Measure ? ← why?

Eg: $P[2^{nd} \text{ year}] = 10\% = \dfrac{10}{|S|}$ ← # of second year student; $|S|$ ← all people sitting in the room

you are measuring something.

Axioms

① $P[A]$ has to be non-negative
② $P[\varnothing] = 0$
③ $P[S] = 1$

Ex $P[S] = P[\text{uni students}] = 1$

④ $P[\text{Additive}] = P[A_1 \cup A_2 \cup \dots] = P[A_1] + P[A_2] \dots$

Eg A_1 = first year, A_2 = second year } disjoint because nothing in common

3 = H H H
2 = H H T
2 = H T H
1 = H T T
2 = T H H
1 = T H T
1 = T T H
0 = T T T

} sample space

$P[H] = \frac{1}{2}$

$P[T] = \frac{1}{2}$

$P[3H] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Random variable

Maps sample space

$S \rightarrow \mathbb{R}$, $X = \#$ of heads

X:	0	1	2	3
$p[X]$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

table called pmf or probability mass function.

or add probabilities

To find $E(X) = \sum x \, p[x] \rightarrow$ discrete

$\int x \, p[x] \, dx \rightarrow$ continuous

To find $E[x^2] = \sum x^2 p[x] \rightarrow$ discrete

$E(x^2] = \int_{-\infty}^{\infty} x^2 p[x] \rightarrow$ continuous.

Bernoulli:

$$p_x(x) = \begin{cases} \theta \\ 1-\theta \end{cases}$$

Ex/ $x = \#$ of H $\begin{cases} 1 \\ 0 \end{cases}$

$P[H] = \theta$

$P[T] = 1-\theta$

poisson mean and variance is exactly the same $\Rightarrow \lambda$

Continuous

Uniform, Exp, gamma, normal, Beta.

Next class review function, mean, cdf, pdf etc....

Marginal prob example

	1	2	3	4	5	6	...	
H	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	!	$= \frac{1}{2}$
T								$= \frac{1}{2}$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$		

$A = H$
$B = 1$

is $A \perp B$? Yes

Proof $P(\text{Joint } (A \cap B)) = \frac{1}{12}$

$P(A] = \frac{1}{2}$

$P(B) = \frac{1}{6}$

$$P(A \cap B) = P(A)\,P(B) \quad - ' - \quad \frac{1}{}$$

$$P[1 \mid H] = \frac{P(1 \cap H)}{P(H)} = \frac{\frac{1}{12}}{\frac{1}{2}} = \frac{1}{6}$$

X:	0	1	2	3
$P[X]$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	

Treatment Heart Transplant

Control group

P1 - survied 49 days after died
P28 - surived 118 days - but he is alive

P - Indicator

P1 - wait time for trasplant is 0 days → Survived 15 days - dead

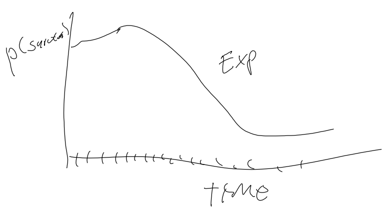
P52 - wait time 5 days → Survived 43 days - alive

30 ppl → mean(t)
Treatment

Control
52 ppl → mean(c)

compare the two values.

Assume 30 is coming from a distribution
eg.



if mean of treatment group is bigger > mean of control group

Interpertation about the population:

=> if you go through the treatment you are expected to live longer.

mean(T) > mean(c) are samples, interpertation we want to make is about the entire population.

assume

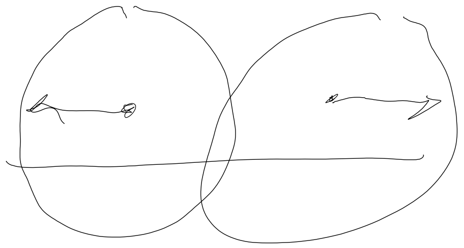
① $X \sim Exp(\lambda) \rightarrow$
② estimate of λ from these #s (parameter)

In real life you never know the distribution

Recall
$X \sim exp(\lambda)$
$f_X(x) = \lambda e^{-\lambda x}$

mean is the center of the distribution

Treatments better



radius also matters.

Data - Circle

mean - center of data

Standard deviation - radius

look if they are overlapping or far apart.

5.1.1 $mean(c) = 93.2$
$= \frac{sum(30 \text{ numbers})}{30}$

$mean(t) = 356.2$

you can't just compare the mean, look at SD and then mean.

Ex 5.1.8

$X \sim Exp(\lambda)$

$x_1, x_2, \ldots x_n$

$E(x) = \int_0^\infty x f(x)\,dx$

$= \int_0^\infty x \lambda e^{-\lambda x}\,dx$

$= \lambda \int_0^\infty x e^{-\lambda x}\,dx$

difference between X and x_i is

Eg $\lambda = 2$

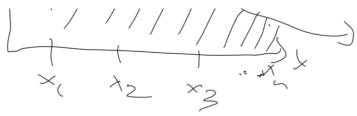
$f_X(x) = 2e^{-2x}$

$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$= \frac{\lambda}{\lambda^2} \underbrace{\int_0^{\infty} x^{2-1} e^{-\lambda x} \cdot \frac{\lambda^2}{\Gamma(2)}}_{\text{gamma density} = 1}$$ ← divide by λ^2

$\Gamma(2)$ ← this is 1

$$= \frac{1}{\lambda}(1) = \frac{1}{\lambda}$$



can take any value X or y. is the random variable that can take any value

x_1 is sample value

eg $x_1 = 100$ days

$x_2 = 200$ days

weak law of large #'s save

$x_1, x_2, \ldots, x_n$ — $\frac{1}{\lambda} \ldots$

$$\Rightarrow \frac{x_1 + x_2 + \ldots + x_n}{n} \xrightarrow{\text{converge}} \frac{1}{\lambda} = E(x)$$

$$= \bar{x} \to \frac{1}{\lambda}$$ ← estimate of mean

$$= \frac{1}{\bar{x}} \to \lambda$$

↑ estimate of lambda

Recall: Gamma

$$y \sim G(\alpha, \beta)$$

$$f(y) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \quad y \geq 0$$

problems

$\bar{x} = 0.05 \quad \hat{\lambda} = 20$ ← estimate

$\bar{x} = 0.06 \quad \hat{\lambda} = 16.66$

if λ is really big eg 200 to contain it you need a large sample size

R - vector: c(1,3,5) => $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

c(1,3,5) + 2 = $\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

variable: x = c(5, 7, 9, 10)

printing: > x

[1] 5 7 9 10

log(x) => log $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ log

$\log(x) \Rightarrow \begin{bmatrix} \log & 5 \\ \log & 7 \\ \log & 9 \\ \log & 10 \end{bmatrix}$

1.60943 1.94590...

log10(x)

$\left.\begin{array}{l} \sin(x) \\ \cos(x) \\ x\hat{}2 \end{array}\right\}$ does it for all elements

y =

5.1.1/ $x \sim N(0,1)$

$y = x^4 + 2x^3 - 3$

$P(y \in (1,2)) \leftarrow$ prob y between (1, 2)

$x^2 \rightarrow$ Chi-sq

$-\infty$ —————————— ∞

<u>Ex</u> 5.2.1

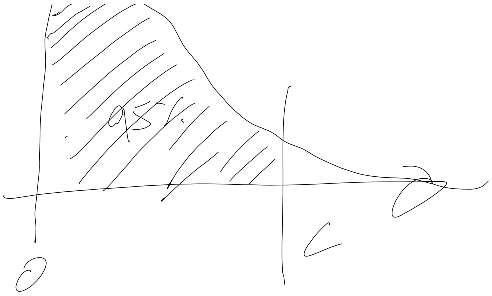
x = life lenght of a machine

$x \sim Exp(1)$

mean life lenght ? => asking for E(x)

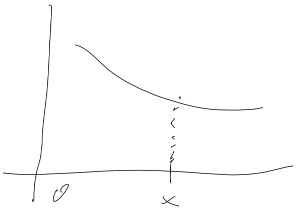
$Exp(\lambda) = \frac{1}{\lambda} \Rightarrow Exp(1) = \frac{1}{1} = 1$

what is the timelenght that 90% of phones will die what is the timelength that 90% of phones will die



$$0.95 = \int_0^c x f(x)\, dx$$

$$F(x) = \int_0^x f(x)\, dx \leftarrow \text{cdf}$$



$$1 - e^{-c} = 0.95$$

$$e^{-c} = \frac{0.95 - 1}{-1}$$

$$\ln(e^{-c}) = \ln(0.05)$$

$$c = -\ln(0.05)$$

$$\int_0^x e^{-x}\, dx$$

Conditional Distribution

x:	0	1	2	3
$P[X=x]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\text{mean} = (0)\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\left(\tfrac{3}{8}\right) + 3\left(\tfrac{1}{8}\right)$$
$$= 1.5$$

Is this valid Probability Distribution?

Yes because

1) $0 \le P(X=x) \le 1$
2) $\sum P[X=x] = 1$

Condition: cellphone survived 6 month then you won't have the same distribution

x:	~~0~~	1	2	3
$P[X=x]$	$\cancel{\frac{1}{8}}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

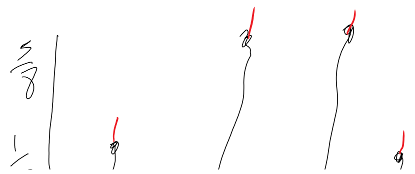
Then 0 isn't an option

This is not a valid probability distribution since it does not add up to $\underline{1}$.

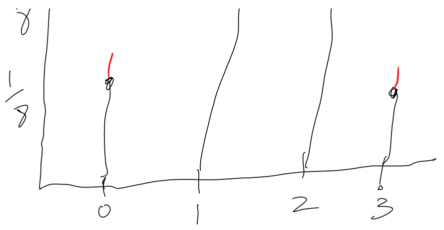
First make it a valid Probability distribution

$P(X=1 \mid X>0)$?

$$\frac{P(X=1 \cap X>0)}{P(X>0)} = \frac{P(X=1)}{P(X>0)} = \frac{\frac{3}{8}}{\frac{7}{8}} = \frac{3}{7}$$



make it a bit bigger



make it a bit bigger for PDF = 1

if $x \sim Exp(1)$

PDF: $f_X(x) = e^{-x}$

$E[X \mid X>1] = \int x \cdot$ conditional function

$$f_{X|X>1}(x) = \frac{f_X(x)}{P[X>1]}$$ ← condition needed for $\sum$ to $= 1$

$$= \frac{e^{-x}}{\int_1^{\infty} e^{-x}\,dx}$$

$$= \int_1^{\infty} x\, e^{-(x-1)}\,dx$$

Review 5.1

Example → Samples
→ Distribution
→ Inference → commenting on the calculation

* If x = life length
what is mean/expected life length?
→ asking for expected value $E(x)$
→ by what time 95% of the machines

↳ by what time 95% of the machines will fail.

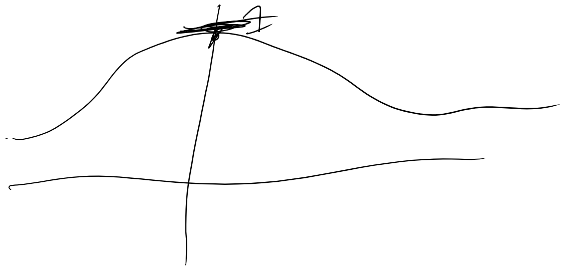
calculate cdf, equal it to 0.95* or value given and solve it.

2nd question: conditional, given $X > 1$

↳ mean/expected life?

$E(x \mid x > 1)$

predict life time distribution; mean or mode



mode = highest point

To find # take derivative set it = 0

max

min

5.2.7 example where you have to calculate the mode.

5.2.8 $x \sim Po(\lambda)$

* Predict the future value? mean / mode

mode

$$P[X=x] = \frac{\lambda^x e^{-\lambda}}{x!}$$, # if discrete no derivative

$$P[X=x+1] = \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!}$$

$$\frac{P[X=x+1]}{P[X=x]} = \frac{\lambda^{\cancel{x+1}} \cancel{e^{-\lambda}}}{(x+1)!} \div \frac{\cancel{\lambda^x} \cancel{e^{-\lambda}}}{x!}$$

$$= \lambda \frac{x!}{(x+1)!} = \frac{\lambda}{x+1}$$

$$\Rightarrow \frac{P[X=x+1]}{P[X=x]} = \frac{\lambda}{x+1}$$

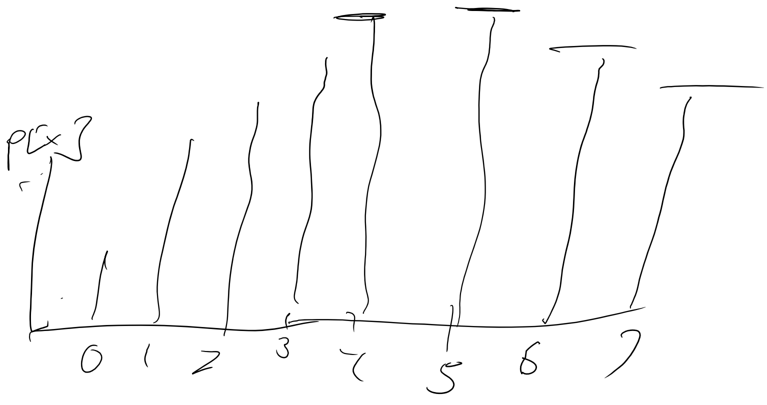
$x=0 \Rightarrow \frac{P[X=1]}{P[X=0]} = \lambda$ → if $\lambda = 5$ then its 5x ratio

$x=1 \Rightarrow \frac{P[X=2]}{P[X=1]} = \frac{\lambda}{2}$ → if $\lambda = 5$ then its 2x ratio

$$\frac{P[X=3]}{P[X=2]} = \frac{5}{3} = 1.666$$

$$\frac{P[X=4]}{P[X=3]} = \frac{5}{4} = 1.25$$

$$\frac{P[X=5]}{P[X=4]} = \frac{\lambda}{5}$$



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5.3 Statistical Models

① Population: a combination of all the subjects in your subspace. All the outcome.

Sample: A small subset of the population
Sample ⊆ Population

Parameter: Any characteristics of a population is a parameter

Statistic: summary of sample

populations are too big, samples are easier to work with.

Study sample => make inference about population. That is statistics

Eg $Exp(1)$ parameter , $P_\theta(x)$

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

x is r.v, λ is the parameter.

Statistical model

$\{ P_\theta : \theta \in \Omega \}$ parameter space

function e.g. poisson normal etc...

θ can take any value in the sample space

$$\frac{e^{-5}5^x}{x!} \sim P_o(5)$$

$$\frac{e^{-100}(100)^x}{x!} \sim P_o(100)$$

Eg $x \sim P_o(\lambda = ?)$

$\bar{X} \to \frac{1}{\lambda}$

we use $\bar{x}$ to estimate $\frac{1}{\lambda}$. This is an example of point estimation.

Interval estimation - review

Example: Pg 264

Ex 5.3.2

$\underbrace{P_\theta(x)}_{\text{discrete}} = \underbrace{f_\theta(x)}_{\text{continuous}}$

Sample $(x_1, x_2, \ldots, x_5)$

In this case, we know the parameters as the mean is already there.

$\leftarrow$ we should be able to distinguish between if a function has a mean of 1 or 1.5.

① $(x_1, \ldots, x_5) = (5.0, 3.5, 3.3, 4.1, 2.8)$

② $(x_1, \ldots, x_5) = (2.0, 2.5, 3.0, 3.1, 3.1)$

} which one belongs with which sample?

① $\Rightarrow$ Exp(1.5)

② $\Rightarrow$ Exp(1)

Parameter space in this case is $\Omega = \{1, 1.5\}$

$\Omega = \{A, B\}$

Ex 5.3.1

① Exp $\leftarrow$ know function
② $\Omega = \{1, 1.5\} \leftarrow$ know parameter space
③ sample $\leftarrow$ see sample and pick

Real life

① observe sample
② find the distribution
- known or assumed

③ sample ⊂ sel'sample and pick — known or assumed

③ Parameter - use sample to make inference about parameter

5.3.2

a)

parameter

θ	$P_\theta(x=1)$	$P_\theta(x=2)$	$P_\theta(x=3)$
A	0.5	0.5	0
B	0	0.5	0.5

Ω space $\{A, B\}$

b) if we observe $x=1$ then x is coming from A

if we observe $x=3$ then x is coming from B

if we observe $x=2$ then either a or b.

parameter

θ	$P_\theta(x=1)$	$P_\theta(x=2)$	$P_\theta(x=3)$
A	$\frac{1}{4}$	$\frac{3}{4}$	0
B	0	0.5	0.5

what if $x=2$?
probably A

Notation

$\{f_\theta : \theta \in \Omega\}$

for one sample value what is the statisc model

Ex Pois(x)

$x = 2$

$\left\{ \frac{e^{-\lambda} \lambda^2}{2!} \right\}$

Ex/ $x \sim Exp(\theta)$

$f_\theta(x) = \theta e^{-\theta x}$

$x = 5 = \theta e^{-\theta(5)}$ ← statistical model of one sample.

Joint density of sample

Ex/ $P[x_1, x_2, x_3 \ldots x_n]$ if ⊥ break it down

$\Rightarrow f_\theta(x_1) f_\theta(x_2) \ldots f_\theta(x_n)$

⊥ is an assumption

$x = \{2, 4, 9\}$

$x = \{2, 4, 9\}$

$\theta e^{-2\theta} \times \theta e^{-4\theta} \times \theta e^{-9\theta}$

Ex 5.3.3

$x \sim Bin(\theta)$

$$p_X(x) = \begin{cases} 1 & , \theta \\ 0 & , 1-\theta \end{cases}$$

$$p_X(x) = \theta^x (1-\theta)^{1-x}$$

$x_1, x_2, \ldots x_n$

$\theta^{x_1}(1-\theta)^{1-x_1} * (\theta)^{x_2}(1-\theta)^{1-x_2} * \ldots \theta^{x_n}(1-\theta)^{1-x_n}$

$= \theta^{x_1+x_2+x_3 \ldots x_n} (1-\theta)^{n-(x_1+x_2 \ldots x_n)}$

$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{(n-\sum_{i=1}^{n} x_i)}$

Ex 5.3.4

$(x_1, \ldots, x_n) \sim N(\mu, \sigma^2)$

$\theta = (\mu, \sigma^2) \in R^1 \times R^+ \quad , \quad R^+ = (0, \infty)$

$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2} \cdot \ldots \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_n-\mu)^2}$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{\left[-\frac{1}{2\sigma^2} \underbrace{\sum (x_i-\mu)^2}\right]}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}$$

$$\sum_{i=1}^{n}(x_i-\mu)^2$$ $-a-b$ / $a-c+c-b$

$$= \sum (x_i-\bar{x}+\bar{x}-\mu)^2$$

$$= \sum\{(x_i-\bar{x})^2+2(x_i-\bar{x})(\bar{x}-\mu)+(\bar{x}-\mu)^2\}$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + \cancel{2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\bar{x})} + \underbrace{\sum_{i=1}^{n}(\bar{x}-\mu)^2}_{n(\bar{x}-\mu)^2}$$

$$\sum(x_i-\bar{x})$$

$$= \sum x_i - \sum \bar{x}$$

sum of deviation of mean = 0

$$= n\bar{x} - n\bar{x} = 0$$

$$\sum_{i=1}^{n}(\bar{x}-\mu)^2 = n(x-\mu)^2$$

$$s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1}$$ ~ sample variance

sample mean

x_i	$\bar{x}$	$x_i-\bar{x}$	$(x_i-\bar{x})^2$
2		-2	4
4	4	0	0
6		2	4
		= 0	= 8

$$s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{8}{2} = 4$$

$$\frac{1}{var} = \text{precision}$$

$$\lambda = \frac{1}{\sigma^2}$$

precision

$$(2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

$$= (2\pi)^{-\frac{1}{2}}(\lambda)^{\frac{1}{2}} \exp\left[-\frac{1}{2} * \lambda * (x-\mu)^2\right]$$ ← still normal but in precision instead of variance

Reparameterization: only do it if its a one to one function. precision is $\frac{1}{\text{variance}}$) the change. old parameter to new parameter has to no one to one

old param —func— new param.

Try ex 3.3.5.

Population

Population CDF

$F_X(x) = \frac{\text{Count of } X \le x}{N}$

Ex 5.4.1

$N = 20$

$\underline{\underline{min}} = 3$

$P(X < 3) = 0$

$P(X < 4) = \frac{3}{20}$

$P(X<4) - P(X<3) = P(X=3)$

5.4 = Population = finite
= Sample — distribution known

Unknown distribution

Parameter

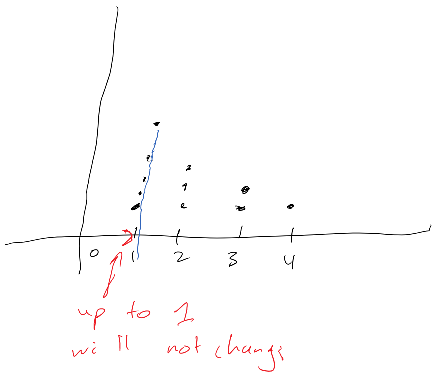
If a population is finite, do you need money / time to study them?

Ex

i	1	2	3	4	5	6	7	8	9	10
$X(\pi_i)$	1	1	2	1	2	3	3	1	2	4

Sort

i	1	1	2	1	2	3	3	1	2	4
$X(\pi_i)$	1	1	1	1	2	2	2	3	3	4



$P[x \le 0] = 0$

$P[x \le 0.999] = 0$

$P[x \le 1] = \frac{4}{10} = 0.4$

$P[x \le 1.999] = \frac{4}{10} = 0.4$

$P[x \le 2] = \frac{7}{10} = 0.7$

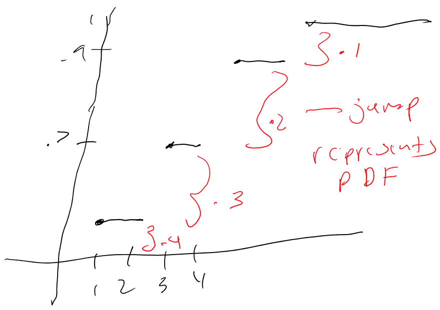
$P[x \le 2.99] = \frac{7}{10} = 0.7$

$P[x \le 3] = \frac{9}{10} = 0.9$

$P[x \le 3.99] = 0.9$

$P[x \le 4] = 1$

$$\text{CDF} = F_X(x) = \begin{cases} 0 & x < 1 \\ .4 & 1 \le x \\ .7 & 2 \le x < 3 \\ .9 & 3 \le x \le 4 \\ 1 & x \le 4 \end{cases}$$



CDF on a graph

PMF:

$$f_X(x) = \begin{cases} .4, & x = 1 \\ .3, & x = 2 \\ .2, & x = 3 \\ .1, & x = 4 \\ 0, & o/w \end{cases} \Rightarrow$$

X	1	2	3	4
$P[x]$	.4	.3	.2	.1

To calculate small f, pdf just calculate the proportion.

$$CDF = F_X(x) = \begin{cases} 0 & x < 1 \\ .4 & 1 \le x \\ .7 & 2 \le x < 3 \\ .9 & 3 \le x \le 4 \\ 1 & x \le 4 \end{cases}$$

population

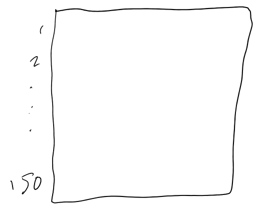
Same calculation but if its sampled its called Emperical distribution function

Emperical distribution: Same calculation but put $\hat{F}_X(x)$

5.4.1 ex/ ← do it on ya own dawg.

Simple random sampling:

Ex, in a class, blindly picking #'s



To do it in R: sample(1:150, size=1)

for a random sample betwee 1 - 150

$\{A, B, C, D, E\}$

$N = 5$ - draw 2 samples $n = 2$

$P(A) = \frac{1}{5}$

$P(A \text{ being selected}) = \frac{1}{N} = \frac{1}{5}$

$P(A \text{ being selecte} \mid b \text{ is selected}) = \frac{1}{N-1} = \frac{1}{4} = 0.25$

Samples are not $\perp$ because one being in the sample changes prob of the other one.

After picking < with replacement — pick and put it back → samples are not $\perp$
without replacement — pick # or name don't replace → samples are $\perp$

$\frac{1}{N}$ $\frac{1}{N-1}$ $N = 100{,}000$

$0.2 \qquad 0.25 \qquad \underbrace{0.00000}_{\text{something like that}}1 \approx$ next numbers are closer.

large samples even though they are dependent because N is large change is insignificant, so we treat as $\perp$

Conditions

$N \to$ large

$n \to$ small relative to N

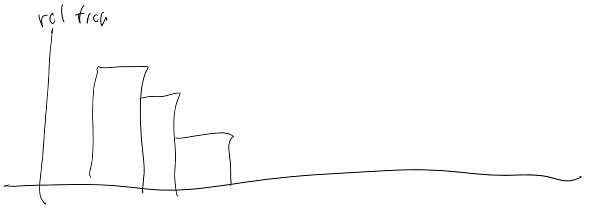
$\therefore \underbrace{\hat{F}_X(x)}_{\text{cdf calculated based on sample}} \to F_X(x)$

Histogram

Ex height

$(h_1, h_2] \quad (h_2, h_3] \quad (h_3, h_4] \quad \dots$

5   5.6   5.6   6   6   6.5



relative frequency = proportion

Density histogram

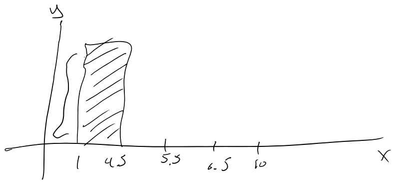
$h_X(x) = \dfrac{\text{proportion}}{\text{length}}$

5.4.5

a) $(1.2 \quad 1.8 \quad 2.3 \quad 2.5 \quad 3.1 \quad 3.4 \quad 3.7 \quad 3.7 \quad 3.9 \quad 4.3 \quad 4.4 \quad 4.5 \quad 4.5]$

$(4.8 \quad 4.8] \; (5.6 \quad 5.8] \quad (6.9 \quad 7.2 \quad 8.5]$

$$h_X(x) = \frac{\frac{13}{20}}{(4.5-1)} \qquad (1, 4.5]$$



$$h_X(x) = \frac{\frac{13}{20}}{(4.5-1)} \quad , (1, 4.5)$$

$$= \frac{13}{20}$$

Question on Midterm on this.

$$(4.5, 5.5] \quad , (4.5, 5.5]$$

$$h_X(x) = \frac{\frac{2}{20}}{(5.5-4.5)}$$

Ex

$$f_X(x) = \begin{cases} .4 & , x = 1 \\ .3 & , x = 2 \\ .2 & , x = 3 \\ .1 & , x = 4 \\ 0 & , o/w \end{cases}$$

from this list make the table

look for unique numbers

1, 2, 3, 4

count pop by total to get proportion

loop

Prof uploaded R code. look at the table.

Statistics

Descriptive Inferential

* falls into two categories

Descriptive: describes any summary eg mean of sample
standard deviation
median

Inferential: moment you use those #'s to make a inference about the data.

Recall: $f_x(x)$ is the proportion of the population members whose X measurements equal x.

$\Rightarrow f_X(x) = P[X = x]$

$F_X(x)$ is the proportion of population members whose x measurements is less than or equal to x

Ex $\{1.2, 2.1, 0.4, 3.3, -2.1, 4.0, -0.3, 2.2, 1.5, 5.0\}$

\# Put in ascending order

$\{-2.1, -0.3, 0.4, 1.2, 1.5, 2.1, 2.2, 3.3, 4.0, 5.0\}$

\# Plug in each data point to get pdf

$f_X(x)$

$f_X(-3) = P[X = -3] = 0$

$f_X(-2.1) = P[X = -2.1] = \frac{1}{10}$

$f_X(-0.3) = P[X = -0.3] = \frac{1}{10}$

$\vdots$

$f_X(5) = P[X = 5] = \frac{1}{10}$

CDF same thing but everything up to the point

$F_X(-2.1) = P[X \leq -2.1] = \frac{1}{10}$

$F_X(-0.3) = P[X \leq 0.3] = \frac{2}{10} = \frac{1}{5}$

A natural estimate of $F_X(x)$ is given by $\hat{F}_X(x)$

$\hat{F}_X(x) = n^{-1} \sum_{i=1}^{n} I_{(-\infty, x]}(x_i)$ thats the formal definition for what we did above

This is also called emperical distribution function of x

↳ means sample.

Calculating Population Quantiles

Given value calculate percentile.

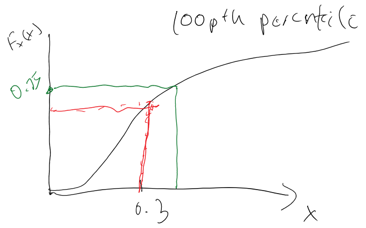
p-quantile Note .75 quantile $\xrightarrow{\times 100}$ = 75th percentile

Given value calculate percentile. $\times 100$

p-quantile   Note $.75$ quantile $=$ 75^{th} percentile

100pth percentile $\frac{1}{100}$

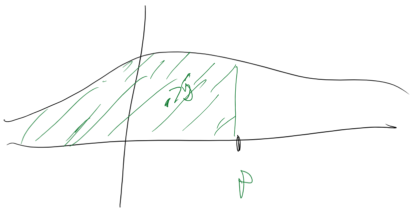
80^{th} percentile $= 0.8$ quantile

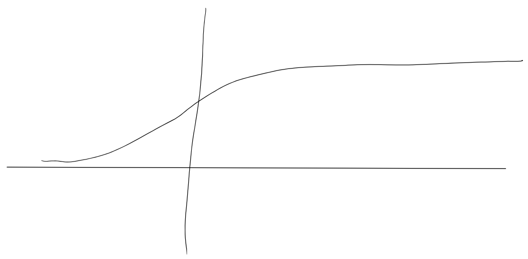


$f_x(0.3) = P[x \le 0.3]$

Percentile is opposite

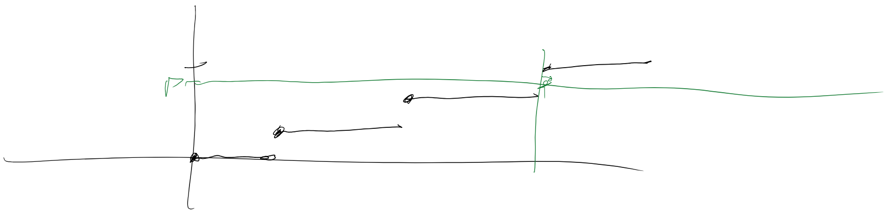
$f_x(x) = P[x \le x] = 0.75$





$F(x_p) = p$

$x_p = ?$ if continuous cdf



$x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_n$

$\{1.2, 2.1, 0.4, 3.3, -2.1, 4.0, -0.3, 2.2, 1.5, 5.0\}$

Put in ascending order   ← (x) bracket means ordered (n)

$x_{(1)} \quad x_{(2)}$

$\{-2.1, -0.3, 0.4, 1.2, 1.5, 2.1, 2.2, 3.3, 4.0, 5.0\}$

$\frac{1}{10} \quad \frac{1}{10} \quad \frac{1}{10} \quad \frac{1}{10} \quad \frac{1}{10} \quad \frac{1}{10} \quad \frac{1}{10} = \frac{7}{10}$

70^{th} percentile? 2.2

$P[x \le 2.2] = 0.7$

* if you don't have equal then go to the next one

$p = 0.75$? its a # between $2.2 - 3.3$

2.2 (70% ... 75% ...) 80% 3.3

2.2 (10% of data) 3.3

& take $\frac{1}{2}$

2.2 (5% of data) $= 2.2 + \frac{(3.3 - 2.2)}{2}$ ← interpolation

$\therefore \frac{i-1}{n} \le p \le \frac{i}{n}$

$7 \le p \le 8$

10 10

Continuous one always a solution, if discrete no solution sometimes!

$$x = x_{(i-1)} + (x_i - x_{i-1}) n \left(p - \frac{i-1}{n}\right) \overset{\text{in this case}}{\Longrightarrow} 2.2 + (3.3-2.2)\,10\,(0.75-0.7)$$

$$= 2.2 + (1.1)(0.5)$$

$$= 2.75$$

25th percentile ⟵ midterm / final question

$$\frac{i-1}{n} < p \le \frac{i}{n}$$

$$0.2 < p \le 0.3 \quad , i = 3$$

$$x = x_{(i-1)} + (x_i - x_{i-1}) n \left(p - \frac{i-1}{n}\right)$$

$$= -0.3 + (0.4+0.3)\,10\,(.25-.2)$$

$$= 0.05$$

✱ unconfirmed

if calculate percentile use this formula on midterm and final

25 th percentile = Q_1 = 1st quartile

75 th percentile = Q_3 = 3rd quartile

$P[x \le 1.5] = 0.5$ $P(x \ge 1.5) = 0.6$

{1.2, 2.1, 0.4, 3.3, -2.1, 4.0, -0.3, 2.2, 1.5, 5.0}

Put in ascending order

{-2.1, -0.3, 0.4, 1.2, 1.5, 2.1, 2.2, 3.3, 4.0, 5.0}

median

In one definition $P[x \le 1.5] = 0.5$ is enough to call a median but in another definition $P[x \ge 1.5) = 0.6$ will not since it != 0.5.

Recall if n is odd → $\frac{n+1}{2}$ th term

even → $\dfrac{\frac{n}{2}\text{th} + \frac{n}{2}+1 \text{ term}}{2}$

$P(x \ge 1.8) = 0.5$ ✓

$P(x \le 1.8) = 0.5$ ✓

1.5 | 2.1

1.8

Two definition of median, use easiest one.

1, 3, 5 median is $\frac{n+1}{2} = \frac{3+1}{2} = 2$nd term

Interquartile range: (width of data)

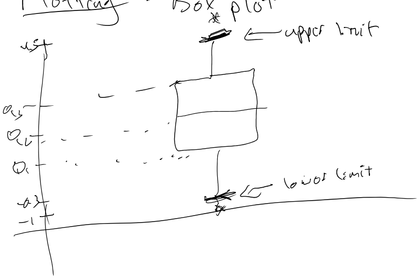
$IQR = Q_3 - Q_1$

replacement of SD.

extreme outliers median doesn't suffer

Skew → median
symmetric → mean.

Plotting - Box plot



lower limit = $Q_1 - 1.5 * IQR$

upperlimit = $Q_3 + 1.5 * IQR$

* Box plots also for continuous cases.

lower-limit = $Q_1 - 1.5 * (Q_3 - Q_1) = 1$ ← means line goes down to -1

stop at -0.3   * — outliers

upper limit = $Q_3 + 1.5 \times (Q_3 - Q_1)$
= 4.5

stops at 4.5

$\{-2.1 \,|\, -0.3, 0.4, 1.2, 1.5, 2.1, 2.2, 3.3, 4.0 \,|\, 5.0\}$ (-1, 4.5)

4.5 is also an outlier.

whiskers = *

adjacent value: 4.5, -1

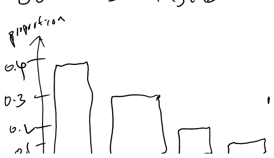
Ex/

(1) Cuc 0.42
(2) Van 0.28
(3) BS 0.22
(4) St 0.08

} categorial variable has no order.

if categorical stuff above does not apply.

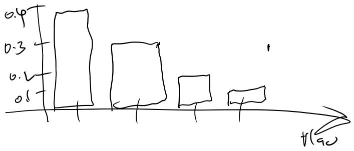
Do this instead



>steps

\# put shit in order

\# determine region



determine region

calculate percentile

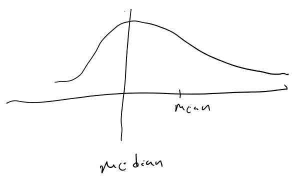
Recap

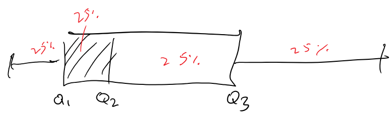
$$\tilde{x}_p = x_{(i-1)} + \left(x_{(i)} - x_{(i-1)}\right) n \left(p - \frac{i-1}{n}\right)$$

odd: $\left(\frac{n+1}{2}\right)$th term

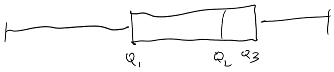
even: $\dfrac{\frac{n}{2}\text{th} + \left(\frac{n}{2}+1\right)\text{th}}{2}$

skewness:



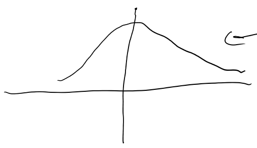


← right skewed distribution
↳ skew part is in the right



← left skewed distribution

if skewed use median and __not__ the mean



← mean = median = mode in standard normal table.

5.2 $X \sim Exp(1)$

① Predict future value
② $P[X>5]$
③ $P[X \le \underline{a}] = 0.95$

} These are inferences about the future X. (observation)

5.3 inference is about distribution

In this chapter $Exp(\theta) \rightarrow$ estimate θ

chi
↓
$\psi(\theta) = \frac{1}{\theta}$ mean $= \frac{1}{\theta}$ $var(\theta) = \frac{1}{\theta^2}$

$\psi(\theta) = \frac{1}{\theta^2}$ median $= \int_0^M f_\theta(x)\,dx = 0.5$

$$\Rightarrow F(m) = 0.5$$
$$\Rightarrow m = F^{-1}(0.5)$$

$$\boxed{\psi(\theta) = F_\theta^{-1}(0.5)}$$ median

25^{th} percentile: $\psi(\theta) = F_\theta^{-1}(0.25)$

Inverse ex/ Exp(1)

$f_\theta(x) = e^{-x}$

$F_\theta(x) = \int_0^x e^{-x}\,dx = 1 - e^{-x}$

$F_\theta(m) = 1 - e^{-m} = 0.5$

$e^{-m} = 0.5$

$-m = \ln(0.5)$

$\Rightarrow \quad m = -\ln(0.5)$

$\Rightarrow \quad F_\theta^{-1}(0.5)$

$f_\theta(x) \longrightarrow \text{Normal}(\mu, \sigma^2)$

$\psi(\theta) \longleftarrow$ Parameter of interest

$\psi(\theta) = \mu$

$x_1, x_2 \ldots x_n \leftarrow$ can you use this to get mean?

yea $\rightarrow \bar{x}$ use sample mean

$T(s) = \frac{1}{n} \sum x_i$

$T(s)$ is an estimate of $\psi(\theta)$

Variance: $\psi(\theta) = \sigma^2$

$\underset{\substack{\| \\ s^2}}{T(s)} = \frac{\sum (x_i - \bar{x})^2}{n-1}$ is an estimate of $\underset{\substack{\| \\ \psi(\theta)}}{\sigma^2}$

$\bar{x} \longrightarrow E(x)$

$n \longrightarrow$

5.5.8

3rd moment $N(\mu_0, \sigma_0^2)$

mean not

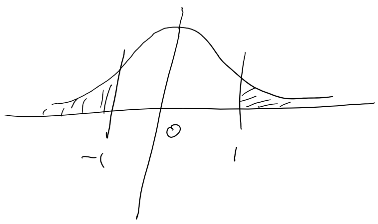
sigma not squared

$$\Psi(\theta) = E(x^3) = E[x-\mu+\mu]^3$$

$$= E[(x-\mu)^3 + 3(x-\mu)^2\mu + 3(x-\mu)\mu^2 + \mu^3]$$

$$= E[(x-\mu)^3] + E[3(x-\mu)^2\mu] + E[3(x-\mu)\mu^2] + E[\mu^3]$$

$$= \underbrace{E[(x-\mu)^3]}_{\text{3rd central moment}} + 3\mu\,\underbrace{E[(x-\mu)^2]}_{\substack{Var(x)\\ \| \\ \sigma^2}} + 3\mu^2 E[(x-\mu)] + \mu^3$$



$$\int_{-\infty}^{0} x f(x) + \int_{0}^{\infty} x f(x)$$

$$x \le 1 = x \ge 1$$

if the distribution is symmetric around (μ)

$$E[x-\mu]^1 = 0$$

$$E[x-\mu]^3 = 0$$

$$E[x-\mu]^5 = 0$$

$$\underbrace{E[(x-\mu)^3]}_{\substack{\text{3rd central moment}\\ \| \\ 0}} + 3\mu\,\underbrace{E[(x-\mu)^2]}_{\substack{Var(x)\\ \| \\ \sigma^2}} + 3\mu^2\,\underbrace{E[(x-\mu)]}_{\substack{\| \\ 0}} + \mu^3$$

$$= 3\mu\sigma^2 + \mu^3$$

if $N(\mu, \sigma^2)$, whats θ? $\theta = \{\mu, \sigma^2\}$

means all list of parameter

$$G(\alpha, \beta) \Rightarrow \theta = \{\alpha, \beta\}$$

October 1, 2018 3:10 PM

Section: 5.2

- Predicting future value
- $X \sim (\lambda)$
  - → mean
  - → mode: predicting x
  - → $P[x>5]$, $P[x>5 \mid x>1]$
  - → $E[x \mid x>1]$

Section 5.3

$S = x_1, x_2, x_3 \ldots x_n$ (Lets say you have a sample, what is $f_\theta(s) \leftarrow$ probability function of the sample)

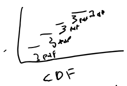
only if ⊥ $f_\theta(s) = f_\theta(x_1) \times f_\theta(x_2) \ldots f_\theta(x_n)$

- reparameterization: $\theta^2 - \theta$, one to one then we can reparameterize

Section: 5.4

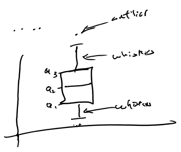
$F_x(x) \rightarrow f_x(x)$

population CDF



CDF

- Once you know the small f, can calculate everything.
- $\hat{F}_x(x) \leftarrow$ empirical distribution function
- histogram
- $\hat{f}_x(x)$
  - mean
  - median $\hat{x}_p = \ldots$
- Q_1, Q_3
- IQR
- $Q - 1.5(IQR)$
- box plot



- $\theta = \mu, \sigma^2$

  $E(x^3) = \ldots$ (refer to last class notes)

6.1 Likelihood Function

$X \sim Bern(\theta)$ ⊥ $\theta \in [0,1]$

$S = \{x_1, x_2, \ldots x_n\}$

$[f_\theta(s) = f_\theta(x_1) \times f_\theta(x_2) \ldots f_\theta(x_n)]$

Toss coins 5 times:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$[1, 0, 1, 1, 0] = S$$

$$\left.\begin{array}{l} f_\theta(s) = \theta(1-\theta)\theta\theta(1-\theta) \\ L(\theta|s) = \theta^3(1-\theta)^2 \end{array}\right\} \text{likelihood of } \theta$$

likelihood function of theta given the sample.

* Not the probability of observing θ. It is the probability of observing the sample for a given true θ.

$\theta \rightarrow$ fixed true value, $\theta \in \Omega$

θ_2 any other member from Ω

believe $f_{\theta_1}(s) > f_{\theta_2}(s)$

The more data the smaller the likelihood.

Ex / 6.1.1

Suppose $S = \{1, 2, \ldots\}$ and that the statistical model is $[P_\theta : \theta \in \{1,2\}]$, where P_1 is the uniform distribution on the integers $\{1, \ldots, 10^3\}$ and P_2 is the uniform distribution on $\{1, \ldots, 10^6\}$

Dist 1 $\rightarrow f_{\theta_1} \rightarrow$ Unif $\{1, 2, 3, \ldots 1000\}$

$\rightarrow$ Unif $\{1, 2, 3, \ldots \ldots 1000000\}$

Dist 1 → 001

Dist 2 → f_{θ_2} → Unif {1, 2, 3, ... 1000000}

$S = \{1001\}$ → comes from distribution 2. Since distribution 1 stops at 1000.

$S = \{10\}$ → # calculate likelihood for both and compare

$$\frac{L(\theta_1|S) = \frac{1}{1000}}{L(\theta_2|S) = \frac{1}{1000000}} \Rightarrow$$ sample 10 is a 1000 x more likely to come from θ_1

Example 6.1.4 $X \sim N(\theta, \sigma_0^2)$, σ_0^2 known?

1 unknown parameter θ, mean

$S = x_1 \cdot x_2 \cdot \cdot \cdot x_n$

$$f_\theta(s) = \prod_{i=1}^{n} \underbrace{(2\pi\sigma_0^2)^{-\frac{1}{2}}}_{\text{constant}} \exp\left[-\frac{1}{2\sigma_0^2}(x_i-\theta)^2\right]$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\theta)^2\right]$$

recognize this & expand it, Do it yourself.

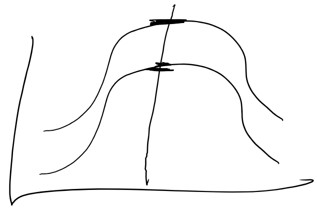
$$= \underbrace{(2\pi\sigma_0^2)^{-\frac{n}{2}}}_{\text{constant}} \exp\left[-\frac{n}{2\sigma_0^2}(\bar{x}-\theta)^2\right] \underbrace{\exp\left[-\frac{1}{2\sigma_0^2}(n-1)s^2\right]}_{\text{constant}}$$

sample variance

Since constant rewrite

$$L(\theta|s) \propto \exp\left[-\frac{n}{2\sigma_0^2}(\bar{x}-\theta)^2\right] \Rightarrow$$

proportional

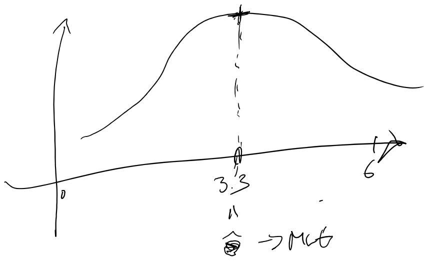


Same graph but just stretched as a result you can make inference about mean

$n = 25$, $\sigma_0^2 = 1$, $\bar{x} = 3.3$

$$L(\theta|S) \propto \exp\left[-\frac{25}{2}(3.3-\theta)^2\right]$$

recall
$y = 5x$
$y \propto x$ } proportional to x



$\hat{\theta}$ → MLE

① Point of estimate of θ

$\underbrace{\hat{\theta}}_{\text{estimate}} \rightarrow \theta$

MLE → maximum likelihood estimator of θ

$L(\hat{\theta}|s) \geq L(\theta|s)$

Note you can have multiple MLE for ex/

Ex 6.2.1

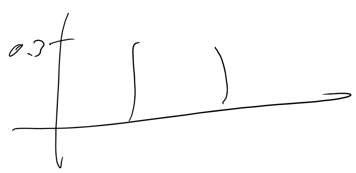
	s=1	s=2	s=3
f_1	0.3	0.4	0.3
f_2	0.1	0.7	0.2

$\{s=1\} \Rightarrow$ probably came from 1^{st} one

$\{s=2\} \Rightarrow$ probably came from 2^{nd} one

$\{s=3\} \Rightarrow$ probably came from 1^{st} one

	s=1	s=2	s=3
f_1	0.3	0.4	0.3
f_2	0	0.7	0.3



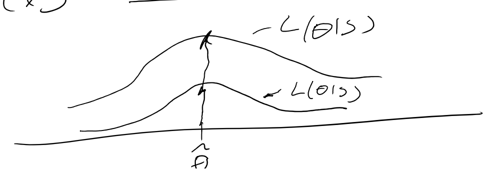
$\{s=3\} \Rightarrow$ MLE is probably Θ_1 or Θ_2

$L(\theta | s) = \prod_{i=1}^{n} f_\theta(x_i)$

$\Downarrow$ # taking the ln allows us to change to summation

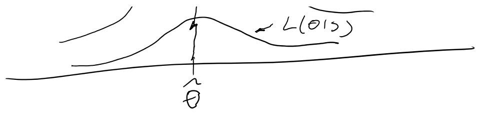
$\ln(L(\theta|s))$

$\ln(x)$ $\xrightarrow{1-1 \text{ function}}$ x



$\ln(L(\theta|s))$ Summa...

$= \sum_{i=1}^{n} \ln f_\theta(x_i)$



Now maximize

$$L(\theta|s) \propto \exp\left[\frac{-n}{2\sigma_0^2}(\bar{x}-\theta)^2\right]$$

proportional

(1) $\ell(\theta|s) = \frac{-n}{2\sigma_0^2}(\bar{x}-\theta)^2 \longleftarrow$ ln likelihood of θ

Differentiate

(2) Score function $= \frac{\partial \ell(\theta|s)}{\partial\theta}$

(3) score equation $= \frac{\partial \ell(\theta|s)}{\partial\theta} = 0$

$\hat{\theta} = \boxed{\text{something}}$

To check if max or min, second derivative.

(4) $$\left.\frac{\partial^2 \ell(\theta|s)}{\partial^2\theta}\right|_{\theta=\hat{\theta}} < 0$$

$$\ell(\theta|s) = \frac{-n}{2\sigma_0^2}(\bar{x}-\theta)^2$$

$$\frac{\partial \ell(\theta|s)}{\partial\theta} = -\frac{2n}{2\sigma_0^2}(\bar{x}-\theta)(-1)$$

$$= \frac{n}{\sigma_0^2}(\bar{x}-\theta) \longleftarrow \text{score}$$

$$\Rightarrow \frac{n}{\sigma_0^2}(\bar{x}-\theta) = 0$$

$$\theta = \bar{x}$$

check MLE

$$\text{Score} = \frac{n}{\sigma_0^2}(\bar{x}-\theta)$$

n sample size

$$\frac{\partial^2 \ell(\theta|s)}{\partial^2(\theta)} = -\frac{n}{\sigma_0^2} \longleftarrow \sigma^2 \text{ positive}$$

n, σ

$\partial^2(\theta)$

$$\therefore \frac{-n}{\sigma_0^2} \leq 0$$

$\therefore \bar{X}$ is the MLE of θ

Do it for exp distribution

Question midterm or final or both, learn to calculate MLE of everything distribution we known.

Invariance

$f_\theta(x)$ $\theta \to \hat{\theta}$ is MLE

$\underbrace{\psi(\theta)}_{\text{function of } \theta}$

one to one

how do you find MLE?

$\psi(\theta) \to \psi(\hat{\theta})$

6.2.2 If $(x_1, \dots, x_n)$ is a sample from $Ber(\theta)$ distribution where $\theta \in [0,1]$ is unknown, then determine the MLE of θ

$X \sim Ber(\theta) \longleftarrow$ mle ?

$S = x_1, x_2, \dots, x_n$

write the distribution plug $x_1, x_2 \dots x_n$

Likelihood $L(\theta | s) = \theta^{x_1}(1-\theta)^{1-x_1} \cdot (\theta)^{x_1}(1-\theta)^{1-x_2} \cdot \dots \cdot (\theta)^{x_n}(1-\theta)^{1-x_n}$

$$= \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

get ln likelihood

$n - \sum x_i$

ln likelihood $\ell(\theta|s) = \ln\left(\theta^{\sum_{i=1} x_i}\right) + \ln(1-\theta) \cdots$

$$= \sum_{i=1}^{n} x_i \ln\theta + \left(n - \sum_{i=1}^{n} x_i\right)\ln(1-\theta)$$

get first derivative

$$\frac{\partial \ell(\theta|s)}{\partial\theta} = \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta} - \left(n - \sum_{i=1}^{n} x_i\right)\cdot\frac{1}{1-\theta} \Big\} \text{ score}$$

set it to zero to get score equation

$$\frac{\partial \ell(\theta|s)}{\partial\theta} = \sum_{i=1}^{n} x_i \cdot \frac{1}{\theta} - \left(n - \sum_{i=1}^{n} x_i\right)\cdot\frac{1}{1-\theta} = 0$$

$$\Rightarrow \frac{\sum x_i}{\theta} = \frac{n - \sum x_i}{1-\theta}$$

$$\Rightarrow \frac{1-\theta}{\theta} = \frac{n - \sum x_i}{\sum x_i}$$

$$\Rightarrow \frac{1}{\theta} - 1 = \frac{n}{\sum x_i} - 1$$

$$\Rightarrow \frac{1}{\theta} = \frac{n}{\sum x_i}$$

$$\sum x_i = n\theta$$

$$\hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

check second derivative then you can say its MLE

Ex/ $\theta \to \bar{x}$ MLE $\Big\}$ since one to one on Ber(θ), [0, 1]

$\theta^2 \to \bar{x}^2$

$\theta^2 \to \bar{X}^2$

6.2.5 Suppose $(x_1, \dots, x_n)$

$X \sim \text{Uniform}[0, \theta]$

$f_\theta(x) = \frac{1}{\theta} I[x \in [0,\theta]]$

$x_1, x_2, \dots, x_n$

$L(\theta|S) = \frac{1}{\theta} \cdot \frac{1}{\theta} \cdots \frac{1}{\theta}$

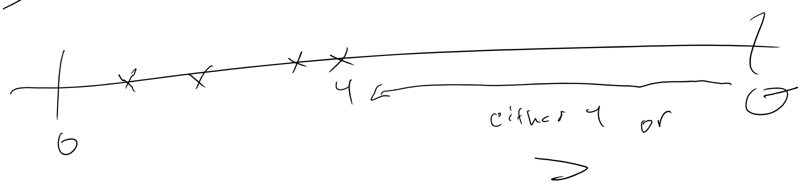
$= \frac{1}{\theta^n}$

#Differentiation part doesn't work

$\frac{1}{\theta^n} \to$ this is maximized when bottom is minimized.

$\boxed{\therefore \text{ when } \theta \text{ is minimum } L(\theta) \text{ is maximum}}$

Ex assume 1, 1.2, 2.3, 4



minimum θ = max value in sample

$\hat{\theta} = \max(S)$

Midterm: November 3rd
1–4 pm
IC #130

Recap:

Likelihood
↓ take ln of likelihood to get MLE
MLE $\frac{\partial}{\partial\theta} = 0 \Rightarrow \hat{\theta}$
$\frac{\partial^2 \ell}{\partial\theta^2} < 0$

Ex $X \sim Bern(\theta)$ $1 \Rightarrow H$
$s_1 = (1,0,1,1,0,0,1)$
$s_2 = (0,1,0,1,0,1,1)$

$L(\theta \mid s_1) = \theta(1-\theta)\theta\theta(1-\theta)(1-\theta)\theta$
$= (1-\theta)^3\theta^4$

$L(\theta, s_2) = (1-\theta)\theta(1-\theta)\theta(1-\theta)\theta\theta$
$= (1-\theta)^3\theta^4$

Ex/ $n = 7$
# of heads $= 4$
sol $\binom{7}{4}\underbrace{\theta^4(1-\theta)^3}_{L(\theta \mid s)}$

All 3 give you the same likelihood functions, because $\sum x_i = 4$, sum of random variables is 4.

$\boxed{\text{Sufficient statistics}}$ $\boxed{\text{Important stuff}}$
taking sum of sample
$T(s_1) = T(s_2)$
$\Rightarrow L(s_1) = L(s_2)$

may or may not have constant in the front. Constant doesn't matter when maximizing it.

T = sum of all your sample.
$f_\theta(s) = h(s) * \underbrace{g_\theta(T(s))}_{\theta,\ T(s)}$
$h(s)$: free of θ, only function of sample
$T(s)$: sufficient stat

Ex
$\underbrace{\binom{7}{4}}_{h(s)}\theta^4(1-\theta)^3$

In general
$\underbrace{\theta^{\sum x}(1-\theta)^{n-\sum x}}_{g_\theta(\ )}$

Ex $X \sim Exp(\theta)$

$x_1, x_2, \ldots \ldots x_n$

Sol
$f_\theta(s) = \theta e^{-\theta x_1} \cdot \theta e^{-\theta x_2} \cdots \cdots \theta e^{-\theta x_n}$
$= \theta^n e^{-\theta \sum x_i}$
$\Rightarrow h(s) = 1$ because both terms are of θ

$= \underbrace{1}_{h(s)} \cdot \underbrace{\theta^n e^{-\theta \sum x_i}}_{g_\theta(\ )}$ ← sufficient statistic

Ex $X \sim Pois(\theta)$

$s = x_1, x_2, \ldots x_n$

$$f_\theta(s) = \frac{e^{-\theta}\theta^{x_1}}{x_1!} * \frac{e^{-\theta}\theta^{x_2}}{x_2!} \times \cdots \frac{e^{-\theta}(\theta)^{x_n}}{x_n!}$$

likelihood function

$$= \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$

$$h(s) = \frac{1}{\prod_{i=1}^{n} x_i!}$$

$$g_\theta = \left(e^{-n\theta}\theta^{\sum x_i}\right)$$

$$T(s) = \sum x_i$$

if $[2,3,4,2,1]$ is # of accidents on HW 401 and follows $Po(\theta)$ then

$$\frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i} = e^{-5\theta}\theta^{12}$$

, don't need to know 5 different #, you just need the sum

take ln

$\frac{12}{5}$

, sum is sufficient stat of data

from likelihood you can get scenarios of x but not sample.

T(s) can be retain back from your likelihood is called a sufficient statistic.

sufficient means data reduction.

$T(s)$

$\uparrow$

minimal sufficient stat means you can't reduce further

if $s_1 = [x_1, x_2, \ldots x_n]$

$s_2 = [y_1, y_2, \ldots y_n]$

$$\frac{\theta^n e^{-\theta \sum x}}{\theta^n e^{-\theta \sum y}} = e^{-\theta(\sum x - \sum y)} \Rightarrow \text{if } \theta \; \sum x = \sum y$$

$\Rightarrow$ minimal statistic

Likelihood Review

For some data $S = (x_1, \dots, x_n)$ observed which follows a join distribution $f_\theta(s)$, write likelihood

$L(\theta|S) = f_\theta(s) = \prod_{i=1}^{n} f_\theta(x)$ when $x_i \sim$ iid

One of the things we can do with likelihood is sufficient statistic.

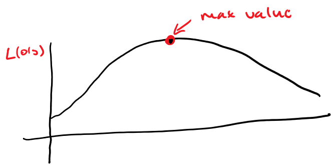
Sufficient statistic of θ ($T(s)$)

↳ To find use factorization theorem.
- This helps identify $T(s) \leftarrow$ sufficient statistic of θ

$$\boxed{f_\theta(s) = h(s)\, g_\theta(T(s))}$$

Maximum Likelihood Estimation (MLE)

- MLE ($\hat{\theta}$) value that maximizes $L(\theta|s)$ namely $L(\hat{\theta}|s) \geq L(\theta|s) \quad \forall \theta \in \Omega$



Steps for finding MLE

1. write out $L(\theta/s) \leftarrow$ likelihood function
2. take ln of $L(\theta|s) \leftarrow$ ln likelihood function
3. take the derivative of $\ln(L(\theta|s))$ with respect to θ
4. Set to 0 and solve for θ

Review chapter 4.6 for background

Inference based on MLE

- for $T(s)$ an estimate for $\psi(\theta)$, measure "closeness" using

$MSE_\theta(T(s)) = E_\theta[T(s) - (\psi(\theta))^2]$

mean squared error; $T(s)$: sufficient statistic of θ; E_θ: Expectation

= average squared distance between T and unknown $\psi(\theta)$

Recall, if $T(s) = \bar{x}$ then sampling distribution of $T(s)$ when S is $N(\mu, \sigma^2)$ is $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

- Easier MSE formula

$$\boxed{MSE_\theta(T(s)) = Var_\theta(T(s)) + (E_\theta(T(s)) - \psi(\theta))^2}$$

Proof

$E_\theta[(T(s) - \psi(\theta))^2]$

$= E_\theta[[T(s) - E_\theta(T(s))] + [E_\theta(T(s)) - \psi(\theta)]^2]$, cheap trick prof gonna use, add and subtract same

* update prof said look at the

value. Figure out why before the midterm!

ending answer and used only for proofs

$$= E\left[(T(s)-E[T(s)])^2\right] + \underbrace{2E\left[(T(s)-E[T(s)])(E[T(s)]-\psi(\theta))\right]}_{0} + (E[T(s)]-\psi(\theta))^2$$

, expand it $E(k)=0$ where k is a constant

$$= Var(T(s)) + \underbrace{(E[T(s)]-\psi(\theta))}_{\text{bias}}{}^2 + 0$$

Bias of T when $E_\theta[T]$ exists

$$Bias(T) = E_\theta[T(s)] - \psi(\theta) = \begin{cases} 0 & \text{then } T \text{ is unbiased} \\ & \text{otherwise } T \text{ is biased} \end{cases}$$

or equivalently, if $E[T(s)] = \psi(\theta)$ then T is unbiased.

Ex 6.3.1

$(x_1, \dots, x_n) \sim N(\mu, \sigma_0^2)$ with unknown μ and known σ_0. Want to calculate the MSE

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + (E[\hat{\theta}]-\mu)^2$$

$$= Var\left(\tfrac{1}{n}\textstyle\sum x_i\right) + (E(\bar{x})-\mu)^2$$

$$= \frac{1}{n^2}\sum Var(x_i) + \left(\frac{1}{n}\sum E(x_i) - \mu\right)^2$$

$$= \frac{1}{n^{\cancel{2}}}(\cancel{n}\sigma_0^2) + \left(\frac{1}{\cancel{n}}\underbrace{\cancel{n}\mu - \mu}_{0}\right)^2$$

$$= \frac{\sigma_0^2}{n} + \underbrace{(0)}_{}{}^2$$

This implies its unbiased

Previous section

MLE of $\underset{\text{mean}}{\mu}$ $\hat{\theta} = T(s) = \bar{x}$

Recall from 4.6

$\bar{x} \sim N(\mu, \frac{\sigma_0^2}{n})$

Example 6.3.4

Now $(x_1 \dots x_n) \sim N(\mu, \frac{\sigma^2}{n})$, (μ, σ^2) are <u>unknown</u>

We have MLE for (μ, σ^2) is $(\bar{x}, \frac{n-1}{n}s^2)$

still $\bar{x} \sim N(\mu, \frac{\hat{\sigma}^2}{n})$ so $E[\bar{x}] = \mu$

$$MSE(\bar{x}) = Var(\bar{x}) + Bias(\bar{x})$$

$$= Var(\bar{x}) + 0$$

$$= \frac{\hat{\sigma}}{n}$$

$$= \frac{1}{n^2}(n-1)s^2 \approx \frac{s^2}{n}$$

← for large values of n, we can approximate it to $\frac{s^2}{n}$ because as $n\uparrow$ $\frac{n-1}{n^2}$ becomes more insignificant

Bias

Variance of statistic is often called standard error $= \sqrt{\sigma^2}$

Confidence Intervals

* Copy slide 7, 8, 9 → she read off the slide really fast.

• "γ" is referred to as the confidence level of the interval
gamma

Example 6.3.6 $(x_1, \dots, x_n) \sim N(\mu, \sigma_0^2)$, μ unknown and σ_0^2 known
mean, variances

want to develop an interval for μ using our data that isn't too wide but has high confidence.

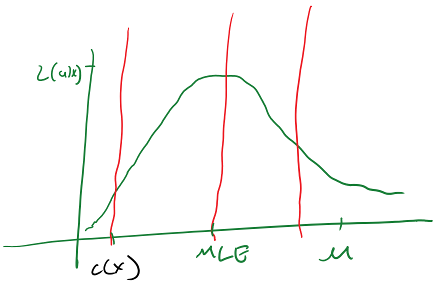
Start from likelihood

$$L(\mu \mid \underline{x}) \propto \exp\left\{\frac{-n}{2\sigma_0^2}(\bar{x}-\mu)^2\right\}$$
proportional

$\mathbb{R}$ – high confidence
– very wide

Define $C(x_1, \dots, x_n)$ such that $\mu_1 \in C(x)$ and
This is a "C" — This is a "C"

$L(\mu_2 \mid \underline{x}) \geq L(\mu \mid \underline{x})$ then $\mu_2 \in C(x)$



Thus, we can write

$$C(\underline{x}) = \{\mu : L(\mu \mid x_1, \dots, x_n) \geq k(x_1, \dots, x_n)\}$$

$$= \{\mu : \exp\left(\frac{-n}{2\sigma_0^2}(\bar{x}-\mu)^2\right) \geq k(x_1, \dots, x_n)\}$$

$$= \{\mu : \frac{-n}{2\sigma_0^2}(\bar{x}-\mu)^2 \geq \ln k(x_1, \dots, x_n)\}$$

$$= \{\mu \ (\bar{x}-\mu)^2 \leq \frac{-2\sigma_0^2}{n}\ln k(x_1, \dots, x_n)\}$$

$$= \{\mu \ \bar{x} - k^*(\underline{x})\frac{\sigma_0}{\sqrt{n}} \leq \mu \leq \bar{x} + k^*(\underline{x})\frac{\sigma_0}{\sqrt{n}}\}$$

where $k^*(x) = \sqrt{-2\ln(k(x))}$

Now we need to choose $k(x)$ or equivalently $k^*(x)$
we know

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma_0}{\sqrt{n}}} \sim N(0,1)$$

can rewrite $C(\underline{x})$ as a probabalistic statement by

$$\gamma \leq P(\mu \in C(x)) = P\left(\bar{x} - k^*\frac{\sigma_0}{\sqrt{n}} \leq \mu \leq \bar{x} + k^*\frac{\sigma_0}{\sqrt{n}}\right)$$

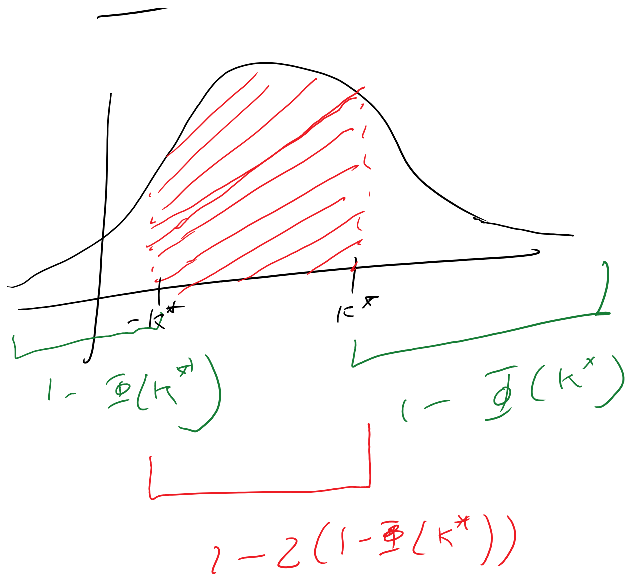
$$= P(-k^* \leq \bar{x} - \mu \ \ldots$$

· Visually

$$\ldots \le \mu \le \bar{x} + k^* \frac{\sigma_0}{\sqrt{n}}$$

$$= P\left(-k^* \le \frac{\bar{x}-\mu}{\frac{\sigma_0}{\sqrt{n}}} \le k^*\right)$$

$$= P\left(\left|\frac{\bar{x}-\mu}{\frac{\sigma_0}{\sqrt{n}}}\right| \le k^*\right)$$



$$\Rightarrow P(\underbrace{|Z|}_{N(0,1)} \le k^*) \ge \gamma$$

$$\Rightarrow 1-2(1-\Phi(k^*)) \ge \gamma$$

$$\Rightarrow \Phi(k^*) = \frac{1+\gamma}{2}$$

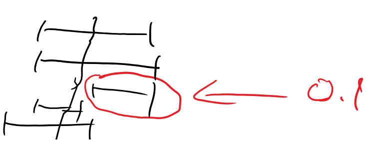
$$\Rightarrow \boxed{k^* = Z\left(\frac{1+\gamma}{2}\right)}$$ from $N(0,1)$ table

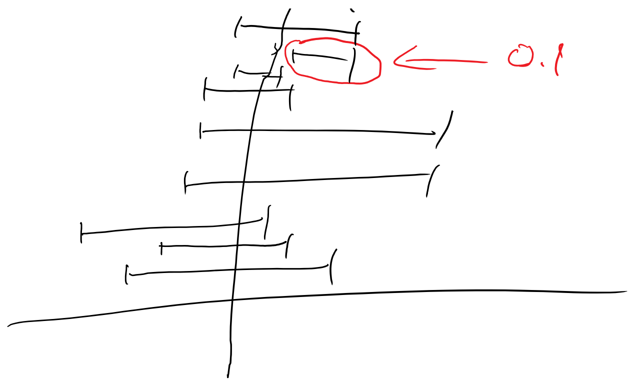
$$\boxed{C(S) = \left[\bar{x} - Z\left(\frac{1+\gamma}{2}\right)\frac{\sigma_0}{\sqrt{n}},\ \bar{x} + \underbrace{Z\left(\frac{1+\gamma}{n}\right)\frac{\sigma_0}{\sqrt{n}}}_{\text{margin of error}}\right]}$$

* based on the likelihood construction, proper CI interpretation is that for repeated sampling with CI constructed each time, then $100\gamma\%$ of these CI contain true $\psi(\theta)$

(!) write this down on exam if you want marks

Ex $\gamma = 0.9$

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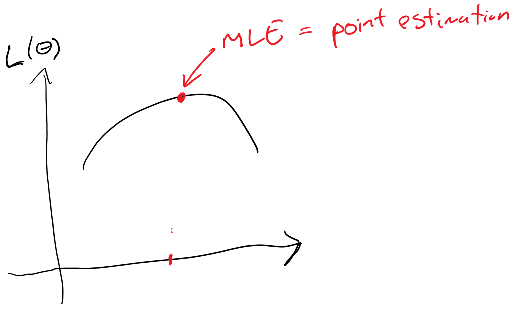
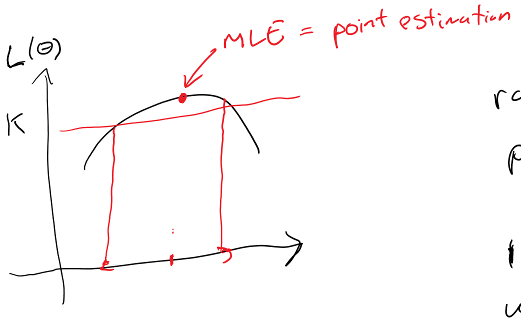




range of θ

$P[\text{likelihood} \geq k] \geq \gamma$

looking for a range of thetas where there is a gamma chance it will be above some threshold.

$P[\text{likelihood} \geq k] \geq \gamma$

$$\underbrace{P[\log \text{likelihood} \geq k^*] \geq \gamma}_{\text{likelihood interval for any distribution}}$$

Ex a) $N(\mu, \underbrace{\sigma_0^2}_{\text{known}})$, is $-\left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$ the log likelihood? yes.

b) $P\left[-\left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \geq \underbrace{k^*}_{\text{constant}}\right] \geq \gamma$

$P\left[\left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \leq \underbrace{k^{**}}_{\text{constant with neg}}\right] \geq \gamma$

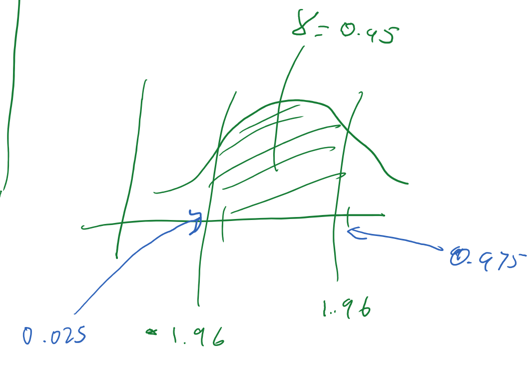
$P\left[-\sqrt{k^{**}} \leq \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq \sqrt{k^{**}}\right] \geq \gamma$

Only cause of CLT

$P\left[-z_{\frac{1+\gamma}{2}} \leq \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq z_{\frac{1+\gamma}{2}}\right] \geq \gamma$

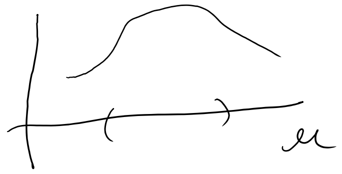
Central limit thm

$\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$



qnorm(0.025) = -1.96

$\sqrt{n}$



interval for μ since μ is unknown

$qnorm(0.025) = -1.96$

$qnorm(0.975) = 1.96$

if $\gamma = 0.9$.

$\left(\frac{1+\gamma}{2}\right) = \frac{1.9}{2} = 0.95$

Transform the equation to have μ on one side.

$$P\left[\bar{x} - z_{\frac{1+\gamma}{2}} \cdot \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\frac{1+\gamma}{2}} \frac{\sigma}{\sqrt{n}}\right] \Rightarrow \left(\bar{x} \pm z_{\frac{1+\gamma}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

Ex $X \sim N(\mu, \sigma^2 = 16)$, $n = 9$, $\bar{x} = 5$, calc 95% confidence interval

σ^2 is known ✓

whats gamma? 0.95

what quantile should I look at? $\frac{1+\gamma}{2} = 0.975 = \Phi(1.96)$

$$\Rightarrow 5 \pm 1.96 * \frac{4}{\sqrt{9}} \Rightarrow \begin{cases} 7.61 \\ 2.38 \end{cases}$$

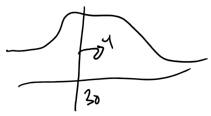
→ interpretation lower and upperbound contain the μ

$P[2.38 \le \mu \le 7.61] = 0.95$ ✗

likelihood is not a pdf, just a function.

$\bar{X}$ is a variable, because

lets say



Sample 2
$28 - 32$

30	31	29	30

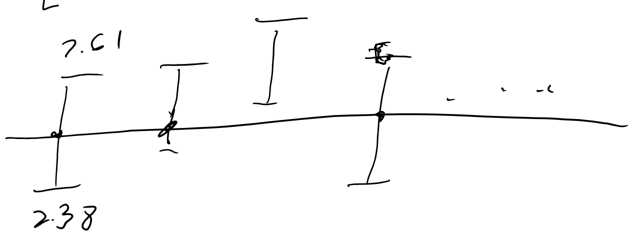
Sample 2
$x_2 - x_3$

32	28	25	μ_2

it changes every sample so its a variable

confidence interval is a variable.

$P[2.38 < \mu < 7.61] = 0.95$ ✗



look at the whole thing not just one case

correct

$P[lb \le \mu \le ub] = 0.95$ ✓

Ex $N(\mu, \sigma^2 \to \text{unknown})$

$$P\left[-k^{**} \le \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \le k^{**}\right] \ge \gamma$$

/ we can't use C/T since σ^2 is unknown

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$ (chi-square distribution) ← don't worry you will learn this later from

parameter

degree of freedom (df)

no parameter in standard normal

$N(0,1) \to$ parameter free $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\frac{\text{std Normal}}{\sqrt{\frac{\chi^2}{df}}} \sim t$$ ← will not be tested

magic

$$\frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{(n-1)s^2}{\sigma^2}}{(n-1)}}} = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \sim t\text{-distribution}$$

when you know sigma use standard normal

When you don't know, use t-distribution.