

12.1 Why STUDY PROBABILITY

- Used extensively in multiple areas by CSE, EE, Mech, Civil engineers, also has extensive applications in Physics and other sciences.
- Software: QuickSort $\rightarrow O(n^2)$ worst case time complexity.
On randomly choosing the pivot we can get $O(n \log n)$ time complexity.
- There are probabilistic Data Structures and algorithms, we use these Data Structures and Algorithms to use to minimize the time complexity of the algorithms.
- Machine learning, Data science, Artificial intelligence :-
A lot of algorithms based on probability \rightarrow Bayes theorem \rightarrow Main Bayesian
 \downarrow
Bayesian Inference.
- Build / simulate Especially at defence organizations such as DRDO / ISRO / CERN / BARC.
 - We try to simulate real world systems like Internet traffic, Road traffic etc.

- Two ways to introduce Probability

1. Set Theoretic (Axiomatic) - intuitive, Venn diagrams, Combinatorics.
2. Measure Theoretic - More Mathematical.

Sample Space (S)

Experiment : Flipping 2 coins (distinct).

$$\text{Outcomes} = \{ (H,H), (H,T), (T,H), (T,T) \} = S.$$

Sample Space :- Set of all possible outcomes of the experiment

Experiment :- Conduct a 7-way horse race.

Outcomes: ordering of 7 horses.

$$1, 2, 3, 4, 5, 6, 7 -$$

$$S = \{ 7! \text{ possible orderings} \}$$

Experiment :- 2 distinct dice are rolled,

$$\text{Outcomes } S = \{ (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)$$

$$(2,1) (2,2) - - - - (2,6)$$

$$(3,1) (3,2) - - - - (3,6)$$

36 total outcomes.

Experiment:- light Bulb, Measuring the #hrs the light bulb works (3)
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$$S = \{ n : 0 \leq n < \infty \}$$

$n \in \mathbb{R}$

here the sample space is of infinite size.

Event (E) : Any subset of S is an event -

(eg) $E = \{ n : 0 \leq n \leq 5 \} \Rightarrow$ light bulb works for 0 to 5 hours.

$$E \subseteq S$$

(eg) If two dice are rolled. Event that sum = 6.

$$E = \{ (1,3), (2,4), (4,2), (1,5), (5,1) \}$$

(eg) E = first house wins out of the 7 houses.

$E = \{ 6! \text{ possible orderings where } 1 \text{ is the first} \}$

(eg) E = 1st coin is always H.

$$E = \{ (H, T), (H, H) \}$$

(4)

12.3 Axioms Of Probability, PROPERTIES AND EXAMPLES

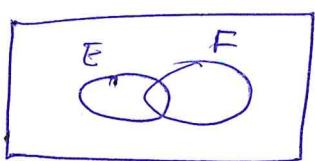
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- For a given experiment

- S: The sample space is the set of all possible outcomes similar to universal set in the set theory.

- E: Event :- set of outcomes



U/S.

e.g.: Throwing/Tossing 2 coins (distinct)

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

E = All outcomes where first coin is H.

$$E = \{(H,H), (H,T)\}$$

F = $\{(H,H), (T,T)\}$ all outcomes with both coins of the same value.

$$E \cap F = \{(H,H)\}$$

$$E \cup F = \{(H,H), (T,T), (H,T)\}$$

$$E^C = \{(T,T), (T,H)\}$$

Frequentist Vs Bayesian

way of thinking about probability. (5)
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- The Bayesian approach is more related to the Bayes theorem
- The frequentist approach

1. Conduct the experiment n times

2. $n(E) = \text{no of times the outcome of the experiment } \in E$

$$P(E) = \lim_{n \rightarrow \infty} \frac{n(E)}{n}$$

e.g. fair coin

$$S = \{H, T\}$$

$$\bar{E} = \{H\}$$

exp: - Toss of a fair coin.

$$P(H) = ?$$

$$\begin{array}{ccc} 10 & \xrightarrow{\hspace{1cm}} & 6H \\ \text{times} & \xrightarrow{\hspace{1cm}} & 4T \end{array} \quad P(H) = \frac{6}{10}$$

$$\begin{array}{ccc} 100 & \xrightarrow{\hspace{1cm}} & 506H \\ & \xrightarrow{\hspace{1cm}} & 494T \end{array} \quad P(H) = \frac{506}{1000}$$

$$\infty \qquad \qquad \qquad \underline{0.5}$$

$$1. \quad 0 \leq P(E) \leq 1$$

$$2. \quad P(S) = 1$$

3. if E_1, E_2, E_3 are mutually exclusive events $E_i \cap E_j = \emptyset$

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n P(E_i)$$

Example : A fair die is rolled.

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$E_1 = \{2\} \quad P(E_1) = \frac{1}{6}$$

$$E_2 = \{3\} \quad P(E_2) = \frac{1}{6}$$

$$E_3 = \{6\} \quad P(E_3) = \frac{1}{6}$$

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \underline{\underline{\frac{1}{2}}} \end{aligned}$$

$$4. \quad P(E^c) = 1 - P(E)$$

5. If $E \subseteq F$ (if E is a subset of F) then $P(E) \leq P(F)$

6. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ (Can be proved using principle of Inclusion & Exclusion).

$$7. \quad P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) = P(E_1) + P(E_2) + \dots + P(E_n)$$

$$= (P(E_1 \cap E_2) + P(E_2 \cap E_3) + \dots + P(E_n \cap E_1))^{2 \text{ way}}$$

$$+ (P(E_1 \cap E_2 \cap E_3) + P(E_2 \cap E_3 \cap E_4) + \dots)^{3 \text{ way}}$$

... n way.

- If we have a sample space with equally likely outcomes then.

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$$P(E) = \frac{|E|}{|S|}$$

for example if we have a fair die $S = \{1, 2, 3, 4, 5, 6\}$.

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\})$$

$$P(S) = 1 = P(\{1, 2, 3, 4, 5, 6\}) = \sum_i P(\{i\}) = 1$$

$$\therefore P(\{i\}) = \frac{1}{6}.$$

→ example: If 2 dice are thrown.

$$E = \text{sum of dice} = 7$$

$$|S| = 36 \quad (1,1), (1,2), \dots, (6,6)$$

$$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$P(E) = \frac{6}{36} = \frac{1}{6}.$$

→ example if we have 6 white and 5 black balls, if we randomly pick 3 balls what is the probability that we get 1 white and 2 black balls

No of ways we can choose 1 white and 2 black balls = ${}^6C_1 \times {}^5C_2$

Total no of ways we can choose 3 from 11 balls = ${}^{11}C_3$

$$\therefore P(E) = \frac{{}^6C_1 \times {}^5C_2}{{}^{11}C_3}$$

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(eg) If we have n balls and one ball is special among them.



Experiment :- K balls are drawn one at a time randomly.

Event : Special ball is picked.

$$|S| = {}^n C_K$$

$$|E| = {}^1 C_1 \cdot {}^{n-1} C_{K-1}$$

Matching Problem

If we have n men who throw/keep their hats on the table and then pick the hat from the table randomly, the probability that all get a hat other than their hat.

E_i = Event that the i th person has picked the correct hat.

$P(E_1 \cup E_2 \cup E_3 \dots \cup E_n)$ = Probability that atleast one person picks his hat

$1 - P(E_1 \cup E_2 \dots \cup E_n)$ = Prob. that no person picks his hat.

$$\begin{aligned} &= P(E_1) + P(E_2) + \dots + P(E_n) = \frac{n}{n} = 1 \\ &= (P(E_1 \cap E_2) + P(E_2 \cap E_3) + \dots) - 2 \text{ way } \frac{1}{2!} \end{aligned}$$

$$+ (P(E_1 \cap E_2 \cap E_3) + \dots) - 3 \text{ way. } - \frac{1}{3!}$$

-

:

$$P\left(\bigcup_{i=1}^n E_i\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{n!}$$

$$1 - P\left(\bigcup_{i=1}^n E_i\right) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-)^n \frac{1}{n!}$$

If we have infinitely many people the expression comes

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots - \dots + \frac{1}{\infty}$$

$$= e^{-n} = 0.36788$$

12.4 CONDITIONAL PROBABILITY AND EXAMPLES

Example:- Two dice are rolled.

$$S = \{(1,1), (1,2), \dots, (6,6)\}$$

$$F = \{\text{outcomes of the first die}\} \cdot B = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$$

$$E = \{\text{sum of two dice} = 7\} = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

Probability that event E has occurred given that event F has already occurred.

$$\boxed{P(E|F) = \frac{P(E \cap F)}{P(F)}} = \frac{1/36}{1/6} = \frac{1}{6}$$

if $P(F) \neq 0$.

- (Q) Student taking one hour exam, P .(Student finishes the exam under n hours) = $\frac{n}{2}$
 Given that the student is working at 0.75 hours
 what is the probability that the student the full 1 hour.

F = Student uses the full one hr

F^C = Student finishes exam under 1 hour.

$$P(F^C) = \frac{1}{2}, \quad F \cup F^C = S$$

$$P(F) = 1 - \frac{1}{2} = \frac{1}{2}$$

L_n = Student finishes in n hours.

L_n^C = Student is still working at n hours.

$L_{0.75}^C$.

$$P(F | L_{0.75}^C) = \frac{P(F \cap L_{0.75}^C)}{P(L_{0.75}^C)} = \frac{P(F)}{\frac{1}{2}} = \frac{1}{2}$$

$$\frac{\frac{1}{2}}{1 - \frac{0.75}{2}} = \underline{\underline{0.8}}$$

12.5 MULTIPLICATION THEOREM

- Example on conditional probability.
- Select n balls sequentially and randomly without replacement from an urn contains r red balls, b blue balls $n \leq r+b$, find the probability that k out of n balls are blue, what's the probability that the 1st ball is blue.

Let's take Event B = Event that 1st picked ball is blue.

B_K = Event that k out of n balls picked are blue.

$$P(B|B_K) = \frac{P(B \cap B_K)}{P(B_K)}$$

$$P(B_K) = \frac{\binom{b}{k} \times \binom{r}{n-k}}{\binom{b+r}{n}} = \frac{n(B_K)}{n(S)}$$

$$P(B|B_{k'}) = \frac{P(B \cap B_k)}{P(B_k)} = \frac{P(B_k|B) \cdot P(B)}{P(B_k)}$$

$$P(B) = \frac{b}{n+b}$$

$$P(B_k|B) = \frac{\binom{b-1}{k-1} \cdot \binom{n}{n-k}}{\binom{n+b-1}{n-1}}$$

$$P(B|B_k) = \left(\frac{\binom{b-1}{k-1} \cdot \binom{n}{n-k}}{\binom{n+b-1}{n-1}} \right) \cdot \frac{b}{n+b}$$

$$\left(\frac{\binom{b}{k} \cdot \binom{n}{n-k}}{\binom{b+n}{n}} \right)$$

$$= \left(\frac{k}{n} \right)$$

Multiplication Rule:

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We have $P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$ if $P(E_2) \neq 0$

$$\Rightarrow P(E_1 \cap E_2) = P(E_1 | E_2) \cdot P(E_2)$$

Can also be written as

$$P(E_1, E_2) = P(E_1 | E_2)P(E_2)$$

Generalization

$$P(E_1, E_2, E_3, \dots, E_n) = P(E_1)P(E_2 | E_1)P(E_3 | E_2, E_1)P(E_4 | E_3, E_2, E_1) \dots P(E_n | E_{n-1}, E_{n-2}, \dots, E_1)$$

(example) Matching problem: n people throw their hats and randomly pick back their hats

$$P(\text{no one picks their hat back}) = \sum_{i=0}^n (-1)^i / i!$$

Q. $P(\text{Exactly } K \text{ persons have picked correctly}) = ?$

Let $A = \{\text{ } K \text{ persons would pick correctly}\}$ - can be done in n_C_k ways

$E = \text{Everyone in } A \text{ has picked correctly}$

$G = \text{Everyone other than people in set } A \text{ have picked}$

We need to calculate $P(G \cap E) = P(G|E) \cdot P(E)$ (14)

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$$P(\bar{G}|E) = -\text{Probability that } n-k \text{ people have not picked the correct hat.}$$

$$= \sum_{i=0}^{(n-k)} \frac{(-1)^i}{i!}$$

$P(E) = F_1$ = event that 1st person in set A has picked the hat correctly.

F_2 = event that 2nd person in set A has picked the hat correctly.

F_3 - " 3rd

$$P(E) = P(F_1 F_2 \dots F_k) = P(F_1) P(F_2 | F_1) P(F_3 | F_2 F_1) P(F_4 | F_3 F_2 F_1) \dots P(F_k | F_{k-1} \dots F_1)$$

$$= \frac{1}{n} \cdot \frac{1}{(n-1)} \cdot \frac{1}{(n-2)} \dots \frac{1}{(n-k+1)}$$

$$\frac{(n-k)!}{n!}$$

$$P(G \cap E) = P(G|E) \cdot P(E)$$

$$= \left[\sum_{i=0}^{(n-k)} \left(\frac{(-1)^i}{i!} \right) \right] \left[\frac{(n-k)!}{n!} \right]$$

\times no of ways we can form
set A
 $({}^n C_k)$

- Example

Experiment: Toss a coin & throw a die.

E = the coin is H

F : The die is 3.

$$P(E \cap F) = P(E|F)P(F). \text{ - definition of conditional probability}$$



The outcome of H given that the outcome of the die is 3.
 the outcome of the coin is not dependent on the outcome of the die therefore we can replace $P(E|F)$ by $P(E)$ itself

$$\Rightarrow P(E \cap F) = P(E)P(F).$$

such events where outcome of one does not impact the outcome of other are known as independent events.

NOTE :- Independent Events and Mutually Exclusive events are not the same, independence of events means that the outcome of one event does not have any impact on the outcome of the other event, whereas if two or more events are mutually exclusive then the occurrence of one prevents the occurrence of the other event or events.

Independent Events $P(E \cap F) = P(E) \cdot P(F)$ **Phone: +91 844-844-0102**

Mutually Exclusive Events $P(E \cap F) = 0$ as $E \cap F = \emptyset$

example. A Deck of 52 cards is available.

experiment :- A card is picked ^{randomly} and replaced.

The second card is also picked ^{randomly}.

$P(\text{the first card is jack and the second is 8})$

E_1

E_2

→ E_1 and E_2 are independent events if we get first card as jack we do not have any impact on E_2 because it is replaced, hence these events are independent events.

→ If after the 1st card is picked, it is not replaced then the prob of the second card being 8 is no longer same as it depends and depends on what the outcome of the 1st event was and also the number of cards in the deck have decreased.

(example). 2 distinct coins are tossed

E : 1st coin is H

F : 2nd coin is T

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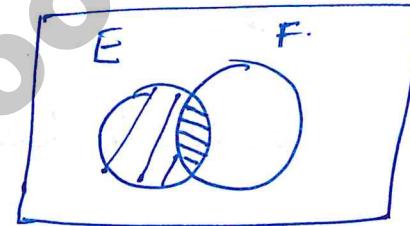
$$P(E \cap F) = P(E) \cdot P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(E_1 E_2 E_3 \dots E_n) = P(E_1) \cdot P(E_2) P(E_3) \dots P(E_n)$$

$$= \prod_{i=1}^n P(E_i)$$

Note If E and F are independent events then $E \& F^L$ are also independent events.

Proof $P(E) = P(E \cap F) + P(E \cap F^L)$



$$= P(E) P(F) + P(E \cap F^L)$$

$$P(E \cap F^L) = P(E) [1 - P(F)]$$

$$= P(E) \cdot P(F^L)$$

Which means E and F^L are independent events.

(example) An infinite sequence of trials are performed.

success probability - p .

Trail : Throwing a die

success - outcome = 1

$p = 1/6$

(a) Prob. of atleast one success in n trials.

$$= 1 - P(\text{no success in } n \text{ trials})$$

$$= 1 - P(F_1 F_2 \dots F_n)$$

$$= 1 - P(F_1)P(F_2) \dots P(F_n)$$

$$= 1 - (1-p)^n$$

(b) Prob. of exactly k successes in n trials.

- We need to have k successes and $(n-k)$ failures.

$$= \frac{n!}{k!} p^k (1-p)^{n-k}$$

Note: S_1, S_2, \dots, S_n are all independent events
 $S_i = \text{Success in the } i\text{th trial.}$

(c) Prob of all n trials are successful.

$$= \underline{\underline{P}}^n$$

12.7 LAW OF TOTAL PROBABILITY

(Example). Manufacturing

- We have two factories X & Y for manufacturing bulbs.
- Bulbs made in X work for 5000 hours in 99% cases.
- Bulbs made in Y work for 5000 hours in 95% cases.
- We create a package which have 60% bulbs from X and 40% of the bulbs from Y, now what is the probability that the bulb will work for 5000 hours?

Let A be the event that the bulb works for 5000 hours.

B_x : Event that a bulb is made at X

B_y : Event that a bulb is made at Y,

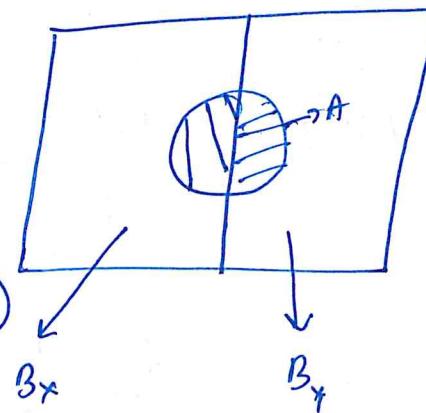
$$P(A) = P(A \cap B_x) + P(A \cap B_y)$$



$$= P(B_x) \cdot P(A|B_x) + P(B_y) P(A|B_y)$$

$$= (0.6 \times 0.99) + (0.4 \times 0.95)$$

$$= \underline{0.974}$$



$$S = B_x \cup B_y$$

$$B_x \cap B_y = \emptyset$$

A Generalization of above example is as follows

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If $B_1, B_2, B_3, \dots, B_n$ are events such that -

$$B_i \cap B_j = \emptyset \quad \text{Mutually Exclusive.}$$

$$B_1 \cup B_2 \cup \dots \cup B_n = S \quad \text{Mutually Exhaustive Events.}$$

$$\text{Then } P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)$$

$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$$

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

12.8 BAYES THEOREM

From definition of conditional probability we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) \neq 0$$

$$= \frac{P(B \cap A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{if } P(B) \neq 0$$

$P(B)$ and $P(A)$ are known as marginal probability. (21)
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$P(A|B)$ and $P(B|A)$ are the conditional probabilities.

Alternate way of writing Bayes theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$= \frac{P(B|A) P(A)}{P(B \cap A) + P(B \cap A^c)}$$

$$P(A|B) = \frac{P(B|A) P(A)}{P(B|A) P(A) + P(B|A^c) P(A^c)}$$

Applications of Bayes theorem

- Medical Diagnosis
 - Detection of Breast cancer.

Approx 1% of women in 40-50 have breast cancer.

Mammogram (X-ray) \rightarrow cheap but (not perfect)

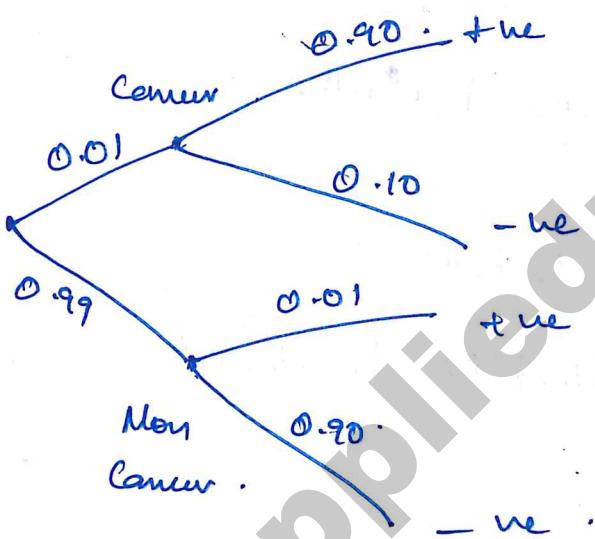
- If a woman has breast cancer, the test will result in true value 90% of the time.
- If a woman does not have breast cancer, then the test will result in \rightarrow 10% of the times.

If the doctor wants to predict, what is the prob that the woman has cancer given the result is +ve

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$$P(\text{Cancer} | +ve) = \frac{P(+ve | \text{Cancer}) P(\text{Cancer})}{P(+ve)}$$

$$= \frac{0.9 \times 0.01}{P(+ve)}$$



$$P(+ve) = P(\text{Cancer} \cap +ve) + P(\text{NonCancer} \cap +ve)$$

$$= P(\text{Cancer}) P(+ve | \text{Cancer}) + P(\text{NonCancer}) P(+ve | \text{NonCancer})$$

$$= (0.9)(0.01) + (0.1)(0.99)$$

$$= 0.108$$

$$P(\text{Cancer} | \text{tue}) = \frac{0.9 \times 0.01}{0.108} = \frac{9}{108} = \underline{\underline{8.3\%}} \quad \text{Phone: } \underline{\underline{844-844-0102}}$$

Actual probability = 8.3%. people learn age and get turned but it is not fair.

In such a case the doctor will take a more advanced test.

- Odds :- Mostly used in betting.

$$\text{Odds of } A \text{ happening} = \frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

The odds in favor of A are 2 to 1 $\rightarrow \frac{P(A)}{P(A^c)} = \frac{2}{1}$

12.9 Solved Problems.

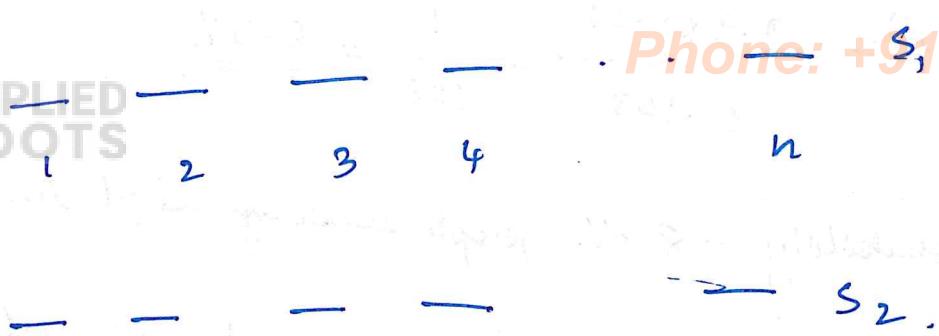
A random bit string of length n is constructed by tossing a fair coin n times and setting a bit to 0 or 1 depending on outcomes head and tail, respectively. The probability that two such randomly generated strings are not identical is -

A. $\frac{1}{2^n}$

C. $\frac{1}{n!}$

B. $1 - \frac{1}{n}$

D. $1 - \frac{1}{2^n}$



Prob that the two strings are different = 1 - Prob that both are same.

$$= 1 - \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \dots \text{n terms} \right)$$

$$= 1 - \frac{1}{2^n}$$

Q. For each element in a set of size $2n$, an unbiased coin is tossed. The $2n$ coin tosses

A. $\frac{2n}{C_n}$

D. $\frac{1}{2}$.

B. $\frac{2^n}{C_n}$

2^n

C. $\frac{1}{2^n C_n}$

Solution

Given that we have a set of the $2n$ elements $\{a_1, a_2, \dots, a_{2n}\}$

We can choose n out of $2n$ coins in ${}^{2n}C_n$ ways.

Getting heads on n coins we need to have $(\frac{1}{2}) \times (\frac{1}{2}) \dots n$ times
 $= (\frac{1}{2})^n$ probability.
 and

getting tails on n coins we need to have $= (\frac{1}{2}) \times (\frac{1}{2}) \dots n$ times
 $= (\frac{1}{2})^n$ probability.

$$\therefore \text{Total probability} = {}^{2n}C_n \times \left(\frac{1}{2}\right)^n \times \left(\frac{1}{2}\right)^n$$

$$= {}^{2n}C_n \times \frac{1}{4^n} \quad (\text{option A}).$$

Q) Let A and B be two arbitrary events, then, which of the following is true?

A. $P(A \cap B) = P(A)P(B)$

B. $P(A \cup B) = P(A) + P(B)$

C. $P(A|B) = P(A \cap B) / P(B)$

D. $P(A \cup B) \leq P(A) + P(B)$.

Solution

- A. $P(A \cap B) = P(A)P(B)$ - it is true only when the events are independent.

X

- B. $P(A \cup B) = P(A) + P(B)$ it is only true when $P(A \cap B) = 0$

- C. $P(A|B) = \frac{P(A \cap B)}{P(B)}$ - From conditional probability we know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- D. $P(A \cup B) \leq P(A) + P(B)$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$\therefore D$ is true.

Q. Let $P(E)$ denote the probability of the event E . Given $P(A)=1$, $P(B)=\frac{1}{2}$, the values of $P(A|B)$ and $P(B|A)$ respectively are?

A. $(\frac{1}{6}), (\frac{1}{2})$.

B. $(\frac{1}{2}), (\frac{1}{4})$.

C. $(\frac{1}{2}), 1$.

D. $1, (\frac{1}{2})$.

50m



$P(A|B) = \text{Prob of } A \text{ given } B \text{ occurs.}$

$$= P(A)=1 \quad P(B)=\frac{1}{2}$$

Will occur always therefore irrespective of occurrence of B

~~A~~ A will occur. $\therefore P(A|B)=1$

Also by using the defn of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \quad \text{As } P(A)=1$$

it is like the complete sample space, therefore.

$$A \cap B = B$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\frac{1}{2}}{1} = \frac{1}{2} .$$

Option D. is correct.

- Q) Box P has 2 red balls and 3 blue balls and box Q has 3 red balls and 1 blue balls. A ball is selected as follows (i) Select a box (ii) choose a ball from the selected box such that each ball in the box is equally likely to be chosen. The probabilities of selecting boxes P and Q are $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Given that a ball selected in the above process is a red ball, the probability that it

Come from the book Pis :-

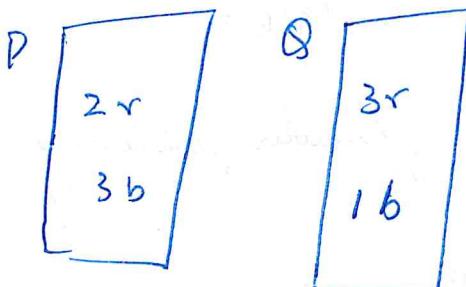
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$$A. \frac{4}{19} \quad B. \frac{5}{19} \quad C. \frac{2}{9} \quad D. \frac{19}{13}$$

Solution



$$P(P) = \frac{1}{3}$$

$$P(Q) = \frac{2}{3}$$

$$P(P| \text{not } Q) = \frac{P(P \cap \text{not } Q)}{P(\text{not } Q)} = \frac{\frac{1}{3} \cdot \frac{2}{5}}{\frac{2}{3}} \Rightarrow \frac{\frac{2}{15}}{\frac{19}{30}} \Rightarrow \frac{4}{19}$$

12.10 RANDOM VARIABLE : AN INTRODUCTION

- A random variable is a mapping from an event to \mathbb{R}

example :- Throw 3 distinct coins // $8 = 2^3$ outcomes = $|S|$.

Random variable X - denotes the number of heads

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$$P(X=2) = {}^3C_2 \cdot \frac{1}{2^2} \cdot \frac{1}{2}$$

In the above example the random variable X can take only values $\{0, 1, 2, 3\}$, example of uniform of random variables known as discrete random variables.

- ② X = amount of rainfall on a given day.

$$P(X \geq 2 \text{ cm}) = 0.95$$

$$P(X \leq 1 \text{ cm}) = 0.99$$

This information is very useful for many people like farmers, people residing in that area.

$$\text{hence } X \in [0, \infty)$$

- ③ X = height of students.

$$P(X \geq 180 \text{ cm}) = 1\%$$

Useful for designing costumes/T-shirt for a public event.
designing benches/chairs.

- ④ X = time spent on a website

let $P(X \geq 10 \text{ min}) = 80\%$

A very good sign

let $P(X \leq 1 \text{ min}) = 90\%$.

Guaranteed
We can't there is something wrong on
our website.

⑤ $X = \# \text{ visitors to a website on a given day}$

$$P(X \geq 1000) = 0.1\%.$$

$$\frac{1}{1000} = \frac{1}{3 \text{ years}} \text{ once in 3 years.}$$

⑥ $X = \# \text{ children in a family}$

$$P(X \leq 2) = 95\%. \text{ India (population control)}$$

$$P(X=0) = 90\%. \text{ Japan (new policies can be decided by governments).}$$

- This example here describes a discrete random variable.

Discrete & Continuous Random Variables

- If # of possible values the random variable can take is countable then the R.V. is discrete
- If the possible values the random variable can take are uncountable then the R.V. is continuous

Probability Mass Function

- Discrete random variable

$X = \# \text{ children in a family}$.

$$X = \{0, 1, 2, 3, 4, \dots, 6\}$$

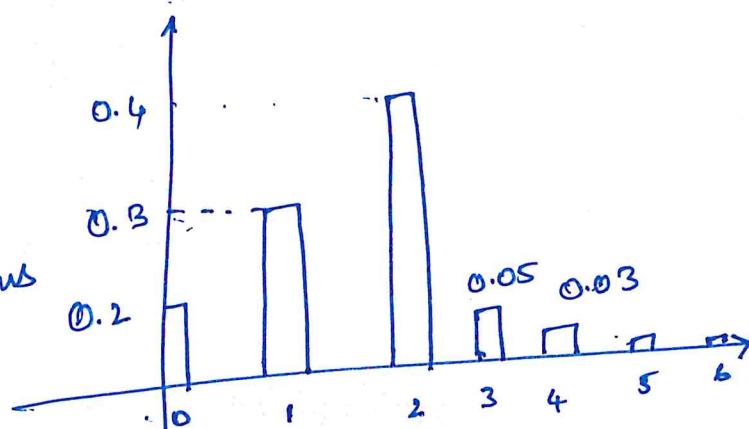
- If a survey is done and we have data for large amount of families.

$$H_1 \rightarrow 0$$

$$H_2 \rightarrow 2$$

$$H_3 \rightarrow 1$$

A graph can be plotted using this data, known as Histogram.



Many of the important questions can be answered by looking at this graph.

$$\text{For ex } P(X=4) = ? - 0.03$$

$$\begin{aligned} P(X \leq 3) &= P(X=0) + P(X=1) + P(X=2) \\ &\quad + P(X=3) = 0.51 \end{aligned}$$

- The probability mass function is a function which maps from a value of a random variable to its probability.

e.g. $P(X=a) \rightarrow p(a)$

example.

X is a discrete random variable.

$$X \in \{0, 1, 2, 3, \dots\}$$

PMF $P(X=i) = p(i) = c \frac{\lambda^i}{i!}$ for some λ : the value

$$\textcircled{a} \quad P(X=0) = p(0) = c \frac{\lambda^0}{0!} = c.$$

$$\textcircled{b} \quad P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{c\lambda^0}{0!} + \frac{c\lambda^1}{1!} + \frac{c\lambda^2}{2!}$$

$$= c + \lambda c + \frac{\lambda^2}{2} c$$

$$\textcircled{c} \quad p(0) + p(1) + \dots = 1$$

$$\sum_{i=0}^{\infty} c \frac{\lambda^i}{i!} = c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = c e^\lambda = 1.$$

$$c = \frac{1}{e^\lambda} = e^{-\lambda}$$

If x is a discrete R.V

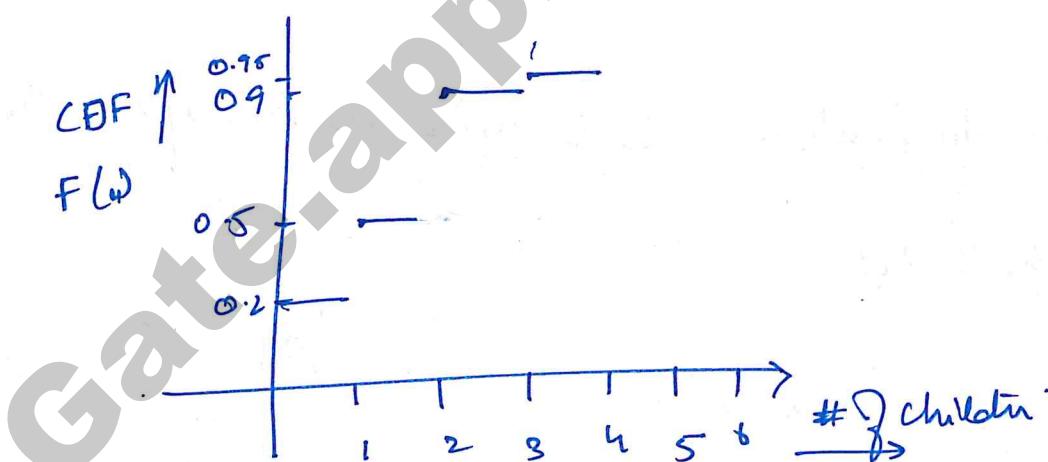
$$x \in \mathbb{Z} \{ 1, 2, 3, \dots \}$$

$$\begin{aligned} F_x(a) &= F(a) = P(x=a) + P(x=a-1) + P(x=a-2) + \dots \\ &= P(x \leq a) \end{aligned}$$

$$= \sum_{n \leq a} p(x=n) = \sum_{n \leq a} p(n)$$

$\uparrow \quad \uparrow$
r.v value

We can also plot the LDF



CDF for a continuous random variable

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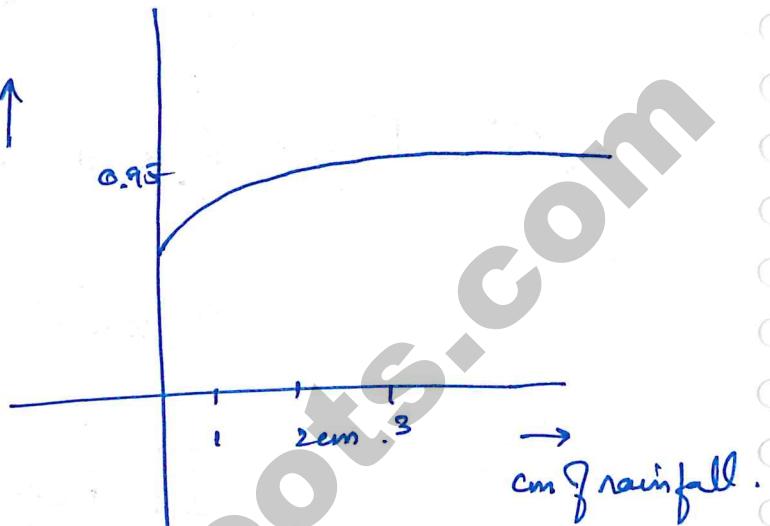
x = rainfall on a particular day.

$$F_n(a) = F(a) = P(X \leq a)$$

$$X \in [0, \infty)$$

cdf ↑

We can answer various questions about the rainfall like

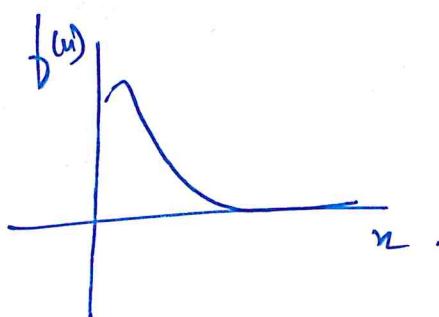


$$(a) P(X \leq 2 \text{ cm}) = 0.95$$

$$\begin{aligned} (b) P(1 < X \leq 3 \text{ cm}) &= P(X \leq 3) - P(X \leq 1) \\ &= 0.96 - 0.90 = 6\%. \end{aligned}$$

Probability density function :- It is defined as the slope of the curve represented in the CDF.

$$f_n(a) = \left[\frac{d F_n(u)}{du} \right]_{u=a}$$



It is the slope of the curve at $u=a$.

12.12 EXPECTATION

If x is a discrete random variable then expectation $E[x]$ is defined as

$$E[x] = \sum_n n p(n) \xrightarrow{\text{Pmf.}} P(x=n)$$

of children in a family example.

$$\text{Avg # of children in a family} = \frac{d+0+0+\dots+3+4}{10} = \frac{11}{10} = 1.1$$

The above can be written as - $\Sigma n p_n$

$$\Rightarrow 2 \times \frac{3}{10} + 0 \left(\frac{3}{10} \right) + 4 \times \left(\frac{1}{10} \right) \frac{48}{49} = 0$$

$$\sum_{i=0}^4 c \times p(i) = 4$$

- Expectation is the mean or average of the random variable.

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$$\text{Ans} \Rightarrow \mu_n = E[x] = \sum_n np(x) \text{ in case of discrete r.v.}$$

$$\mu_n = E[x] = \int n f(x) dx \text{ in case of a continuous random variable.}$$

Variance

→ # of children per household problem.

City 1

$$H_1 \rightarrow 2$$

$$H_2 \rightarrow 2$$

$$H_3 \rightarrow 2$$

⋮

$$\frac{H_{10} \rightarrow 2}{\mu = 2}$$

City 2

$$H_1 \rightarrow 0$$

$$H_2 \rightarrow 0$$

$$H_3 \rightarrow 0$$

⋮

$$\frac{H_9 \rightarrow 4}{H_{10} \rightarrow 4}$$

$$\underline{\mu = 2}$$

- Mean gives us an idea about the average of the data in the distribution.

- Variance gives us idea about how spread away the data is from the mean.

- In the above example both the city 1 and city 2 have the same mean, but in case of city 1 the data is all $x_i = 2$ whereas in city 2 it is spread.

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Variance is defined as $\text{Var}(x) = \sum_i \frac{(x_i - \mu)^2}{n}$

In the city, Variance = 0

In the city Variance = $\frac{40}{10} = 4$.

Variance is also defined as $E((x-\mu)^2)$

- In case of discrete random variable

$$E[g(x)] = \sum_n g(n)p(n)$$

- In case of continuous random variable

$$g(n) = y, \quad \int_n g(n)f(n)dn$$

$$\rightarrow \text{Variance}(x) = \sum_i \frac{(x_i - \mu)^2}{n}$$

$$= E\left[\frac{g(x)}{(x-\mu)^2}\right]$$

$$= E[x^2 - 2\mu x + \mu^2] - ①$$

Properties of Expectation

$$1. E[aX + b] = aE[X] + b.$$

$$\int_{-\infty}^{\infty} (an+b)f(u)du = a \int_{-\infty}^{\infty} f(u)du + b \int f(u)du.$$

[These properties can be applied on eq ①]

$$E(C) = \int_{-\infty}^{\infty} c f(u)du = c \int_{-\infty}^{\infty} f(u)du = c \cdot 1$$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - 2\mu^2 + \mu^2 = \underline{E[X^2] - \mu^2}$$

PMF OF MULTIPLE VARIABLES

example :- x - rainfall on a particular day y : temperature on that day.

X - # of children, Y - # of females.

We can define CDF, PDF defined on two variables

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$$F_{x,y}(x,y) = P(X \leq x, Y \leq y)$$

2cm 40°C

- If we have two random variables X and Y then.

$$\begin{aligned} 1. E[X+Y] &= \text{avg(temperature + rainfall)} \\ &= E[X] + E[Y]. \end{aligned}$$

$$2. \text{ If } X \text{ and } Y \text{ are independent then } P(X=n \cap Y=y) = P(X=n)P(Y=y).$$

$P(Y=y)$
↳

$$\underline{\underline{P(X=n) \times P(Y=y)}}$$

$$E[XY] = \sum_{n,y} ny P(n,y)$$

$$= \sum_{n,y} n \cdot y P(X=n) P(Y=y)$$

$$= \sum_n n P(n) \sum_y y P(y)$$

- Mathematicians and statisticians have observed natural phenomena and modelled their probability in the form of distributions.

eg 1. - Coin Toss $\begin{cases} T = 0 \\ H = 1 \end{cases}$

X : - discrete random variable $X \in \{0, 1\}$

$$P(X=1) = \frac{1}{2} = p$$

$$P(X=0) = \frac{1}{2} = \frac{(1-p)}{p}$$

2. Will it rain tomorrow? $\begin{cases} T(1) \\ F(0) \end{cases} \quad X \in \{0, 1\}$

$$P(X=1) = 0.15 = p$$

$$P(X=0) = 0.85 = q$$

3. Will a customer purchase a product?

$$\hookrightarrow Y(1) \quad P(X=1) = 0.05 = p$$

$$\hookrightarrow N(0) \quad P(X=0) = 0.95 = q$$

4. What will be the gender of the new born baby $\begin{cases} M(0) \\ F(1) \end{cases}$

$$P(X=1) = 0.5 = p$$

$$P(X=0) = 0.5 = q$$

In general $x \in \{0,1\}$

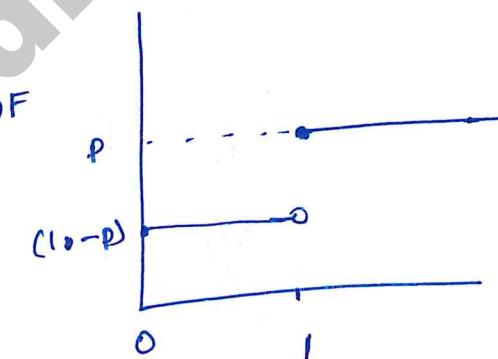
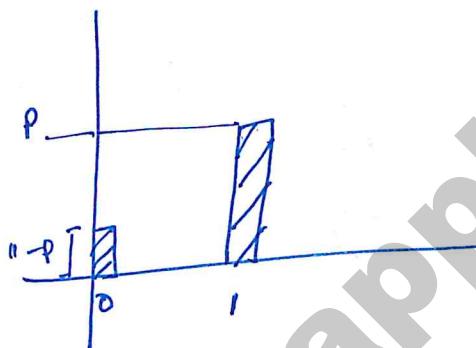
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$P(X=1) = p$ success probability

$P(X=0) = 1-p = q$ failure probability.

- If X is a random variable which follows Bernoulli distribution, then $(X \sim \text{Bernoulli}(p))$, X has two outcomes $X \in \{0,1\}$ and its parameter p represents its probability of success $p = P(X=1)$.
- p is the parameter of the distribution. Using p we can plot the the P.M.F & the CDF of the distribution also.



$$\Rightarrow \text{Mean} = E(X) = \mu$$

$$= \sum x p(x)$$

$$= 0 \cdot q + 1 \cdot p$$

$$= p.$$

$$\begin{aligned}
 - \text{Variance}(X) &= E[X^2] - \mu^2 \\
 &= \sum n^2 p(n) - p^2 \\
 &= 0 \cdot q + 1 \cdot p - p^2 \\
 &= p(1-p) = \underline{\underline{pq}}
 \end{aligned}$$

BINOMIAL DISTRIBUTION

Example 1: If we want to test a new drug/medicine, we would like to test for how many patients will benefit from this drug.

$$\begin{array}{ccccccc}
 P_1 & P_2 & \cdots & P_n \\
 0/1 & 0/1 & \cdots & 0/1
 \end{array}$$

Prob(K patients will benefit out of n patients) = ?

→ Out of n patients how many will benefit will be answered by a Binomial random variable

2. Toss n coins at a time and prob of getting a single head is p . Probability of getting R heads in n tosses is given by binomial distribution.

i) All the n trials are independent.

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ii) $P(X=1) = 0.5 = p$, for each of the trials the probability of success remains the same.

iii) Each of the trial is a Bernoulli(p).

$$P(\text{K heads in } n \text{ trials}) = {}^n C_k p^k q^{(n-k)}$$

$$X \in \{0, 1, 2, \dots, n\}$$

$$P(X=k) = P(k) = {}^n C_k p^k (1-p)^{n-k} \quad \text{— PMF. (Prob. Mass functions)}$$

$$P(X \leq k) = \sum_{i=0}^k {}^n C_i p^i (1-p)^{n-i} \quad \text{— CDF. (Cumulative density function).}$$

\rightarrow If X is a Binomial random variable it is represented as $X \sim \text{Binomial}(n, p)$

$$X = X_1 + X_2 + X_3 + \dots + X_n$$

↑
Bernoulli
Binomial

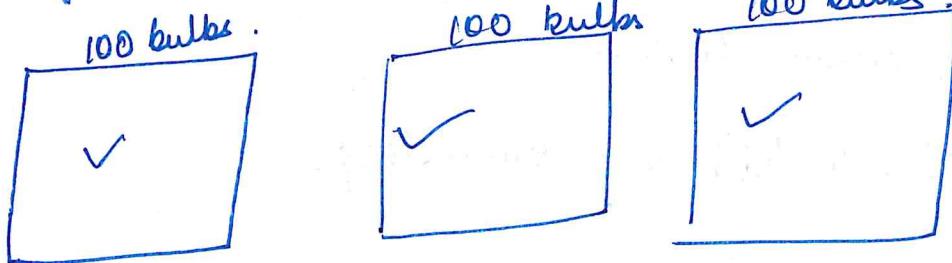
Bernoulli (Each is a Bernoulli random variable).

$$\rightarrow \text{Mean } M = E(X) = \sum_{k=0}^n k \cdot p(X=k) = \sum_{k=0}^n k {}^n C_k p^k (1-p)^{n-k}$$

$$\rightarrow \text{Variance} = \text{Var}(X_1 + X_2 + \dots + X_n) \quad \text{As all } X_1, X_2, \dots, X_n \text{ are independent.}$$

$$\begin{aligned} &= \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n) \\ &= pq + pq + \dots + pq \text{ (n times)} \Rightarrow \underline{\underline{n pq}} \end{aligned}$$

**APPLIED
Manufacturing** example of a bulb factory



$$P(\text{A bulb is faulty in a lot}) = 0.01 p.$$

Bernoulli ($p=0.01, n=100$)

$$P(K \text{ out of } 100 \text{ bulbs in a pack are faulty}) = {}^{100}C_K p^k q^{n-k}.$$

- How estimation for p is done.

$$n=100$$

$$P_1 = 2 \text{ faulty } n_1$$

$$P_2 = 0 \text{ faulty } n_2$$

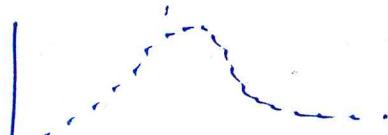
 \vdots

$$P_{10} = 2 \text{ faulty } n_{10}$$

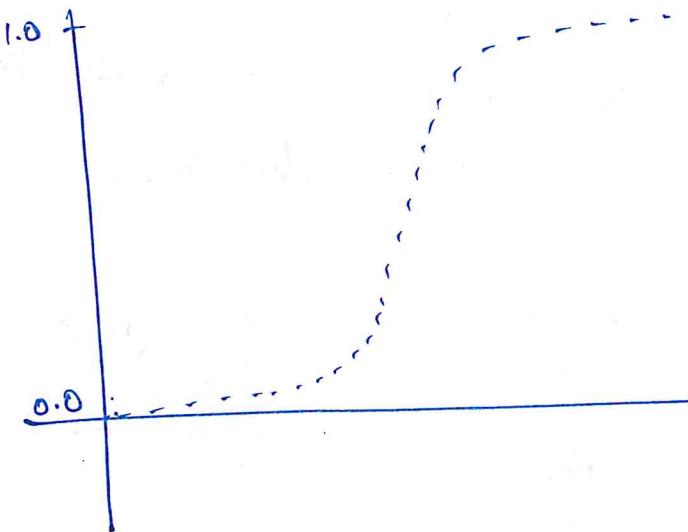
$$\hat{p} = \frac{\sum_{i=1}^m x_i}{n \times m}$$

To estimate the value of \hat{p} .

→ The PMF of Binomial Distribution



But not continuous.



5 - Shaped curve but not continuous.

12.14 Poisson DISTRIBUTION

Example :- 1. Number of calls received at a call center per hour

- ① Calls happen at a constant rate (λ)
 - ② Occurrence of one call does not affect another.
 - ③ Two or more events (calls cannot occur simultaneously)
2. No of patients arriving at the emergency between 10PM - 11PM .
3. No of customers at the counter per hour.
4. No of insurance claims in a year .
5. No of goals in a sport event in between 2 teams .
6. # of visitors on our website per minute .

→ The random variable x is said to be poission distributed if Phone: +91 841-844-0102 (4)

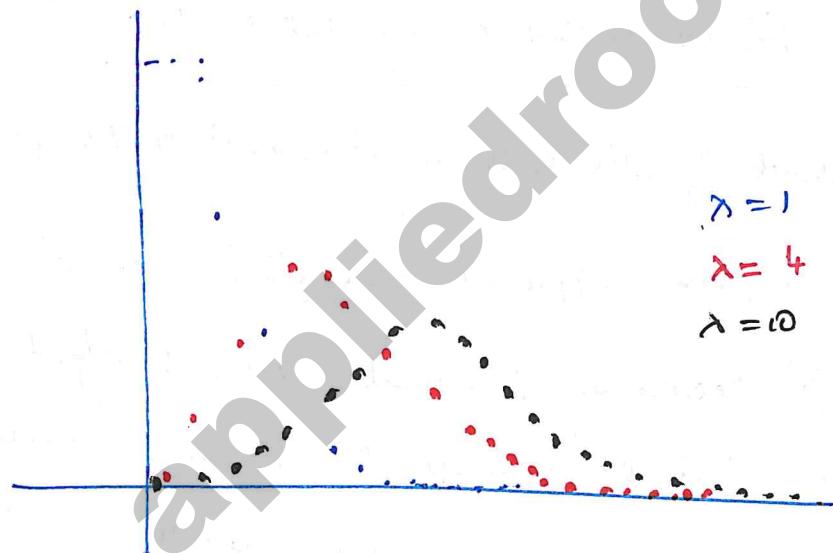


$X \sim \text{Poisson}(\lambda)$ λ is the rate and also the parameter of the distribution.

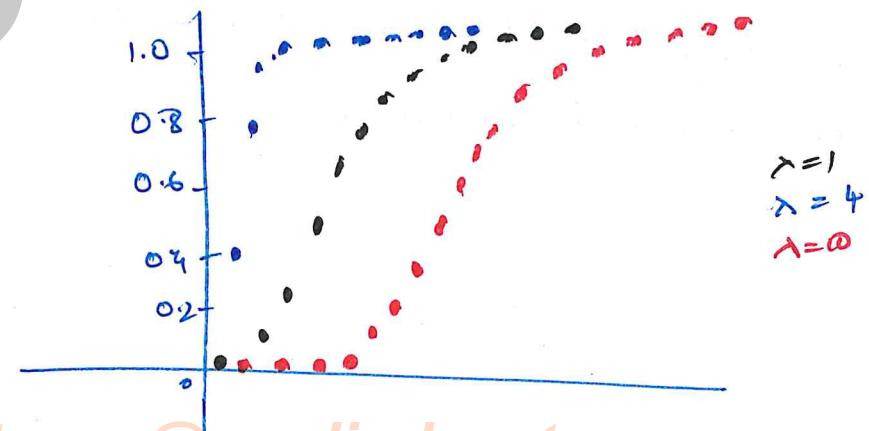
PMF: $P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $\sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = \frac{e^{-\lambda} e^{\lambda}}{=} = 1$

CDF: $P(X \leq k) = \sum_{n=0}^k \frac{\lambda^n e^{-\lambda}}{n!}$

PMF Curve



CDF Curve



→ Mean of poission random variable = $\mu = E[x] = \sum_{n=0}^{\infty} n e^{-\lambda} \frac{\lambda^n}{n!}$

$$= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$$

$$\Rightarrow \lambda e^{-\lambda} \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} \dots \right]$$

$$\Rightarrow \lambda e^{-\lambda} e^{\lambda}$$

$$\Rightarrow \underline{\lambda}$$

Estimation parameter is the mean of all the observations $\Rightarrow \lambda = \frac{1}{n} \sum_{i=1}^n x_i$

→ Variance :-

$$\text{Var}(x) = E(x^2) - \mu^2$$

$$\Rightarrow E(x(x-1) + x) - \mu^2$$

$$\Rightarrow E(x(x-1)) + E(x) - \mu^2$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda} \lambda^n}{n!} \Rightarrow x^2 e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!}$$

$$\Rightarrow x^2 e^{-\lambda} e^{\lambda} + \lambda - \lambda^2$$

$$\Rightarrow \underline{\lambda}$$

∴ Mean and Variance of poission R.V are same = λ .

→ If $X_1 \sim \text{Poisson}(x_1)$



$X_2 \sim \text{Poisson}(x_2)$

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- X_1 & X_2 are independent

$(X_1 + X_2)$ is also a poisson random variable with parameter $\lambda_1 + \lambda_2$ i.e. $\sim \text{Poisson}(\lambda_1 + \lambda_2)$

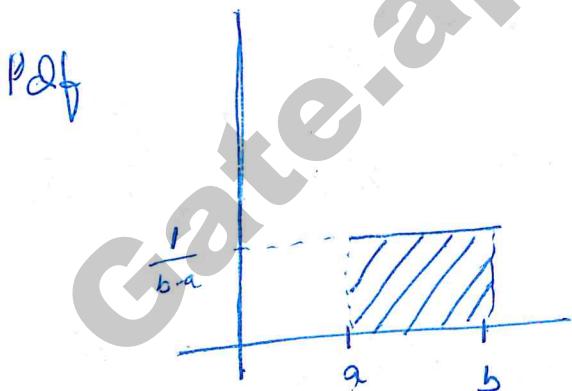
12.15 UNIFORM (CONTINUOUS DISTRIBUTION)

→ $X \sim \text{Uniform}(a, b)$. means that $x \in [a, b]$.

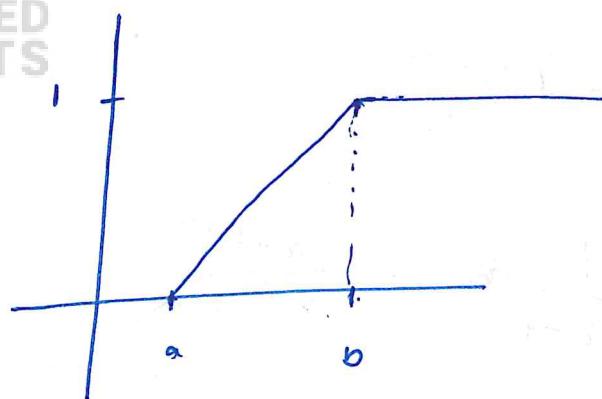
a = minimum value -

b = maximum value .

→ X is a continuous random variable which is in between a and b .



$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$



$$F(n) = P(X \leq n)$$

$$= \int_{-\infty}^n f(u) du.$$

$$F(n) = \begin{cases} 0 & \text{if } n < a \\ \frac{n-a}{b-a} & \text{if } n \in [a, b] \\ 1 & \text{if } n > b \end{cases}$$

$$\text{Mean} = \mu = E[X] = \int_{-\infty}^{\infty} n f(n) dn = \int_{-\infty}^a n \cdot 0 dn + \int_a^b n \cdot \frac{1}{(b-a)} dn + \int_b^{\infty} n \cdot 0 dn$$

$$\Rightarrow \left[\frac{n^2}{2(b-a)} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(a+b)}{2}$$

$$\text{Variance}(x) = E(x^2) - [E(x)]^2$$

$$= \int_a^b n \cdot \frac{1}{b-a} dx - \left(\frac{a+b}{2} \right)^2$$

$$\Rightarrow \left[\frac{n^3}{3} \right]_a^b \times \frac{1}{(b-a)} + \frac{(a+b)^2}{4}$$

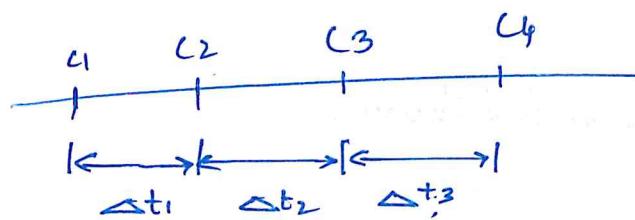
$$\Rightarrow \frac{b^3 - a^3}{(b-a) \cdot 3} + \frac{(a+b)^2}{4}$$

$$\Rightarrow \underline{\underline{\frac{1}{12} (b-a)^2}}$$

- Some applications of uniform random variables is in randomized algorithms like Randomized QuickSort.

- Continuous distribution

- Let's take an example of a poission process for example the calls received at a call center.

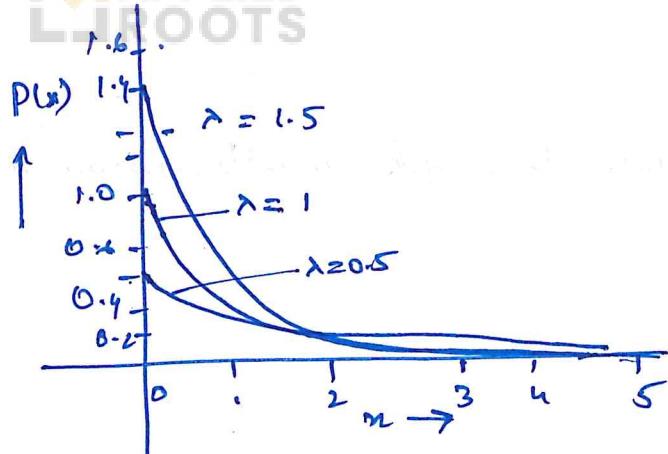


If C_1, C_2, C_3, \dots follow the poission process/distribution then the time interval in between the two poission events follows exponential distribution, in the above figure $\Delta t_1, \Delta t_2, \Delta t_3$ etc follow exponential distribution.

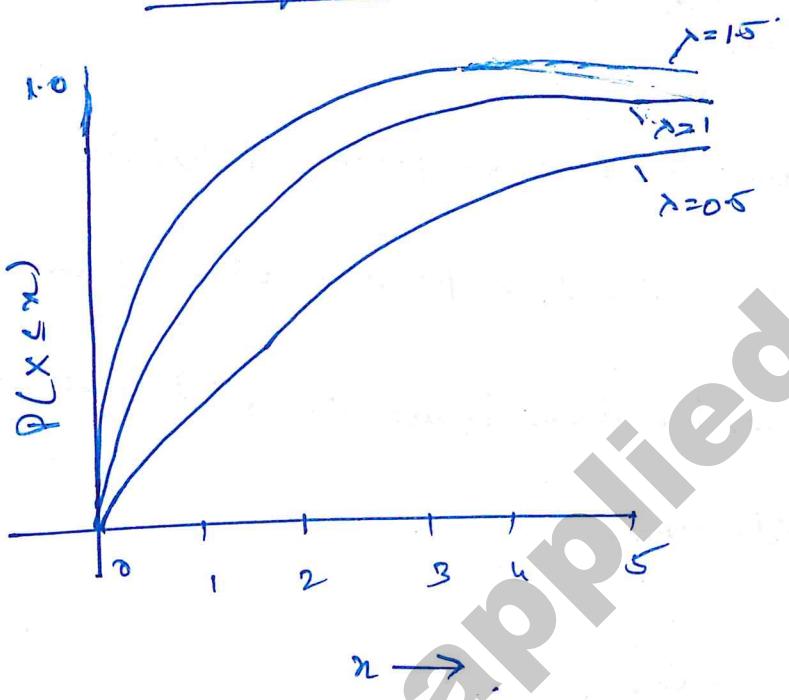
en we can know $P(\Delta t_2 < 2 \text{ min})$.

$$\text{Pof} f(n) = \begin{cases} \lambda e^{-\lambda n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\text{CDF} = F(n) = P(X \leq n) = \int_0^n \lambda e^{-\lambda t} dt = \lambda \left[\frac{1}{-\lambda} e^{-\lambda t} \right]_0^n = \begin{cases} 1 - e^{-\lambda n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$



CDF for an Exponential distribution



$$\text{Mean} \cdot \mu = E(x) = \int_0^{\infty} n(n e^{-\lambda n}) d n$$

applying integration by parts .

$$u = n \quad dv = dn$$

$$v = \lambda e^{-\lambda n} \quad du = \lambda e^{-\lambda n}$$

$$\left[-n e^{-\lambda n} \right]_0^\infty + \int_0^\infty n e^{-\lambda n} d\lambda$$

$$\Rightarrow 0 + \left[\frac{1}{\lambda} e^{-\lambda n} \right]_0^\infty$$

$$\Rightarrow \frac{1}{\lambda}$$

$$\boxed{\therefore \mu = \frac{1}{\lambda}}$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$= E(x^2) - \mu^2$$

$$= \int_0^\infty n^2 x e^{-\lambda n} - \frac{1}{\lambda^2}$$

on doing integration by parts we get.

$$\Rightarrow \frac{1}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

→ How to estimate λ given values of intervals $n_1, n_2, n_3, \dots, n_n$,
 if we are given the observations we know μ (mean) = $\frac{1}{\lambda}$

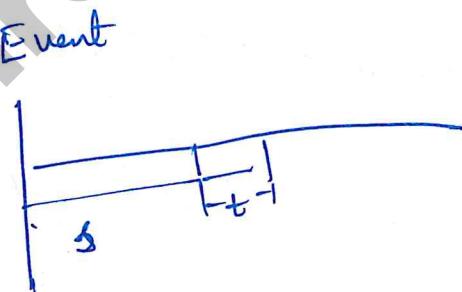
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$$\therefore \frac{n_1 + n_2 + \dots + n_n}{n} = \frac{1}{\lambda} \quad \therefore \boxed{\lambda = \frac{n}{\sum_{i=1}^n n_i}}$$

→ Memoryless Property of Exponential distribution

$$P(x \geq s+t | x \geq s) = P(x \geq t).$$

$$= \frac{P(x \geq s+t \cap x \geq s)}{P(x \geq s)}$$



$$= \frac{P(x \geq s+t)}{P(x \geq s)}$$

$$= \frac{1 - P(x \leq s+t)}{1 - P(x \leq s)} \Rightarrow \frac{e^{-x(s+t)}}{e^{-xs}} = e^{-xt}$$

$$= \underline{\underline{P(x \geq t)}}$$

Mail: gatecse@appliedroots.com

12.17 Normal Distribution

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- Also known as Gaussian distribution, most used and popular distributions.

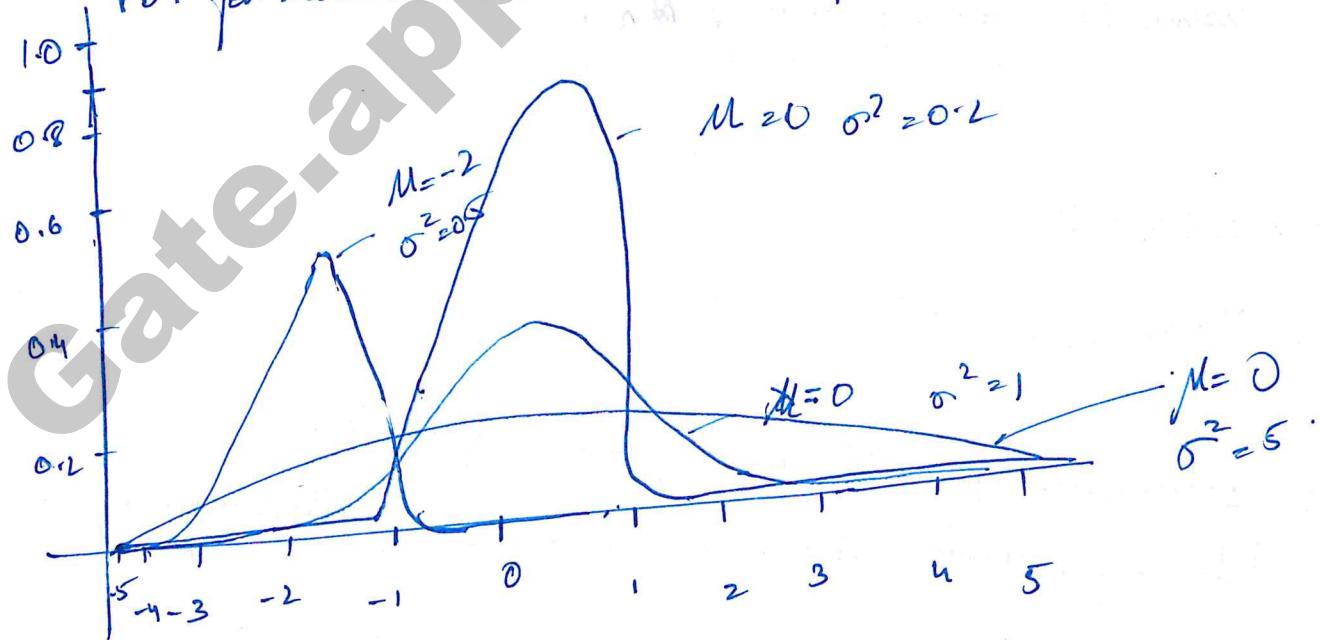
→ It is a continuous distribution

- Exs :- heights, lengths of leaves, weights of people in a set of population if not normally distributed they are approximately Normal or Gaussian distributed.
- Represented as $X \sim N(\mu, \sigma^2)$ μ and σ^2 should be finite.

$$\text{Standard deviation} = \sqrt{\text{Variance}}$$

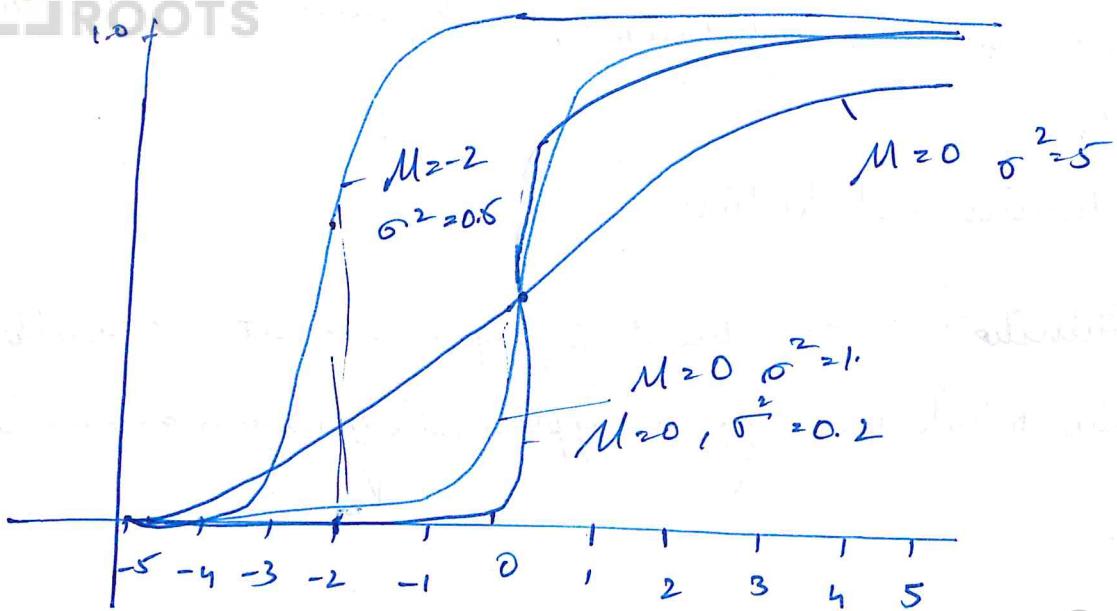
$$\text{PDF} = f(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(n-\mu)^2}{2\sigma^2}\right\} \quad \sigma \neq 0.$$

PDF for Normal distribution - Bell shaped curve.



→ As variance increases the peakedness of the curve decreases in PDF.

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(More accurate figures can be found on the Wiki Page for Normal distribution)

→ A Normal distribution can be thought of as a ^{logistic} extension of binomial distribution for large values of n .

$$\rightarrow E(x) = \mu$$

$$\rightarrow \text{Var}(x) = \sigma^2$$

→ Estimation of parameters from given sample data.
We can estimate μ and σ^2 for them using the following formulae

$$X \sim N(\mu, \sigma^2)$$

$$\hat{\mu} = \frac{\sum_{i=1}^n x_i}{n}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (x_i - \hat{\mu})^2}{n}$$

Standard Normal distribution

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If $- X \sim N(\mu, \sigma^2)$ then we can convert it to another normal variable with 0 mean and $\sigma^2 = 1$, by the following transformation

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

12.18 Mean, Median & Mode

Suppose if we are given few observations, for example heights of the students in a classroom.

n-observation

 $n_1, n_2, n_3, n_4, n_5, n_6, \dots, n_n$

$$\text{Mean} = \frac{1}{n} \sum_{i=1}^n n_i = \frac{n_1 + n_2 + \dots + n_n}{n}$$

Median = ① Sort all the values and select the middle value.

 $150, 160, 161, 163, 165, 166, 180$

1 2 3 4 5 6 7

If we have even no of observations, we need to take average of middle n elements

 $150, 160, 161, 162, 163, 165, 167, 168$

$$= \frac{(162+163)}{2} = \underline{\underline{162.5}}$$

Median is not much affected by extreme value / or one error.

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150, 160, 161, 162, 163, 165, 168 → By mistake later as 198

still median remains 162.

Mode: The value that is observed most frequently

Let say following is the list of observations.

Height	Frequency
150	10
155	20
160	20
170	15
180	4

160 - is the Mode of the data set.