Programming Project idea's

n-body simulator

- Simulate the action of gravity on n-planets

Gradient Descent Optimizer

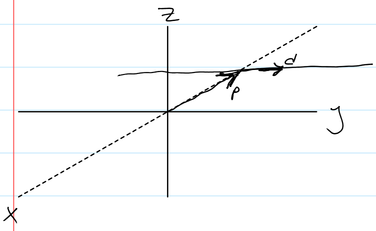
- Finds min/max of a function by following the path of steepest growth.

Lines & Planes

Recall the parametric equation of a line:

$\vec{x} = \vec{p} + t\vec{d}$

$\vec{p}$ = initial point
$\vec{d}$ = direction
$\vec{t}$ = parameter



Ex/ Find the line passing through: $(1,2,3)$ & $(4,5,6)$

Decide the initial point, in this case $(1,2,3)$.

$\vec{p} = (1,2,3)$

Calculate the direction vector.

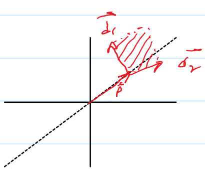
$\vec{d} = (4,5,6) - (1,2,3) = (3,3,3)$

Thus the line is:

$\vec{x} = (1,2,3) + t\,(3,3,3)$
$\quad = (1,2,3) + s\,(1,1,1)$

Definition: The parametric equation for a plane

$\vec{x} = \vec{p} + s\vec{d_1} + t\vec{d_2}$ where $\vec{d_1}$ and $\vec{d_2}$ are non zero and not parallel



Ex/ Find the parametric equation for the plane containing
(1,0,0) (0,1,0) & (0,0,1)

Pick the initial point

$\vec{p} = (0,1,0)$

calculate $\vec{d_1}$

$\vec{d_1} = (1,0,0) = (1,-1,0)$

Calculate $\vec{d_2} = (0,0,1) - (0,1,0) = (0,-1,1)$

$\vec{x} = (0,1,0) + s(1,-1,0) + t(0,-1,1)$

The normal form of plane

We can imagine planes as the set of points orthogonal to a given vector

Ex/ Write the x-y plane in normal form

observe the z-axis is orthogonal to xy

$xy\text{-plane} = \{\vec{v} : \vec{v} \cdot (0,0,1) = 0\}$

(!) if a plane π (not pi, plane variable!) is defined by:

$\pi = \{\vec{v} : \vec{v} \cdot \vec{n} = 0\}$ Then $\vec{v} = \vec{0}$ is always in π (our plane)

($\vec{v} \cdot \vec{n} = 0$ for any $\vec{n}$)

Ex/ Show that (0 -2 -1) (1 4 0) (2 10 1) do not lie on a unique plane.

observation: If they lie on the line then we get no unique plane

consider: $\vec{x} = (0\ -2\ -1) + t((2\ 10\ 1) - (1\ 4\ 0))$
$= (0\ -2\ -1) + t(1\ 6\ 1)$

$\boxed{t=0} \Rightarrow \vec{x} = (0\ -2\ -1)$

$\boxed{t=1} \Rightarrow \vec{x} = (1\ 4\ 0)$

$\boxed{t=2} \Rightarrow \vec{x} = (2\ 10\ 1)$

Any plane containing this line will contain all three point thus, all three points lie on a line

Definition: The normal form of a plane π

$\pi = \{ \vec{v} : (\vec{v} - \vec{p}) \cdot n = 0 \}$

Where $\vec{p}$ is a point in π and $\vec{n}$ is the normal of π
($\vec{n}$ orthogonal to all $\vec{v} - \vec{p}$)

Ex/ Express the plane containing $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in normal form.

we have points but lack a normal

make a normal

$\vec{d_1} = (0,1,0) - (1,0,0) = (-1,1,0)$

$\vec{d_2} = (0,0,1) - (1,0,0) = (-1,0,1)$

To make the normal find something orthogonal to these

Find $\vec{n}$ so that $\vec{n} \cdot \vec{d_1} = 0$ & $\vec{n} \cdot \vec{d_2} = 0$

Assume $\vec{n} = (x,y,z)$

$0 = \vec{n} \cdot \vec{d_1} \Rightarrow 0 = -x + y \Rightarrow x = y$

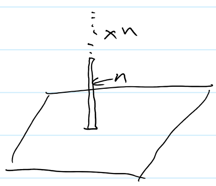
$0 = \vec{n} \cdot \vec{d_2} \Rightarrow 0 = -x + z \Rightarrow x = z$

(!) non zero !

We get $x = y = z$

we pick $\vec{n} = (1,1,1)$

(we could pick $\vec{n} = (42, 42, 42)$ because we can scale it. Only condition is non zero. Think of it as an extension of the plane and vectors



Thus, the normal form is: $\pi = \{ \vec{v} : (\vec{v} - (1,0,0)) \cdot (1,1,1) = 0 \}$

Lines & Planes

In 3D two lines can be:

- co-incident: same line

- intersect: at a point
- parallel: never touch
- skew: non parallel, never touch

Ex/ Determine whether these lines intersect?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

① $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+3t \\ -1+t \\ 1+6 \end{bmatrix}$ ② $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1+3s \\ -2+s \\ 0+s \end{bmatrix}$

③ # Set the equations equal

$$\begin{bmatrix} 2+3t \\ -1+t \\ 1+6t \end{bmatrix} = \begin{bmatrix} -1+3s \\ -2+s \\ s \end{bmatrix}$$

$$\begin{bmatrix} 3t-3s \\ t-s \\ 6t-s \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

④ # Use linear Algebra

$$\left[\begin{array}{cc|c} 3 & -3 & 3 \\ 1 & -1 & -1 \\ 6 & -1 & -1 \end{array}\right] \xrightarrow{R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & -3 & 3 \\ 6 & -1 & -1 \end{array}\right] \xrightarrow[R_3-6R_1]{R_2-3R_1} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ we obtain $(t,s) = (0,1)$

Check your work!

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Ex/ Determine where the line meets the plane

$\ell: \vec{x} = (2,1,0) + t(1,1,0)$

$\pi: x + 2y + 3z = 0$

Normal form: $\boxed{(x,y,z)\cdot(1,2,3)=0}$

Find $\vec{x} = (x,y,z)$ for the line and plug it into the plane.

$(x,y,z) = (2,1,0) + t(1,1,0) = (2+t, 1+t, 0)$

From $x + 2y + 3z = 0$ we get:

$$(2+t) + 2(1+t) + 3\cdot 0 = 0$$
$$2+t+2+2t = 0$$
$$3t = -4$$

Thus, $t = -\frac{4}{3}$ $\Rightarrow$ Thus, the line and plane meet at

$$\left(2-\frac{4}{3}, 1-\frac{4}{3}, 0\right) = \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$$

Lecture 1

Course Website

http: //pgadey.ca

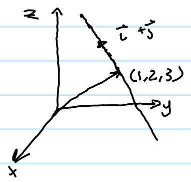
Vectors

In this course, there are 3 notations:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = (x,y,z) = \underbrace{x\vec{i} + y\vec{j} + z\vec{k}}_{\text{standard basis}}$$

Ex/ $\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\vec{j} = \begin{bmatrix}1\\2+3\\3\end{bmatrix} = \begin{bmatrix}1\\5\\3\end{bmatrix}$

Ex/ Write the parametric equation for a line passing through $(1,2,3)$ in the direction $\vec{i} + \vec{j}$



$(x,y,z) = (1,2,3) + t(\vec{i} + \vec{j})$

start at $(1,2,3)$ add a stretched copy of $\vec{i} + \vec{j}$

$= (1,2,3) + t(1,1,0)$

$= (1+t, 2+t, 3)$

* Note this equation is parametric because it is dependant on t.

Lengths & Angles

The pythagorean Theorem says: $\|(x,y,z)\|^2 = x^2 + y^2 + z^2$

Notation: $\|\vec{v}\|$ is the Length of vector $\vec{v}$

Definition: The dot product

$$\begin{bmatrix}a\\b\\c\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = ax + by + cz$$

Fact: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$ where
θ = angle between $\vec{u}$ and $\vec{v}$

Two vectors are parallel if $\vec{u} = \lambda \vec{v}$ for some number λ

Ex/ For what values of x are $\vec{u} = (5,x,3)$ & $\vec{v} = (x,x,2)$ orthogonal?

$\vec{u} \cdot \vec{v} = 5x + x^2 + 6 = x^2 + 5x + 6 = (x+3)(x+2)$

$\therefore$ we get $\vec{u} \cdot \vec{v} = 0$ when $x = -2$ or $x = -3$

Ex/ Find when two nonparallel vectors are orthogonal to $(1,2,3)$

Suppose $\vec{u} = (x,y,z)$ orthogonal to $(1,2,3)$

$\vec{u}\,(1,2,3) = 1x + 2y + 3z = 0$

Pick (x,y,z) to get orthogonality

$\vec{u} = (3,0,-1)$ $\quad \vec{v} = (0,3,-2)$

we have $(1,2,3) \cdot \vec{u} = 0$ and $(1,2,3) \cdot \vec{v} = 0$
but $\vec{u}$ and $\vec{v}$ are not parallel.

Plan for this week

- → Matricies ⊛
- → Determinants
- → Cross Product

Matrix Multipication

Recall from last week

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

We are going to define Matrix Multipication

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 = \vec{u}\cdot\vec{v}$$

For a matrix $M = [m_{ij}]$ we define the <u>transpose</u> of M

$$M^T = [m_{ji}]$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

We obtain for vectors $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$$

Definition: If M is an $n \times k$ matrix and N is a $k \times \ell$ matrix

$$M = \underbrace{\begin{bmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ \text{---} & r_n & \text{---} \end{bmatrix}}_{\text{rows vectors}} \quad \text{and} \quad N = \underbrace{\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_n \\ | & | & | \end{bmatrix}}_{\text{column vectors}}$$

We define $MN = [r_i \cdot c_j]$

Ex/ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{cases} r_1 = [1\ 2] \\ r_2 = [3\ 4] \end{cases}$

$$\begin{bmatrix} \cdots \\ 3\ 4 \end{bmatrix} \begin{bmatrix} \cdots \\ 0\ -1 \end{bmatrix} \qquad \begin{aligned} r_1 &= [\,1\ 2\,] \\ r_2 &= [\,3\ 4\,] \\ c_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ c_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 \\ r_2 \cdot c_1 & r_2 \cdot c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$$

Another interpertation of $A\vec{x}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Try and formulate $A\vec{x}$ using the dot product.

$A\vec{x}$

$$= \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_n - \end{bmatrix} \begin{bmatrix} | \\ c \\ | \end{bmatrix} = \begin{bmatrix} r_1 \cdot c_1 \\ r_2 \cdot c_1 \\ r_n \cdot c_1 \end{bmatrix}$$

dot product

Matrix Inverses

Definition: The inverse of a matrix M is M^{-1} so that

$$MM^{-1} = M^{-1}M = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix inversion algorithm

Goal: To invert M

① Setup the augmented matrix $[M \mid I]$
② Row reduce $[M \mid I]$ to $[A \mid \tilde{M}]$

If $A = I \Rightarrow \tilde{M} = M^{-1}$

otherwise:

M is not invertible

Ex/ Find the inverse of $M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$[M \mid I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 - R_2} \left[\,\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \;\middle|\; \begin{matrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix}\,\right]$$

Thus, $M^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

Check our work

$$MM^{-1} = I$$

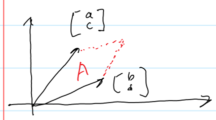
$$MM^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Determinants

Definition: The determinant of a 2×2 matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Fact: In all dimensions the determinant measures the volume of a parallel-piped



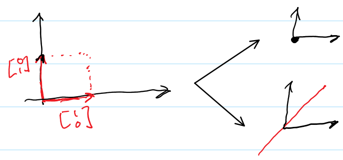
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ = Area of A

if non invertible matrix there is no area since the basis vectors becomes parallel

Theorem: if $\det(A)=0$ Then A is not invertible



$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ everything maps to $\vec{0}$

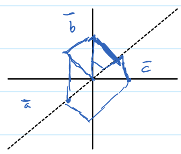
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and basis vectors becomes parallel

Definition: The Determinant of a 3×3 matrix is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Definition: The parallel piped formed by the vector $\vec{a}, \vec{b}, \vec{c}$ emanating from the origin of $\mathbb{R}^3$ is:

$$P(\vec{v} \ldots \vec{v}_n) = \{ \vec{x} = \sum_{i=1}^{n} t_i \vec{v}_j,\ t_j \in [0,1] \}$$



All possible combination of $s\vec{a} + t\vec{b} + u\vec{c}$ where $s \in [0,1]$, $t \in [0,1]$ and $u \in [0,1]$

Fact: (Restatement of Def)

$$\det(\vec{v}_1 \ldots \vec{v}_n) = \text{vol}(P(\vec{v}_1 \ldots \vec{v}_n))$$

The cofactor expansion

Goal: To calculate $\det(M)$, let $M=[m_{ij}]$ and $\tilde{M}_{ij}$ = matrix M with row i and column j removed.

Define $\det([a])=a$

$$\det(M)=\sum_{j=1}^{n}(-1)^{1+j}\, m_{1j}\det(\tilde{M}_{1j}) \quad \Leftarrow \text{memorize this}$$

Ex/ Calculate $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = (-1)^{1+1}(1)\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{0} & 4 & 5 \\ \cancel{0} & 0 & 6 \end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 0 & \cancel{4} & 5 \\ 0 & \cancel{0} & 6 \end{vmatrix} + (-1)^{1+3}(3)\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 0 & 4 & \cancel{5} \\ 0 & 0 & \cancel{6} \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 0 & 6 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} = (46-0-0)+0+0 = 24$$

Algebraic Facts about determinants

det plays nicely with row operations

① $A \xrightarrow{R_i \leftrightarrow R_j} B$ Then $\det(A) = -\det(B)$

② $A \xrightarrow{kR_i} B$ Then $\det(B) = k\det(A)$

③ $A \xrightarrow{R_i + R_j} B$ Then $\det(A) = \det(B)$

Fact (Hamilton)

$$\det(AB)=\det(A)\det(B)$$

Ex/ Calculate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix} \xrightarrow{R_3 - R_1} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{vmatrix} \xrightarrow{R_3 - R_2} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} \overset{R_1 \leftrightarrow (-1)R_3}{=} \begin{vmatrix} 0 & 0 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = (-1)(0) = 0$

Fact: if A is invertible then $\det(A) \neq 0$

we know $AA^{-1}=I$

Thus $\det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$

and $\det(A^{-1}) = \frac{1}{\det(A)}$, thus $\det(A)\neq 0$.

Lemma

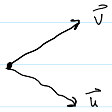
$\det(I) = 1$

The Cross Product

Definition: The cross product of $\vec{u}, \vec{v}$, of $\vec{u}, \vec{v} \in \mathbb{R}$ satisfies:

① $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \sin\theta$

② $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$



$\vec{u} \times \vec{v}$ is out of the paper

"Right hand rule. line your fingers against u curl to $\vec{v}$. The way your thumb is pointing is the cross product.

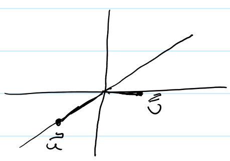
Right thumb = cross product.

Fact: If $\vec{u} = (u_1\ u_2\ v_3)$ & $\vec{v} = (v_1\ v_2\ v_3)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

* Note: $\vec{i}, \vec{j}, \vec{k}$ are formal variables

Ex/ Calculate $\underbrace{(1,0,0)}_{u} \times \underbrace{(0,1,0)}_{v}$



out of the paper

$\therefore$ guess $(0,0,1)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= i \cdot 0 - j \cdot 0 + 1k = K$$

Ex/ (§ 1.3 0 3) Calculate

$(\vec{i} - 2\vec{j} + \vec{k}) \times (2\vec{i} + \vec{j} + \vec{k})$

$= (1\ -2\ 1) \times (2\ 1\ 1)$

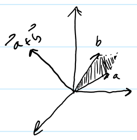
$$= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = i \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} + \vec{j} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = (-3)\vec{i} + \vec{j} + 5\vec{k}$$

check our work $(\vec{a}\times\vec{b})\cdot\vec{a}=0$

$(1\ -2\ 1)\cdot(-3\ 1\ 5) = -3 - 2 + 5 = 0$

$(2\ 1\ 1)(-3\ 1\ 5) = -6 + 1 + 5 = 0$

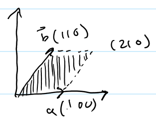
Fact: The area of a parallelogram spanned by $\vec{a}$ $\vec{b}$ is
$\|\vec{a}\times\vec{b}\|$



$A = \|\vec{a}\times\vec{b}\|$

① cross product gives you a normal
② It gives you an area.

Ex/ Calculate the area of the parallelogram
$\vec{a}=(1\,0\,0)$ $\vec{b}=(1\,1\,0)$



Area $= \|\vec{a}\times\vec{b}\| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$= i\begin{vmatrix}0&0\\1&0\end{vmatrix} + j\begin{vmatrix}1&0\\1&0\end{vmatrix} + k\begin{vmatrix}1&0\\1&1\end{vmatrix} = 0\cdot i - 0\cdot j + 1\vec{k} = \|(0,0,1)\| = 1$

Summary of this week:

Determinants
- ↳ Cofactor expansion
- ↳ volume of parallel piped
- ↳ $\det(AB) = \det(A) + \det(B)$
- ↳ row operations
- ↳ invertibility

Cross product
- ↳ right hand rule
- ↳ formula
- ↳ area of parallelogram
- ↳ $\vec{a}\times\vec{b}$ orthogonal to $\vec{a}$ and $\vec{b}$

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Matrix Multiplication
- ↳ dot product
- ↳ matrix inversion algorithm
- ↳ transpose

Ex (1.3 Q 13): Show that the points $(0,-2,-1)$ $(1\,4\,0)$ and $(2\,10\,1)$
do not lie on a unique plane

Find the volume of parallelpiped

$\begin{vmatrix} 0 & 1 & 2 \\ \end{vmatrix}$ $\begin{vmatrix} 0 & 1 & 2 \\ \end{vmatrix}$ $\begin{vmatrix} 1 & 0 & -1 \\ \end{vmatrix}$ $\begin{vmatrix} 1 & 0 & -1 \\ \end{vmatrix}$ $\begin{vmatrix} 1 & 0 & -1 \\ \end{vmatrix}$ $\begin{vmatrix} 0 & 0 & 0 \\ \end{vmatrix}$

$$\begin{vmatrix} 0 & 1 & 2 \\ -2 & 4 & 10 \\ -1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 1 & 2 \\ -2 & 4 & 10 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 4 & 10 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

Since the parallel piped has a volume of zero, they lie on a line and we get no unique plane.

$\det(A) = \det(A^T)$

July 10, 2018 8:06 PM

Remark: Available Friday July 13^{th} IC200, 4:00pm-6:00pm
o/w attend office hours.

Mock Final: Friday July 27^{th} - SY110 at 12:00-3:00pm

for mock final, stay for the whole thing.

Constrained Optimization

So far: "Minimize / maximize $f(\vec{x})$ on its domain"

We get interesting behaviour when we restrict the domain

Ex/ Maximize $f(x) = x^2$ on $[1,2]$

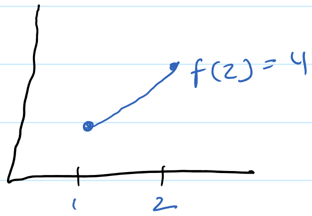
Find the critical points

$f'(x) = 2x = 0 \Rightarrow x = 0$

(!) 0 is outside of $[1,2]$

check the end points

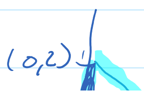
$f(1) = 1 \qquad f(2) = 4$



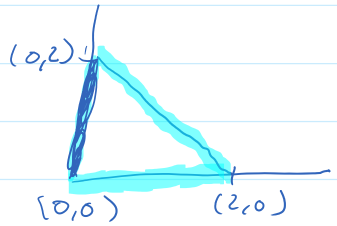
Ex Minimize $f(x,y) = x^2 + y^2$ on the triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$

Find the critical points

$f_x = 2x = 0 \qquad \Rightarrow (x,y) = (0,0)$



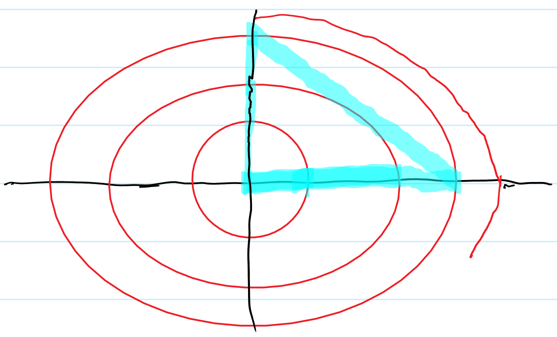
$f_x = 2x = 0$
$f_y = 2y = 0$ $\Rightarrow (x,y) = (0,0)$



\# Apply second derivative test

$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow$ minimum

\# check boundary ("end points")



Look for the largest level set that still touches the boundary, we got $f(0,2) = f(2,0) = 4$

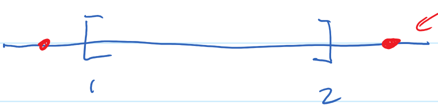
The maximum of $f(x,y)$ on the triangle is $f(0,2) = f(2,0) = 4$

Definition $S \subseteq \mathbb{R}^n$ is OPEN if: $\forall\, x \in S\ \exists\, \varepsilon > 0\ D_\varepsilon(x) \subseteq S$

"Any point can be surrounded by a disk"

Definition A set $T \subseteq \mathbb{R}^n$ is CLOSED if $S = \mathbb{R}^n \setminus T$ is open

"Every point outside of T can be surrounded by a disk."



can be surrounded by a disk

Definition A set $U \subseteq \mathbb{R}^n$ is bounded if there is N such that $U \subseteq D_N(\vec{0})$

" U can be surrounded by a disk "

Thrm (Extreme Value)

If S is closed and bounded then any continuous $f: S \to \mathbb{R}$ achieves a maximum and minimum number

$f(x_0) = M \quad f(x_1) = m$

Why closed and bounded?

	Yes	No
Yes	☺	$[0,\infty)$ $f(x) = \frac{1}{1+x^2}$
No	$(0,1)$ $f(x) = x$	$(0,\infty)$ $f(x) = \frac{1}{1+x^2}$

\# max but no min

\# No max / No min

\# No max or min

Ex Final 2018

Find the maximum of $f(x,y,z) = x^2 - 3x + y^2 - 2y + z^2 - 4z + 1$ on the ball $x^2 + y^2 + z^2 \le 9$

\# Find the critical points

$$\left.\begin{aligned} f_x &= 2x - 3 = 0 \\ f_y &= 2y - 2 = 0 \\ f_z &= 2z - 4 = 0 \end{aligned}\right\} \Rightarrow (x,y,z) = \left(\tfrac{3}{2}, 1, 2\right)$$

"This point $(\frac{3}{2}, 1, 2)$ is in the ball"

\# Apply second derivative test

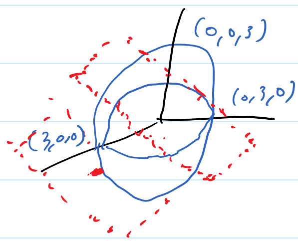
$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{critical point is a minimum}$$

$f(x,y,z) = x^2 - 3x + y^2 - 2y + z^2 - 4z + 1$

use the constraint $x^2 + y^2 + z^2 = 9$

$f(x,y,z) = 9 - 3x - 2y - 4z + 1$

$= 10 - 3x - 2y - 4z$



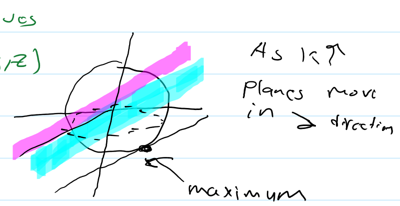
who ever is reading this - sorry can't draw

July 14, 2018 7:14 PM

Optimization

Maximize $f(x,y,z) = 10 - 3x - 2y - 4z$

on the sphere: $x^2 + y^2 + z^2 = 9$



As we increase the value of k, the planes move through space in a particular direction

when the maximum occurs, the level set is tangent to the constraint

The maximum should be tangent to the constraint.

Note: planes are parallel and have the same normal

Find the point of tangency

All the planes $f(x,y,z) = k$ have the same normal
$\vec{n} = (-3,-2,-4)t$

Find where $l: (-3,-2,-4)t$ meets sphere.

$(-3t)^2 + (-2t)^2 + (-4t)^2 \iff 9t^2 + 4t^2 + 16t^2$

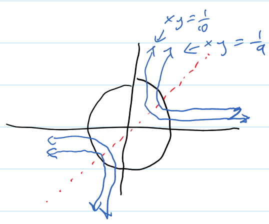
$29t^2 = 9 \Rightarrow t = \pm \frac{3}{\sqrt{29}}$

We pick: (x,y,z)

$= +\frac{3}{\sqrt{29}}(-3,-2,-4)$

Note: negative root corresponds to minimum.

Ex Find the maximum of $f(x,y) = xy$ on $x^2 + y^2 = 1$



$xy = \frac{1}{10}$

$x^2 + y^2 = 1$

Draw constraints

Draw some level sets

We need to find the point of tangency.

Observe both curves are symmetric about $y = x$.

Assume $\boxed{y = x}$

$x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1$

$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\sqrt{\frac{1}{2}} = \frac{1}{\pm\sqrt{2}}$$

We obtain: $f(x,y) = xy$ is tangent to $x^2+y^2=1$
at $(x,y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

We get $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} \rightarrow$ seems to be a maximum

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \rightarrow$ seems to be the maximum

Look at gradient at the point of tangency

$f(x,y) = xy \Rightarrow \nabla f = (y, x)$

$g(x,y) = x^2+y^2 \Rightarrow \nabla g = (2x, 2y)$

At the point of intersection the gradients are parallel. This means that they have the same normal for there tangent plane, so same tangent plane.

The method of Lagrange Multiplier (Parallel Gradients)

To optimize $f(\vec{x})$ subject to $g(\vec{x}) = 0$

① Define $\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) - \lambda g(\vec{x})$

② Find the critical points of $\mathcal{L}(\vec{x}, \lambda)$

③ Select the critical points that minimize/maximize $f(\vec{x})$

Observations:

① $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0$ $\quad$ $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ $\quad$ Parallel gradients

$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \iff \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \iff \nabla f = \lambda \nabla g$

② $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff -g(x) = 0$

$\iff g(x) = 0$

We satisfy the constraint.

Ex Optimize $f(x,y) = xy$ on $x^2 + y^2 = 1$

Rewrite the constraint

$g(x,y) = x^2 + y^2 - 1 = 0$

Set up Lagrang function

$\mathcal{L}(x,y,\lambda) = xy - \lambda(x^2+y^2-1)$

Find the critical points

$\mathcal{L}_x = y - \lambda(2x) = 0$ $\qquad\qquad y = \lambda(2x) = \mathcal{L}_x$

$\mathcal{L}_y = x - \lambda(2y) = 0 \quad \Rightarrow \quad x = \lambda(2y) = \mathcal{L}_y$

$\mathcal{L}_\lambda = (x^2+y^2-1) = 0$ $\qquad\qquad x^2+y^2 = 1 = \mathcal{L}_\lambda$

$x = 2\lambda y$

$x = 2\lambda(2\lambda x)$

Either $\boxed{x = 0}$ or $\boxed{1 = 4\lambda^2}$

$\Downarrow$ $\qquad\qquad \Downarrow$

$y = 0$ $\qquad\qquad \lambda = \pm\frac{1}{2}$

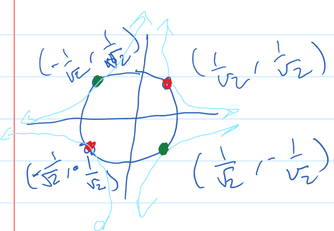
$(x,y) = (0,0)$ not on circle since $x^2+y^2=1$, $0^2+0^2 \neq 1$

$\boxed{y = -x}$ $\qquad \boxed{y = x}$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Solve $x^2+y^2=1$



● min

● max

Ex Maximize $f(x,y,z) = \ln(x) + \ln(y) + \ln(z)$
subject to $x + y + z = 1$.

re-write constraints

$g(x,y,z) = x + y + z - 1$

Set up Lagrange function

$\mathcal{L} = [\ln(x)+\ln(y)+\ln(z)] - \lambda(x+y+z-1)$

Find critical points # Tip: isolate for λ

$\mathcal{L}_x = \frac{1}{x} - \lambda = 0 \iff \lambda = \frac{1}{x}$

$\mathcal{L}_y = \frac{1}{y} - \lambda = 0 \iff \lambda = \frac{1}{y}$

$\mathcal{L}_z = \frac{1}{z} - \lambda = 0 \iff \lambda = \frac{1}{z}$

$\Rightarrow \boxed{\frac{1}{x} = \frac{1}{y} = \frac{1}{z}} \Rightarrow \boxed{x = y = z}$

$\mathcal{L}_\lambda = x+y+z-1 = 0$

Solve

$x+y+z = 1$
$x+x+x = 1$
$3x = 1$
$x = \frac{1}{3} = y = z$

you can make $f(x,y,z)$ negative by $y = 0.01$ or $z = e^{-1000}$

Ex (§ 3.4 Q 30)

Suppose $S \subseteq \mathbb{R}^3$ is a surface defined by $f(x,y,z) = 1$. Suppose P is a point on S. So that the distance to the origin is maximal.

Claim: $\vec{OP}$ has to be $\perp$ to S

Consider maximizing $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ on the surface $f(x,y,z) = 1$

The Lagrange function is

$\mathcal{L}(x,y,z,\lambda) = d - \lambda(f-1)$

if we are a critical point to the Lagrange function Then:

$\mathcal{L}_x = d_x - \lambda f_x = 0$

$\mathcal{L}_y = d_y - \lambda f_y = 0$

$\mathcal{L}_z = d_z - \lambda f_z = 0$

In terms of coordinates

$\kappa x = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \lambda f_x$

$$\frac{\quad}{2\sqrt{x^2+y^2+z^2}}$$

$$k_y = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \lambda f_y$$

$$k_z = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \lambda f_z$$

$$k = \frac{2}{2\sqrt{x^2+y^2+z^2}}$$

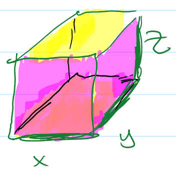
We obtain: $k\,\overrightarrow{OP} = \nabla f$ and we know ∇f is $\perp$ to the surface

Ex Suppose you make boxes. The top and bottom cost \$2/m³
sides cost \$3/m²

Design the least expensive box with volume 1 m³

when setting up physical optimization, imagine what shapes could arise
eg. thin long,

Draw pictures with variables



want to minimize cost

$f(x,y,z) = \text{cost}$
$= ?$

constraint

subject to $1 = V = xyz$

$$f(x,y,z) = \$2(2xy) + \$3(2xz) + \$3(2yz)$$

$$= 4xy + 6xz + 6yz$$

Set up Lagrange

$$\mathcal{L}(x,y,z,\lambda) = [4xy + 6xz + 6yz] - \lambda(xyz - 1)$$

Find critical points

① Tricky move, multiply the equation by x, y, z

$$\mathcal{L}_x = 4y + 6z - \lambda yz = 0 \overset{*x}{\Longrightarrow} 4xy + 6xz - \lambda(\underbrace{xyz}_{1}) = 0$$

$\mathcal{L}_y$ ... $*y$

$\mathcal{L}_y = 4x + 6z - \lambda xz = 0 \overset{\cdot y}{\Rightarrow} 4xy + 6yz - \lambda \underbrace{(xyz)}_{1} = 0$

$\mathcal{L}_z = 6x + 6y - \lambda xy = 0 \overset{\cdot z}{\Rightarrow} 6xz + 6yz - \lambda \underbrace{(xyz)}_{1} = 0$

So: $\lambda = 4xy + 6xz$ (1)

$\lambda = 4xy + 6yz$ (2)

$\lambda = 6xz + 6yz$ (3)

$4xy + 6xz = 4xy + 6yz$

$6xz = 6yz \Rightarrow$ either $z = 0$ ✗ since if $z=0$ no volume

or $x = y$ ✓

$4xy + 6yz = 6xz + 6yz$

$4xy = 6xz \Rightarrow \boxed{x = 0}$ or $\boxed{y = \frac{3}{2}z}$ ✓

✗ No volume

We get $x = y = \frac{3}{2} z$

Excercise: Solve for (x, y, z) using volume constraint

Summary of this week

- Constrained optimization
- "Check the end points" / boundary
- Bounded sets
- Closed sets
- Extreme value therom
- Lagrange multiplier and parallel gradients
- Find max/min using level curves

July 17, 2018 8:10 PM

Admin

Homework 5 available

Homework 6 will be available next week.

Integration (5.2)

Recall,

$\int_a^b f(x)dx$ calculates an area for us.

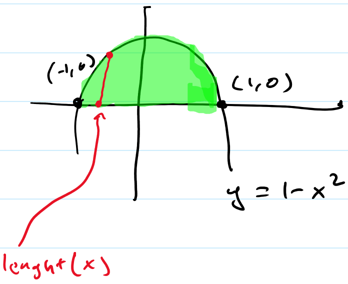
"Signed area of $f(x)$ on $[a,b]$

Fact (Fundamental theorem of calculus)

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where}$$

$F'(x) = f(x)$ is an antiderivative of $f(x)$

Ex Find the area bounded by $y = 1-x^2$ and $y=0$



$$A = \int_{-1}^{1} \text{length}(x)\,dx$$

$$= \int_{-1}^{1} (1-x^2)\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

#Using FTC

$$= \left(1 - \tfrac{1}{3}(1)^3\right) - \left(-1 - \tfrac{1}{3}(-1)^3\right)$$

$$= \frac{2}{3} - \left(-1 + \frac{1}{3}\right)$$

$$= \frac{4}{3}$$

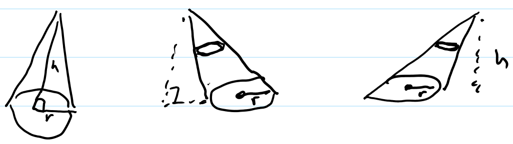
Cavalleri's Principle

"The integral of area is volume"

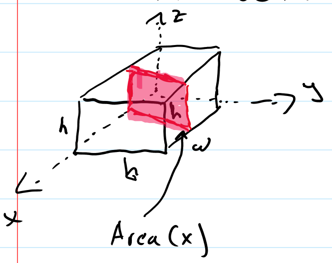
Corollary: If two bodies have the same horizontal slice area functions then they have the exact same

Corollary: If two bodies have the same horizontal slice area functions then they have the exact same volume.

Ex All three cones have the same volume.



Find the volume of a rectangular prism.



Let h = height, b = base, w = width

Set up Integral

$$Vol = \int_0^w Area(x)\,dx$$

$$= \int_0^w bh\,dx$$

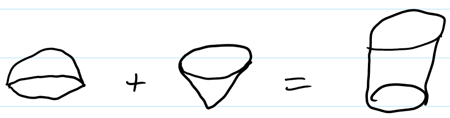
$$= \left[bhx \right]_0^w$$

$$= bhw - bh0$$

$$= bhw$$

Application of Cavalieri

Claim:



The volume of a hemi-sphere of radius r and a cone of radius r and height r equals the volume of a cylinder of height and radius r.

The hemisphere is $x^2+y^2+z^2=r^2$

The cone is $z=\sqrt{x^2+y^2}$

Find the area of cross sections

At $z=k$, we get: $x^2+y^2+k^2=r^2 \iff x^2+y^2=r^2-k^2$

At $z=k$, we get: $x^2+y^2+k^2=r^2 \iff x^2+y^2=r^2-k^2$

A circle of radius $\sqrt{r^2-k^2}$