 \Rightarrow x = 25$ cp

$A''(x) = -4 < 0$ $\quad x = 25$ max

Also $A(0) = A(50) = 0$

$\therefore A(25) = 25(100-50) = 1250\ m^2$ is the biggest area we may enclose.



1.2 Mathematical Models

Example 1.1: A produce grower is purchasing fertilizer containing three nutrients, A, B, and C. The minimum needs are 160 units of A, 200 units of B, and 80 units of C. There are two popular brands of fertilizer on the Market.
Fast Grow: contains 3 units of A, 5 units of B, and 1 units of C, costing $8 a bag.
Easy Grow: contains 2 units each nutrient costing $6 a bag.
If the grower wishes to minimize cost while still maintaining the nutrients required, how many bags of each brand should be bought?

Set up x, y, z ... all the variables
Assume that we are going to buy x bags of fast growth and y bags of easy grow

Cost $= 8x + 6y \longrightarrow$ what we want to minimize

Make table if possible

	A	B	C
fast grow → F.G x	3	5	1
easy grow → E.G y	2	2	2
	160	200	80

$3x+2y \ge 160$ $\quad 5x+2y \ge 200$ $\quad x+2y \ge 80$

Model: min $C = 8x + 6y$ subject to

$3x + 2y \ge 160$
$5x + 2y \ge 200$
$x + 2y \ge 80$
$x, y \ge 0$

1

Example 1.2: A furniture maker has a line of four types of desks. They vary in the manufacturing process and their profitability. The furniture maker has available 6000 hours of time in the carpentry shop each six months, and 4000 hours of time in the finishing shop. Each desk of type 1 requires 4 hours of carpentry and 1 hour of finishing. Each desk of type 2 requires 9 hours of carpentry and 1 hour of finishing. Each desk of type 3 requires 7 hours of carpentry and 3 hours of finishing. Each desk of type 4 requires 10 hours of carpentry and 40 hours of finishing. The profit is $12 for each desk of type 1, $20 for each desk of type 2, $28 for each desk of type 3, $40 for each desk of type 4. How should the production be scheduled to maximize the profit?

4 variables

Type	Carpentry	Finishing	Profit $\underbrace{\$\ \$}$
1 x_1	4	1	\$12
2 x_2	9	1	\$20
3 x_3	7	3	\$28
4 x_4	10	40	\$40
	6000	4000	

Max $P = 12x_1 + 20x_2 + 28x_3 + 40x_4$

s.t $4x_1 + 9x_2 + 7x_3 + 10x_4 \le 6000$

$x_1 + x_2 + 3x_3 + 40x_4 \le 4000$

$x_i \ge 0 \quad i = 1, 2, 3, 4 \ldots$

Example 1.3: A woman has $10,000 to invest. Her broker suggests investing in two bonds, A and B. Bond A is a rather risky bond with an annual yield of 10%, and bond B is rather safe bond with an annual yield of 4%. After some consideration, she decides to invest at most $6000 in bond A, at least $3000 in bond B, and the total annual yield should be better than 6%. How should she invest her $10000 in order to maximize her annual yield?

① $x + y \le 10,000$

② $x + y = 10,000$

$\boxed{x \cdot 1\% + y \cdot \%}$

max: $\frac{x}{10000} \cdot 10\% + \frac{y}{10000} \cdot 4\%$ s.t $x \le 6000$

≥ 3000

$\frac{x}{10000} \cdot 10\% + \frac{y}{10000} \times 4\% \ge 6\%$

$x + y = 10,000$

$x, y \ge 0$

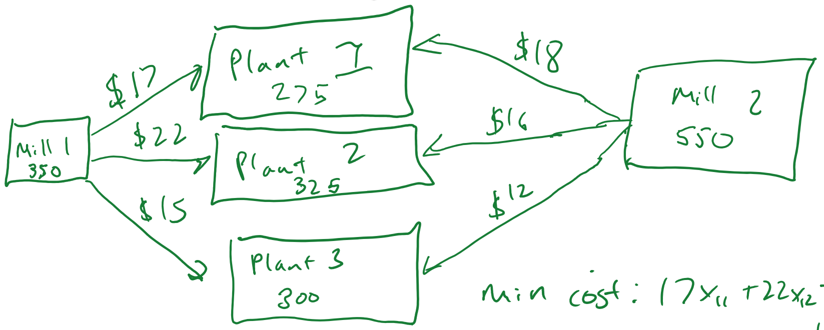
Similar to first and second example. You write this form!

Example 1.4: (The diet problem) A nutritionist is planning a menu consisting of two main foods A and B. Each ounce of A contains 2 units of fat, one unit of carbohydrates, and 4 units of protein. Each ounce of B contains 3 units of fat, 3 units of carbohydrates, and 3 units of protein. The nutritionist wants the meal to provide at least 18 units of fat, at least 12 units of carbohydrates, and at least 24 units of protein. If an ounce of A costs $2 and an ounce of B costs $2.5, how many ounces of each food should be served to minimize the cost of the meal yet satisfy the nutritionist's requirements?

Example 1.5: A paper manufacturer having two mills must supply weekly three printing plants with newsprint. Mill 1 produces 350 tons of newsprint a week and Mill 2 550 tons. Plant 1 requires 275 tons/week, plant 2 325 tons, and plant 3 300 tons. The shipping costs, in dollars per ton, are as follows:

	Plant 1	Plant 2	Plant 3
Mill 1	17	22	15
Mill 2	18	16	12

The problem is to determine how many tons each mill should ship to each plant so that the total transportation cost is minimal.



x_{ij}

$i = 1, 2$ mills

$j = 1, 2, 3$ plants

min cost: $17x_{11} + 22x_{12} + 15x_{13} + 18x_{21} + 16x_{22} + 12x_{23}$

subjective to:

$x_{11} + x_{12} + x_{13} \le 350$

$x_{21} + x_{22} + x_{23} \le 550$

$x_{11} + x_{22} \ge 275$

$x_{12} + x_{22} \ge 325$

$x_{13} + x_{23} \ge 300$

$x_{ij} \ge 0$

$i = 1, 2,\ j = 1, 2, 3$

General LP Problem:

Can we convert it into LP Problem in canonical form.

1.3 Linear programming problems

General LP problem

For value of $x_1, x_2, \ldots, x_n$,

maximize or minimize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le (\ge)(=) b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le (\ge)(=) b_2 \\
\vdots \qquad \vdots \qquad\quad & \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le (\ge)(=) b_m
\end{aligned}
$$

A LP problem in *__standard form__*

For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\
x_j \ge 0, j = 1, 2, \ldots, n. &
\end{aligned}
\tag{1}
$$

← variables should be non-negative

A LP problem in *__canonical form__*

difference between canonical form and standard?

make it equations

For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\
x_j \ge 0, j = 1, 2, \ldots, n. &
\end{aligned}
\tag{2}
$$

How to convert a GLP to a standard LP:

1) Minimization problem as a maximization problem

Min $z = \ldots$ Max $-z = \ldots$

2) Reversing an inequality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n \ge b$$

$$-k_1x_1 - k_2x_2 - \ldots - k_nx_n \le -b$$

3) Changing an equality to an inequality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n = b$$

$$\begin{cases} k_1x_1 + k_2x_2 + \cdots + k_nx_n \le b \\ k_1x_1 + k_2x_2 + \cdots + k_nx_n \ge b \end{cases}$$

4) unconstrained variables

If there is no nonnegative constrain on x_j,

Let $x_j^+, x_j^- \ge 0$

$x_j = x_j^+ - x_j^-$

5) Changing an inequality to an equality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n \le b$$

$k_1x_1 + k_2x_2 + \cdots + k_nx_n + s = b$

$s \ge 0$ slack variable

To convert a standard LP problem (1) to a canonical form (2) of the problem: For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_nx_n$

subject to the restrictions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \qquad (1)$$

$x_j \ge 0, j = 1, 2, \ldots, n.$

Example 1.6: Let LP be min f(x, y) = 3x+2y subject to the constraints

$$\begin{cases} y \ge x + 2 \\ y \le -x + 3 \\ y \ge \dfrac{2}{5}x + \dfrac{1}{5} \end{cases}$$

convert it to a standard LP problem then a canonical form.

Min $f(x,y) = 3x + 2y$

Max $z = -3x - 2y$

$x - y \le -2$

$x + y \le 3$

$\frac{2}{5}x - y \le -\frac{1}{5}$

$2x - 5y \le -1$

Set $x = x^+ - x^-$

$y = y^+ - y^-$

$x^+, x^- \ge 0$ & $y^+, y^- \ge 0$ the LP in the standard form. Max $z = -3x^+ + 3x^- - 2y^+ + 2y^-$

s.t. $x^+ - x^- - y^+ + y^- \le -2$

$x^+ - x^- + y^+ - y^- \le 3$

$2x^+ - 2x^- - 5y^+ + 5y^- \le -1$

$x^+, x^-, y^+, y^- \ge 0$

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the LP in the Canocical form

max

Example 1.7: Convert the following LP problem into standard form:

Max $\leftarrow$ Min $z = -x_1 + 2x_2 - 3x_3$

$z' = x_1 - 2x_2 + 3x_3$, s.t. $x_1 - x_2 + x_3 = -2$

$|x_1 - x_3| \le 1$

$x_2 \ge 0$

① $x_1 - x_2 + x_3 \le -2$

$x_1 - x_2 + x_3 \ge -2$

$\Downarrow$

② $-x_1 + x_2 - x_3 \le 2$

$-1 \le x_1 - x_3 \le 1$

$\downarrow$

$x_1 - x_3 \ge -1 \rightarrow$ ③ $-x_1 + x_3 \le 1$

④ $x_1 - x_3 \le 1$

Set $x_1 = x_1^+ - x_1^-$ $\quad x_1^+ \ge 0$

$x_3 = x_3^+ - x_3^-$ $\quad x_3^+ \ge 0$

Then max

$z' = x_1^+ - x_1^- - 2x_2 + 3x_3^+ - 3x_3^-$

s.t.

$x_1^+ - x_1^- - x_2 + x_3^+ - x_3^- \le -2$

$-x_1^+ + x_1^- + x_2 - x_3^+ + x_3^- \le 2$

$-x_1^+ + x_1^- + x_3^+ - x_3^- \le 1$

$x_1^+ - x_1^- - x_3^+ + x_3^- \le 1$

$x_1^\pm, x_2, x_3^\pm \ge 0$

Exercise

M2. Set $x_1 = x_1' - y$

$x_3 = x_3' - y$

$y \ge 0$

$x_1', x_3' \ge 0$

Set $x_1 = x_1' - y$

$x_2 = x_2' - y$

1.4 Matrix notation

The standard form LP: For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ $= c^T x$

subject to the restrictions

$$Ax \le b \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 = b \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \tag{1}$$

$x_j \ge 0, j = 1, 2, \ldots, n.$

max $z'' = x_1' - y - 2x_2 + 3x_3' - 3y$

$= x_1' - 2x_2 + 3x_3' - 4y$

s.t $x_1' - x_2 + x_3' - 2y \le -2$

$-x_1' + x_2 - x_3' + 2y \le 2$

$-x_1' + x_3' \le 1$

$x_1' - x_3' \le 1$

$x_1', x_2, x_3', y \ge 0$

Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The standard form LP in matrix form

maximize $z = c^T x$ for x

subject to $Ax \le b$

$x \ge 0$

The canonical form LP in matrix form

Maximize $z = c^T x$ for x

subject to $Ax = b$

$x \ge 0$

The LP in canonical form

max $z'' = x_1' - 2x_2 + 3x_3' - 4y$

s.t

$x_1' - x_2 + x_3' - 2y + s_1 = -2$

$-x_1' + x_2 - x_3' + 2y + s_2 = 2$

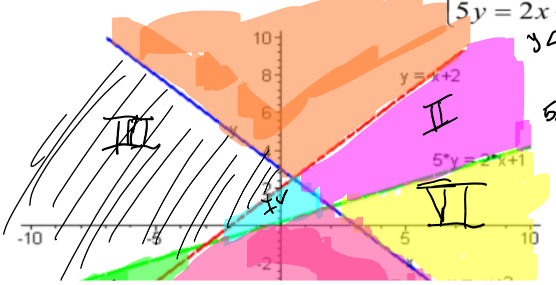
$-x_1' + x_3' + s_3 = 1$

$x_1' - x_3' + s_4 = 1$

$x_1', x_2, x_3', y, s_i \ge 0, i = 1, 2, 3, 4$

1.5 Theorem of LP problems

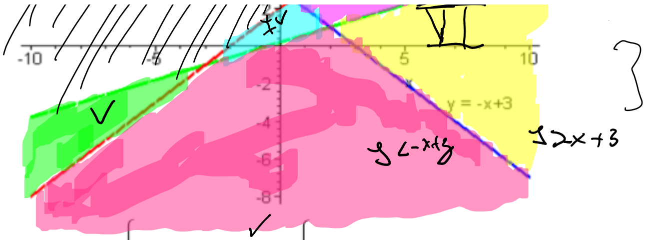
Example1.8: Consider three different non parallel $\begin{cases} y = x + 2 \\ y = -x + 2 \\ 5y = 2x + 1 \end{cases}$ lines



$y < x + 2$

$5y > 2x + 1$? region

? work on it later



} work on it later dude

$$\text{I.}\begin{cases} y \ge x+2 \\ y \ge -x+3 \\ y \ge \frac{2}{5}x+\frac{1}{5} \end{cases} \quad \text{II.}\begin{cases} y \le x+2 \\ y \ge -x+3 \\ y \ge \frac{2}{5}x+\frac{1}{5} \end{cases} \quad \text{III.}\begin{cases} y \ge x+2 \\ y \le -x+3 \\ y \ge \frac{2}{5}x+\frac{1}{5} \end{cases}$$

$$\text{IV.}\begin{cases} y \le x+2 \\ y \le -x+3 \\ y \ge \frac{2}{5}x+\frac{1}{5} \end{cases} \quad \text{V.}\begin{cases} y \ge x+2 \\ y \le -x+3 \\ y \le \frac{2}{5}x+\frac{1}{5} \end{cases} \quad \text{VI.}\begin{cases} y \le x+2 \\ y \ge -x+3 \\ y \le \frac{2}{5}x+\frac{1}{5} \end{cases} \quad \text{VII.}\begin{cases} y \le x+2 \\ y \le -x+3 \\ y \le \frac{2}{5}x+\frac{1}{5} \end{cases}$$

A *feasible solution* to a LP problem

$x_{n+1} = s_1$, $x_{n+2} = s_2$, $x_{n+m} = s_m$

$\{x_1, x_2, \ldots, x_n\}$ is a feasible solution of (1)

$\leftrightarrow$ $\{x_1, x_2, \ldots, x_{n+m}\}$ is a feasible solution of (5)

feasible region = {All the feasible solutions}

If a feasible region can be contained within a circle then it is bounded. Otherwise it is called unbounded. If it contains at least one point then it is non empty, otherwise it is called empty.

Objective function: a function we wish to maximized or minimized under some system of constrains

Optimum solution: a solution to the system of constrains that gives the maximum or minimum value of the objective function.

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Linear Programming Theorem (LP):

Let f be a linear function. Let U be a nonempty region in R^2 such that U is defined by linear inequalities and it includes its boundaries.

a) If U is bounded then f has a maximum and a minimum on U and these values occur at corner points of U.

b) If U is unbounded and if f has a maximum or a minimum then this occurs at a corner point of U.

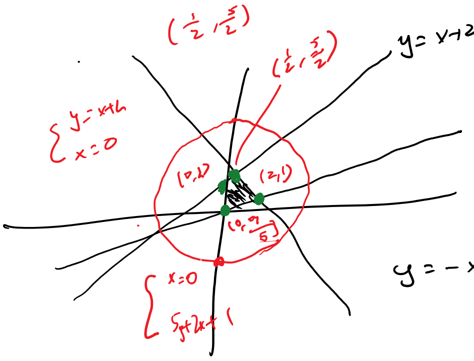
1.6 Graphic solution of LP problems

Example 1.9: Find the maximum and minimum of f(x, y) = 3x+2y subject to the constraints

$$\begin{cases} y \le x+2 \\ y \le -x+3 \\ y \ge \frac{2}{5}x + \frac{1}{5} \end{cases} \quad \text{with } x \ge 0 \text{ and } y \ge 0.$$

$\begin{cases} y = x+2 \\ y = -x+3 \end{cases}$

$\left(\frac{1}{2}, \frac{5}{2}\right)$



$\begin{cases} y = -x+3 \\ 5y = 2x+1 \end{cases}$

$f\left(\frac{1}{2}, \frac{5}{2}\right) = 3\cdot\frac{1}{2} + 2\cdot\frac{5}{2} = \frac{13}{2}$

$f(0,2) = 3\cdot 0 + 2(2) = 4$

$f(2,1) = 3\cdot 2 + 2\cdot 1 = 8$

$f\left(0, \frac{1}{5}\right) = 3\cdot 0 + 2\cdot\frac{1}{5} = \frac{2}{5}$

$f = 8$ at $(2,1)$ is max value

$f = \frac{2}{5}$ at $\left(0, \frac{1}{5}\right)$ is the min value

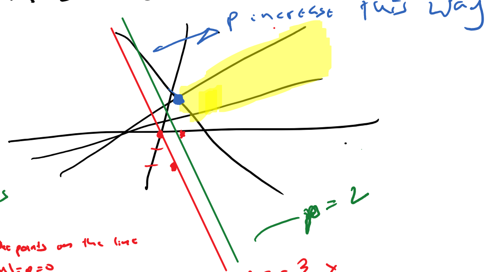
How about $f(x,y) = 3x+2y$ subject to the constraint

$\begin{cases} y \le x+2 \\ y \ge -x+3 \\ y \ge \frac{2}{5}x + \frac{1}{5} \end{cases}$ with $x \ge 0$, $y \ge 0$

set $P = 3x+2y$ ← we want to maximize this

$y = -\frac{3}{2}x + \frac{P}{2}$

$P = 0$, $y = -\frac{3}{2}x$



All the points on the line $f(x,y) = P = 0$ - isoprofit

A family of lines are parallel to the isoprofit line

$P = 2 \quad y = -\frac{3}{2}x + 1$

$P = 10 \quad y = -\frac{3}{2}x + 5$

Since no bound there is no max ∴ infinite

min val is $f\left(\frac{1}{2}, \frac{5}{2}\right) = \frac{13}{2}$

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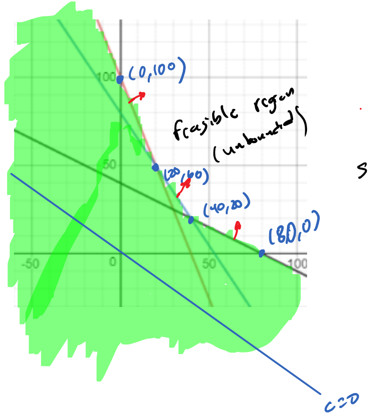
Example 1.1:

I'm too lazy to draw



min $c = 8x + 6y$

s.t. $3x + 2y \geq 160$

$5x + 2y \geq 200$

$x + 2y \geq 80$

$x, y \geq 0$

set $8x + 6y = c$

$y = -\frac{8}{6}x + \frac{c}{y}$

$c = 0$

is profit line

$c = 0 \quad y = -\frac{4}{3}x$

$\therefore$ the optimal solution is $C(40, 20)$

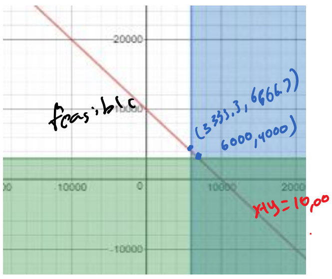
$= 8 \cdot 40 + 6 \cdot 20$

$= \$440$

$\therefore$ the grower should buy 40 F.G and 20 E.G to minimize the cost.

check: $C(20, 50) = 8 \cdot 20 + 6 \cdot 50 = \$460$

Example 1.4:



$x + y = 10\,000$

s.t. $\begin{cases} x \leq 6000 \\ y \geq 3000 \\ \frac{x \cdot 10\% + y \cdot 4\%}{10000} \geq 6\% \end{cases}$

$x, y \geq 0$

max $AT = \frac{x \cdot 10\% + y \cdot 4\%}{10\,000}$

$10x + 4y = 60000$

$x + 4y = 10000$

$6x = 50\,000$

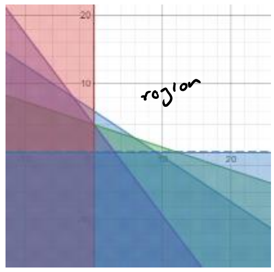
$x = 3333.3$

$y = 6666.7$

$AY_1 = \frac{3333.3 \times 10\% + 6666.7 \times 4\%}{10{,}000}$

$AY = \frac{6000 \times 10\% + 4000 \times 4\%}{10{,}000}$ max

Example 1.3:

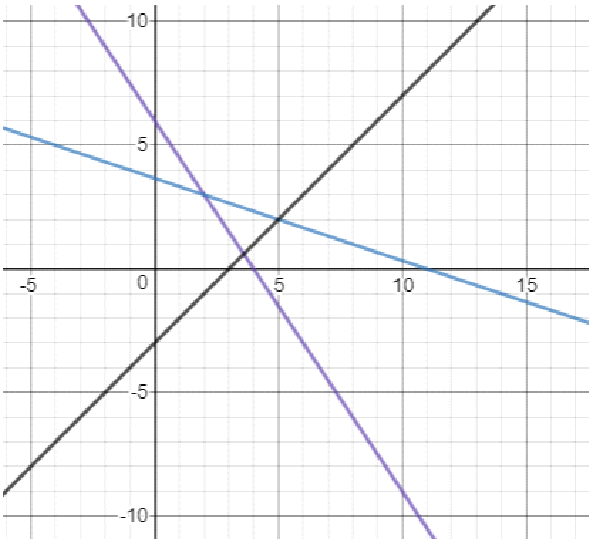


Example 1.10: Minimize Z = 3x + 9y

Subject to

$$\begin{cases} y \le -\dfrac{3}{2}x + 6 \\ y \ge -\dfrac{1}{3}x + \dfrac{11}{3} \\ y \ge x - 3 \\ x, y \ge 0 \end{cases}$$

$\frac{3}{2}x + y \le 6$



$Z(0,6) = 3 \cdot 0 + 9 \cdot 6 = 54$

$Z(2,3) = 3 \cdot 2 + 9 \cdot 3 = 33$

$Z(0, \frac{11}{3}) = 3 \cdot 0 + 9 \cdot \frac{11}{3} = 33$

$\therefore$ min $Z = 33$ at line segment on $y = -\frac{1}{3}x + \frac{11}{3}$ between $(0, \frac{11}{3})$ and $(2,3)$

1.7 Geometry of LP problems

Geometry of a constraint of a LP problem:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

or $a^T x \leq b_i$ where $a^T = [a_{i1}, a_{i2}, \cdots, a_{in}]$

The set of points $x = (x_1, x_2, \ldots, x_n)$ in R^n that satisfy this constraint is called a

________________________.

The set of points $x = (x_1, x_2, \ldots, x_n)$ in R^n that satisfy $a^T x = b_i$ is called a _____________.

A hyperplane is a ______________of a closed half-space.

The set of feasible solutions to a LP problem is the intersection of all the closed half-spaces determined by the constraints.

Example 1.11: $\begin{cases} x + y + z \geq 1 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$

Geometry of the objective function

An objective function

Let k be a constant.

Geometrically, the optimal solution is the hyperplane that

Geometry of the set of feasible solutions

Let x_1 and x_2 be feasible solutions.
The *line segment* connecting x_1 and x_2

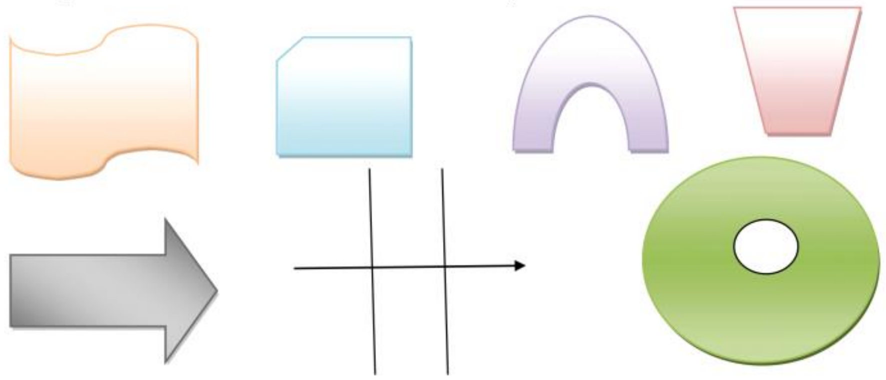
$$= \{ x \in R^n | \, x = \lambda x_1 + (1 - \lambda) x_2 , \; 0 \le \lambda \le 1 \}$$

If $a^T x \le b_i$ is a constraint of the problem, and $a^T x_1 \le b_i$ and $a^T x_2 \le b_i$, for any interior point of the line segment,

Interior point is a __________ solution but not an ___________ solution.
Def: A sub set K of R^n is _______________ if for any $x_1, x_2 \in K$

$$x = \lambda x_1 + (1 - \lambda) x_2 \in K, \qquad 0 \le \lambda \le 1$$

Example 1.12: Determine whether or not the following sets are convex.



Example 1.13. A hyperplane $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

A closed half-space $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$

$\{ \|x\| \geq 1 \mid x \in R^n\}$ & $\{ \|x\| = 1 \mid x \in R^n\}$

Thm: The intersection of a finite collection of convex sets is convex.

1.8 The extreme point theorem

Def. A point $x \in R^n$ is a ______________________ of the points $x_1, x_2, \ldots, x_r$ in R^n if for some real numbers $c_1, c_2, \ldots, c_r$ which satisfy

Thm. The set of all convex combinations of a finite set of points in R^n is a convex set.

Def. A point x in a convex set S is called an ___________________ of S if it is not an interior point of any line segment in S.

Thm. Let S be a convex set in R^n. A point x in S is an extreme point of S if and only if x is not a convex combination of other points.

Thm. Let S be the set of feasible solutions to a general LP problem.

1) If S is nonempty and bounded, then an optimal solution to the problem exists and occurs at an extreme point.
2) If S is nonempty and unbounded, and if an optimal solution to the problem exists, then it occurs at an extreme point.
3) If an optimal solution to the problem does not exist, then either S is empty or S is unbounded.

1.9 Basic solutions

Consider a LP problem in canonical form

$$\text{maximize } z = c^T x \quad \text{for } x \in R^s \qquad (1)$$

subject to

$$Ax = b \qquad (2)$$

$$x \geq 0 \qquad (3)$$

where A is an $n \times s$ matrix and b is an $n \times 1$ matrix.

Let S be the convex set of all feasible solutions of (2).

Thm. Suppose that the last m columns of A, which denote by A'_1, A'_2, ..., A'_m are linearly independent and suppose that

$$x'_1 A'_1 + x'_2 A'_1 \cdots + x'_m A'_m = b \qquad (5)$$

where $x'_i \geq 0$ for i = 1, 2, …, m. Then the point

$$x = (0, 0, \ldots, 0, x'_1, x'_2, \ldots, x'_m)$$

is an extreme point of S.

Thm. If $x = (x_1, x_2, \ldots, x_m)$ is an extreme point of S, then the columns of A that correspond to positive x_i form a linearly independent set of vectors in R^n

Corollary. If x is an extreme point and $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ are the r positive components of x, then $r \leq m$, and the set of columns $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ can be extended to a set of m linearly independent vectors in R^n by adjoining a suitably chosen set of m – r columns of A.

Thm. At most m components of any extreme point of S can be positive. The rest must be zero.

A ____________ x to Ax = b is the solution of it obtained from solving this system along with the s – m zeros form x .

Def. A ____________________ to the LP problem given by (1) –(3) is a ______ solution that is also a _______ solution of (2).

Thm. For the LP problem given by (1) – (3), every ____________ solution is an ______ point, and, conversely, every _________ point is a _______________ solution.

Thm. The LP problem given by (1) – (3) has a finite number of ______________ solutions.

Thm. Every extreme point of S yields an extreme point of S' when slack variable are added. Conversely, every extreme point of S', when truncated, yields an extreme point of S.

Example 1.14: Maximize f(x, y) = 3x+2y

Subject to

$$-x + y \le 2$$
$$x + y \le 3$$
$$\frac{2}{5}x - y \le -\frac{1}{5}$$
$$x, y \ge 0$$