

4.1 WHY LEARN CALCULUS: +91 844-844-0102

- Calculus is widely used in computer science in the following

1. Analysis of Algorithms :- limits, Taylor Series

2. Probability & Statistics :- Expectation \rightarrow Integration

3. Numerical Optimization :- ML, AI, Robotics \rightarrow differentiations
(Maxima, Minima)

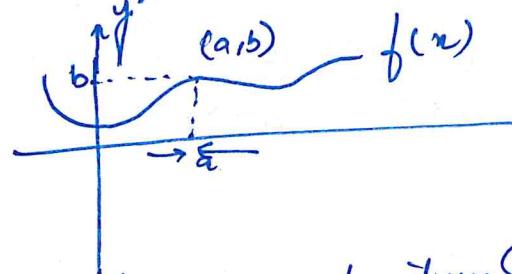
4. Simulation :- Any physical system, Branches \rightarrow Physics \rightarrow Calculus.

↓
Weather, Nuclear, Turbines, etc

4.2. LIMITS: AN INTRODUCTION

→ limit at a point $n=a$ for a function $f(n)$ is written as

$$\lim_{n \rightarrow a} f(n)$$



- As n approaches "a" what value does $f(n)$ approach, it need not be equal to $f(a)$.

for example $f(n) = \frac{\sin n}{n}$. $f(0) = \frac{\sin 0}{0} = \frac{0}{0}$ not defined

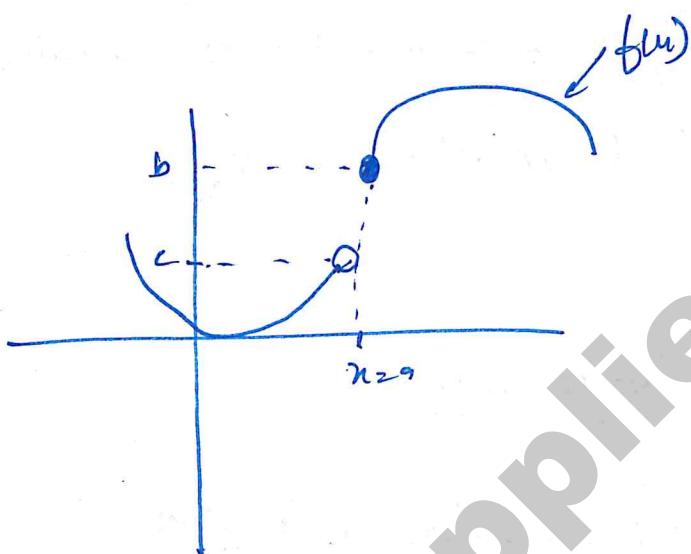
but $\lim_{n \rightarrow 0} \frac{\sin n}{n} = 1$

$$\lim_{n \rightarrow n_0} f(n) = L$$

if $\forall \epsilon > 0 \exists \delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow$

$$|f(n) - L| < \epsilon$$

One Side limits

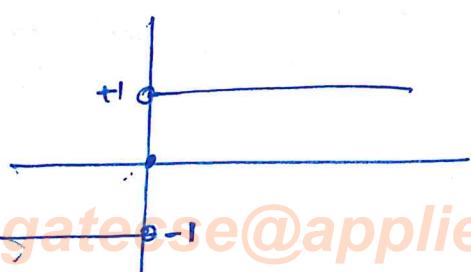


$$\lim_{n \rightarrow a^+} f(n) = b$$

$$\lim_{n \rightarrow a^-} f(n) = c$$

If $\lim_{n \rightarrow a^+} f(n) \neq \lim_{n \rightarrow a^-} f(n)$, the limit does not exist at $n=a$.

e.g. $f(n) = \text{sign}(n) = \begin{cases} \frac{n}{|n|} & \text{if } n \neq 0 \\ 0 & \text{if } n=0 \end{cases}$



$$\lim_{n \rightarrow 0^+} f(n) = 1$$

$$\lim_{n \rightarrow 0^-} f(n) = -1.$$

$\therefore \lim_{n \rightarrow 0}$ Does not exist.

Why limits are important

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In computer science Analysis of Algorithms makes use of limits

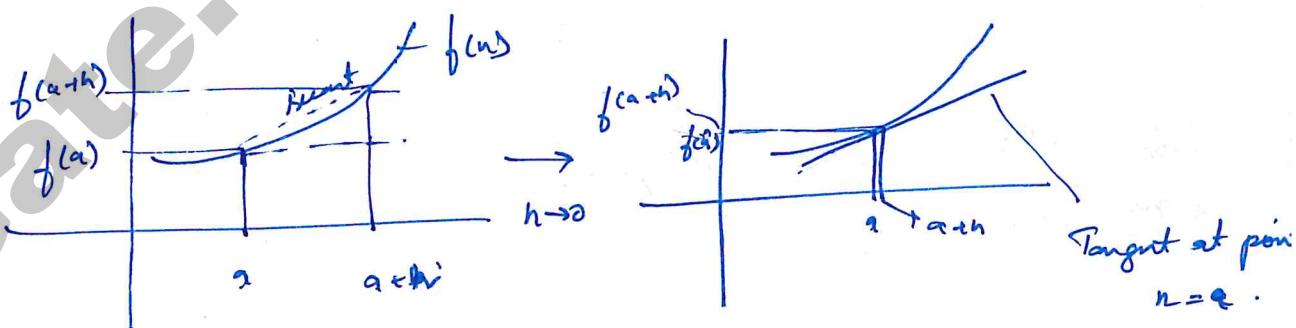
$$f(n) = O(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

little Oh notation.

- Many other areas in engineering like Mech, Civil, Physics and also C.S. make use of differentiation which can be defined with the help of limits

$$\left(\frac{df}{dn} \right)_{n=a} = f'(a) = \lim_{\substack{h \rightarrow 0 \\ n=a}} \frac{f(a+h) - f(a)}{h}$$

- Derivative of any function at a point "a" is the slope of the tangent to the function at that point. As $h \rightarrow 0$ the secant becomes or reduces to tangent.



↓

slope of Secant $\sim \frac{f(a+h) - f(a)}{h}$

\therefore Slope of Tangent is given by $\frac{f(a+h) - f(a)}{h}$

Properties of limits

If $\lim_{n \rightarrow a} f(n) = M$

$\lim_{n \rightarrow a} g(n) = N$

If the above limits exist, then

$$\textcircled{1} \quad \lim_{n \rightarrow a} (f(n) + g(n)) = M + N.$$

$$\textcircled{2} \quad \lim_{n \rightarrow a} (f(n) - g(n)) = M - N$$

$$\textcircled{3} \quad \lim_{n \rightarrow a} f(n) \cdot g(n) = M \cdot N$$

$$\textcircled{4} \quad \lim_{n \rightarrow a} \frac{f(n)}{g(n)} = \frac{M}{N} \quad \text{if } N \neq 0$$

$$\textcircled{5} \quad \lim_{n \rightarrow a} f(n)^k = M^k \quad (\text{if } M, k \geq 0).$$

$$\textcircled{6} \quad \lim_{n \rightarrow a} c = c \quad (\text{if } c \text{ is a constant}).$$

(7) $\lim_{n \rightarrow a} n = a$

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(5)

(8) Polynomials: $f(n), g(n)$, let $g(a) \neq 0$ then

$$\lim_{n \rightarrow a} \frac{f(n)}{g(n)} = \frac{f(a)}{g(a)}$$

INDETERMINATE Forms

(1) If $\lim_{n \rightarrow a} \frac{f(n)}{g(n)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ for example $\lim_{n \rightarrow \infty} \frac{1}{n}$ is of $\frac{0}{0}$ form

then these are called as Quotient Indeterminate forms. (Can be solved using L'Hopital's rule).

ex. $\lim_{n \rightarrow 0} \frac{n \cdot \sin \frac{1}{n}}{n} = \lim_{n \rightarrow 0} \sin(\frac{1}{n}) = \text{Does not exist.}$

(2) $0 \cdot \infty$ form or $0 \cdot -\infty$ form. Known as product indeterminate form.

example $\lim_{n \rightarrow 0^+} n \log(n)$

$\downarrow \quad \downarrow$

$\rightarrow 0 \quad \rightarrow -\infty$

This form can also be solved using L'Hopital's rule.

(3) $\infty - \infty$ (Subtraction Indeterminate Form)

ex. $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 3n + 5} - n)$

1

④ Exponential indeterminate Form.

 $0^0, \infty^0, 1^\infty$ Non Indeterminate Forms

$$\textcircled{1} \quad \lim_{n \rightarrow 0^+} \frac{1}{n} = \infty$$

$$\lim_{n \rightarrow 0} \frac{1}{n} \cdot \text{Does Not Exist.}$$

$$\textcircled{2} \quad \lim_{n \rightarrow 0^-} \frac{1}{n} = -\infty$$

$$\textcircled{3} \quad \frac{1}{\infty} \stackrel{\text{or}}{=} \frac{0}{\infty} = 0.$$

$$\textcircled{4} \quad \infty \cdot \infty = \infty.$$

$$\textcircled{5} \quad 0^{\infty} = 0$$

$$\textcircled{6} \quad \infty^{\infty} = \infty$$

$$\textcircled{7} \quad 0^{-\infty} = \infty$$

$$\textcircled{8} \quad \infty^{-\infty} = \frac{1}{\infty^\infty} = 0$$

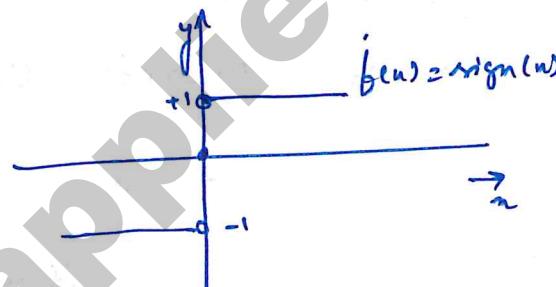
① Substitution Method

$$\textcircled{a} \lim_{n \rightarrow 1} n^2 = (n^2 \text{ at } n=1) = 1.$$

Note * Substitution method works only when the function is continuous for which the limit is computed.

$$\textcircled{b} f(n) = \text{sign}(n) = \begin{cases} \frac{n}{|n|} & \text{if } n \neq 0 \\ 0 & \text{if } n=0 \end{cases}$$

$$\lim_{n \rightarrow 0} \text{sign}(n) = 0 \times \text{incorrect}$$



$$\lim_{n \rightarrow 0^+} f(n) = 1$$

∴ limit D.N.E (Does not exist).

$$\lim_{n \rightarrow 0^-} f(n) = -1$$

① $\lim_{n \rightarrow \infty} \frac{1}{n+2}$ as $n \rightarrow \infty$ $P(n+2) \rightarrow \infty$ Phone: +91 844-844-0102

$$\frac{1}{\lim_{n \rightarrow \infty} (n+2)} = \frac{1}{\infty} = 0$$

② Tableau & Approximation Method

③ $\lim_{n \rightarrow 2} \frac{(n-2)}{(n^2-4)}$ $\frac{0}{0}$ form

n	1.9	1.99	1.9999	2.0	2.001	2.01
$f(n)$	0.2564	0.2506	0.25001	Undefined	0.24999	0.2494

→ 0.25 ←

It approaches 0.25.

Note: Not always $f(a) = \lim_{n \rightarrow a} f(n)$, we need to evaluate the limit

on both sides and then we can say that limit exists and value can be decided by observing the limits from both the sides if they come as close as possible as the value is approaching a in limit $n \rightarrow a$.

③ Factoring Method

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(6)

$$\textcircled{a} \lim_{n \rightarrow 3} \frac{n^2 - 9}{n - 3} \quad \frac{0}{0} \text{ form}$$

$$\lim_{n \rightarrow 3} \frac{(n-3)(n+3)}{(n-3)} = n+3.$$

$$\lim_{n \rightarrow 3} (n+3) = \underline{\underline{6}}$$

$$\textcircled{b} \lim_{n \rightarrow 5} \frac{n^2 - n - 20}{n - 5} = \frac{(n-5)(n+4)}{(n-5)}$$

$$= \lim_{n \rightarrow 5} (n+4)$$

$$= \underline{\underline{9}}$$

$$\textcircled{c} \lim_{n \rightarrow 0} \frac{(n+4)^2 - 16}{h} = \frac{h^2 + 16 + 8h - 16}{h} = \underset{h \rightarrow 0}{\text{lt}} \frac{h^2 + 8h}{h}$$

$$= \underset{h \rightarrow 0}{\text{lt}} \frac{h(h+8)}{h} \Rightarrow \underline{\underline{8}}$$

(d) $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 4n} - n)$

Here we use the conjugate

$$\lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + 4n} - n)(\sqrt{n^2 + 4n} + n)}{(\sqrt{n^2 + 4n} + n)}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\cancel{n^2 + 4n} - \cancel{n^2}}{\sqrt{n^2 + 4n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{4n}{n(\sqrt{1 + 4/n} + 1)} \end{aligned}$$

$$= \frac{4}{\sqrt{1+0} + 1} = \frac{4}{2} = 2$$

(e) $\lim_{n \rightarrow 8} \frac{\sqrt[3]{n} - 2}{n - 8}$ $\frac{0}{0}$ form.

$$= \lim_{n \rightarrow 8} \frac{\sqrt[3]{n} - 2}{(\sqrt[3]{n} - 2)(\sqrt[3]{n^2} + 2\sqrt[3]{n} + 4)}$$

Here $(n-8)$ is factorized as

$$(\sqrt[3]{n} - 2)(\sqrt[3]{n^2} + 2\sqrt[3]{n} + 4)$$

$$\text{as. } (a^3 - b^3) = (a-b)(a^2 + ab + b^2)$$

$$\lim_{n \rightarrow 8} \frac{1}{(\sqrt[3]{n^2} + 2\sqrt[3]{n} + 4)} = \frac{1}{4+4+4} = \underline{\underline{\frac{1}{12}}}$$

(f) $\lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$\frac{0}{0}$ form.

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{(x+h)x}}{h} = \lim_{h \rightarrow 0} \frac{-\frac{h}{(x+h)x}}{h} \Rightarrow \lim_{h \rightarrow 0} \frac{-1}{(x+h)x}$$

$$\Rightarrow \underline{\underline{-\frac{1}{x^2}}}$$

4.5. SOLVING LIMITS - 2

(4) Rationalization Method

- In this method we use the conjugate entity

(2) $\lim_{n \rightarrow \infty} \frac{\sqrt{1+n} - 1}{n}$

Conjugate $(\sqrt{1+n} - 1)$ is $(\sqrt{1+n} + 1)$ multiply and divide by its conjugate

$$\lim_{n \rightarrow \infty} \frac{\sqrt{1+n} - 1}{n} \times \frac{(\sqrt{1+n} + 1)}{(\sqrt{1+n} + 1)}$$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{x + \sqrt{x}}{x(\sqrt{1+n} + 1)}$$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{1}{(\sqrt{1+n} + 1)} = \frac{1}{\sqrt{1+1}} = \frac{1}{2}.$$

(b) $\frac{d}{dx} f(x); f(x) = \sqrt{x} = x^{1/2}$

$$\hookrightarrow \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{(x+h) - x}{h \times \sqrt{x+h} + \sqrt{x}}$$

$$\Rightarrow \frac{1}{2\sqrt{x}}$$

$$\textcircled{c} \quad \lim_{n \rightarrow 0} \frac{\sec n - 1}{x^2}$$

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$$= \frac{\sec n - 1}{n^2} \cdot \frac{(\sec n + 1)}{(\sec n + 1)}$$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{\sec^2 n - 1}{n^2 (\sec n + 1)}$$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{\tan^2 n}{n^2 (\sec n + 1)} = \lim_{n \rightarrow 0} \frac{\sin^2 n / n^2}{n^2 \cos^2 n (\sec n + 1)}$$

$$\Rightarrow \lim_{n \rightarrow 0} \left(\frac{\sin n}{n} \right)^2 \left(\frac{1}{\cos n} \right)^2 \cdot \frac{1}{\sec n + 1}$$

$$\text{As } \lim_{n \rightarrow 0} \frac{\sin n}{n} = 1$$

$$\Rightarrow (1) (1) \left(\frac{1}{1+1} \right)$$

$$\Rightarrow \underline{\underline{\frac{1}{2}}}$$

⑤ Expansion Method

APPLIED
ROOTS

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(a) $\lim_{n \rightarrow 0} \frac{e^n - 1}{n}$

$$e^n = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \dots$$

$$= \lim_{n \rightarrow 0} \frac{x + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \dots - x}{n}$$

$$= \lim_{n \rightarrow 0} 1 + \frac{n}{2!} + \frac{n^2}{3!} + \dots$$

$$= 1$$

(b) $\lim_{n \rightarrow 0} \frac{e^n \sin n - n - n^2}{n^3}$

$$= \lim_{n \rightarrow 0} \frac{\left(1 + \frac{n}{1!} + \frac{n^2}{2!} + \frac{n^3}{3!} + \dots\right) \left(n - \frac{n^3}{3!} + \frac{n^5}{5!} - \dots\right) - n - n^2}{n^3}$$

↑ expansion for sin n

$$\lim_{n \rightarrow 0} \frac{n^3 \left(-\frac{1}{3!} + \frac{1}{1!}\right) + n^4 \left(\dots\right) + n^5 \left(\dots\right) - \dots}{n^3}$$

$$-\frac{1}{3!} + \frac{1}{2!}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{6}$$

$$\frac{3-1}{6} = \frac{2}{6} = \underline{\underline{\frac{1}{3}}}$$

4.6 L'HÔPITAL'S RULE

L'Hôpital's Rule - If we have two functions such that

If $\lim_{n \rightarrow a} f(n) = 0$ and If $\lim_{n \rightarrow a} g(n) = 0$ (OR) If $\lim_{n \rightarrow a} f(n) = \pm\infty$ and $\lim_{n \rightarrow a} g(n) = \pm\infty$

then if (1) If $\lim_{n \rightarrow a} \frac{f(n)}{g(n)}$ is either $\frac{0}{0}$ or $\frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{0}{-\infty}$ or $\frac{-\infty}{\infty}$ form.

and (2) $g'(n) \neq 0$ and $f'(n)$ and $g'(n)$ exists,

↓
i.e.
 $\pm \frac{\infty}{\infty}$

(3) $\lim_{n \rightarrow a} \frac{f'(n)}{g'(n)}$ exists

then $\lim_{n \rightarrow a} \frac{f(n)}{g(n)} = \lim_{n \rightarrow a} \frac{f'(n)}{g'(n)}$

(a) $\lim_{n \rightarrow 0} \frac{\sin x}{x} = \lim_{n \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{n \rightarrow 0} \frac{\cos x}{1} = 1$
 (L'Hopital's rule)

(b) $\lim_{n \rightarrow 0} \frac{\sqrt{n+9} - 3}{n} = \lim_{n \rightarrow 0} \frac{1}{2\sqrt{n+9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$

Reference link for standard formulae for derivatives of standard functions
https://en.wikipedia.org/wiki/Differentiation_rules

(c) $\lim_{n \rightarrow 0} \frac{e^n - 1}{n^2 + n} = \frac{0}{0}$ form.

$$\Rightarrow \lim_{n \rightarrow 0} \frac{e^n}{2n+1} = \frac{1}{2} = 1$$

(d) $\lim_{n \rightarrow \infty} n^x e^{-x} = \frac{x^n}{e^x} \stackrel{(n=2)}{=} \frac{\infty}{\infty}$ form.

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^x}{e^x} = \frac{\infty}{\infty} \text{ form again}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n(n-1)x^{n-2}}{e^n} = \frac{2x}{e^x} \Rightarrow 0$$

(e) $\lim_{n \rightarrow 0^+} n \ln x$

0.00 form

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$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \quad (\frac{\infty}{\infty} \text{ form})$$

$$\Rightarrow \lim_{n \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{n^2}} = \lim_{n \rightarrow 0^+} -n^2 = 0$$

(f)

$$\lim_{n \rightarrow 0} \frac{1 - \cos n^2}{n^4}$$

$$= \lim_{n \rightarrow 0} \frac{-(\Delta x)(-\sin n^2)}{4n^3}$$

$$\Rightarrow \lim_{x^2 \rightarrow 0} \frac{\sin x^2}{x^2}$$

$$y = x^2$$

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Proof for L'Hopital's rule can be found at https://en.wikipedia.org/wiki/L%27Hospital%27s_rule (18)
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OR Google for "L'Hopital's rule Wiki"

Special case

$$\textcircled{1} \lim_{n \rightarrow \infty} \frac{n + \sin n}{n} = \lim_{n \rightarrow \infty} \frac{1 + \cos n}{1} \rightarrow \text{oscillates in between } [0, 2].$$

$\frac{\infty}{\infty}$ form

\therefore This limit does not exist.

$$\textcircled{2} \lim_{n \rightarrow \infty} \sin n = \text{Does not exist as } \sin n \text{ oscillates in between } [-1, 1].$$

$$\textcircled{3} \lim_{n \rightarrow \infty} \frac{e^n + e^{-n}}{e^n - e^{-n}} \quad \frac{\infty}{\infty} \text{ form.}$$

Applying L'Hopital's rule.

$$\lim_{n \rightarrow \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} \quad \frac{\infty}{\infty}$$

Applying L'Hopital's rule again.

we get back. $\lim_{n \rightarrow \infty} \frac{e^n + e^{-n}}{e^n - e^{-n}}$

$$y = e^u$$

$u \rightarrow \infty \Rightarrow e^u \rightarrow \infty \Rightarrow y \rightarrow \infty$

$$\lim_{y \rightarrow \infty} \frac{y + \frac{1}{y}}{y - \frac{1}{y}} = \lim_{y \rightarrow \infty} \frac{1 + y^{-2}}{1 - y^{-2}} = \frac{1}{1} = 1$$

4.7 STANDARD LIMITS

Reference link : https://en.wikipedia.org/wiki/List_of_limits

If $\lim_{n \rightarrow c} f(n) = L_1$ and $\lim_{n \rightarrow c} g(n) = L_2$ then

$$1. \lim_{n \rightarrow c} |f(n) \pm g(n)| = L_1 \pm L_2$$

$$2. \lim_{n \rightarrow c} |f(n) \cdot g(n)| = L_1 \times L_2$$

$$3. \lim_{n \rightarrow c} \frac{f(n)}{g(n)} = \frac{L_1}{L_2} \text{ if } L_2 \neq 0$$

4. $\lim_{n \rightarrow \infty} f(n)^n = L_1^n$, if n is a positive integer.

5. $\lim_{n \rightarrow \infty} f(n)^{\frac{1}{n}} = (L_1)^{\frac{1}{n}}$.

6. $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$ if $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} g(n) = 0$ or $\pm \infty$ (L'Hopital's Rule).

Polynomials and functions of the form x^a

1. $\lim_{n \rightarrow \infty} a = a$.

2. $\lim_{n \rightarrow \infty} n = \infty$.

3. $\lim_{n \rightarrow \infty} n^n = c^n$

4. $\lim_{n \rightarrow \infty} \frac{n}{a}$ $\left\{ \begin{array}{ll} \infty & \text{if } a > 0 \\ \text{Does not exist, } a=0 \\ -\infty & , a < 0. \end{array} \right.$

5. If $p(n)$ is a polynomial then by the continuity of polynomials

$$\lim_{n \rightarrow \infty} p(n) = p(c)$$

Functions of the form n^a .

$$1. \lim_{n \rightarrow c} n^a = c^a$$

$$2. \lim_{n \rightarrow \infty} n^a = \begin{cases} \infty, & a > 0 \\ 1, & a = 0 \\ 0, & a < 0 \end{cases}$$

$$3. \lim_{n \rightarrow c} n^{1/a} = \lim_{n \rightarrow \infty} \sqrt[a]{n} = \infty \text{ for any } a > 0.$$

$$4. \lim_{n \rightarrow 0^+} n^{-n} = \lim_{n \rightarrow 0^+} \frac{1}{n^n} = +\infty$$

$$5. \lim_{n \rightarrow 0^-} n^{-n} = \lim_{n \rightarrow 0^-} \frac{1}{n^n} = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even.} \end{cases}$$

$$6. \lim_{n \rightarrow \infty} a^{n^{-1}} = \lim_{n \rightarrow \infty} \frac{a}{n} = 0 \text{ for any real } a.$$

Exponential Functions

Functions of the form $a^{g(n)}$.

$$1. \lim_{n \rightarrow c} e^n = e^c$$

2. $\lim_{n \rightarrow \infty} a^n = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \\ 0, & 0 < a < 1 \end{cases}$

3. $\lim_{n \rightarrow \infty} a^{-n} = \begin{cases} 0, & a > 1 \\ 1, & a = 1 \\ \infty, & 0 < a < 1 \end{cases}$

4. $\lim_{n \rightarrow \infty} \sqrt[n]{a} = \lim_{n \rightarrow \infty} a^{\frac{1}{n}} = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ \text{does not exist}, & a < 0. \end{cases}$

- Functions of the form $f(n)^{g(n)}$

1. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} (n)^{\frac{1}{n}} = 1$

- Functions of the form $f(n)^{g(n)}$

1. $\lim_{n \rightarrow \infty} \left(\frac{n}{n+k} \right)^n = \frac{1}{e^k}$.

2. $(1+n)^{\frac{1}{n}} = e$

3. $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e$

4. $\lim_{n \rightarrow -\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$

5. $\lim_{n \rightarrow \infty} \left(1 + \frac{m}{n}\right)^n = e^{mk}$

6. $\lim_{n \rightarrow 0} \left(1 + a(e^{-n} - 1)\right)^{-\frac{1}{n}} = e^a$

Sums, Products & Composites

1. $\lim_{n \rightarrow 0} n e^{-n} = 0$

2. $\lim_{n \rightarrow \infty} n e^{-n} = 0$

3. $\lim_{n \rightarrow 0} \left(\frac{a^n - 1}{n}\right) = \ln a \quad \forall a > 0.$

$\lim_{n \rightarrow e} \ln n = \ln e$, due to the continuity of $\ln x$.

$$1. \lim_{n \rightarrow 0^+} \ln n = -\infty$$

$$2. \lim_{n \rightarrow \infty} \ln n = \infty$$

$$3. \lim_{n \rightarrow 1} \frac{\ln(n)}{n-1} = 1$$

$$4. \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n} = 1$$

$$5. \lim_{n \rightarrow 0} \frac{-\ln(1+\alpha(e^{-n}-1))}{n} = \alpha \quad (\text{By using L'Hopital's rule}).$$

$$6. \lim_{n \rightarrow 0} n \ln n \geq 0$$

$$7. \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

Logarithms to arbitrary bases



For $a > 1$

$$1. \lim_{n \rightarrow 0^+} \log_a^n = -\infty$$

$$2. \lim_{n \rightarrow \infty} \log_a^n = \infty$$

For $a < 1$

$$3. \lim_{n \rightarrow 0^+} \log_a^n = \infty$$

$$4. \lim_{n \rightarrow \infty} \log_a^n = -\infty$$

Trigonometric Functions

If n is expressed in radians

$$1. \lim_{n \rightarrow a} \sin n = \sin a.$$

$$2. \lim_{n \rightarrow a} \cos n = \cos a.$$

As $\sin n$ and $\cos n$ are continuous functions.

$$3. \lim_{n \rightarrow 0} \frac{\sin n}{n} = 1 \quad (a \neq 0)$$

$$4. \lim_{n \rightarrow 0} \frac{\sin ax}{n} = a \quad (a \neq 0)$$

(25) (26)

3. $\lim_{n \rightarrow \infty} \frac{\sin n}{bn} = \frac{a}{b}$ (a, b ≠ 0) Phone: +91 844-844-0102

4. $\lim_{n \rightarrow \infty} n \sin\left(\frac{1}{n}\right) = 1$

5. $\lim_{n \rightarrow \infty} \frac{1 - \cos n}{n} = 0$

6. $\lim_{n \rightarrow \infty} \frac{1 - \cos n}{\cos^2 n} = \frac{1}{2}$.

4. $\lim_{n \rightarrow n \pm} \tan\left(\pi n + \frac{\pi}{2}\right) = \mp \infty$ for integer n.

5. $\lim_{n \rightarrow \infty} \underbrace{\sin \sin \sin \dots \sin(n\pi_0)}_{n \text{ times}} = 0$ where π_0 is an arbitrary real number.

6. $\lim_{n \rightarrow \infty} \cos \cos \dots \cos(n\pi_0) = d$, where d is Dotsits number, π_0 is any real number.

4.8 Solved Problems

(Q1) $\lim_{n \rightarrow \infty} \frac{n(e^n - 1) + 2(\cos n - 1)}{n(1 - \cos n)} = ?$

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$$\frac{0(1-1) + 2(1-1)}{0(1-1)} = \frac{0}{0} \text{ form}$$

Applying L'Hopital's rule.

$$\Rightarrow \lim_{n \rightarrow 0} \frac{(e^n - 1) + n(e^n) + 2(-\sin n)}{(1-\cos n) + n \sin n}$$

$\frac{0}{0}$ form

Applying L'Hopital's rule again

$$\lim_{n \rightarrow 0} \frac{ne^n + e^n + e^n - 2\cos n}{n\cos n + \sin n + n\sin n}$$

$\frac{0}{0}$ form

Applying L'Hopital's rule

$$\lim_{n \rightarrow 0} \frac{ne^n + e^n + e^n + e^n + 2\sin n}{n\sin n + \cos n + \cos n + \cos n} = \frac{3}{3} = 1$$

(Q2)

$$\lim_{n \rightarrow 0} \frac{n - \sin n}{n + \cos n}$$

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(28)

- A. 1 B. -1 C. 0 D. ∞

\div Numerator and Denominator by n .

$$\lim_{n \rightarrow 0} \frac{1 - \frac{\sin n}{n}}{1 + \frac{\cos n}{n}} \Rightarrow \frac{1 - 0}{1 + 0} = 1.$$

(Q3)

What is the value of $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{2n}$

A. 0

B. $= e^{-2}$ C. $e^{-1/2}$

D. 1

solution

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{2n}$$

1^{∞} indeterminate form.

$\lim_{n \rightarrow \infty} f(n)^{g(n)}$ is 1^{∞} form we can solve using $\frac{\lim_{n \rightarrow \infty} g(n) \cdot f(n)-1}{e^{f(n)-1}}$

$$\lim_{n \rightarrow \infty} \frac{2^n}{e} \left(1 - \frac{1}{n} - 1 \right) \Rightarrow \underline{\underline{e^{-2}}}$$

Q 4

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}}$$

- A. 0
- B. 0
- C. 1
- D. Not defined

Solution

$$\text{let } m = \lim_{n \rightarrow \infty} n^{\frac{1}{n}}$$

$$\log m = \lim_{n \rightarrow \infty} \frac{1}{n} \log n$$

$$\log m = \lim_{n \rightarrow \infty} \frac{\log n}{n}$$

As we know $\log n$ grows very slow when compared to n .

therefore

$$\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0.$$

$$\log m = 0$$

$$\Rightarrow m = 1$$

$\therefore C$ is the correct option

5.

$$\lim_{n \rightarrow \infty} (1+n^2)^{e^{-n}}$$

A 0

B $\frac{1}{2}$

C 1

D ∞

let us take

$$y = \lim_{n \rightarrow \infty} (1+n^2)^{e^{-n}}$$

$$\log y = \lim_{n \rightarrow \infty} e^{-n} \log (1+n^2)$$

$$\log y = \lim_{n \rightarrow \infty} \frac{\log(1+n^2)}{e^n}$$

= We know that e^n is an exponential function and it grows very fast when compared to polynomial functions.

$$\therefore \lim_{n \rightarrow \infty} \frac{\log(1+n^2)}{e^n} = 0$$

$$\log y = 0$$

y^{21} is the correct option.

(6)

$$\lim_{n \rightarrow 4} \frac{\sin(n-4)}{(n-4)} = \underline{\hspace{2cm}}$$

Take $y = (n-4)$

$$= \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Q7

The value of the limit

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$$\lim_{n \rightarrow 1} \frac{n^7 - 2n^5 + 1}{n^3 - 3n^2 + 2}$$

- A. is 0
 B. is -1
 C. is 1
 D. does not exist

Soh

$$\lim_{n \rightarrow 1} \frac{n^7 - 2n^5 + 1}{n^3 - 3n^2 + 2} = \frac{0}{0} \text{ form L'Hopital rule}$$

$$= \lim_{n \rightarrow 1} \frac{7n^6 - 10n^4}{3n^2 - 6n} = \frac{7-10}{3-6} \Rightarrow \frac{-3}{-3} = \underline{\underline{1}}$$

4.9 CONTINUITY: An INTRODUCTION | Phone: +91 844-844-0102

Definition 1:- A function $f(n)$ is continuous at $n=a$ if and only if

- i. $f(a)$ exists
- ii. $\lim_{n \rightarrow a} f(n)$ exists (which means $\lim_{n \rightarrow a^-} f(n) = \lim_{n \rightarrow a^+} f(n) = f(a)$)
- iii. $\lim_{n \rightarrow a} f(n) = f(a)$.

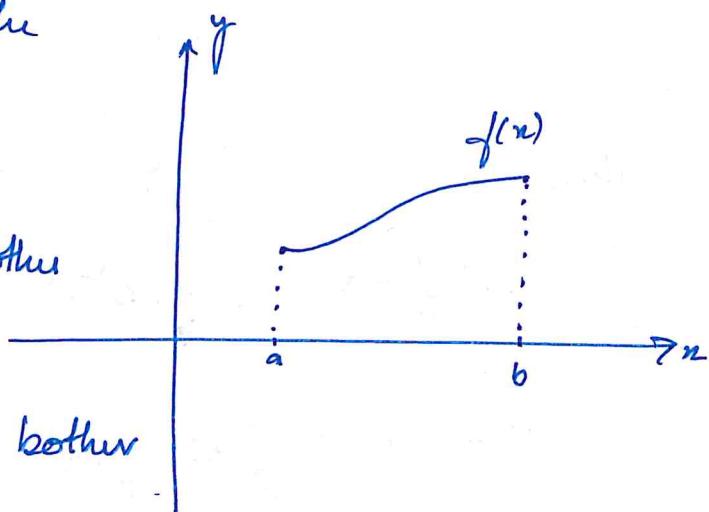
A function'

Definition 2:- $f(n)$ is continuous $[a, b]$ if $f(n)$ is continuous at all points in the given interval. At the

We do not have to bother at the extreme points $n=a, b$.

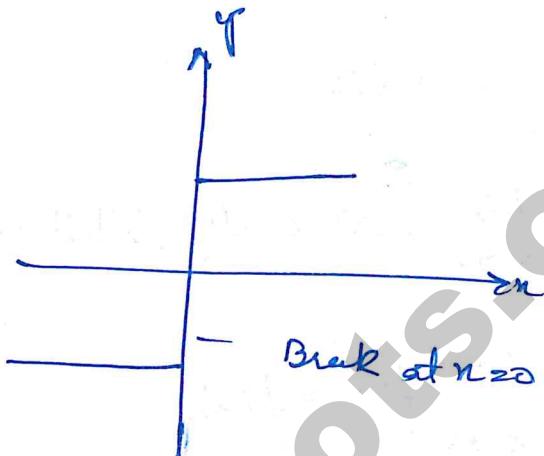
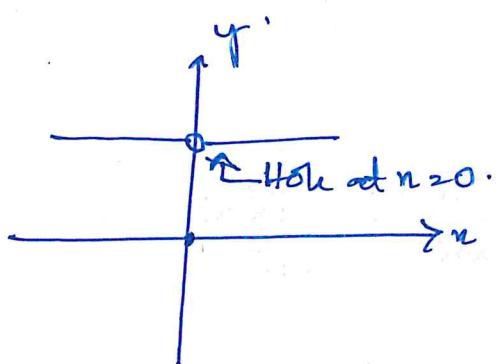
At $n=a$, we don't have to bother about the left hand limit and at $n=b$ we do not have to bother about the right hand limit.

i.e. we do not need to worry about $\lim_{n \rightarrow a^-} f(n)$ and $\lim_{n \rightarrow b^+} f(n)$.



Properties of Continuity

- Intuitively:- The graph for the function does not have no breaks and no holes.



→ If f and g are continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$.

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

then

$$1. f+g = f(n)+g(n)$$

$$5. f \circ g = f(g(n))$$

$$2. f-g = f(n)-g(n)$$

Are all continuous functions.

$$3. f \times g = f(n) \times g(n)$$

$$4. \frac{f}{g} = \frac{f(n)}{g(n)} \text{ if } g(n) \neq 0.$$

(a) $f(n) = \begin{cases} 2n+1 & n < 3 \\ 3n-2 & n \geq 3 \end{cases}$

$\forall x \in \mathbb{R}$

Note :- All polynomials are continuous in nature.

$2n+1$ and $3n-2$ are continuous we need to check for $n=3$

$$\lim_{\substack{n \rightarrow 3^- \\ n \rightarrow 3^+}} f(n) = 2n+1 = 7.$$

$$\lim_{n \rightarrow 3^+} f(n) = 3n-2 = 7.$$

$$\therefore \lim_{n \rightarrow 3} f(n) = 7. \quad \therefore f(3) = 7. \quad \therefore \text{The fn } f(n) \text{ is continuous for } n \in \mathbb{R}.$$

(b) $f(n) = \begin{cases} n+1 & n < 2 \\ n^2 & n = 2 \\ 2n-1 & n > 2 \end{cases}$

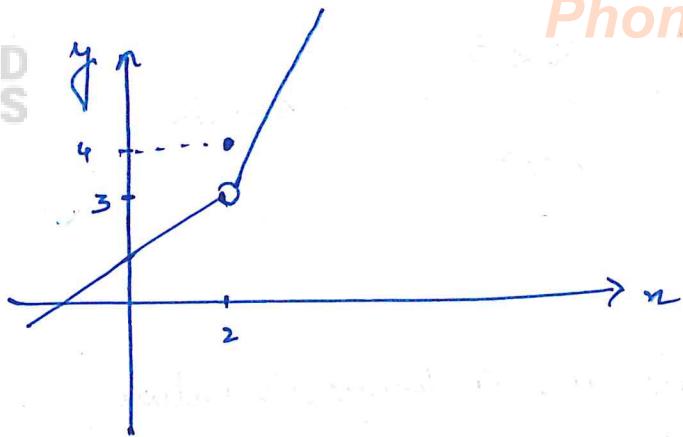
$$A \ n=2.$$

$$\lim_{\substack{n \rightarrow 2^- \\ n \rightarrow 2^+}} f(n) = \lim_{\substack{n \rightarrow 2^- \\ n \rightarrow 2^+}} n+1 = 3$$

$$\lim_{n \rightarrow 2} f(n) = 3$$

$$\lim_{n \rightarrow 2^+} f(n) = \lim_{n \rightarrow 2^+} 2n-1 = 3$$

$$f(2) = 4, \quad \lim_{n \rightarrow 2} f(n) \neq f(2)$$



The graph looks like this. There is a hole at $n=2$.

(2)

$$f(n) = \begin{cases} -\cos(n) & n < 0 \\ e^{n-2} & n \geq 0 \end{cases} \quad n \in \mathbb{R}$$

$\cos n$ is a continuous fn also e^{n-2} is a continuous function, we need to check @ $n=0$ how $f(n)$ is i.e. continuous or discontinuous.

$$\lim_{n \rightarrow 0^-} f(n) = -1$$

$$\lim_{n \rightarrow 0^+} f(n) = -1$$

$$\therefore \lim_{n \rightarrow 0} f(n) = -1.$$

$$f(0) = -1$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{n \rightarrow 0^+} f(n) = 0$$

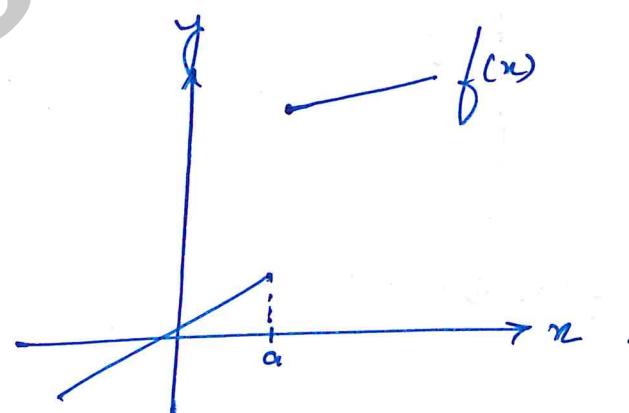
$$\lim_{n \rightarrow 0^-} f(n) = 0 \quad \left. \right\} \quad \lim_{n \rightarrow 0} f(n) = 0.$$

$$f(0) = 0.$$

\therefore The fn is continuous at $x=0$.

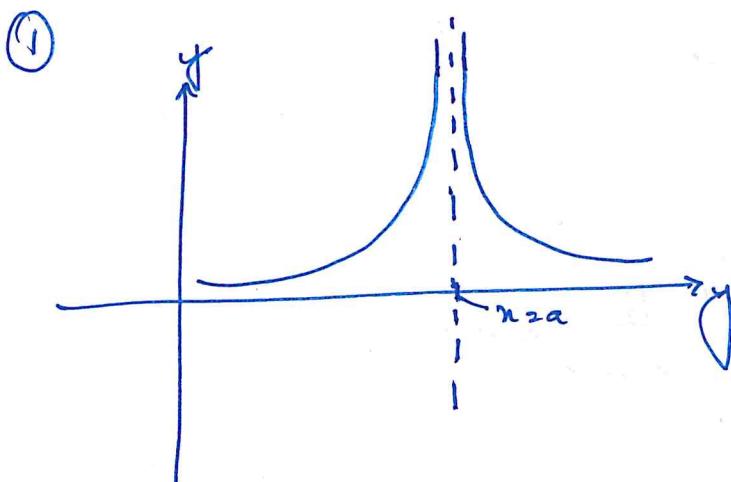
4.10 Discontinuities .

① Jump Discontinuity

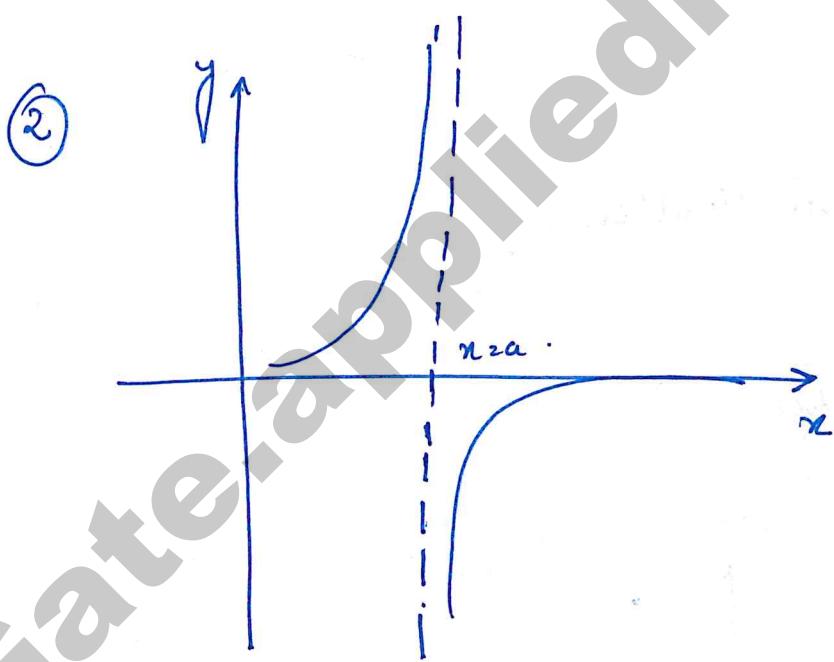


$$\lim_{n \rightarrow a^+} f(n) \neq \lim_{n \rightarrow a^-} f(n).$$

Q) Infinite discontinuity 3 types

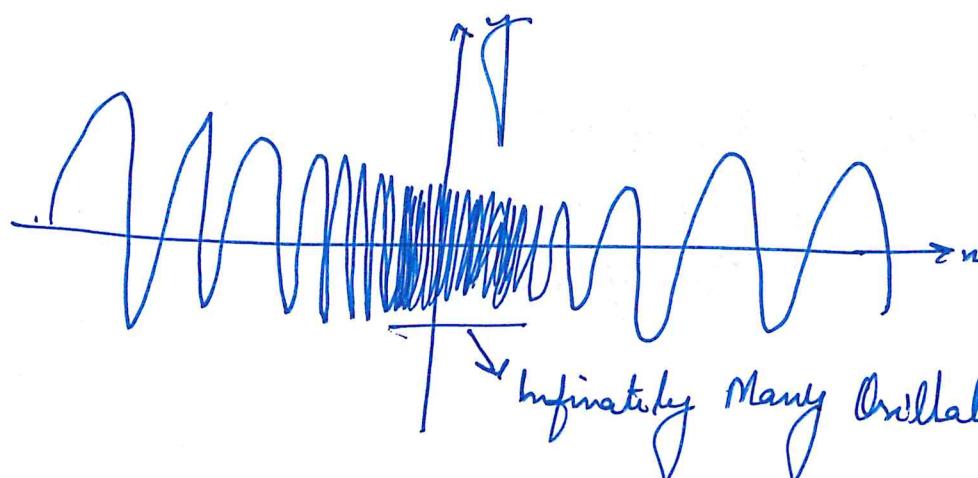


$f(a)$ is finite



$f(a)$ is finite

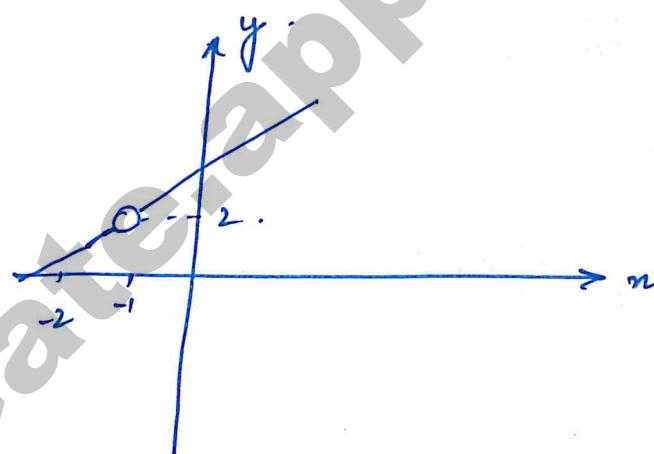
③ $f(n) = \sin(\frac{1}{n})$ Infinitely Many Oscillations at $n=0$.
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② Removable Discontinuities:

$$f(n) = \frac{(n+3)(n+1)}{(n+1)}$$

At $n=1$ Denominator = 0



At $n = -1$ if we define $f(-1)$ as 2 our function $f(n)$ will become continuous.

Q1

Which of the following functions are continuous at $x=3$? **Phone: +91 944344-0102**

A. $f(n) = \begin{cases} 2, & \text{if } n=3 \\ n-1, & \text{if } n>3 \\ \frac{n+3}{3}, & \text{if } n<3. \end{cases}$

B. $f(n) = \begin{cases} 4, & \text{if } n=3 \\ 8-n, & \text{if } n \neq 3. \end{cases}$

C. $f(n) = \begin{cases} n+3 & \text{if } n \leq 3 \\ n-4 & \text{if } n > 3. \end{cases}$

D. $f(n) = \begin{cases} \frac{1}{n^3-27} & \text{if } n \neq 3 \\ \dots & \end{cases}$

Solution For continuous $LHL = RHL = f(3)$.

A. $f(3) = 2$. $\lim_{n \rightarrow 3^-} f(n) = \lim_{n \rightarrow 3^-} \frac{n+3}{3} = 2$; $\lim_{n \rightarrow 3^+} f(n) = \lim_{n \rightarrow 3^+} n-2 = 2$.

Mail: gatecse@appliedroots.com. A is the correct option.

(Q2)

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(41)

A function $f(x)$ is continuous in the interval $[0, 2]$.

It is known that $f(0) = f(2) = -1$ and $f(1) = 1$. Which of the following statements must be true?

A. There exists a y in the interval $(0, 1)$ such that

$$f(y) = f(y+1)$$

B. For every y in the interval $(0, 1)$, $f(y) = f(2-y)$.

C. The maximum value of the function in the interval $(0, 2)$ is 1.

D. There exist a y in the interval $(0, 1)$ such that $f(y) = -f(2-y)$.

Solution.

By looking at the 1st option

let us define a function $g(u) = f(y) - f(y+1)$

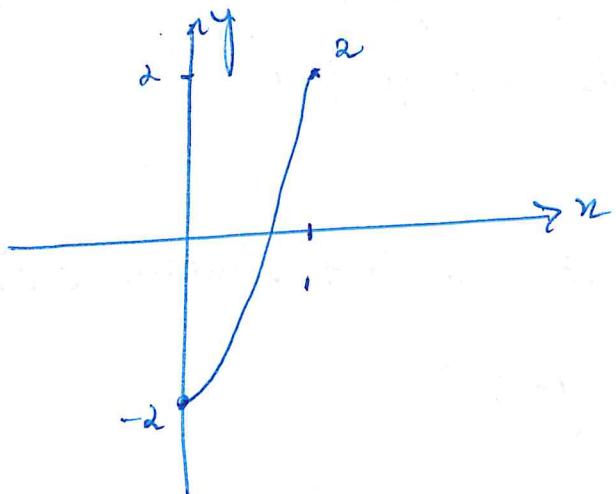
Since $f(x)$ is continuous in $[0, 2]$ $g(u)$ is continuous in $(0, 1)$.

as if f, g are continuous then $f-g$ is also a continuous function.

$$g(0) = f(0) - f(1) = -1 - 1 = -2$$

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If we try to plot $g(n)$



It may be of different shape but it has to cross 0 and $g(n)$ will be < 0 at some point.

$$g(n) > 0 \Rightarrow f(y) - f(y+1) > 0$$

$$\underline{f(y) - f(y+1)} \quad \text{option 1 is correct.}$$

- Q3 Let $f(n) = n^{-\frac{1}{3}}$ and A is the area of the region bounded by $f(n)$ and the x-axis when n varies from -1 to 1.

Which of the following statements is/are TRUE?

I. f is continuous in $[-1, 1]$

II. f is not bounded in $[-1, 1]$

III. A is non-zero and finite Phone: +91 844-844-0102



- A. II only.
- B. III only.
- C. II and III only.
- D. I, II and III.

Solution.

$$\text{I. } f(n) = n^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{n}}$$

At $n=0$ the function is not defined.

$\therefore f(n)$ is not continuous in $[-1, 1]$.

II $f(n)$ is not bounded in $[-1, 1]$.

As $n \rightarrow \infty$ $\frac{1}{\sqrt[3]{n}} \rightarrow 0 \therefore f(n)$ is not bounded.

Statement II is true.

III

A is the area can be calculated by definite integral $\int_{-1}^1 f(n) dx$ using

which is finite. \therefore option C is correct.

Two ways we can discuss about differentiability

1. Mathematical (Rigorous)

2. Geometric way.

1. Mathematical Approach :-

A function $f(n)$ is differentiable ($\text{if } f: \mathbb{R} \rightarrow \mathbb{R}$) at $n=a$

if $f'(a)$ exists i.e. $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists

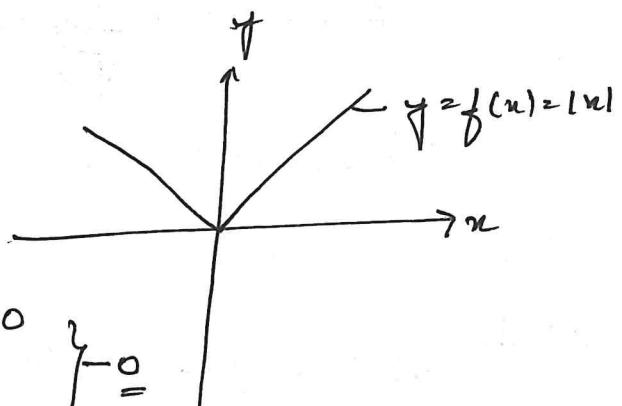
Note → if f is differentiable at $n=a$ then f is continuous at $n=a$.

→ if f is continuous at $n=a$ then f need not be differentiable at $n=a$.

Example $f(n) = |n| = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n < 0 \end{cases}$

At $n=0$

$$\lim_{n \rightarrow 0} f(n) = \left\{ \begin{array}{l} \lim_{n \rightarrow 0^+} |n| = 0 \\ \lim_{n \rightarrow 0^-} |n| = 0 \end{array} \right\} = 0$$



Mail: gatocse@appliedroots.com and the fn is continuous at $n=0$.

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

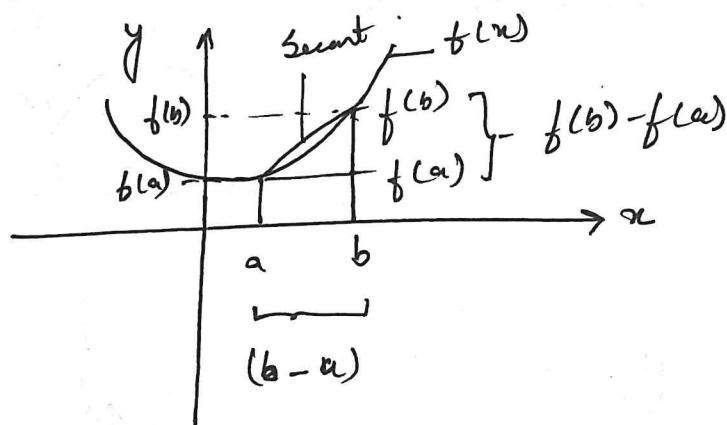
left hand limit $\lim_{h \rightarrow 0^+} \frac{f(h)}{h} = \frac{h}{h} = 1$

right hand limit $\lim_{h \rightarrow 0^-} \frac{-f(h)}{h} = \frac{-h}{h} = -1$

$\therefore LHL \neq RHL$. \therefore limit does not exist at $x=0$.

$\therefore f'(n)$ does not exist at $n=0$ and $f(n)$ is not differentiable at $n=0$.

Geometrically what does derivative mean?



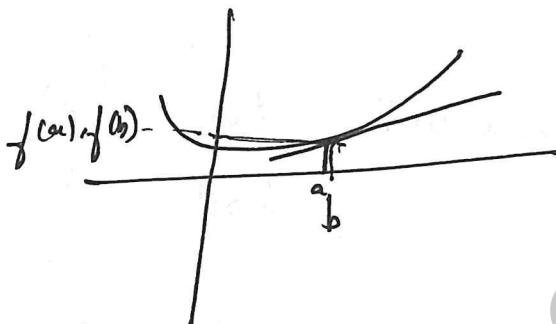
Slope of the secant $= \frac{f(b) - f(a)}{b - a}$ but $h = b - a$
 then

$$= \frac{f(h+a) + f(a)}{h}$$

As point b approaches point a the secant reduces to a tangent

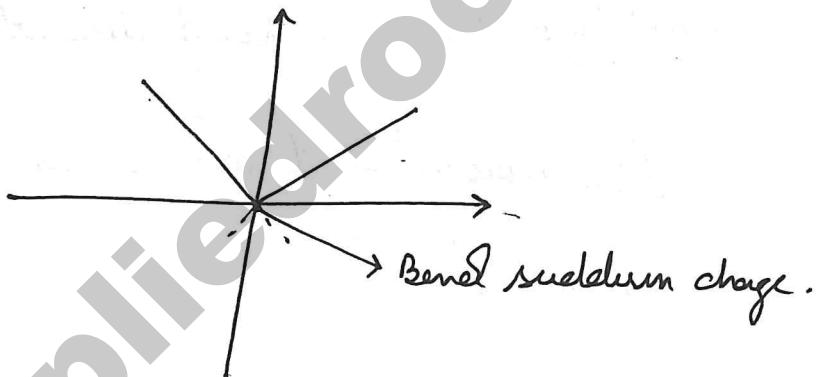
Secant \rightarrow tangent

$$\text{Slope } f' = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

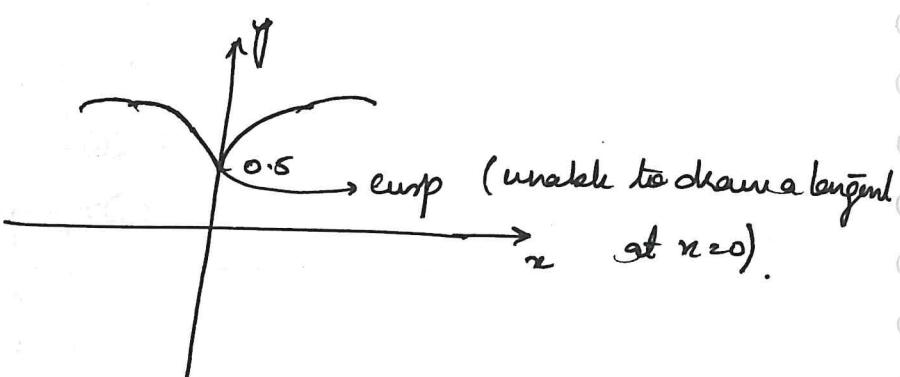


Non-differentiability

(a) $f(x) = |x|$



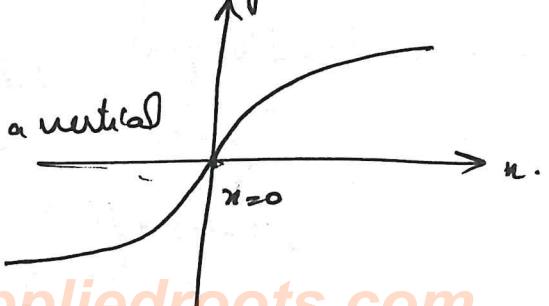
(b) $f(x) = \sqrt{|x|} + 0.5$



(c) $f(x) = \sqrt[3]{x}$

At $x=0$ the tangent is a vertical line, i.e undefined

Slope i.e. $f'(x)$ is not differentiable at $x=0$.



4.13 DIFFERENTIABILITY

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SOLVED PROBLEMS

Q1) The function $f(n) = |2 - 3n|$

- A. is continuous $x \in \mathbb{R}$ and differentiable $\forall n \in \mathbb{R}$.
- B. is continuous $n \in \mathbb{R}$ and differentiable $\forall n \in \mathbb{R}$ except $n = 2/3$.
- C. is continuous $n \in \mathbb{R}$ and differentiable $\forall n \in \mathbb{R}$ except $n = 2/3$.
- D. is continuous $n \in \mathbb{R}$ except $n = 3$ and differentiable $n \in \mathbb{R}$.

Ans.

$$f(n) = |2n - 3| = \begin{cases} (2n - 3) & n \geq 2/3 \\ -(2n - 3) & n < 2/3 \end{cases}$$

LHL at $n = 2/3$ $\lim_{n \rightarrow 2/3^-} f(n) = -(2n - 3) = 0$.

RHL at $n = 2/3$ $\lim_{n \rightarrow 2/3^+} f(n) = (2n - 3) = 0$.

$$f(2/3) = 0.$$

∴ The fn is continuous at $n = 2/3$

Right Hand derivative at $n = 2/3$ $\lim_{n \rightarrow 2/3^+} = +3$; $\lim_{n \rightarrow 2/3^-} = +3$.

∴ fn is not differentiable at $n = 2/3$ ∴ C is the correct option.

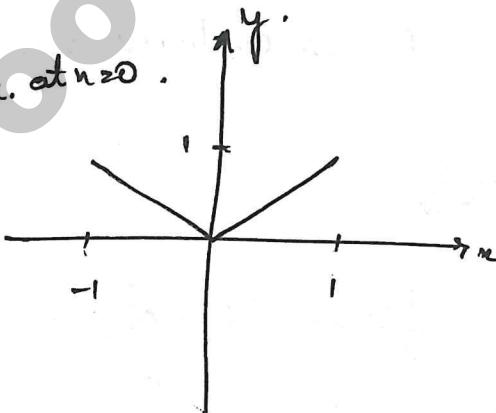
(Q2) Consider the function $y = f(x) = |x|$ in the interval $[-1, 1]$. In this interval, the function is **Phone: +91 844-844-0102**

- A. Continuous and differentiable
- B. Continuous but not differentiable
- C. Differentiable but not continuous.
- D. neither continuous nor differentiable

Ans

in the interval $[-1, 1]$ $f(x) = |x|$ looks like the below plot

$|x|$ is continuous but not differentiable at $x=0$.



(Q3) Consider the following two statements about the function $f(x) = |x|$

P. $f(x)$ is continuous for $\forall x \in \mathbb{R}$.

Q. $f(x)$ is differentiable for $\forall x \in \mathbb{R}$.

Which of the following is true?

A. P is true & Q is false.

B. P is false & Q is true.

C. Both P and Q are true

D. Both P and Q are false

Solution

We know that $f(n)$ is continuous & $n \neq 0$ but it is not differentiable at $x=0$. Therefore A is the correct option.

(Q4)

The function $f(n) = n \sin n$ satisfies the following equation

$$f''(n) + f(n) + t \cos n = 0. \text{ The value of } t \text{ is } -.$$

Solution

$$f'(n) = \sin n + n \cos n$$

$$\begin{aligned} f''(n) &= \cos n - n \sin n + \cos n \\ &= 2 \cos n - n \sin n. \end{aligned}$$

$$\begin{aligned} f''(n) + n \sin n - 2 \cos n &= 0 \\ \downarrow & \quad \downarrow \\ f(n) &+ t \end{aligned}$$

$$\underline{\underline{t = -2}}$$

(Q5)

Let $f(n)$ be a polynomial and $g(n) = f'(n)$ be its derivative.

If the degree of $f(f(n) + f(-n))$ is 10 then the degree of $g(n) - g(-n)$ is _____.

lets assume $f(u)$ is of degree 10

$$f(u) = k u^{10} \quad f(-u) = k(-u)^{10} = k u^{10}$$

$$f(u) + f(-u) = 2k u^{10}$$

$$g(u) = 10k u^9$$

$$\begin{aligned} g(-u) &= 10k(-u)^9 \\ &= -10k u^9 \end{aligned}$$

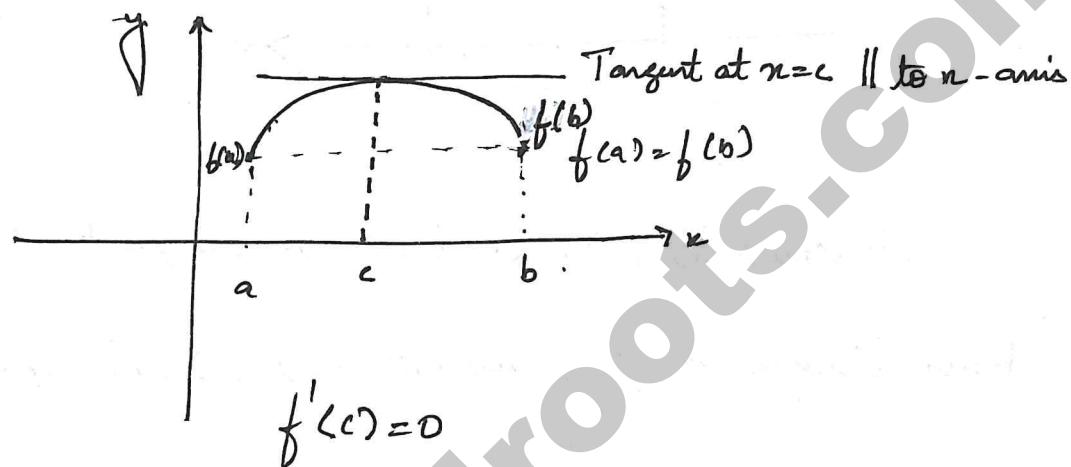
$$\begin{aligned} g(u) - g(-u) &= 10k u^9 - (-10k u^9) \\ &= 20k u^9 \text{ is of degree } 9 \end{aligned}$$

4.14 MEAN VALUE THEOREM

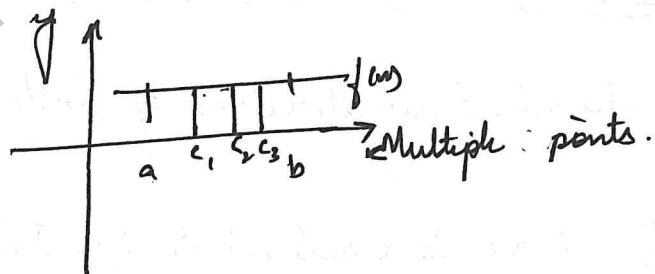
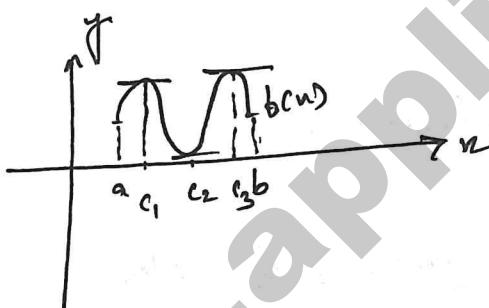
- Mean Value theorems are used for :
 1. Proving other theorems : 1. Taylor Series.
 2. L'Hopital's Rule.
- 2. Real World Applications : example track speedup cars.

I. ROLLE'S THEOREM : If $f(x)$ is a real valued continuous in $[a, b]$ and differentiable in (a, b) then $\exists c \in (a, b)$ such that $f'(c) = 0$.

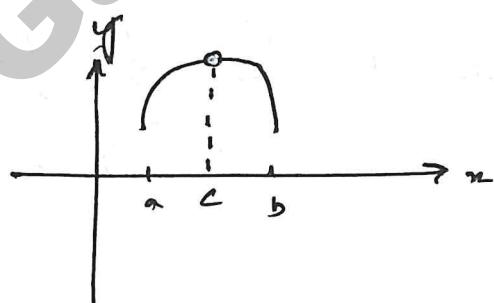
Geometrically



Note It could also be possible to have multiple such points having $f'(x) = 0$.



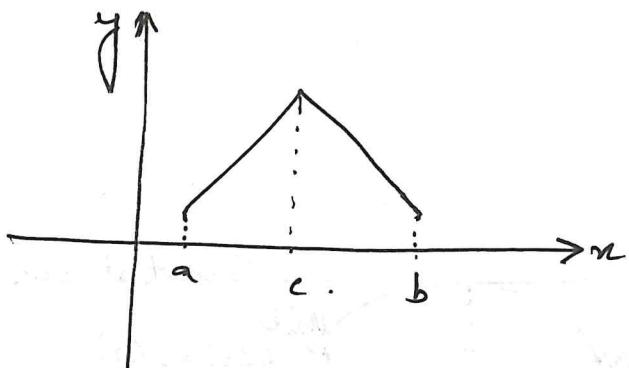
- What happens if the function is not continuous?



There could be a hole at the point where $f'(x)$ is going to be $= 0$, then we will have no such point c .

What happens when the function is not differentiable? **Phone: 91 844-844-0102**

Consider the following example

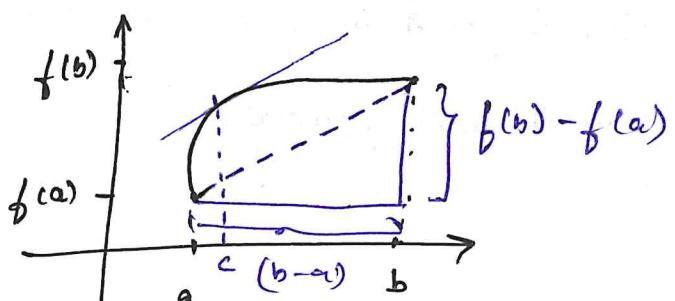


We could have the point $x=c$ as the point where we cannot draw a tangent as $f'(c)$ does not exist / is not defined.

(a) Lagrange's Mean Value Theorem / MVT

- Extension / Generalization of Rolle's theorem.
- If $f(x)$ is a real valued function $f(x): [a, b] \rightarrow \mathbb{R}$
 1. Continuous in $[a, b]$
 2. Differentiable in (a, b) .

then MVT states that $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{(b - a)}$

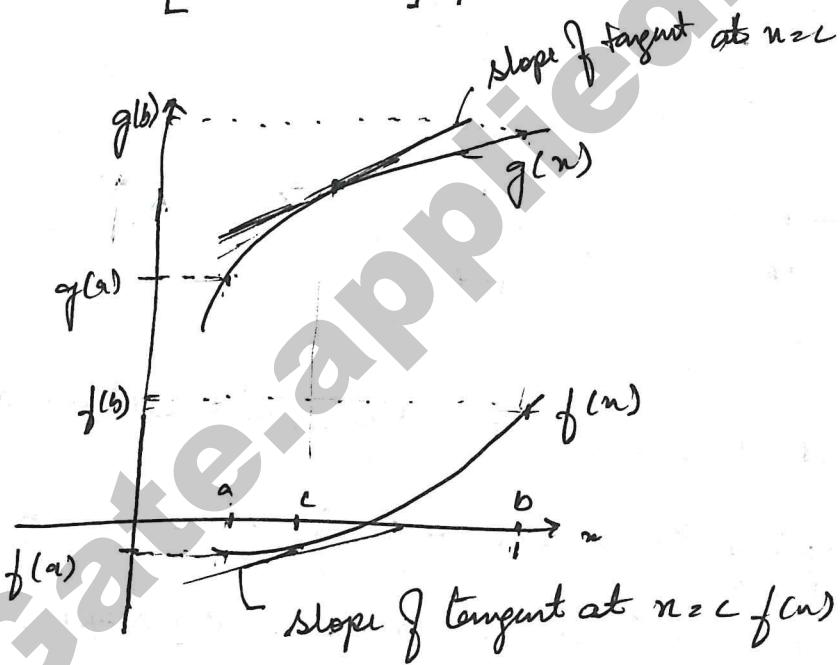


In other words slope of the tangent at $n=c$ = Slope of the secant $(a, f(a)), (b, f(b))$. 53

③ Extended Mean Value Theorem / Cauchy's MVT

- The L'Hopital's rule is a special case of the extended M.V.T.
- If f and g are real continuous in $[a, b]$ and differentiable in (a, b) . then $\exists c \in (a, b)$ such that

$$[f(b) - f(a)] g'(c) = (g(b) - g(a)) f'(c).$$



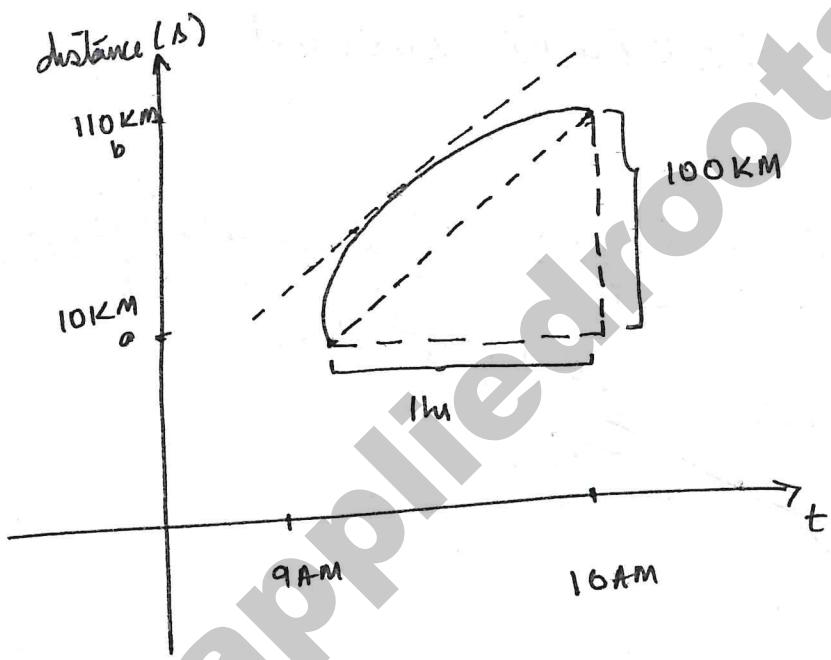
Ratio of difference of the values of the functions is same as the ratio of the derivatives at a point c for the functions

→ A lot of ideas in calculus have arrived from Physics/Mechanics.

→ Consider an example

→ A vehicle moves point A (city centre say) to point A and then it moves to point B.

- It reaches point A at 9 AM and point B at 10 AM.



→ According to MVT, there exist atleast one point between a and b such that the slope of the secant connecting a and b is equal to the slope of the curve at the point m between a and b (here the slope of the curve is the slope of the tangent at that point and it is given by the instantaneous speed at that point).

→ If we have a speed limit of 70 km/h. if slope of the secant > 70 km/h then we can say for sure the driver has exceeded atleast once in between the points a and b, if we know the time of arrival at points a and b we can easily track the speeding vehicles.

Q

$$f(\theta) = \begin{vmatrix} \sin \theta & \cos \theta & \tan \theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

Where $\theta \in [\frac{\pi}{6}, \frac{\pi}{3}]$ and $f'(\theta)$ denote the derivative of f with respect to θ .

D. Which of the following statements are true?

I. There exists $\theta \in (\pi/6, \pi/3)$ such that $f'(\theta) = 0$

II. There exists $\theta \in (\pi/6, \pi/3)$ such that $f'(\theta) \neq 0$

A. I only.

B. II only.

C. Both I and II are true.

D. Neither I nor II are true.

Substituting $\theta = \pi/6$,

$$f(\pi/6) = \begin{vmatrix} \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

As the first two rows are identical, $f(\pi/6) = 0$.

for $\theta = \pi/3$ also we will get R_2 and R_3 as identical and therefore $f(\pi/3) = 0$.

- $f(a) = f(b)$ and $f(n)$ is continuous in $[a, b]$ and differentiable in (a, b)

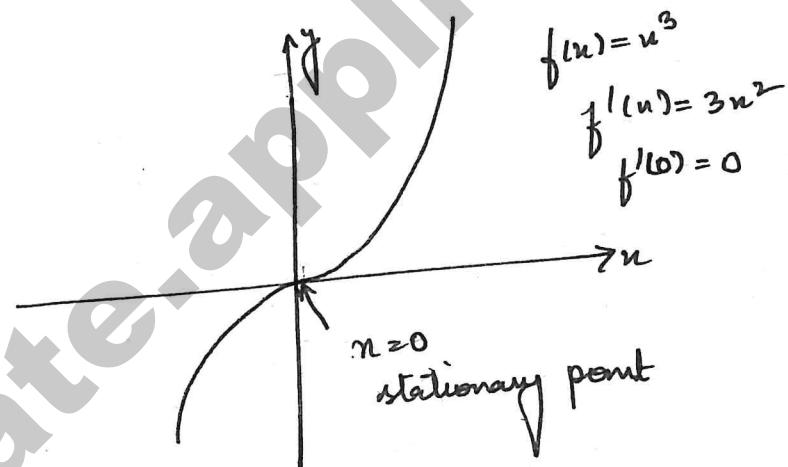
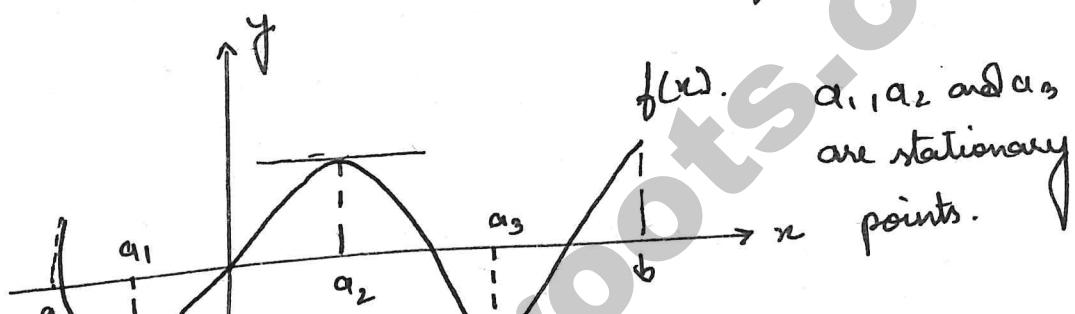
\therefore A point n in (a, b) such that $f'(n) = 0$. \therefore Statement I is true.

- $f(n)$ or $f(\theta)$ is not a constant function, therefore statement II is also true. Option C is true.

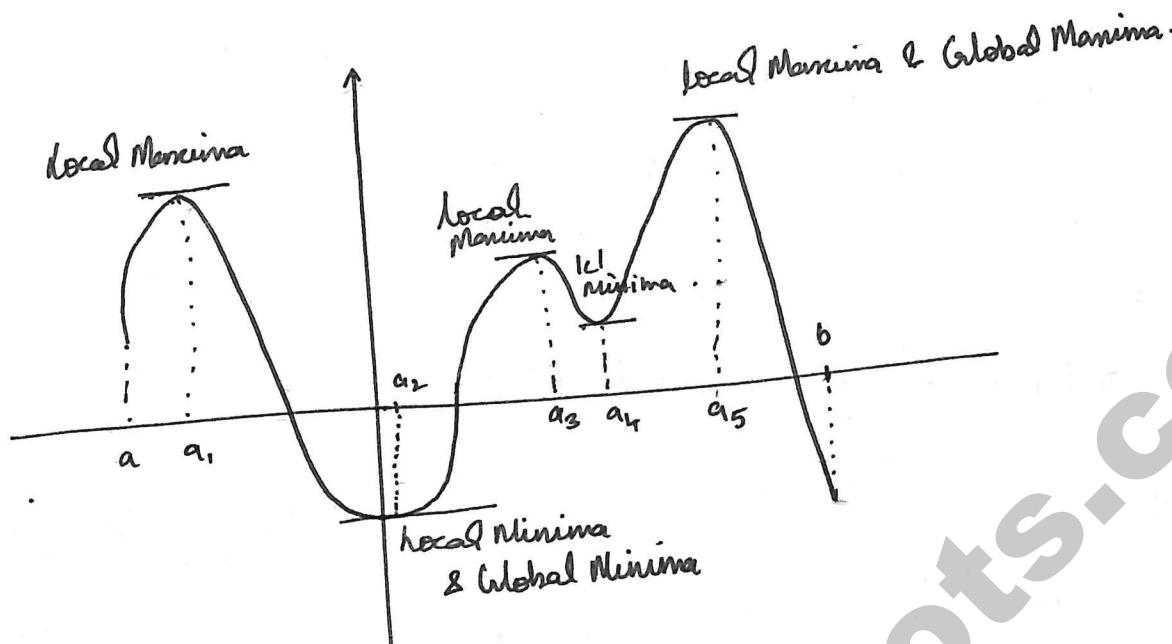
→ Applications in areas of sciences and engineering → ML & AI.

1. Stationary Point : If $f(n)$ is a function $f(n) : \mathbb{R} \rightarrow \mathbb{R}$, $a \in \mathbb{R}$ is a stationary point if $f'(a) = 0$ or in other words, tangent at $n=a$ is parallel to n -axis.
2. Critical Point : A point for which $f'(n)=0$ or $f'(n)$ does not exist.

$$f(n) : [a, b] \rightarrow \mathbb{R}$$



- In the first graph at the stationary point the curve is changing from increasing to decreasing and vice versa but in case of the curve $y = n^3$ there is no such change it was increasing before and afterwards as well. The second curve shows $y = n^3$, at $n=0$ it has a different type of stationary point.

MAXIMA AND MINIMA

$f'(x) = 0$ at $x = a_1, a_2, a_3, a_4, a_5$ they are stationary points

local Maxima / Minima Are those points for which the value of the function $f(x)$ is maximum / minimum in the local neighbourhood.

Global Maxima / Minima :- It is that point C such that $f(c) \geq f(x) \forall x \in [a, b]$
(global maxima)

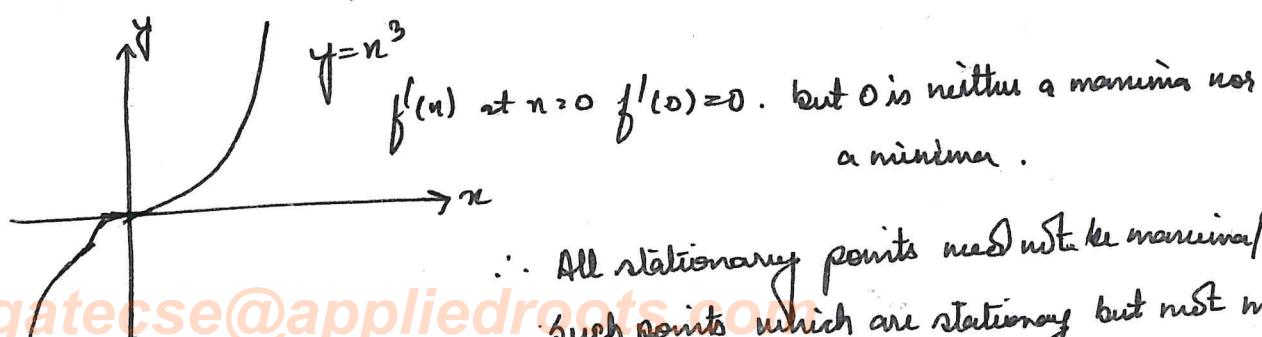
$$f(c) \leq f(x) \forall x \in [a, b]$$

(global minima).

where $c \in [a, b]$

- All the local, Global Minima, Maxima are stationary points.

Exceptional case example.



\therefore All stationary points need not be maxima/minima
such points which are stationary but not max/min
... known as SADDLE POINTS!

(stationary points)

→ Those points where the curve is at one side of the tangent

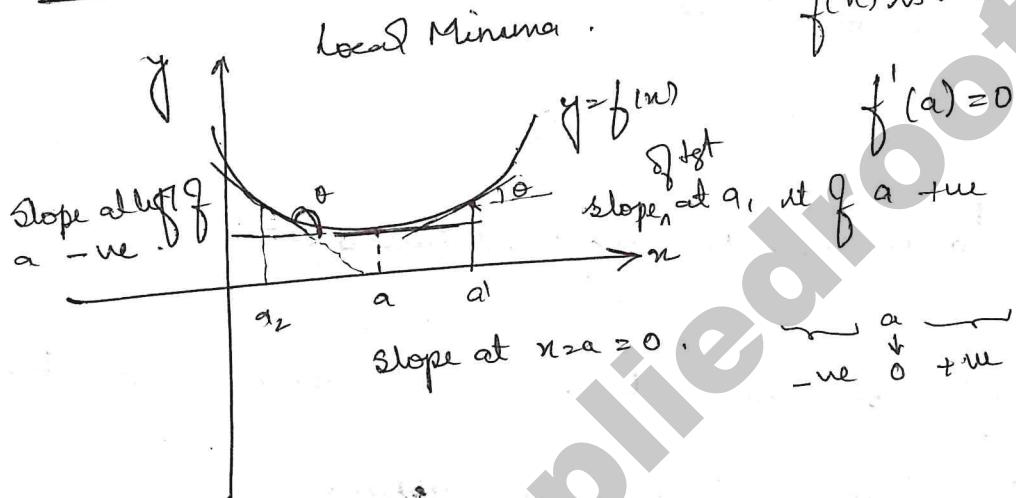
such points are local minima / maxima.

→ The stationary points where we have the curve on both sides of the tangent such points are the Saddle Points.

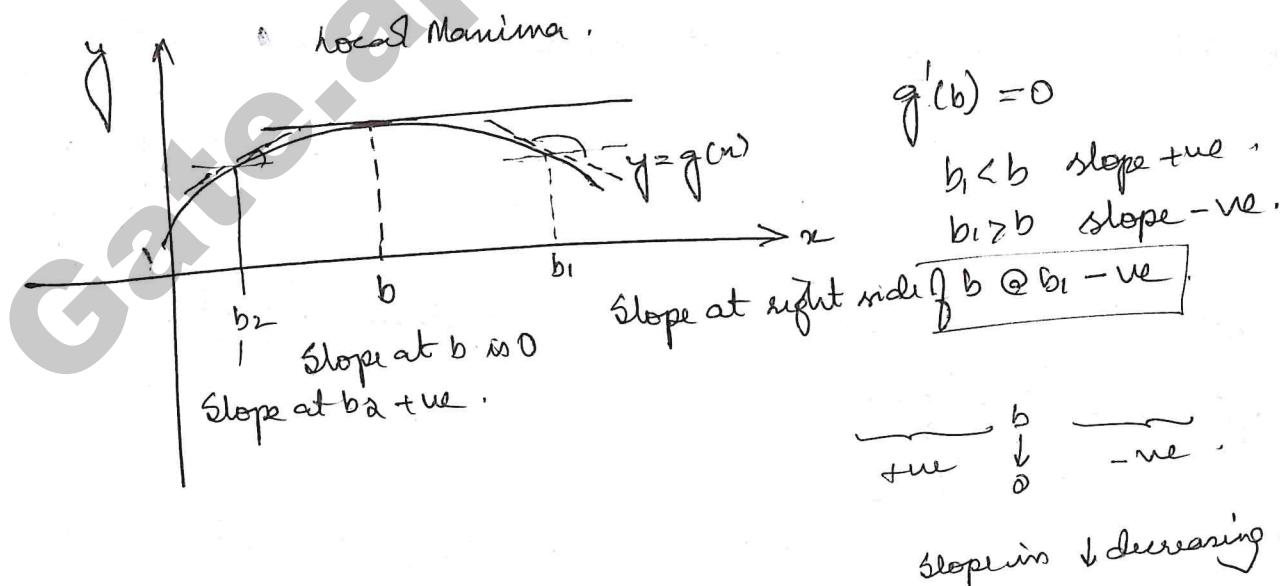
→ For some functions Maxima/Minima do not exist.

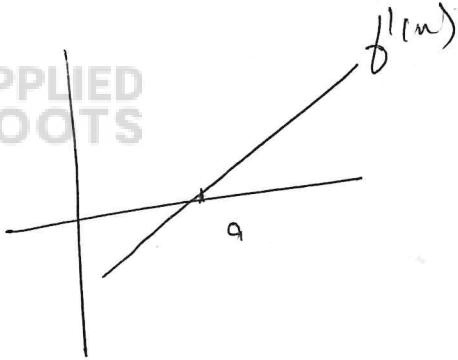
4.17 FINDING MAXIMA AND MINIMA

Geometric Intuition



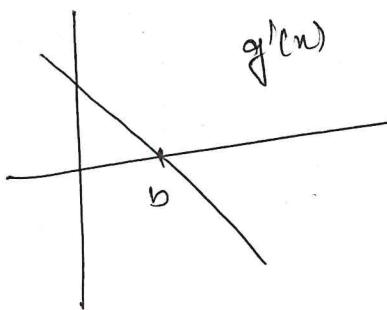
Slope is \uparrow sing
(increasing)





If $f'(n)$ is an increasing function then
at $n = a$, $f(n)$ is minimum (local)

$$f''(n) = \frac{d f'(n)}{dn} > 0 \text{ at } n = a$$



If $g'(n)$ is a decreasing function then
at $n = b$, the function $g(n)$ is maximum (local)

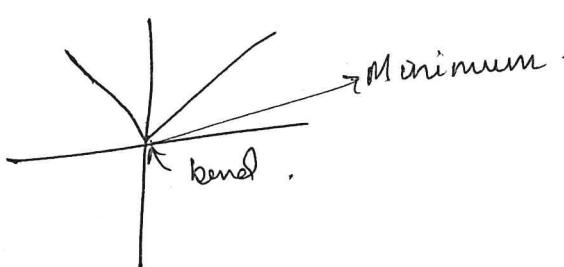
$$g''(n) = \frac{d g'(n)}{dn} < 0 \text{ at } n = b$$

- For local minima / maxima $\rightarrow f'(n) = 0, f''(n) = -ve/ +ve$ respectively
- For global minima / maxima \rightarrow Compute all local minima a_1, a_2, a_3
- If $f(a_1) < f(a_2) < f(a_3)$ then f has a global minima at $\underline{n = a_1}$.
- Similarly we can also calculate Global Maxima using local maxima.

Some Examples

① $f(n) = |n|$

$f'(n)$ at $n = 0$ is not defined
as it is not differentiable at $n = 0$.



$$\textcircled{2} \quad f(n) = n^2$$

**APPLIED
ROOTS**

$$f'(n) = 2n = 0$$

$$n=0$$

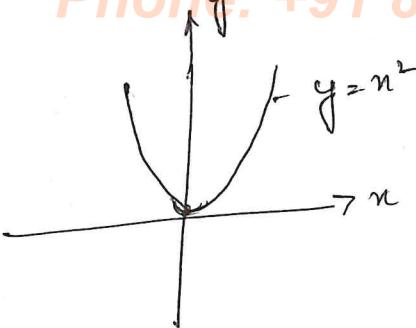
$$f''(n) = 2 > 0$$

$$f'(0)=0$$

$$f''(0) > 0$$

\therefore It is local minima.

As it is the only minima therefore it is also the global minima.



$$\textcircled{3} \quad f(n) = n^3$$

$$f'(n) = 3n^2$$

$$f'(n) = 0 \Rightarrow 3n^2 = 0$$

$$\underline{\underline{n=0}}$$

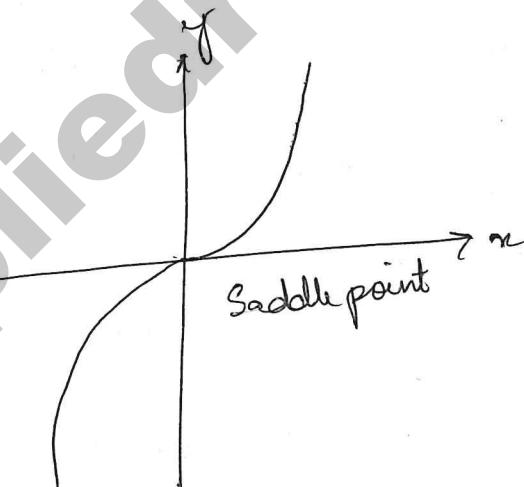
$$f''(n) = 6n$$

$$\text{at } n=0$$

$$f''(0) = 0$$

neither maxima nor minima

$$\neq 0$$



$$(4) f(x) = \frac{x^3}{3} - x$$

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$$f'(x) = x^2 - 1 = 0$$

$$x = \pm 1$$

$$f''(x) = 2x$$

$$f''(-1) = -2 \xrightarrow{<0} \text{Maxima}$$

$$f''(1) = 2 > 0 \rightarrow \text{Minima}$$

$$(5) f(x) = 2 \cos x - x$$

$$f'(x) = -2 \sin x - 1 = 0$$

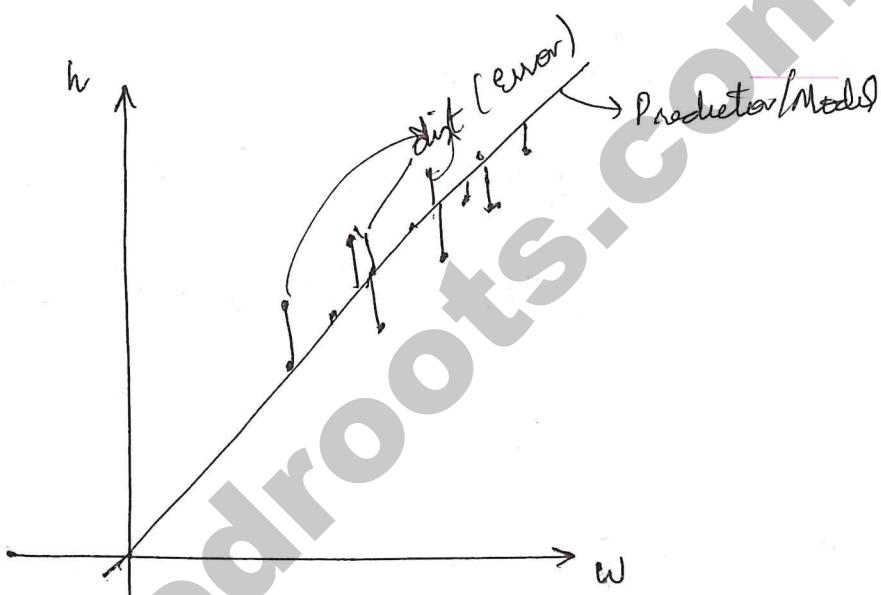
$$\sin x = -\frac{1}{2}$$

There are infinitely many values of x for which $\sin x = -\frac{1}{2}$

$\therefore f(x)$ has infinitely many maxima's and minima's.

To MACHINE LEARNING

- The concept of maxima & minima is used extensively in ML & AI.
- Example:- Given the weights and heights of n people

 (w_1, h_1)
 (w_2, h_2)
 (w_3, h_3)
 \vdots
 (w_n, h_n)


- We would like to have a line or some other function which act as the predictor for heights of a new person if his weight is given.

Such a line can be predicted or derived in such a way that the error with the current points $(w_1, h_1), (w_2, h_2), (w_3, h_3) \dots (w_n, h_n)$ is minimum.

$\sum_{i=1}^n e_i$ should be as small as possible.

then $\sum_{i=1}^n e_i$ should be as small as possible.

$$\Rightarrow \sum_{i=1}^n \left(\frac{h_i}{\uparrow \text{Actual height}} - \frac{m w_i}{\uparrow \text{Predicted height}} \right)^2$$

↑ Parameters
 ↑ $g(m)$ that minimises $\underline{g(m)}$

* We would like to find a parameter m that minimises $\underline{g(m)}$.

→ This problem of predicting/drawing an optimum straight line such that error is minimum, $f(m)$ is minimum is called linear Regression (63) **Phone: +91 844-844-0102**

→ The minima is calculated using the same rules of calculus (the best value of m is estimated this way).

4. 20. MAXIMA AND MINIMA

SOLVED PROBLEMS

(Q1) What is the maximum value of the function $f(n) = 2n^2 - 2n + 6$ in the interval $[0, 2]$?

- A : 6
- B . 10
- C . 12
- D . 5.5

Solution,

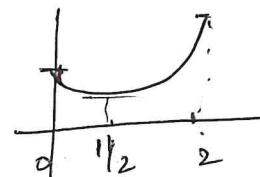
$$f'(n) = 4n - 2 \cdot 20 \Rightarrow n = \frac{2}{4} = \frac{1}{2}$$

$f''(n) = 4 > 0$. ∴ At $n = \frac{1}{2}$ the function $f(n)$ has a minima

The given function is a quadratic function and we need to check the boundaries i.e. at $n=0, 2$ one of them will be the ^{maximum} point.

$$f(0) = 6, f(2) = 4 - 4 + 6 = 6$$

∴ 6 is the minimum.



(Q2) A point on a curve is said to be **extremum** if it is a local minimum or a local maximum. The number of distinct extremum for the curve $3n^4 + 16n^3 + 24n^2 + 37$ is _____.

- A. 0
- B. 1
- C. 2
- D. 3.

Solution

$$f'(n) = 12n^3 - 48n^2 + 48n$$

$$f'(n) = 0$$

$$12n^3 - 48n^2 + 48n = 0$$

$$\Rightarrow n^3 - 4n^2 + 4n = 0$$

$$n(n^2 - 4n + 4) = 0$$

$$n(n-2)(n+2) = 0$$

$n = 0, 2$ 2 unique points C is the correct option.

$$f''(n) = 36n^2 - 96n + 48$$

$$f''(0) = 48 > 0$$

$$\begin{aligned} f''(2) &= 36(4) + 96(2) + 48 \\ &= 144 - 192 + 48 \end{aligned}$$

$= 0$. \therefore only one point is an extremum.

Q3 If $f(n)$ is defined as follows, what is the minimum value of $f(n)$ for $x \in (0, 2]$? (65) Phone: +91 844-844-0102

$$f(n) = \begin{cases} \frac{25}{8n} & \text{when } n \leq 3/2 \\ n + \frac{1}{n} & \text{otherwise.} \end{cases}$$

A. 2.

B. $2\left(\frac{1}{12}\right)$

C. $2\left(\frac{1}{6}\right)$

D. $2\left(\frac{1}{2}\right)$.

Solution

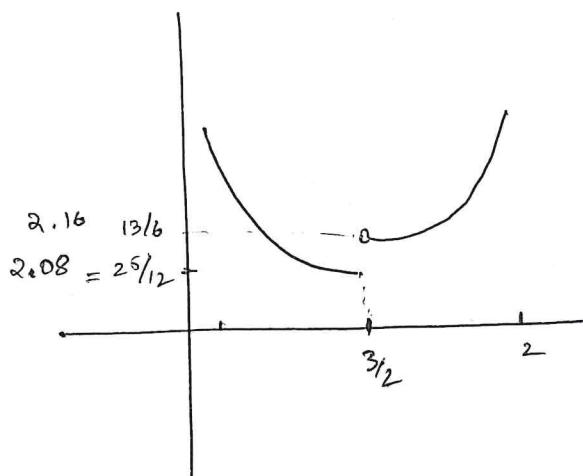
Checking for continuity of this function

$$\lim_{n \rightarrow 3/2^+} f(n) = \lim_{n \rightarrow 3/2^+} n + \frac{1}{n} = \frac{3}{2} + \frac{2}{3} \Rightarrow \frac{9+4}{6} \Rightarrow \frac{13}{6}$$

$$\lim_{n \rightarrow 3/2^-} f(n) = \lim_{n \rightarrow 3/2^-} \frac{25}{8n} = \frac{25}{8 \times \frac{3}{2}} = \frac{25}{12}$$

L.H.lim \neq R.H.lim \therefore the function is not continuous.

Let us draw a rough plot of $f(n)$. Phone: +91 844-844-0102



Min value is at $n = \frac{3}{2}$ by looking at the plot $= \frac{25}{12} = 2\frac{1}{12}$ option B.

Q4) Consider the function $f(n) = \sin(n)$ in the interval $n \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

The number and locations of the local minima are ____.

A. One, at $\frac{\pi}{2}$.

B. One at $\frac{3\pi}{2}$.

C. Two at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

D. Two at $\frac{\pi}{4}$ and $\frac{3\pi}{2}$.

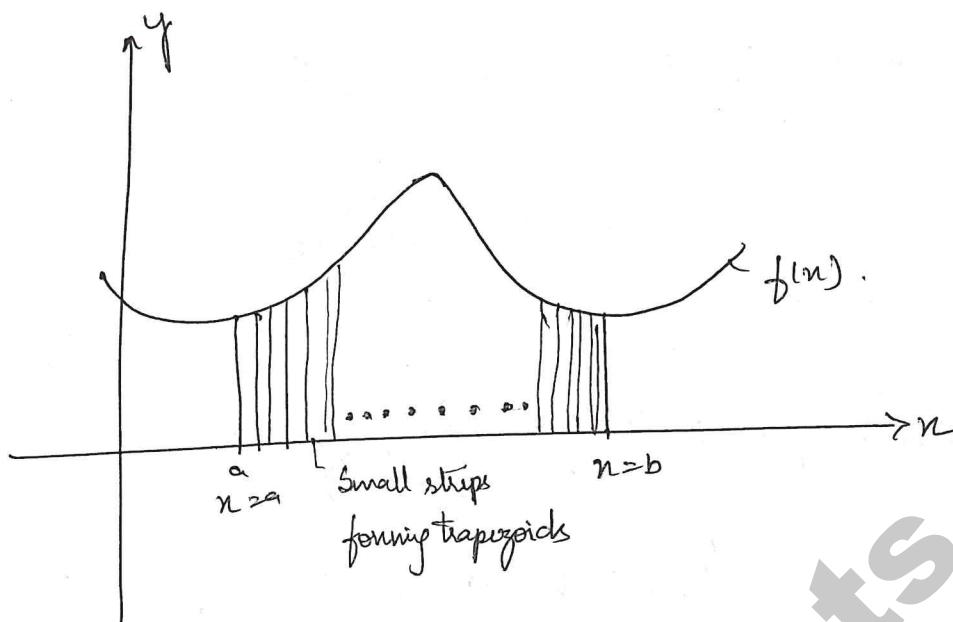
Solution

$$f'(n) = \cos n = 0$$

$$n = \pi/2, 3\pi/2$$

$$f''(n) = -\sin n \quad f''(\pi/2) = -1 < 0 \text{. local max } f''(3\pi/2) = 1 > 0 \text{ local min}$$

→ If our function look like the below **Phone: +91 844-844-0102**



Then

$$\int_a^b f(n) \, dn = \text{area between } f(n) \text{ and } n\text{-axis, in between } n=a \text{ and } n=b.$$

It can be rewritten as

$$\lim_{\Delta n \rightarrow 0} \sum_{n=a}^b f(n) \Delta n = \int_a^b f(n) \, dn \text{ given by } \underline{\text{Riemann sum.}}$$

list of integrals can be found on wikipedia page. http://en.wikipedia.org/wiki/Lists_of_integrals.

4.22 INTEGRATION BY SUBSTITUTION

$$\textcircled{1} \quad \int n^2 \sqrt{n^3 + 1} \, dn$$

Let $u = n^3 + 1 \Rightarrow \frac{du}{dn} = 3n^2 \Rightarrow \frac{du}{3} = n^2 \, dn$

$$\int n^2 \sqrt{n^3 + 1} \, dn = \int \frac{u^{4/2}}{3} \, du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{9} (n^3 + 1)^{3/2} + C$$

If we have the integral in the form $\int f(u) \cdot f'(u) \, du$

then take $u = f(n)$

$$du = f'(n) \, dn$$

$$\int f(n) \cdot f'(n) \, dn = \int u \, du = \frac{u^2}{2} + C$$

$$= \frac{1}{2} (f(n))^2 + C$$

(2) $\int n^2 e^{n^2} \, dn$

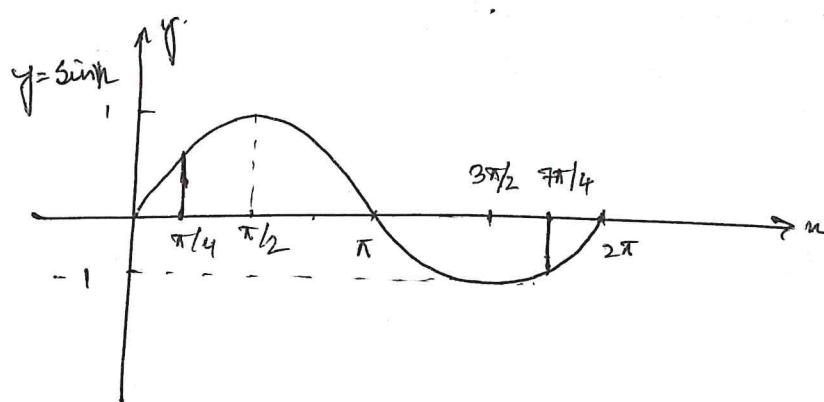
Take $u = n^3$

$$du = 3n^2 \, dn$$

$$\frac{du}{3} = n^2 \, dn$$

$$= \int e^u \frac{du}{3} = \frac{1}{3} e^u + C$$

let us check the graph for this range $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$ **Phone: +91 844-844-0102**



At $u = \pi/4$, it is the local minima because it is the least value in its surroundings.

\therefore The correct answer is $u = \pi/4$.

4. 21 INTEGRATION : AN INTRODUCTION

→ We can think of integration as inverse of derivation or anti-derivative

- Also referred as Indefinite Integral.

If $F(u)$ is differentiable.

$$\text{then } F'(u) = f(u) \cdot \frac{dF(u)}{du} \Rightarrow f(u)du = dF(u).$$

then

$$\int f(u) du = F(u) + C \xrightarrow{\text{constant}}$$

 $F(n) = n^2$

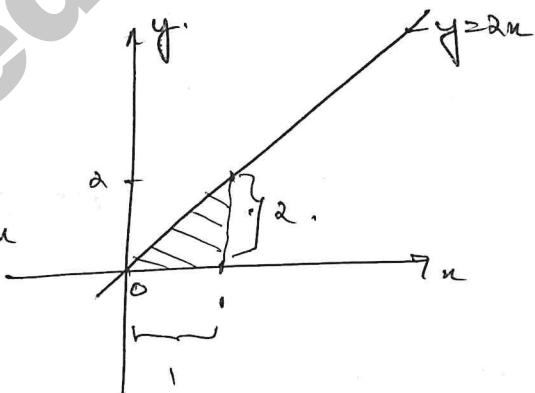
$$F'(n) = 2n^1 = 2n = f(n) = \frac{dF}{dn}$$

$$\int 2n \, dn = n^2 + C$$

Definite Integral

- Represented by $\int_a^b f(n) \, dn$ ex: $\int_0^1 2n \, dn$.
- Geometrically it means area covered by the curve $y=2n$ and n axis in between the values $n=0, 1$

By method of definite integral = $\int_0^1 2n \, dn$



$$= \left[n^{\frac{2}{2}} \right]_0^1$$

$$= [1+1] - [0+0] \quad \text{Area} = \frac{1}{2} \times 2 \times 1 \Rightarrow 1 \text{ unit}$$

$$= 1$$

$$\therefore \int_a^b f(n) \, dn = F(b) - F(a)$$

$$\textcircled{3} \quad \int_0^{\pi/2} \sin^5 x \cos x \, dx$$

$$u = \sin x$$

$$du = \cos x \, dx$$

$$x=0 \quad u=\sin \frac{\pi}{0}=0$$

$$x=\pi/2 \quad u=\sin \frac{\pi}{2}=1$$

$$= \int_0^1 u^5 du = \left[\frac{u^6}{6} \right]_0^1 = \frac{1}{6}$$

\textcircled{4} \quad \ln |f| \text{ form} \rightarrow \text{widely used.}

$$\int \frac{f'(x)}{f(x)} \, dx \quad \text{then take } u=f(x) \\ du = f'(x) \, dx.$$

$$= \int \frac{du}{u} = \ln |u| + C = \ln |f(x)| + C$$

$$\int \frac{e^u - e^{-u}}{e^u + e^{-u}} \, du \quad u = e^u + e^{-u} \\ du = (e^u - e^{-u}) \, du.$$

$$= \int \frac{du}{u} \Rightarrow \log |(e^u + e^{-u})| + C$$

$$\textcircled{5} \quad \int \frac{2n+3}{n^2+3n-4} \downarrow u.$$

$$du = 2n+3.$$

$$\int \frac{du}{u} = \log|u| + C$$

$$= \underline{\log|n^2+3n-4| + C}.$$

$$\textcircled{6} \quad \int \cot n \, dn = \int \frac{\cancel{\cos n} \, dn}{\sin n} \downarrow u = \log|\sin n| + C.$$

$$\textcircled{7} \quad \int \frac{1}{\ln(n^n)} \, dn = \int \frac{1}{n \cancel{\ln n}} \, dn = \frac{-du}{u} = \ln|\ln n| + C$$

4.23 INTEGRATION BY PARTS

→ If we have $u=f(n)$ and $v=g(n)$, then $\int u \, dv = u \cdot v - \int v \, du$.

Proof by product rule of differentiation

$$(fg)' = f'g + fg' \quad (\text{Product rule})$$

Integral

$$fg = \int f'g \, dn + \int fg' \, dn \rightarrow \left\{ \int u \, dv = uv - \int v \, du \right.$$

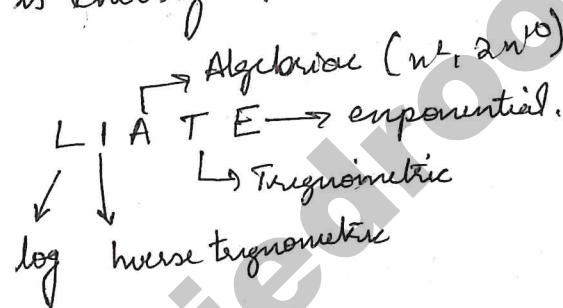
$$\textcircled{1} \int x e^x dx$$

$$\begin{aligned} u &= x & du &= dx \\ dv &= e^x dx & v &= e^x \end{aligned}$$

$$\begin{aligned} \int x e^x dx &= x e^x - \int e^x dx \\ &= x e^x - \underline{\underline{e^x}} + C \end{aligned}$$

Here the challenge is choosing u ?

Rule of thumb:



Another way of remembering.

$$\textcircled{2} \int \frac{\ln x}{x^2} dx \quad \text{LIATE}$$

$$\begin{aligned} u &= \ln x & du &= \frac{1}{x} dx \\ dv &= \frac{1}{x^2} dx & v &= -\frac{1}{x} \end{aligned}$$

$$\int \frac{\ln x}{x^2} dx = (\ln x) \left(-\frac{1}{x} \right) - \int -\frac{1}{x} \frac{1}{x} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx$$

$$\Rightarrow -\frac{\ln x}{x} - \underline{\underline{\frac{1}{x}}} + C$$

$$\textcircled{3} \quad \int \sec^2 n \ln |\sin n| dn \quad \underline{\text{LIATE}}$$

$$U = \ln |\sin n| \quad \left. \begin{array}{l} \\ \end{array} \right| \quad dU = \sec^2 n dn$$

$$dU = \frac{1}{\sin n} \times \cos n dn \quad V = \tan n$$

$$= \int \sec^2 n \ln |\sin n| dn$$

$$= \ln |\sin n| + \tan n - \int \tan n \frac{1}{\tan n} dn$$

$$\Rightarrow \underline{\underline{\tan n \ln |\sin n| - n + C}}$$

$$\textcircled{4} \quad \int n \cos n dn \quad \underline{\text{LIATE}}$$

$$U = n$$

$$dU = dn$$

$$dV = \cos n dn$$

$$V = \sin n$$

$$\int n \cos n dn = n \sin n - \int \sin n dn$$

$$\Rightarrow \underline{\underline{n \sin n + \cos n + C}}$$

$$\textcircled{5} \quad \int e^u \cos u \, du$$

$$U = \cos u$$

$$du = -\sin u \, du$$

$$dv = e^u \, du$$

$$V = e^u$$

$$\int e^u \cos u \, du = e^u \cos u + \int e^u \sin u \, du \quad \begin{matrix} U = \sin u \\ du = \cos u \, du \\ dv = e^u \, du \end{matrix}$$

$$= e^u \cos u + e^u \sin u - \int e^u \cos u \, du$$

original integral lets take it as I .

$$I = e^u \cos u + e^u \sin u - I$$

$$\Rightarrow 2I = e^u \cos u + e^u \sin u$$

$$I = \frac{e^u \cos u + e^u \sin u}{2}$$

$$\textcircled{6} \quad I = \int \ln x \, dx$$

$$U = \ln x$$

$$du = \frac{1}{x} \, dx$$

$$dv = 1 \cdot dx$$

$$V = x$$

$$\Rightarrow I = \int \ln x \cdot 1 \, dx \Rightarrow x \ln x - \int x \cdot \frac{1}{x} \, dx \Rightarrow x \ln x - x + C$$

(7)

$$\int n^3 e^{n^2} dn$$

$$U = n^3 \times$$

$$U = n^2$$

$$du = 3n^2 dn \times \quad du = 2n dn$$

$$dv = n e^{n^2} dn$$

$$V = \frac{e^{n^2}}{2}$$

$$\int n^3 e^{n^2} dn = n^2 \frac{e^{n^2}}{2} - \int \frac{e^{n^2}}{2} \cdot 2n dn$$

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$w = n^2$
 $dw = 2n dn$

Integration by substitution

$$= n^2 \frac{e^{n^2}}{2} - \frac{e^{n^2}}{2} + C$$

$$\int e^w \frac{dw}{2}$$

$$= \frac{e^w}{2} + C$$

$$= \frac{e^{n^2}}{2} + C$$

Plugging in

Mail: gatecse@appliedroots.com

→ Partial fractions have many other applications in mathematics apart from integrations.

Partial Fraction decomposition

- Here the degree of the numerator is less than the degree of the denominator.

$$\text{example } f(n) = \frac{n}{(n^2 - n - 2)} = \frac{n}{(n+1)(n-2)} = \frac{A}{(n+1)} + \frac{B}{(n-2)}$$

↓
Decompose into 2 fractions.

$$\Rightarrow \frac{A(n-2) + B(n+1)}{(n+1)(n-2)} = \frac{n}{(n+1)(n-2)},$$

$$\frac{n(A+B) + (B-2A)}{(n+1)(n-2)} = \frac{n}{(n+1)(n-2)}$$

$$\begin{aligned} \Rightarrow A+B &= 1 \\ A+B &= 0 \end{aligned} \Rightarrow \begin{cases} A = 1/3 \\ B = 2/3 \end{cases}$$

$$\frac{n}{n^2 - n - 2} = \frac{1}{3(n+1)} + \frac{2}{3(n-2)} ; \int \frac{n}{n^2 - n - 2} = \int \frac{1}{3(n+1)} + \int \frac{2}{3(n-2)}$$

$$\frac{1}{3} \ln |n+1| + \frac{2}{3} \ln |n-2| + C$$

① $\int \frac{1}{n(n+1)} dn$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$A(n+1) + Bn = 1$$

$$A+B=0$$

$$A=1$$

$$B=-1$$

$$\Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\int \frac{1}{n(n+1)} = \ln |n| - \ln |n+1| + C$$

② $\int \frac{2n+3}{n^2-9} dn =$

$$\frac{2n+3}{(n-3)(n+3)} = \frac{A}{n-3} + \frac{B}{n+3} \Rightarrow A(n+3) + B(n-3) = 2n+3$$

$$A+B=2$$

$$3A - 3B = 3$$

$$\Rightarrow A = \frac{3}{2}$$

$$B = \frac{1}{2}$$

Plug these values back.

$$\int \frac{2n+3}{n^2-9} dn = \int \frac{1}{2(n-3)} dn + \int \frac{3}{2(n+3)}$$

$$= \frac{1}{2} \ln |n-3| + \frac{3}{2} \ln |n+3| + C$$

$$(3) \int \frac{2n+3}{n^2+3n+9} dn$$

Here we can see that numerator is derivative of denominator
it is better to go by method of substitution in this case

$$U = n^2 + 3n + 9$$

$$du = (2n+3)dn$$

$$\int \frac{du}{U} = \underline{\underline{\ln |n^2 + 3n + 9| + C}}$$

$$(4) f(n) = \frac{n+10}{n^3+n} = \frac{A}{(n)} + \frac{Bn+C}{(n^2+1)}$$

↓

$$\frac{n+10}{n(n+1)}$$

* * * * *
y- we have 2nd order polynomial which
cannot be decomposed using real numbers.
∴ we need to have n term in the numerator.

(B) Decompose into partial fractions

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$$f(n) = \frac{1}{(n^3 - 1)(n^2 - 1)^2}$$

$$= \frac{1}{(n-1)(n^2+n+1)(n-1)(n+1)^2}$$

$$= \frac{1}{(n-1)^3 (n^2+n+1) (n+1)^2}$$

$$= \frac{A}{(n-1)} + \frac{B}{(n-1)^2} + \frac{C}{(n-1)^3} + \frac{D}{(n+1)} + \frac{E}{(n+1)^2} \\ + \frac{Fn+G}{(n^2+n+1)}$$

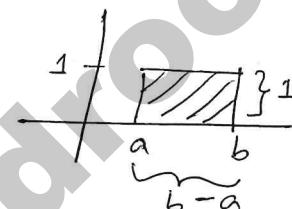
→ When we have a power of a fraction then we have to consider it in all powers from 1 upto its actual exponent.

for example for $(n-1)^3$ we have to write $(n-1)$, $(n-1)^2$ and $(n-1)^3$.

Definite integrals - area under a curve
or
area between a curve and x-axis, when the curve is being represented by the function we are integrating.

Properties of Definite Integrals:

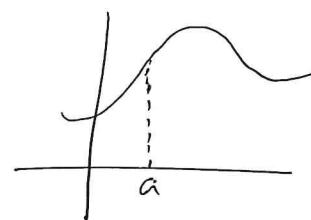
$$\textcircled{1} \quad \int_a^b 1 \, dx = b - a$$



$$\textcircled{2} \quad \int_a^b k \cdot f(x) \, dx = k \int_a^b f(x) \, dx \quad \xrightarrow{\text{Riemann Sums}}$$

$$\textcircled{3} \quad \int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

$$\textcircled{4} \quad \int_a^a f(x) \, dx = 0$$

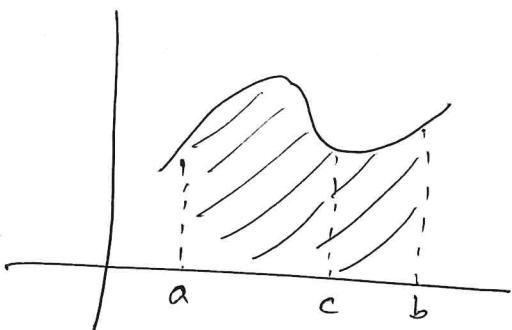


$$\textcircled{5} \quad \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$\begin{aligned} & \downarrow \\ & F(x) \Big|_a^b \quad \Big| \quad F(x) \Big|_b^a \\ & = F(b) - F(a) \quad \Big| \quad F(a) - F(b) \end{aligned}$$

$$\textcircled{6} \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$c \in [a, b]$$



$$\textcircled{7} \quad \int_a^b f(x) dx \geq 0 \quad \text{if } f(x) \geq 0 \quad [a, b]$$

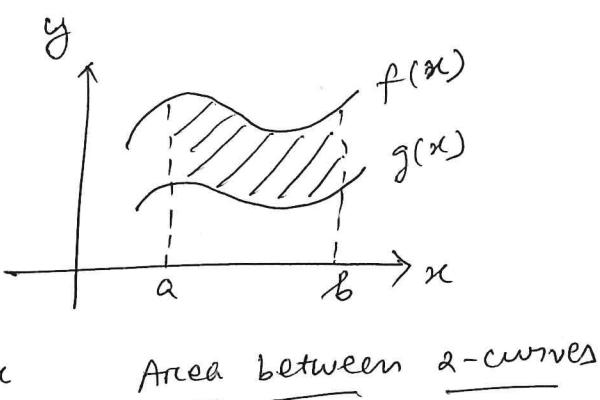
$$\int_a^b f(x) dx \leq 0 \quad \text{if } f(x) \leq 0 \quad \text{in } [a, b]$$

\textcircled{8} Fundamental Theorem of Calculus

$$\left\{ \begin{array}{l} \text{anti-} \\ \text{derivative} \end{array} \right\} \int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a) \\ \text{if } F'(x) = f(x)$$

$$\textcircled{9} \quad S = \int_a^b (f(x) - g(x)) dx$$

$$= \int_a^b f(x) dx - \int_a^b g(x) dx$$



4.26

INTEGRATION | SOLVED PROBLEMS

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Q1.)

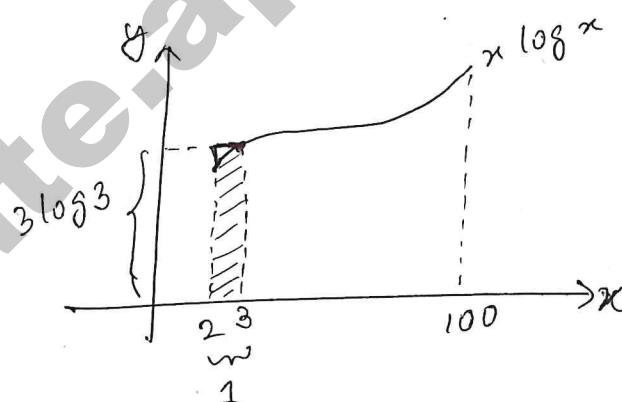
Let $S = \sum_{i=3}^{100} i \log_2 i$, and

$$T = \int_2^{100} x \log_2 x dx.$$

Which of the following statements is true?

- A. $S > T$
- B. $S = T$
- C. $S < T$ and $2S > T$
- D. $2S < T$

Soln. As x increases, $x \log_2 x$ also increases. And is always positive in this interval 2 to 100.



$$\sum_{i=3}^{100} i \log_2 i = 3 \log 3 + 4 \log 4 + 5 \log 5 + \dots + 100 \log 100$$

$$3 \log 3 > \int_2^3 x \log x dx$$

$$4 \log 4 > \int_3^4 x \log x \, dx$$

$$\downarrow \qquad \downarrow$$

$$\vdots \qquad \vdots$$

$$S > T$$

\therefore option (A).

Q2)

$$\int_0^{\pi/4} \frac{(1-\tan x)}{(1+\tan x)} \, dx$$

- A. 0
- B. 1
- C. $\ln 2$
- D. $1/2 \ln 2$

$$\int_0^{\pi/4} \frac{(1-\tan x)}{(1+\tan x)} \, dx$$

$$= \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} \, dx = \int_1^{\sqrt{2}} \frac{du}{u}$$

$$u = \cos x + \sin x$$

$$du = (-\sin x + \cos x) \, dx$$

$$x = 0 ; u = 1$$

$$x = \pi/4 ; \sqrt{2}$$

$$= \left[\ln u \right]_1^{\sqrt{2}}$$

$$= \ln \sqrt{2} - 0$$

$$= \frac{1}{2} \ln 2$$

(Q3)

Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral

$$\int_0^{\pi/2} \frac{\cos x + i \sin x}{\cos x - i \sin x} dx ?$$

- A. 0
- B. 2
- C. $-i$
- D. i

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\int_0^{\pi/2} \frac{e^{ix} dx}{e^{-ix}} = \int_0^{\pi/2} e^{2ix} dx = \frac{e^{2ix}}{2i} \Big|_0^{\pi/2} = \frac{1}{2i} (-1 - 1) = \frac{-2}{2i} = \frac{-1}{i}$$

$$e^{2ix} = \cos 2x + i \sin 2x \quad \begin{cases} x = \pi/2 \rightarrow -1 \\ x = 0 \rightarrow 1 \end{cases}$$

$$= \frac{i^2}{i} = \frac{-1}{i} = i$$

\therefore Option (D)

(Q4)

The value of the integral given below is

$$\int_0^{\pi} x^2 \cos x \, dx$$

- A. -2π
- B. π
- C. $-\pi$
- D. 2π

Soln. We try to use Integration by parts and
LIATE principle.

$$u = x^2$$

$$du = 2x \, dx$$

$$dv = \cos x \, dx$$

$$v = \sin x$$

$$\begin{aligned} \therefore \int_0^{\pi} x^2 \cos x \, dx &= \int u \cdot dv \\ &= x^2 \sin x \Big|_0^{\pi} - \int \frac{2x \sin x}{u} \, dv \\ &= 0 - \int 2x \sin x \, dx \end{aligned}$$

$$u = u$$

$$du = dx$$

$$dv = \sin x \, dx$$

$$v = -\cos x$$

$$\begin{aligned} &= -2 \left\{ -x \cos x \Big|_0^{\pi} - \int -\cos x \, dx \right\} \\ &= 2x \cos x \Big|_0^{\pi} + \int 2 \cos x \, dx \\ &= -2\pi \leftarrow \underline{\underline{\text{Ans}}} \end{aligned}$$

Q5)

If $\int_0^{2\pi} |x \sin x| dx = k\pi$, then the value of k is

equal to _____

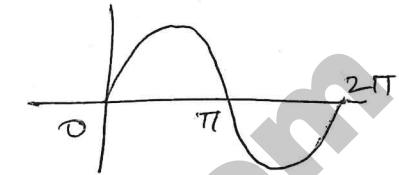
Soln.

$$\int_0^{\pi} \frac{x \sin x}{u} dv - \int_{\pi}^{2\pi} x \sin x dv$$

↓

$$= -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx$$

$$= \pi + \sin x \Big|_0^{\pi} = \pi$$



$$\begin{aligned} u &= x \\ du &= dx \\ dv &= \sin x dx \\ v &= -\cos x \end{aligned}$$

$$-x \cos x \Big|_{\pi}^{2\pi} + \sin x \Big|_{\pi}^{2\pi}$$

$$= -3\pi$$

$$\therefore \pi - (-3\pi) \\ = 4\pi$$

Q6) Compute the value of :

$$\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2} dx$$

$$u = 1/x$$

$$du = -\frac{1}{x^2} dx$$

$$\begin{aligned}
 - \int_{\pi}^{\pi/2} \cos u du &= \int_{\pi/2}^{\pi} \cos u du = \sin u \Big|_{\pi/2}^{\pi} \\
 &= 0 - 1 \\
 &= -1
 \end{aligned}$$

Q7.) The value of $\int_0^{\pi/4} x \cos(x^2) dx$ correct to three decimal places (assuming that $\pi = 3.14$) is _____

$$u = x^2$$

$$du = 2x dx$$

$$\int_0^{\pi^2/16} \frac{\cos u du}{2} = \frac{1}{2} \sin u \Big|_0^{\pi^2/16}$$

$$= \frac{1}{2} \sin(\pi^2/16)$$

$$= \frac{0.5779}{2}$$

$$= 0.289$$

(Q 8)

If $f(x) = R \sin\left(\frac{\pi x}{2}\right) + S \cdot f'\left(\frac{1}{2}\right) = \sqrt{2}$

and $\int_0^1 f(x) dx = \frac{2R}{\pi}$, then the constants R and S are

- A. $\frac{2}{\pi}$ and $\frac{16}{\pi}$
- B. $\frac{2}{\pi}$ and 0
- C. $\frac{4}{\pi}$ and 0
- D. $\frac{4}{\pi}$ and $\frac{16}{\pi}$

Soln.

$$f(x) = R \sin\left(\frac{\pi x}{2}\right) + S$$

$$f'\left(-\frac{1}{2}\right) = \sqrt{2}$$

$$\int_0^1 f(x) dx = \frac{2R}{\pi}$$

$$f'(x) = R \cos\left(\frac{\pi x}{2}\right) * \pi/2$$

$$f'\left(\frac{1}{2}\right) = R \cos\left(\pi/2 \cdot 1/2\right) * \pi/2 = \sqrt{2}$$

$$\Rightarrow R = 4/\pi$$

$$f(x) = \int_0^1 \left(R \sin\left(\frac{\pi x}{2}\right) + S \right) dx$$

$$= \int_0^1 \frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right) dx + S x \Big|_0^1$$

$$= \frac{-4}{\pi} \cos\left(\frac{\pi x}{2}\right) + \frac{2}{\pi} \int_0^1 + 8x \int_0^1 = \frac{8}{\pi^2}$$

$$\Rightarrow \delta = 0$$

Ans (C)