

Programming Project Ideas

n-body simulator

- Simulate the action of gravity on n-planets

Gradient Descent Optimizer

- Finds min/max of a function by following the path of steepest growth.

Lines & Planes

Recall the parametric equation of a line:

$$\vec{x} = \vec{p} + t\vec{d}$$

\vec{p} = initial point
 \vec{d} = direction
 t = parameter

Ex/ Find the line passing through: $(1, 2, 3)$ & $(4, 5, 6)$

Decide the initial point, in this case $(1, 2, 3)$.

$$\vec{p} = (1, 2, 3)$$

Calculate the direction vector.

$$\vec{d} = (4, 5, 6) - (1, 2, 3) = (3, 3, 3)$$

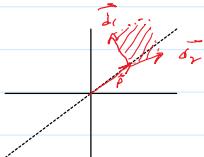
Thus the line is:

$$\begin{aligned}\vec{x} &= (1, 2, 3) + t(3, 3, 3) \\ &= (1, 2, 3) + s(1, 1, 1)\end{aligned}$$

Definition: The parametric equation for a plane

$$\vec{x} = \vec{p} + s\vec{d}_1 + t\vec{d}_2$$

where \vec{d}_1 and \vec{d}_2 are non zero and not parallel



Ex/ Find the parametric equation for the plane containing $(1, 0, 0)$, $(0, 1, 0)$ & $(0, 0, 1)$

Pick the initial point

$$\vec{p} = (0, 1, 0)$$

calculate \vec{d}_1

$$\vec{d}_1 = (1, 0, 0) = (1, -1, 0)$$

$$\# \text{ calculate } \vec{d}_2 = (0, 0, 1) - (0, 1, 0) = (0, -1, 1)$$

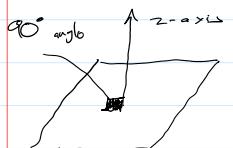
$$\vec{x} = (0, 1, 0) + s(1, -1, 0) + t(0, -1, 1)$$

The normal form of plane

We can imagine planes as the set of points orthogonal to a given vector

Ex/ Write the x-y plane in normal form

observe the z-axis is orthogonal to xy



$$xy\text{-plane} = \{\vec{v} : \vec{v} \cdot (0, 0, 1) = 0\}$$

① if a plane π (not \vec{p} , plane variable!) is defined by:

$\pi = \{\vec{v} : \vec{v} \cdot \vec{n} = 0\}$ Then $\vec{v} = \vec{o}$ is always in π (our plane)

$$(\vec{v} \cdot \vec{n} = 0 \text{ for any } \vec{n})$$

Ex/ Show that $(0, -2, -1)$, $(1, 4, 0)$, $(2, 10, 1)$ do not lie on a unique plane.

observation: If they lie on the line then we get no unique plane

$$\begin{aligned} \text{consider: } \vec{x} &= (0, -2, -1) + t((2, 10, 1) - (1, 4, 0)) \\ &= (0, -2, -1) + t(1, 6, 1) \end{aligned}$$

$$[t=0] \Rightarrow \vec{x} = (0, -2, -1)$$

$$[t=1] \Rightarrow \vec{x} = (1, 4, 0)$$

$$[t=2] \Rightarrow \vec{x} = (2, 10, 1)$$

Any plane containing this line will contain all three points

thus, all three points lie on a line

Definition: The normal form of a plane π

$$\pi = \{ \vec{v} : (\vec{v} - \vec{p}) \cdot \vec{n} = 0 \}$$

Where \vec{p} is a point in π and \vec{n} is the normal of π
(\vec{n} orthogonal to all $\vec{v} - \vec{p}$)

Ex/ Express the plane containing $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in normal form.

we have points but lack a normal

make a normal

$$\vec{d}_1 = (0, 1, 0) - (1, 0, 0) = (-1, 1, 0)$$

$$\vec{d}_2 = (0, 0, 1) - (1, 0, 0) = (-1, 0, 1)$$

To make the normal find something orthogonal to these

Find \vec{n} so that $\vec{n} \cdot \vec{d}_1 = 0$ & $\vec{n} \cdot \vec{d}_2 = 0$

Assume $\vec{n} = (x, y, z)$

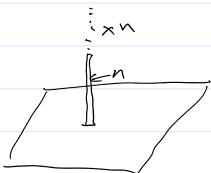
$$0 = \vec{n} \cdot \vec{d}_1 \Rightarrow 0 = -x + y \Rightarrow x = y$$

$$0 = \vec{n} \cdot \vec{d}_2 \Rightarrow 0 = -x + z \Rightarrow x = z$$

We get $x = y = z$

we pick $\vec{n} = (1, 1, 1)$

(we could pick $\vec{n} = (42, 42, 42)$ because we can scale it.
Only condition is non zero. Think of it as an extension of the plane and vectors)



Thus, the normal form is: $\pi = \{ \vec{v} : (\vec{v} - (1, 0, 0)) \cdot (1, 1, 1) = 0 \}$

Lines & Planes

In 3D two lines can be:

- co-incident: same line 

- intersect: at a point ↴
- parallel: never touch ═
- skew: non parallel, never touch ✗

Ex / Determine whether these lines intersect?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\textcircled{1} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+3t \\ -1+t \\ 1+6t \end{bmatrix}$$

$$\textcircled{2} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1+3s \\ -2+s \\ 0+s \end{bmatrix}$$

\textcircled{3} # Set the equations equal

$$\begin{bmatrix} 2+3t \\ -1+t \\ 1+6t \end{bmatrix} = \begin{bmatrix} -1+3s \\ -2+s \\ s \end{bmatrix}$$

$$\begin{bmatrix} 3t-3s \\ t-s \\ 6t-s \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

\textcircled{4} # Use linear Algebra

$$\left[\begin{array}{cc|c} 3 & -3 & 3 \\ 1 & -1 & -1 \\ 6 & -1 & -1 \end{array} \right] \xrightarrow{R_1 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & -3 & 3 \\ 6 & -1 & -1 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 6 & -1 & -1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \therefore \text{ we obtain } (t, s) = (0, 1)$$

Check your work!

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Ex / Determine where the line meets the plane

$$l: \vec{x} = (2, 1, 0) + t(1, 1, 0)$$

$$\pi: x + 2y + 3z = 0$$

Normal form: $\boxed{[(x,y,z) \cdot (1,2,3)] = 0}$

Find $\vec{x} = (x, y, z)$ for the line and plug it into the plane-

$$(x, y, z) = (2, 1, 0) + t(1, 1, 0) = (2+t, 1+t, 0)$$

From $x + 2y + 3z = 0$ we get:

$$\begin{aligned} (2+t) + 2(1+t) + 3 \cdot 0 &= 0 \\ 2+t+2+2t &= 0 \\ 3t &= -4 \end{aligned}$$

thus, $t = -\frac{4}{3} \Rightarrow$ Thus, the line and plane meet at

$$(2 - \frac{4}{3}, 1 - \frac{4}{3}, 0) = (\frac{2}{3}, -\frac{1}{3}, 0)$$

Lecture 1

Course Website

<http://pgn.dey.ca>

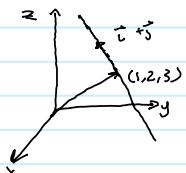
Vector

In this course, there are 3 notations:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$$

Ex/ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\vec{j} = \begin{bmatrix} 1 \\ 2+3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

Ex/ Write the parametric equation for a line passing through $(1, 2, 3)$ in the direction $\vec{i} + \vec{j}$



$$(x, y, z) = (1, 2, 3) + t(\vec{i} + \vec{j})$$

Start at $(1, 2, 3)$ add a stretched copy of $\vec{i} + \vec{j}$

$$\begin{aligned} &= (1, 2, 3) + t(1, 1, 0) \\ &= (1+t, 2+t, 3) \end{aligned}$$

* Note this equation is parametric because it is dependent on t .

Lengths & Angles

The pythagorean Theorem says: $\| (x, y, z) \|^2 = x^2 + y^2 + z^2$

Notation: $\| \vec{v} \|$ is the Length of vector \vec{v}

Definition: The dot product

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$$

Fact: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos \theta$ where
 θ = angle between \vec{u} and \vec{v}

Two vectors are parallel if $\vec{v} = \lambda \vec{u}$ for some number λ

Ex/ For what values of x are $\vec{u} = (5, x, 3)$ & $\vec{v} = (x, y, z)$ orthogonal?

$$\vec{u} \cdot \vec{v} = 5x + x^2 + 6 = x^2 + 5x + 6 = (x+3)(x+2)$$

\therefore we get $\vec{u} \cdot \vec{v} = 0$ when $x = -2$ or $x = -3$

Ex/ Find when two nonparallel vectors are orthogonal to $(1, 2, 3)$

Suppose $\vec{u} = (x, y, z)$ orthogonal to $(1, 2, 3)$

$$\vec{u} \cdot (1, 2, 3) = 1x + 2y + 3z = 0$$

Pick (x, y, z) to get orthogonality

$$\vec{u} = (3, 0, -1) \quad \vec{v} = (0, 3, -2)$$

we have $(1, 2, 3) \cdot \vec{u} = 0$ and $(1, 2, 3) \cdot \vec{v} = 0$
but \vec{u} and \vec{v} are not parallel.

Plan for this week

- Matrices ④
- Determinants
- Cross Product

Matrix Multiplication

Recall from last week

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

We are going to define Matrix Multiplication

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \vec{u} \cdot \vec{v}$$

For a matrix $M = [m_{ij}]$ we define the transpost of M

$$M^T = [m_{ij}]$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

We obtain for vectors \vec{u} and \vec{v}

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

Definition: If M is an $n \times k$ matrix and N is a $k \times l$ matrix

$$M = \left[\begin{array}{c|c|c} r_1 & & \\ \hline r_2 & & \\ \hline \cdots & & \\ \hline r_n & & \end{array} \right] \quad \text{and} \quad N = \left[\begin{array}{c|c|c} & c_1 & \\ \hline & c_2 & \\ \hline & \vdots & \\ \hline & c_n & \end{array} \right]$$

rows
vectors

columns
vectors

We define $MN = [r_i \cdot c_j]$

Ex/ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \left\{ \begin{array}{l} r_1 = [1 \ 2] \\ r_2 = [3 \ 4] \\ \vdots \\ r_n = [F \ I] \end{array} \right.$

$$\begin{aligned}
 & \text{Circled } (34) \quad \left[\begin{matrix} 6 & 1 \\ 0 & 1 \end{matrix} \right] \xrightarrow{\quad} \left\{ \begin{array}{l} c_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ r_2 = \begin{bmatrix} 3 & 4 \end{bmatrix} \\ c_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array} \right. \\
 &= \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 \\ r_2 \cdot c_1 & r_2 \cdot c_2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}
 \end{aligned}$$

Another interpretation of $A\vec{x}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Try and formulate $A\vec{x}$ using the dot product-

$$A\vec{x} = \begin{bmatrix} \underline{-r_1-} \\ \underline{-r_2-} \\ \vdots \\ \underline{-r_n-} \end{bmatrix} \begin{bmatrix} 1 \\ c_1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} r_1 \cdot c_1 \\ r_2 \cdot c_1 \\ \vdots \\ r_n \cdot c_1 \end{bmatrix} \quad \underbrace{\quad}_{\text{dot product}}$$

Matrix Inverses

Definition: The inverse of a matrix M is M^{-1} so that

$$MM^{-1} = M^{-1}M = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix inversion algorithm

Goal: To invert M

- ① Setup the augmented matrix $[M | I]$
- ② Row reduce $[M | I]$ to $[A | \tilde{M}]$

$$\text{If } A = I \Rightarrow \tilde{M} = M^{-1}$$

otherwise:

M is not invertible

Ex) Find the inverse of $M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$\left[M \mid I \right] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array} \right] \underbrace{I}_{I}$$

Thus, $M^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

Check our work

$$MM^{-1} = I$$

$$MM^{-1} = \left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array} \right] \left[\begin{array}{cc} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \checkmark$$

Determinants

Definition: The determinant of a 2×2 matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Fact: In all dimensions the determinant measures the volume of a parallel-piped

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \text{Area of } A$$

If non invertible matrix there is no area since the basis vectors becomes parallel

Theorem: if $\det(A)=0$ Then A is not invertible

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ everything maps to 0

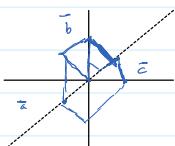
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and basis vectors becomes parallel

Definition: The Determinant of a 3×3 matrix is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Definition: The parallel piped formed by the vector $\vec{a}, \vec{b}, \vec{c}$ emanating from the origin of \mathbb{R}^3 is:

$$P(\vec{v}_1 \dots \vec{v}_n) = \sum_{i=1}^n \vec{x} = \sum_{i=1}^n t_i \vec{v}_i, t_i \in [0,1]$$



All possible combination of $s\vec{a} + t\vec{b} + u\vec{c}$ where $s \in [0,1]$, $t \in [0,1]$ and $u \in [0,1]$

Fact: (Restatement of Det)

$$\det(\vec{v}_1 \dots \vec{v}_n) = \text{vol}(P(\vec{v}_1 \dots \vec{v}_n))$$

The cofactor expansion

Goal: To calculate $\det(M)$, let $M = [m_{ij}]$ and $\tilde{M}_{ij} =$ matrix M with row i and column j removed.

Define $\det([a]) = a$

$$\det(M) = \sum_{j=1}^n (-1)^{i+j} m_{ij} \det(\tilde{M}_{ij}) \quad \leftarrow \text{memorize this}$$

Ex/ calculate

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = (-1)^{1+1}(1) \begin{vmatrix} \cancel{1} & \cancel{2} & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} + (-1)^{1+2}(2) \begin{vmatrix} 1 & \cancel{2} & 3 \\ \cancel{0} & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} + (-1)^{1+3}(3) \begin{vmatrix} 1 & 2 & \cancel{3} \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & 5 \\ 0 & 6 \end{vmatrix} + 3 \begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} = (4 \cdot 6 - 0 \cdot 0) + 0 + 0 = 24$$

Algebraic Facts about determinants

\det plays nicely with row operations

$$\textcircled{1} \quad A \xrightarrow{R_i \leftrightarrow R_j} B \quad \text{Then } \det(A) = -\det(B)$$

$$\textcircled{2} \quad A \xrightarrow{kR_i} B \quad \text{Then } \det(B) = k \det(A)$$

$$\textcircled{3} \quad A \xrightarrow{R_i + R_j} B \quad \text{Then } \det(A) = \det(B)$$

Fact (Hamilton)

$$\det(AB) = \det(A)\det(B)$$

Ex/ calculate

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix} \xrightarrow{R_3 - R_1} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 5 & 6 \end{vmatrix} \xrightarrow{R_3 - 4R_2} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{=} \begin{vmatrix} 0 & 0 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = (-1)(0) = 0$$

Fact: if A is invertible then $\det(A) \neq 0$

we know $A A^{-1} = I$

$$\text{Thus } \det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$$

and $\det(A^{-1}) = \frac{1}{\det(A)}$, thus $\det(A) \neq 0$.

Lemma

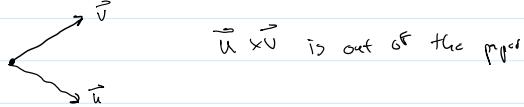
$$\det(I) = 1$$

The Cross Product

Definition: The cross product of \vec{u}, \vec{v} , of $\vec{u}, \vec{v} \in \mathbb{R}$ satisfies:

$$\textcircled{1} \quad \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$$

\textcircled{2} $\vec{u} \times \vec{v}$ is orthogonal to \vec{u} and \vec{v}



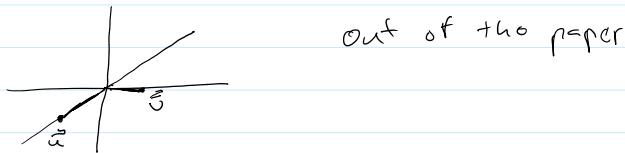
"Right hand rule": line your fingers against \vec{u} until to \vec{v} . The way your thumb is pointing is the cross product.

Right thumb = cross product.

Fact: If $\vec{u} = (u_1, u_2, u_3)$ & $\vec{v} = (v_1, v_2, v_3)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad * \text{Note: } \vec{i}, \vec{j}, \vec{k} \text{ are formal variables}$$

Ex/ Calculate $\underbrace{(1, 6, 0)}_{\vec{u}} \times \underbrace{(0, 1, 0)}_{\vec{v}}$



\therefore guess $(0, 0, 1)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= i \cdot 0 - j \cdot 0 + k \cdot 1 = k$$

Ex/ (E1.3 O3) calculate

$$(\vec{i} - 2\vec{j} + \vec{k}) \times (2\vec{i} + \vec{j} + \vec{k})$$

$$= (1 - 2 \ 1) \times (-2 \ 1 \ 1)$$

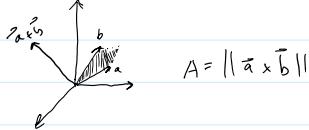
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{vmatrix} = i \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} + \vec{j} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = (-3)\vec{i} + \vec{j} + 5\vec{k}$$

check our work $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$$(1 - 2 1) \cdot (-3 1 5) = -3 - 2 + 5 = 0$$

$$(2 1 1) (-3 1 5) = -6 + 1 + 5 = 0$$

Fact: The area of a parallelogram spanned by \vec{a} , \vec{b} is $\|\vec{a} \times \vec{b}\|$



- ① cross product gives you a normal
- ② It gives you an area.

Ex/ Calculate the area of the parallelogram

$$\vec{a} = (1 0 0) \quad \vec{b} = (1 1 0)$$

$$\text{Area} = \left\| (\vec{a} \times \vec{b}) \right\| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= i |0 0| + j |1 0| + k |1 1| = 0 \cdot i - 0 \cdot j + 1 \vec{k} = \|(0, 0, 1)\| = 1$$

Summary of this week:

Determinants

↳ cofactor expansion

↳ volume of parallel piped

↳ $\det(AB) = \det(A) \det(B)$

↳ row operations

↳ invertibility

Cross product

↳ right hand rule

↳ formula

↳ area of parallelogram

↳ $\vec{a} \times \vec{b}$ orthogonal to \vec{a} and \vec{b}

email: parker-glynn-adley@utoronto.ca

Matrix Multiplication

↳ dot product

↳ matrix inversion algorithm

↳ transpose

Ex (1.3 Q 13): Show that the points $(0, -2, -1)$, $(1 4 0)$ and $(2 1 0)$ do not lie on a consecutive plane

Find the volume of parallelpiped

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ -2 & 4 & 10 \\ -1 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 1 & 2 \\ -2 & 4 & 10 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 4 & 10 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

Since the parallel piped has a volume of zero, they lie on a line and we get no unique plane.

$$\det(A) = \det(A^T)$$

Remark: Available Friday July 13th I C200, 4:00pm-6:00pm
o/w attend office hours.

Mock Final: Friday July 27th - SY10 ext 12:00 - 3:00pm

for mock final, stay for the whole thing.

Constrained Optimization

So far: "Minimize / maximize $f(x)$ on its domain"

We get interesting behaviour when we restrict the domain

Ex/ Maximize $f(x) = x^2$ on $[1, 2]$

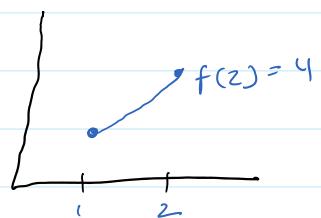
Find the critical points

$$f'(x) = 2x = 0 \Rightarrow x=0$$

(1) 0 is outside of $[1, 2]$

check the end points

$$f(1)=1 \quad f(2)=4$$



Ex Minimize $f(x,y) = x^2 + y^2$ on the triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$

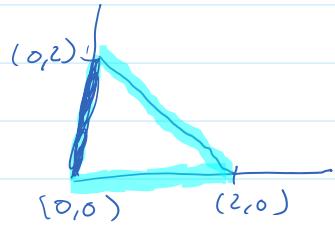
Find the critical points

$$f_x = 2x = 0 \Rightarrow (x, y) = (0, 0)$$

$$(0, 2)$$

$$f_x = 2x = 0 \Rightarrow (x, y) = (0, 0)$$

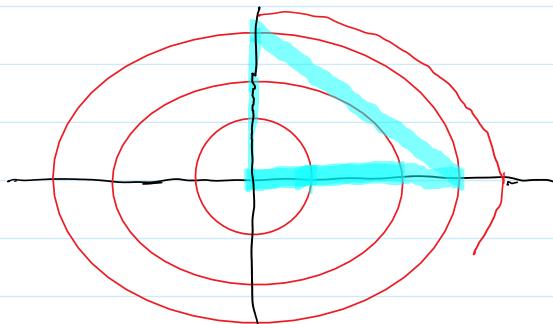
$$f_y = 2y = 0$$



Apply second derivative test

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \text{minimum}$$

check boundary ("end points")



Look for the largest level set that still touches the boundary, we get $f(0,2) = f(2,0) = 4$

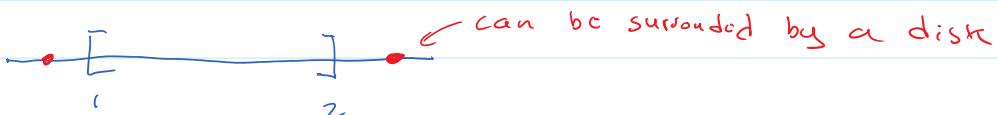
The maximum of $f(x,y)$ on the triangle is $f(0,2) = f(2,0) = 4$

Definition $S \subseteq \mathbb{R}^n$ is OPEN if: $\forall x \in S \exists \varepsilon > 0 D_\varepsilon(x) \subseteq S$

"Any point can be surrounded by a disk"

Definition A set $T \subseteq \mathbb{R}^n$ is CLOSED if $S = \mathbb{R}^n \setminus T$ is open

"Every point outside of T can be surrounded by a disk."



Definition A set $U \subseteq \mathbb{R}^n$ is bounded if there is N such that $u \in D_n(\emptyset)$

"U can be surrounded by a disk"

Thrm (Extreme Value)

If S is closed and bounded then any continuous $f: S \rightarrow \mathbb{R}$ achieves a maximum and minimum number

$$f(x_0) = M \quad f(x_1) = m$$

Why closed and bounded?

	Yes	No
Yes	$\begin{matrix} 0 & 0 \\ \text{---} & \end{matrix}$	$[0, \infty)$ $f(x) = \frac{1}{1+x^2}$
No	$(0, 1)$ $f(x) = x$	$[0, \infty)$ $f(x) = \frac{1}{1+x^2}$

↑
No max or min

← # max but no min
← # No max / No min

Ex Final 2018

Find the maximum of $f(x, y, z) = x^2 - 3x + y^2 - 2y + z^2 - 4z + 1$
on the ball $x^2 + y^2 + z^2 \leq 9$

Find the critical points

$$\left. \begin{array}{l} f_x = 2x - 3 = 0 \\ f_y = 2y - 2 = 0 \\ f_z = 2z - 4 = 0 \end{array} \right\} \Rightarrow (x, y, z) = \left(\frac{3}{2}, 1, 2 \right)$$

"This point $(\frac{3}{2}, 1, 2)$ is in the ball"

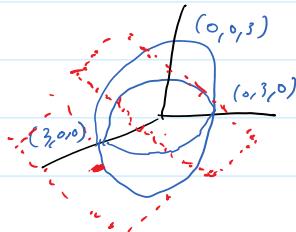
Apply second derivative test

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{critical point is a minimum}$$

$$f(x, y, z) = x^2 - 3x + y^2 - 2y + z^2 - 4z + 1$$

use the constraint $x^2 + y^2 + z^2 = 9$

$$\begin{aligned} f(x, y, z) &= 9 - 3x - 2y - 4z + 1 \\ &= 10 - 3x - 2y - 4z \end{aligned}$$

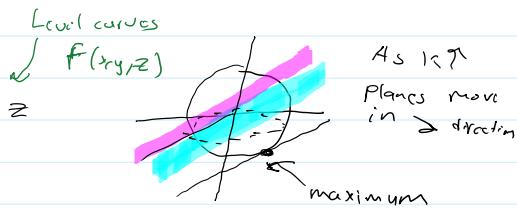


who ever is reading
this - sorry can't draw

Optimization

$$\text{Maximize } f(x, y, z) = 10 - 3x - 2y - 4z$$

$$\text{on the sphere: } x^2 + y^2 + z^2 = 9$$



As we increase the value of t , the planes move through space in a particular direction

when the maximum occurs, the level set is tangent to the constraint

The maximum should be tangent to the constraint.

Note: planes are parallel and have the same normal

Find the point of tangency

All the planes $f(x, y, z) = t$ have the same normal
 $\vec{n} = (-3, -2, -4)t$

Find where $\ell: (-3, -2, -4)t$ meets sphere.

$$(-3t)^2 + (-2t)^2 + (-4t)^2 \Leftrightarrow 9t^2 + 4t^2 + 16t^2$$

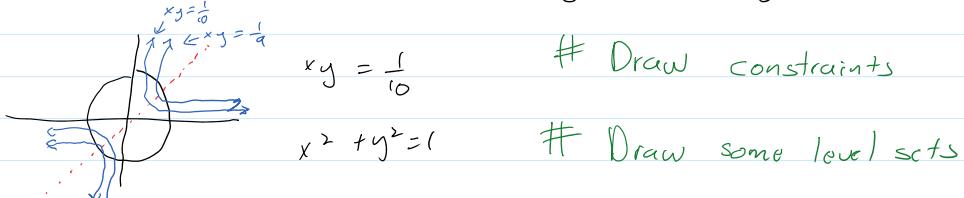
$$29t^2 = 9 \Rightarrow t = \pm \frac{3}{\sqrt{29}}$$

We pick: (x_{xy}, z)

$$= \pm \frac{3}{\sqrt{29}} (-3, -2, -4)$$

Note: negative root corresponds to minimum.

Ex Find the maximum of $f(x, y) = xy$ on $x^2 + y^2 = 1$



We need to find the point of tangency.

Observe both curves are symmetric about $y = x$.

Assume $y = x$

$$x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1$$

$$\begin{aligned}2x^2 &= 1 \\x^2 &= \frac{1}{2} \\x &= \pm \sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}}\end{aligned}$$

We obtain: $f(x,y) = xy$ is tangent to $x^2 + y^2 = 1$
at $(x,y) = (\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})$

We get $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} \rightarrow$ seems to be a maximum

$f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2} \rightarrow$ seems to be the maximum

Look at gradient at the point of tangency

$$f(x,y) = xy \Rightarrow f = (y, x)$$

$$g(x,y) = x^2 + y^2 \Rightarrow g = (2x, 2y)$$

At the point of intersection the gradients are parallel. This means that they have the same normal for their tangent planes, so same tangent plane.

The method of Lagrange Multiplier (Parallel Gradients)

To optimize $f(\vec{x})$ subject to $g(\vec{x}) = 0$

- ① Define $\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) - \lambda g(\vec{x})$
- ② Find the critical points of $\mathcal{L}(\vec{x}, \lambda)$
- ③ Select the critical points that minimize/maximize $f(\vec{x})$

Observations:

$$\begin{aligned}\textcircled{1} \quad \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 & \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} & \text{Parallel gradients} \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 & \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} & \Leftrightarrow f = \lambda g \\ \textcircled{2} \quad \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \Leftrightarrow -g(x) = 0 & \Leftrightarrow g(x) &= 0\end{aligned}$$

We satisfy the constraint.

Ex Optimize $f(x, y) = xy$ on $x^2 + y^2 = 1$

Rewrite the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0$$

Set up Lagrange function

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 1)$$

Find the critical points

$$\mathcal{L}_x = y - \lambda(2x) = 0$$

$$y = \lambda(2x) = \mathcal{L}_x$$

$$\mathcal{L}_y = x - \lambda(2y) = 0 \Rightarrow \lambda = \lambda(2y) = \mathcal{L}_y$$

$$\mathcal{L}_\lambda = -(x^2 + y^2 - 1) = 0$$

$$x^2 + y^2 = 1 = \mathcal{L}_\lambda$$

$$x = 2\lambda y \\ \star = 2\lambda(2\lambda x)$$

Either $\boxed{x=0}$ or $\boxed{1 = 4\lambda^2}$

$$x = 0$$

$$\lambda = \pm \frac{1}{2}$$

$$(x, y) = (0, 0)$$

not on circle

$$\text{since } x^2 + y^2 = 1 \\ 0^2 + 0^2 \neq 1$$

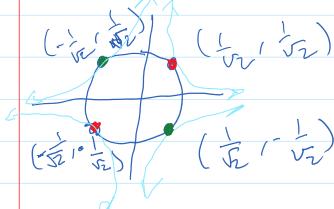
$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\boxed{y=x}$$

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Solutions $x^2 + y^2 = 1$

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$



• min
• max

Ex Maximize $f(x, y, z) = \ln(x) + \ln(y) + \ln(z)$
subject to $x+y+z=1$.

re-write constraint

$$g(x, y, z) = x+y+z-1$$

Set up Lagrange function

$$\mathcal{L} = [\ln(x) + \ln(y) + \ln(z)] - \lambda(x+y+z-1)$$

Find critical points # Tip: isolate for λ

$$\mathcal{L}_x = \frac{1}{x} - \lambda = 0 \iff \lambda = \frac{1}{x}$$

$$\mathcal{L}_y = \frac{1}{y} - \lambda = 0 \iff \lambda = \frac{1}{y}$$

$$\mathcal{L}_z = \frac{1}{z} - \lambda = 0 \iff \lambda = \frac{1}{z}$$

$$\mathcal{L}_\lambda = x + y + z - 1 = 0$$

Solve

$$x + y + z = 1$$

$$x + x + x = 1$$

$$3x = 1$$

$$x = \frac{1}{3} = y = z$$

you can make $f(x, y, z)$ negative by $y=0.01$ or $z=-1000$

Ex (§ 3.4 Q 30)

Suppose $S \subseteq \mathbb{R}^3$ is a surface defined by $f(x, y, z) = 1$. Suppose P is a point on S . So that the distance to the origin is maximal.

Claim: \vec{OP} has to be \perp to S

Consider maximizing $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ on the surface $f(x, y, z) = 1$

The Lagrange function is

$$\mathcal{L}(x, y, z, \lambda) = d - \lambda(f - 1)$$

if we are a critical point to the Lagrange function Then:

$$\mathcal{L}_x = d_x - \lambda f_x = 0$$

$$\mathcal{L}_y = d_y - \lambda f_y = 0$$

$$\mathcal{L}_z = d_z - \lambda f_z = 0$$

In terms of coordinates

$$\kappa_x = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \lambda f_x$$

$$2\sqrt{x^2+y^2+z^2}$$

$$k_y = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \lambda f_y$$

$$k_z = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \lambda f_z$$

$$k = \frac{2}{2\sqrt{x^2+y^2+z^2}}$$

We obtain: $\lambda \vec{OP} = \vec{f}$ and we know $\vec{f} \perp$ to the surface

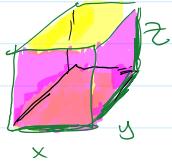
Ex Suppose you make boxes. The top and bottom cost $\$2/m^2$
sides cost $\$3/m^2$

Design the least expensive box with volume $7 m^3$

when setting up physical optimization, imagine what shapes could arise
eg. turn long,



Draw pictures with variables



want to minimize cost

$$f(x, y, z) = \text{cost} \\ = ? \quad \text{constraint}$$

subject to $1 = V = xyz$

$$f(x, y, z) = \$2(2xy) + \$3(2xz) + \$3(2yz)$$

$$= 4xy + 6xz + 6yz$$

Set up Lagrange

$$\mathcal{L}(x, y, z, \lambda) = [4xy + 6xz + 6yz] - \lambda(xyz - 1)$$

Find critical points

$$\mathcal{L}_x = 4y + 6z - \lambda yz = 0 \Rightarrow 4xy + 6xz - \lambda(xyz) = 0$$

① Tricky move, multiply this equation by x, y, z

$$\cancel{x} \quad \cancel{y} \quad \cancel{z}$$

$$4xyz + 6xyz - \lambda(xyz)^2 = 0$$

$$L_y = 4x + 6z - \lambda xy = 0 \Rightarrow 4xy + 6yz - \lambda \underbrace{[xyz]}_{1} = 0$$

$$L_z = 6x + 6y - \lambda xy = 0 \stackrel{*z}{\Rightarrow} 6xz + 6yz - \lambda \underbrace{[xyz]}_{1} = 0$$

$$\text{So: } \lambda = 4xy + 6xz \quad (1)$$

$$\lambda = 4xy + 6yz \quad (2)$$

$$\lambda = 6xz + 6yz \quad (3)$$

$$4xy + 6xz = 4xy + 6yz$$

$$6xz = 6yz \Rightarrow \text{either } z=0 \quad \text{or } x=y$$

$\cancel{z=0}$
Since
no volume

$$4xy + 6yz = 6xz + 6yz$$

$$4xy = 6xz \Rightarrow \boxed{x=0} \quad \text{or} \quad \boxed{y = \frac{3}{2}z} \quad \checkmark$$

\cancel{x}
No volume

$$\text{We get } x = y = \frac{3}{2}z$$

Exercise: Solve for (x, y, z) using volume constraint

Summary of this week

- Constrained optimization
- Check the endpoints / boundary
- Bounded sets
- Closed sets
- Extreme value theorem
- Lagrange multiplier and parallel gradients
- Find max/min using level curves

Admin

Homework 5 available

Homework 6 will be available next week.

Integration (5.2)

Recall,

 $\int_a^b f(x) dx$ calculates an area for us."Signed area of $f(x)$ on $[a, b]$ "Fact (Fundamental theorem of calculus) $\int_a^b f(x) dx = F(b) - F(a)$ where $F'(x) = f(x)$ is an antiderivative of $f(x)$ Ex Find the area bounded by $y = 1 - x^2$ and $y = 0$

$$\begin{aligned}
 A &= \int_{-1}^1 \text{length}(x) dx \\
 &= \int_{-1}^1 (1 - x^2) dx \\
 &= \left[x - \frac{x^3}{3} \right]_{-1}^1 \\
 &= \left(1 - \frac{1}{3}(1)^3 \right) - \left(-1 - \frac{1}{3}(-1)^3 \right) \\
 &= \frac{2}{3} - \left(-1 + \frac{1}{3} \right) \\
 &= \frac{4}{3}
 \end{aligned}$$

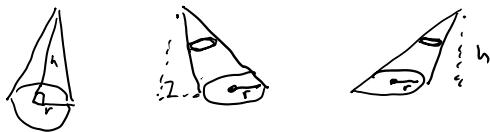
Using FTC

Cavalieri's Principle

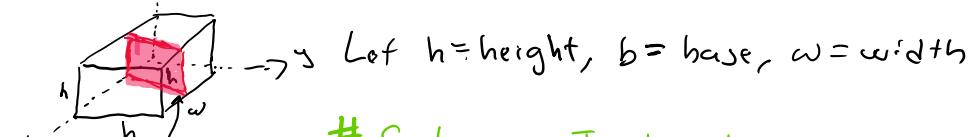
"The integral of area is volume"

Corollary: If two bodies have the same horizontal slice area functions then they have the exact same volume.

Ex All three cones have the same volume



Find the volume of a rectangular prism.



Set up Integral

$$\begin{aligned} \text{Vol} &= \int_0^w \text{Area}(x) dx \\ &= \int_0^w b h dx \\ &= [bhx]_0^w \\ &= bhw - bho \\ &= bhw \end{aligned}$$

Application of Cavalieri

Claim:



The volume of a hemi-sphere of radius r and a cone of radius r and height r equals the volume of a cylinder of height r and radius r .

The hemisphere is $x^2 + y^2 + z^2 = r^2$

The cone is $z = \sqrt{x^2 + y^2}$

Find the area of cross sections

At $z=k$, we get: $x^2 + y^2 + k^2 = r^2 \Leftrightarrow x^2 + y^2 = r^2 - k^2$

A circle of radius $\sqrt{r^2 - k^2}$

At $z=k$ we get

$$k = \sqrt{x^2 + y^2} \iff k^2 = x^2 + y^2 \quad \# A \text{ circle of radius } k$$

This says if we take a cut at $z=k$, then the cross sections are two circles. And the area of the circle



Adding together areas

$$\pi k^2 + \pi (\sqrt{r^2 - k^2})^2 = \pi k^2 + \pi(r^2 - k^2) = \pi r^2$$

Ex / Find the volume under $z = 2 - x^2 - y^2$ above
 $R = \{(x, y) \mid -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\} = [-1, 1] \times [-1, 1]$

Set up integral, using Cavalieri's principle

$$\begin{aligned} V &= \int_{-1}^1 \text{Area}(x) dx \\ &= \int_{-1}^1 \int_{-1}^1 2 - x^2 - y^2 dy dx \\ &= \int_{-1}^1 \left[2y - x^2 y - \frac{y^3}{3} \right]_{-1}^1 dx \\ &= \int_{-1}^1 \left[2 - x^2 - \frac{1}{3} + 2 - x^2 - \frac{1}{3} \right] dx \\ &= \int_{-1}^1 \left[-2x^2 + 4 - \frac{2}{3} \right] dx \\ &= \int_{-1}^1 \left[-2x^2 + \frac{10}{3} \right] dx \\ &= \left[-\frac{2x^3}{3} + \frac{10}{3} x \right]_{-1}^1 \\ &= \left[-\frac{2}{3} + \frac{10}{3} - \left(\frac{2}{3} \right) + \frac{10}{3} \right] \\ &= \left[\frac{20}{3} - \frac{4}{3} \right] = \left[\frac{16}{3} \right] \end{aligned}$$

To find the volume under $z = f(x, y)$ on the region $[a, b] \times [c, d]$

$$1. \quad V = \int_a^b \text{Area}(x) dx = \int_a^b \int_c^d f(x, y) dy dx$$

Double Integrals (5.3)

Ex Calculate $\int_0^1 \int_2^4 x+y \, dx \, dy$

This is the volume under $z = x+y$
above the rectangle $[2, 4] \times [0, 1]$

$$\int_0^1 \int_2^4 x+y \, dx \, dy$$

$$\begin{aligned} &= \int_0^1 \left[\frac{1}{2}x^2 + yx \right]_2^4 \, dy \\ &= \int_0^1 \left[\frac{1}{2}4^2 + 4y \right] - \left[\frac{1}{2}2^2 + 2y \right] \, dy \\ &= \int_0^1 8 + 4y - 2 - 2y \, dy \\ &= \int_0^1 6 + 2y \, dy \\ &= \left[6y + \frac{2y^2}{2} \right]_0^1 \\ &= \left[6y + y^2 \right]_0^1 \\ &= \left[6 + 1 - 0 \right] \\ &= 7 \end{aligned}$$

"What about $dy \, dx$?"

Switch bounds of integration

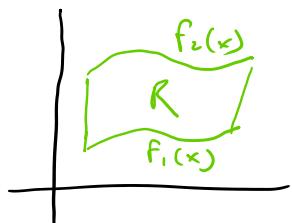
$$\int_2^4 \int_0^1 x+y \, dy \, dx$$

$$\begin{aligned} &\int_2^4 \left[xy + \frac{y^2}{2} \right]_0^1 \, dx \\ &= \int_2^4 \left[x + \frac{1}{2} \right] \, dx \\ &= \left[\frac{x^2}{2} + \frac{x}{2} \right]_2^4 \, dx \\ &= \left[8 + 2 - 2 - 1 \right] \\ &= 7 \end{aligned}$$

" $dxdy$ and $dyydx$ give you the same volume"

Now - We integrate over more general regions in the plane.

Definition: A region R is y -simple if there are continuous functions $f_1(x)$ and $f_2(x)$



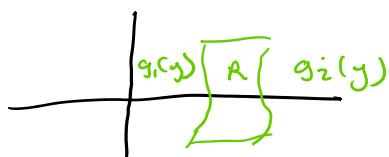
$$\text{S.t. } R = \{(x, y) : x \in [a, b] \Rightarrow f_1(x) \leq y \leq f_2(x)\}$$

y -simple = ("y coordinate is simple")

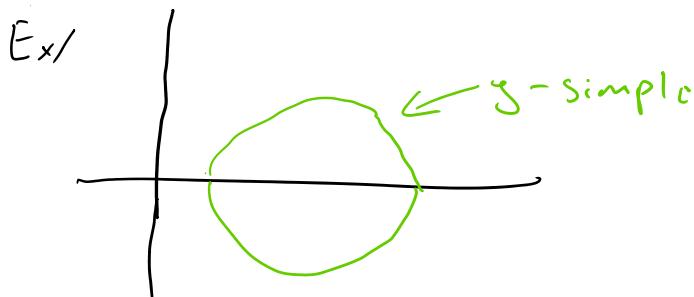
Definition: A region is x -simple

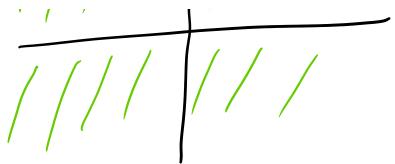
$$R = \{(x, y) : y \in [a, b] \Rightarrow g_1(y) \leq x \leq g_2(y)\}$$

continuous

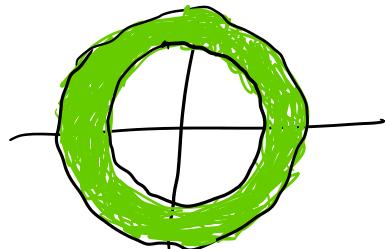


Examples: Simple vs Non-simple





Non-Ex/



Simple regions can't have holes

Integration on g-simple regions

$$\iint_R f(x, y) dA$$

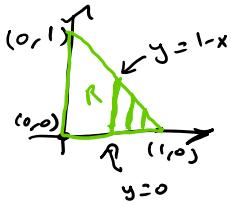
dA means "with respect to area"

$dA = dx dy$ or $dA = dy dx$

$$= \int_a^b \left[\int_{f(x)}^{f(x)} f(x, y) dy \right] dx$$

Ex) Integrate $f(x, y) = xy$ over the triangle with vertices $(0, 0), (0, 1), (1, 0)$

Sketch the region



$$\begin{aligned} & \iint_R f(x, y) dA \\ &= \int_0^1 \int_0^{1-x} xy dy dx \\ &= \int_0^1 \left[\frac{1}{2}x y^2 \right]_0^{1-x} dx \\ &= \int_0^1 \left[\frac{1}{2}x(1-x)^2 \right] dx \\ &= \int_0^1 \frac{1}{2}(x - 2x^2 + x^3) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \frac{1}{2}(x - 2x^2 + x^3) dx \\
 &= \frac{1}{2} \left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right]_0^1 \\
 &= \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] \\
 &= \frac{1}{24}
 \end{aligned}$$

Define R is Simple

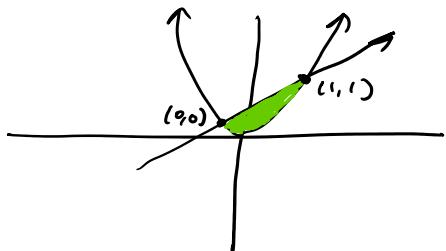
if it is x-simple and y-simple.

① bounded by continuous functions
in both directions.

Ex: Set up the integral over the region bounded by $y = x^2$ and $y = x$ using both $dA = dy dx$ and $dA = dy dx$

$dA = dy dx$

① Sketch the region



② # Solve for where they meet

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$\iint_R f(x, y) dy dx$$

$$\int_R \int f(x, y) dy dx$$

$$\int_0^1 \int_x^{x^2} f(x, y) dy dx$$

$dA = dx dy$

$$\int_R \int f(x, y) dx dy$$

$$= \int_0^1 \left[\int_s^{x^2} dx \right] dy$$

how far does x go at height y ?

$$y = x^2$$

$$\Rightarrow \pm \sqrt{y} = x$$

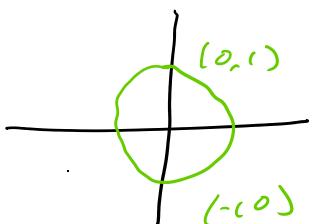
$$\Rightarrow + \sqrt{y} = x$$

$$= \int_0^1 \int_y^{x^2} f(x, y) dx dy$$

Ex Setup the integral of $f(x, y)$ on

$$R = \{(x, y), x^2 + y^2 \leq 1\}$$

using $dA = dx dy$.



Draw the region

$$\int_R \int f(x, y) dx dy = \int_{-1}^1 \left[\int_s^{x^2} dy \right] dx$$

Solve for x in terms of y

$$x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2$$

$$\Rightarrow x = \pm \sqrt{1 - y^2}$$

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx dy$$

Volume:

Ex: Find the volume bounded by

$$x^2 + y^2 = r^2 \text{ and } y^2 + z^2 = r^2$$

Recall

By Cavalieri's principle

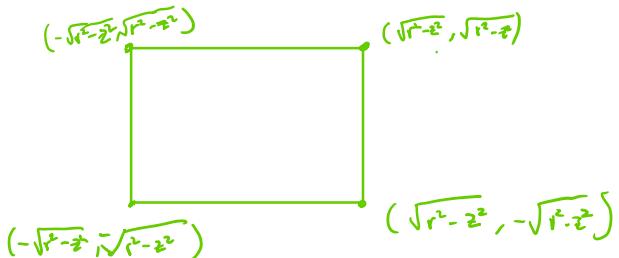
$$\text{Volume} = \int_{-r}^r \text{Area}(z) dz$$

Find $\text{Area}(z)$

$$x^2 = r^2 - z^2 \Rightarrow x = \pm \sqrt{r^2 - z^2}$$

$$y^2 = r^2 - z^2 \Rightarrow y = \pm \sqrt{r^2 - z^2}$$

regions in
the plane at
the slice z



$$\text{Area}(z) = \left[\underbrace{\sqrt{r^2 - z^2} - (-\sqrt{r^2 - z^2})}_{x\text{-direction}} \right] \left[\underbrace{\sqrt{r^2 - z^2} - (-\sqrt{r^2 - z^2})}_{y\text{-direction}} \right]$$

$$= [2\sqrt{r^2 - z^2}] [2\sqrt{r^2 - z^2}] = 4(r^2 - z^2)$$

$$\text{Volume} = \int_{-r}^r 4(r^2 - z^2) dz$$

$$= 4 \int_{-r}^r r^2 - z^2 dz$$

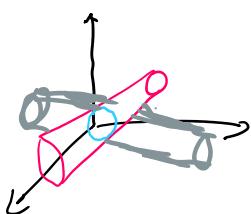
$$= 4 \left[\int_{-r}^r r^2 dz - \int_{-r}^r z^2 dz \right]$$

$$= 4 \left[r^2 z - \frac{z^3}{3} \right]_{-r}^r$$

$$= 4 \left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3} \right]$$

$$= 4 \left[2r^3 - \frac{2}{3}r^3 \right]$$

$$= 8r^3 - \frac{8}{3}r^3 = \frac{16r^3}{3}$$



Area as Integral

We can calculate area as the volume of a prism of height one.

Area



$$\text{Vol} = 1 \text{ Area}$$

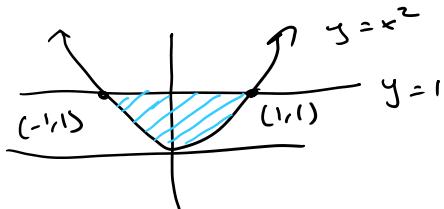
Area



$$\text{Vol} = \int \text{Area}$$

Ex Find the area bounded by $y=1$ and $y=x^2$

Draw the region



$$\begin{aligned}\text{Area} &= \iint_R 1 \, dA \\ &= \int_{-1}^1 \int_{x^2}^1 1 \, dy \, dx \\ &= \int_{-1}^1 [y]_{x^2}^1 \, dx \\ &= \int_{-1}^1 [1 - x^2] \, dx \\ &= \left[x - \frac{x^3}{3} \right]_{-1}^1 \\ &= \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right] \\ &= \left[2 - \frac{2}{3} \right] \\ &= \frac{4}{3}\end{aligned}$$

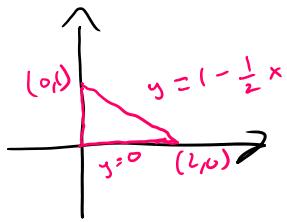
Fubini's Theorem

Fact: If f is c^2 and R is simple then

$$\iint_R f(x,y) \, dx \, dy = \int_R \int f(x,y) \, dy \, dx = \iint_R f(x,y) \, dA$$

Fx Calculate the area under $y = 2 - x - u$ and

Ex Calculate the area under $z = 2 - x - y$ and above the triangle with vertices $(0,0), (2,0), (0,1)$ using $dA = dx dy$ and $dA = dy dx$



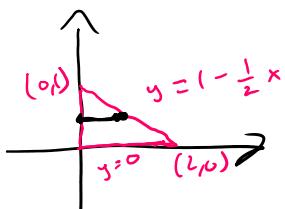
$dA = dy dx$

$$\begin{aligned} \text{Vol} &= \iint_R 2 - x - y \, dA \\ &= \int_0^2 \int_0^{1-\frac{1}{2}x} 2 - x - y \, dy \, dx \end{aligned}$$

$$\begin{aligned} &= \int_0^2 \left[2y - xy - \frac{y^2}{2} \right]_0^{1-\frac{1}{2}x} \, dx \\ &= \int_0^2 \left[2(1 - \frac{1}{2}x) - x(1 - \frac{1}{2}x) - \frac{1}{2}(1 - \frac{1}{2}x)^2 \right] \, dx \\ &= \int_0^2 2 - x - x + \frac{1}{2}x^2 - \frac{1}{2}(1 - x + \frac{1}{4}x^2) \, dx \\ &= \int_0^2 2 - 2x + \frac{1}{2}x^2 - \frac{1}{2} + \frac{1}{2}x - \frac{1}{8}x^3 \, dx \\ &= \int_0^2 \frac{3}{2} - \frac{3}{2}x + \frac{3}{8}x^2 \, dx \\ &= \left[\frac{3}{2}x - \frac{3}{4}x^2 + \frac{3}{24}x^3 \right]_0^2 \\ &= [3 - 3 + 1] \\ &= 1 \end{aligned}$$

By Fubini, $1 = \boxed{\iint_R 2 - x - y \, dx \, dy}$

$$\text{Vol} = \int_0^1 \int_0^{2-y} \boxed{\text{rectangle}} \, dy$$



$$y = 1 - \frac{1}{2}x$$

$$\begin{aligned} \Leftrightarrow 2y &= 2 - x \\ x &= 2 - 2y \end{aligned}$$

Summary

- Double Integrals
- Cavalier's principle ($\text{Vol} = \int \text{Area}$)
- Fubini's Theorem
- Simple regions $\leq_{\text{x-simple}, \text{y-simple}, \text{z-simple}}$
- Archimedes $\bigtriangleup + \Delta = \varnothing$