1.1 Linear Algebra: Why And How

Why?

- Huge Applications in ML/AI
- A lot of applications in Physics (Velocity, acceleration, force)
- In all fields of Engineering (Mechanical, Civil, electrical, etc).
- For example in Computer science
  1. Computer Graphics → screen, games, edit an image.
  2. Linear System of Equations. (solution to linear system of Equations).
  3. Data Compression.
  4. Recommender Systems.
- Also have a lot of applications in Physics, Chemistry, Biology.

How

- Usually L.A. is taught in Numeric/Mathematic way.
- Geometric/Intuitive way helps us visualize/picturize the concept.
- The geometric approach provides us a much deeper understanding and is fun to learn.

Book :- Linear Algebra and Its Applications - Gilbert Strang.

Video Lectures :- MIT Lectures on Youtube by Gilbert Strang.

3blue1brown :- Essence of Linear Algebra, for visualizations.
↓
Youtube Channel

1.2. HOW TO LEARN MATHEMATICS

1. Abstract / Mathematical way of learning → Mathematics
2. Applied way of learning → Scientist Engineering.

eg Prime Numbers → Applications in cryptography :- login on websites securely //
↓
Mathematicians.

Applied Mathematicians / Scientists

- Mathematics - consist of equations but it has a lot of meaning behind it like poetry, but we should understand the concept behind it.

- Geometric / Diagrammatic way → art/painting

- Our focus should be on both the equations and the diagrams and how to relate them both.

1.3 Vectors - 1

- Vectors
  - Encountered in Physics

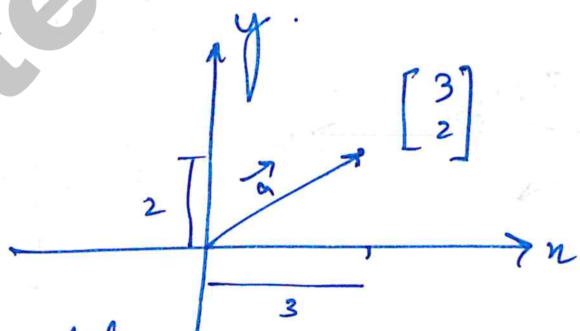
<u>Physics interpretation</u>
  - A vector has length/magnitude & direction (vel, acc, force).

- In terms of a computer science - a vector can be represented by arrays,

$$\begin{bmatrix} 160 \\ 55 \\ 7 \end{bmatrix} \begin{array}{l} - \text{height} \\ - \text{weight} \\ - \text{Shoe Size.} \end{array}$$

- In terms of Mathematicians :-
  - A vector is considered as an abstract concept where we have two operations 1. Vector addition 2. Scalar Multiplication

<u>Visualisation</u>



Vector can be represented as arrow, dot/point, ordered list based on the application.

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2} \qquad \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

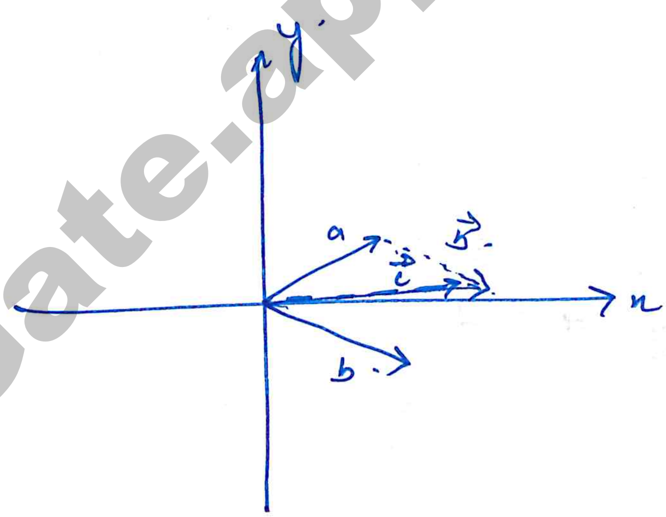
→ The concepts of 2D, 3D can be extended to higher dimension space like 4D, 5D .. 10D or even 1000D.

Addition of Vectors

$$V_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad V_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$ n Dimentional.

Geometric Interpretation.

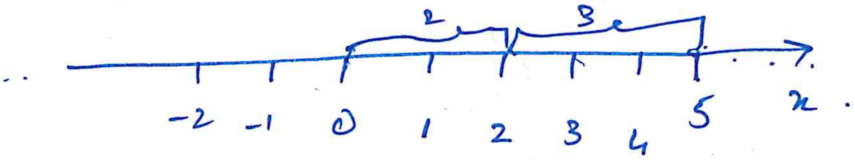


$\vec{c} = \vec{a} + \vec{b}$

$$\vec{c} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

1-D addition

$2+3=5$



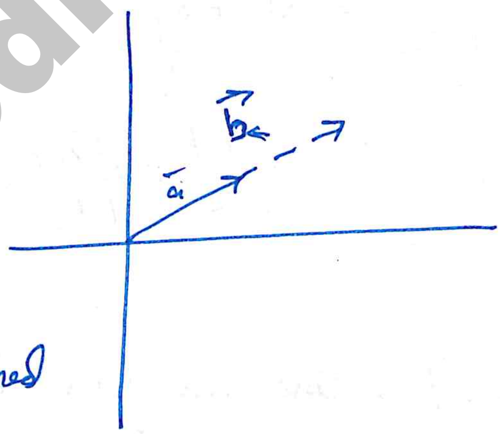
$\vec{a}+\vec{b}$

Scalar Multiplication of a vector or Multiplication of a Vector By a Scalar.

(More appropriate to us)

$$2 \times \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

$$2 \times \vec{a} = \vec{b}.$$



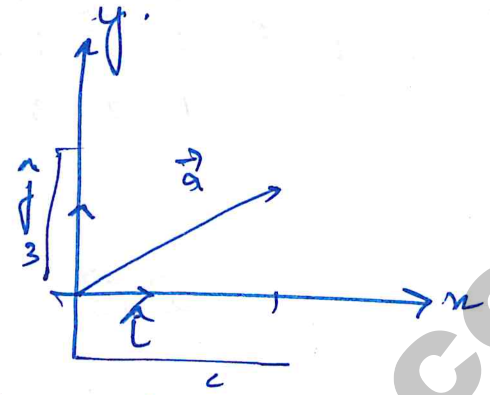
→ The vector b has become streached

→ If the scalar is <0 then direction is reversed.

- Mathematicians say that any operations i.e. vector operations can be written in terms of vector multiplications with a scalar and vector vector addition.

1.4 Vectors - 2

Unit Vectors



- $\hat{i}$ is unit vector in x direction
- $\hat{j}$ is unit vector in y direction
- $\|\hat{i}\| = 1$, $\|\hat{j}\| = 1$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \vec{a} = 2\hat{i} + 3\hat{j}$

Standard basis unit vector.

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\hat{i} + 3\hat{j}$$

$$= 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- Any pair of vectors can form a basis vectors except for 2 cases.

1) $\vec{b} = \vec{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2) $\vec{b} =$ constant $\vec{a}$. (In this case the span of the 2 vectors is a st line).

Span :- The span is the complete space which can be represented by linear combinations of the vector or pair of vectors.

Span of 2 vectors in 3D space :- It is always a plane/sheet.

Span of 3 vectors in 3D space :- is a 3D space.

Linear dependence :-

- If a vector $\bar{a}$ can be expressed sum of two other vectors multiplied by some scalar then $\bar{a}$ is said to be linearly dependent on the other vectors. For example $\vec{a} = x\vec{b} + y\vec{c}$ $\vec{a}$ is linearly dependent on $\bar{b}$ and $\vec{c}$.

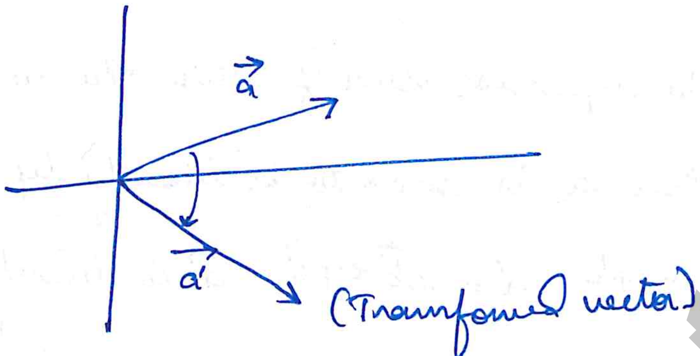
- If a and b are two linearly dependent vectors in the 2D space, then they cannot span the complete 2D space, Otherwise they can span the complete 2D space.

→ If $\bar{a}$ and $\bar{b}$ span a d-dimention space (where $d < n$) n is the dimention of the vectors a and b then if another vector c is added and the space spanned by a and b does not increase in dimention. then c is linearly dependent on a and b.

→ Basis vectors are a set of linearly independent vectors which span the complete space in all dimentions.

1.5 Matrices: Linear Transformations/Functions

Tranformation :- Changing vector from one form to another.



- A transformation could also represent the complete space i.e. the comple space and all the lines/vectors in the space may be transformed.

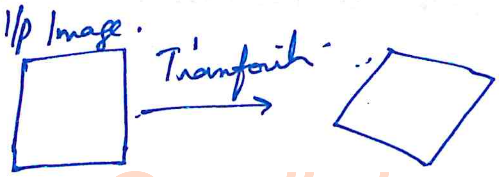
Linear Transformation

1. All the lines must remain lines (not curved)
2. Origin should remain the same.

(or)

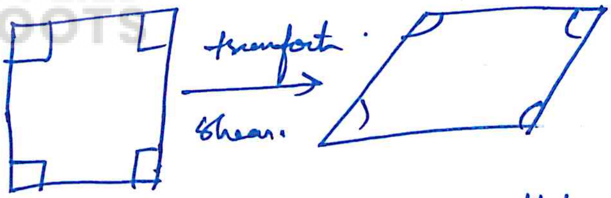
Grid lines should remain parallel and evenly spaced.

example application :- Computer graphics. :- Rotation of given image.



Rotation is a linear transformation.

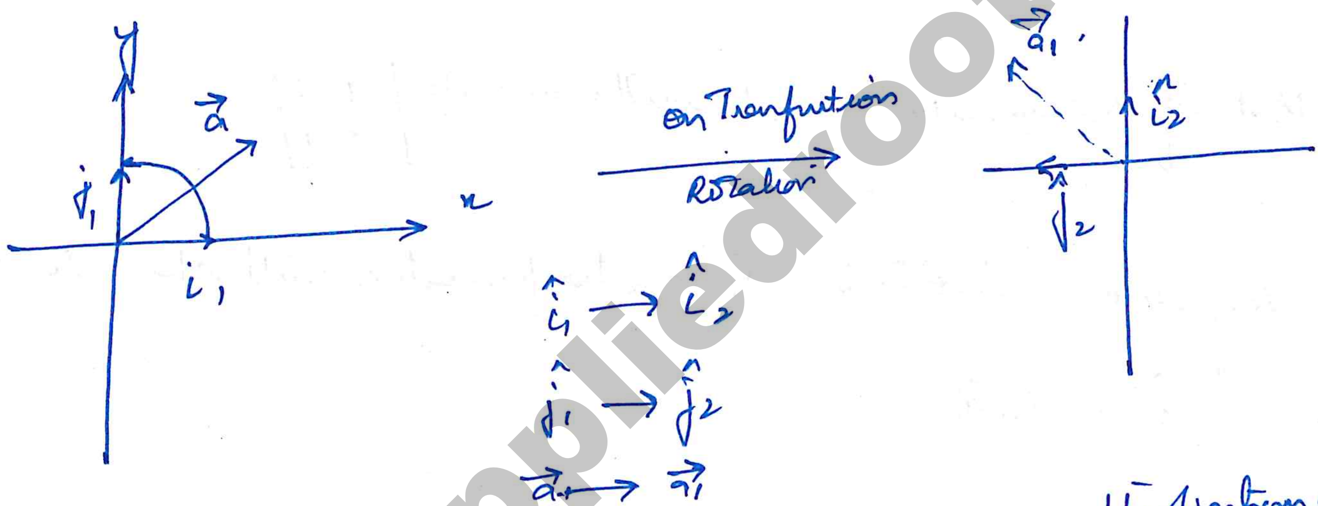
A shear is also another type of a linear transformation



o/p parallelogram.

i/p

— A linear transformation can be expressed in the form of Matrix multiplication.



This can be expressed in the form of matrix multiplication as well.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Transformation matrix

Final position of $\hat{i}_1$ after transformation

Final position of $\hat{j}$ after transformation.

→ We can get $\vec{a}_1$ by.

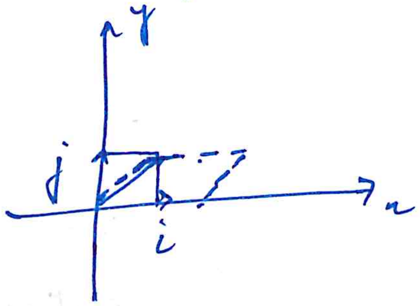
$$= A \cdot \vec{a}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2 \\ a_1 \end{bmatrix}$$

Example of shear linear transformation.



$$i \rightarrow \hat{i}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$j \rightarrow \hat{j}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Matrix of transformation can be written as. $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Any vector can be obtained $\bar{a}_1$ on transformation will change to.

$$\vec{a}_2 = M.\vec{a}_1$$

Order of a Matrix :- A matrix of consists of n no of rows and m number of columns then it is said to be of order $m \times n$.

- Even if a matrix is a non square matrix it represents a linear transformation.

1.6. Matrix - Matrix Multiplication And Addition

Matrix Vector Multiplication

- Linear transformation in any dimension can be expressed in form of Matrix vector multiplication $A \cdot V = U$, in 2D, 3D, n-D.

- Also if A is a non square matrix we may represent a transform

for eg. $A_{3\times2}$ will transform a 3D vector to 2D space in form of a 2D vector.

$$A_{3\times\boxed{2}}\,V_{\boxed{2}\times1} = U_{3\times1}.$$

Matrix Matrix Multiplication

- Similar to composition of functions.

- Composition of matrices like performing 2 linear transforms

$$(A(B\cdot V)) = AU = W.$$

example
$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \qquad ABV = CV$$

$$C = AB.$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\times1 + 2\times1 & 0+0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

Matrix multiplication is a composition of linear transforms.

- For matrix multiplication the orders of the adjacent matrix should match as follows $A_{m \times n} \; B_{n \times k} = C_{m \times k}$.

Matrix-Matrix Addition

- For addition to be done the order of the two matrices should be identical.
- Sum is calculated by adding corresponding elements.
- It can be though as computing the sum of transformations

$M_1 \rightarrow$ Transformation-1

$M_2 \rightarrow$ Transformation-2

$(M_1 + M_2)V =$ Sum of transformations M_1 and M_2 on V.

Properties

1. Commutativity → $M_1 + M_2 = M_2 + M_1$

$M_1 . M_2 \neq M_2 . M_1$

2. Associativity . $A + (B + C) = (A + B) + C$.

$A.(BC) = (AB).C$.

3. Distributive : . $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

$x(AB) = (xA)x = A(Bx)$

1.7 Transpose And Dot Product

- Transpose : Obtained by interchanging the rows and columns of a matrix.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Properties

1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$.

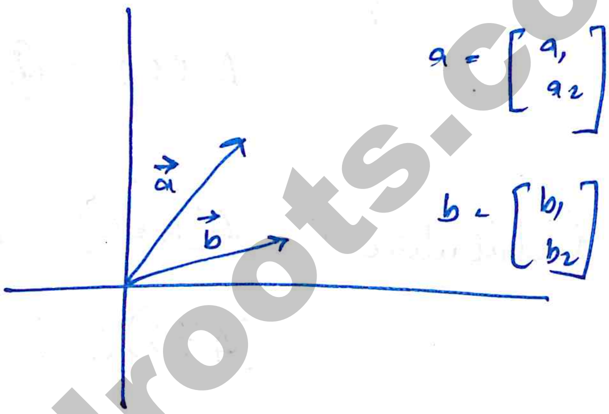
3. $(AB)^T = B^T A^T$

4. $(cA)^T = c \cdot (A^T)$

Dot Product

$$a \cdot b = a^T b$$

$$= [a_1 \;\; a_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

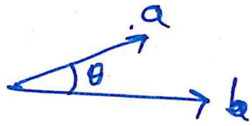


$$a \cdot b = a_1 b_1 + a_2 b_2$$

- Dot product of two matrices is column wise multiplication of corresponding elements.

- Geometrically a.b is the length of vector a × length of vector b and cosine of angle in between both of them.

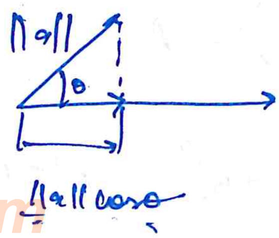
$$a_1 b_1 + a_2 b_2 = ||a|| \, ||b|| \cos\theta.$$



- Another way can be is the projection of a vector in the direction of the other vector.

$$a \cdot b = ||a|| \, ||b|| \cos\theta$$

$$= ||b|| \, proj_b a.$$



1.8 Types Of Matrices

① Row Matrix / Column Matrix :-

Row Matrix having only one row, whereas a column matrix is a matrix having only one column matrix

② Square Matrix :- If the no of rows and columns are equal then the matrix is known as a square matrix $m=m$.

③ Symmetric Matrix :- If $A = A^T$ and A is a square matrix,

$A_{ij} = A_{ji}$

$$\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}_{3\times 3}.$$

1, 4, 6 :- constitute the diagonal of the matrix.

④ Skew Symmetric Matrix :- A square Matrix $A^T = -A$.

⑤ Diagonal Matrix

- Square diagonal Matrix: If all elements on diagonal elements are non zero and remaining all elements are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Rectangular Diagonal Elements :- A rectangular matrix where every non diagonal element is zero and all diagonal elements are non zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

(6) Identity Matrix $A_{ij} = 1$ if $i=j$ and A is a square matrix
$= 0$ otherwise

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3} = I_3.$$

$IA = A$

(7) Scalar Matrix :- $A_{ij} = \lambda$ if $i=j$ A is a square matrix
$= 0$ $i \neq j$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

(8) Zero Matrix :- All elements are zero's.

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A \cdot Z = Z$ (if Multiplication is valid).

9. Upper Triangular Matrix

- If All the elements which are not in the upper triangular matrix are zero then it is said to be an upper triangular matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

10. Lower Triangular Matrix

- If all the elements which are not in the lower triangle of the matrix are zero then it is said to be a lower triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

1.9 Determinant of a Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(A) = |A|$

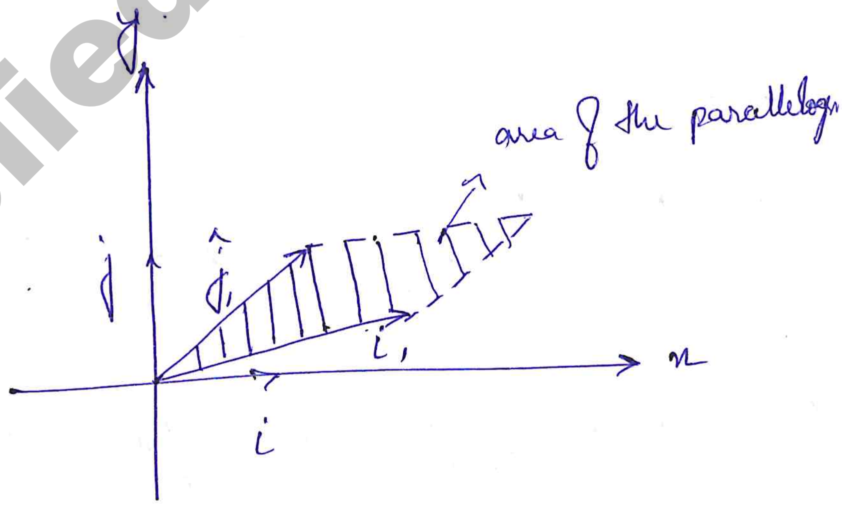
$|A| = (ad - bc)$

If A represents a linear transformation then $|A|$ represents the area of the parallelogram formed by the transformed vectors for $\hat{i}$ and $\hat{j}$.

→ After the linear transformation

$\hat{i} \xrightarrow{A} \hat{i}_1 \begin{bmatrix} a \\ c \end{bmatrix}$

$\hat{j} \xrightarrow{B} \hat{j}_1 \begin{bmatrix} b \\ d \end{bmatrix}$



$\hat{i} \xrightarrow{A} \hat{i}_1$

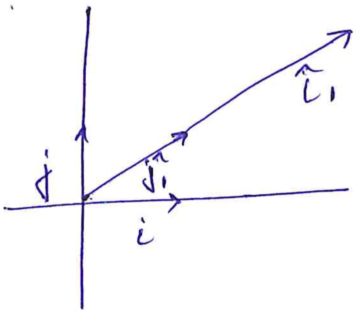
$j \xrightarrow{A} \hat{j}_1$

→ Another way to think about $|A|$. Scale by which the area of any figure changes due to the linear transformation represented by A.

example :

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$\hat{i}$ lands $\hat{j}$ lands.



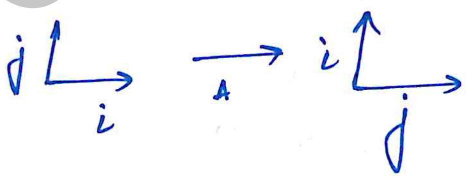
$\hat{i}_1 = 2\hat{j}_1$ They are linearly dependent.

$|A| = 0$. Also the area of the Parallelogram $= 0$.

ex 2

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad |A| = 4 - 6 = -2$$

- If the sign is $-$ve after the transformation the orientation of $\hat{i}$ and $\hat{j}$ gets changed.



- Same concept if extended to 3D

$\Delta_{3\times3}$ represents the volume of the parallelopiped

A still corresponds to a linear transformation.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - gf) + c(dh - eg).$$

$$= \underbrace{a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{\text{Minors of } \Delta}.$$

Minors of Δ.

SARRU'S RULE

$$\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} = aei + bfg + cdh - ceg - bdi - afh.$$

2 cols repeated

Bold lines + sign.

dotted lines - sign.

Properties of determenants

① $|I_n| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \qquad I \cdot V = V$.

② $|A^T| = |A|$

③ $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

④ $|A \cdot B| = |A|\,|B|$.

⑤ $|cA| = c^n |A|$ — A is a n x n matrix.

⑥ Traingular Matrix A_T,

$|A_T| =$ product of the diagonal elements .

Reference Links

1. https://en.wikipedia.org/wiki/Determenant
2. 3Blue1Brown channel Videos.
   1. https//youtu.be//Ip3x9LOh2dK?t=523
   2. ,, ?t=142
   3. ,, t=257

1.10 Minor, Cofactor, Adjoint, Inverse Of A Matrix.

Cofactor..

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

$$C = \text{Cofactor}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Minor M_{11}

$$C_{11} = (-1)^{1+1} \overbrace{\begin{vmatrix} 0 & 5 \\ 9 & 11 \end{vmatrix}}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 4 \\ -1 & 9 \end{vmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Adjoint / Adjugate / Adjunct

1. $$Adj(A) = C^T = \left[Cofactor(A)\right]^T$$

2. $$A \cdot Adj(A) = |A| I.$$

$$A \cdot \frac{Adj(A)}{|A|} = I$$

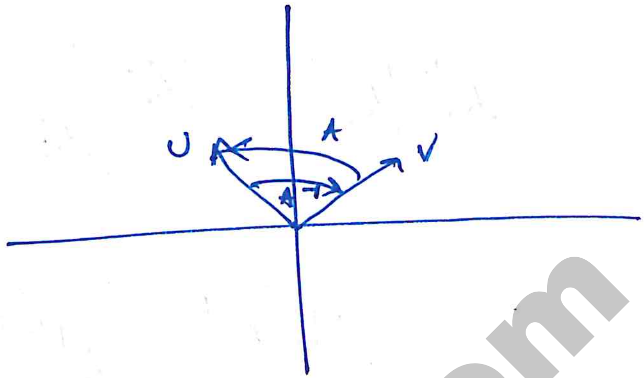
$$\therefore \quad A^{-1} = \frac{Adj(A)}{|A|}$$

If $|A|=0$ inverse of A is not defined, the matrix is non invertible

→ One of the application of inverse of a matrix is to find the solution of system of linear equations.

→ Geometric intuition of inverse of a matrix :-

If vector v on linear transformation we get $\bar{u}$, the inverse transformation helps us or is that transformation that will help us get back vector v from vector u.



→ Geometric Interpretation of Adj(A).

$$\mathrm{Adj}(A) = |A|\,A^{-1}$$

↓ inverse transform of A

Scale of the transformation of A.

= It is scaled transformation of the inverse transform of A, the scale is same as that of scale of A.

<u>Algebraic Properties & Geometric Properties</u>.

① $|A^{-1}| = \dfrac{1}{|A|}$ or $|A|^{-1}$

② $A^{-1} = \dfrac{\mathrm{adj}(A)}{|A|}$.

$\Rightarrow |A|\,A^{-1} = \mathrm{adj}(A)$

$\cancel{|A|\,\mathrm{Adj}(A)}$. $\Rightarrow \big|\,|A|\,A^{-1}\big| = |\mathrm{Adj}(A)|$

$$|A|^n |A|^{-1} = |A|^n \frac{1}{|A|} = |A|^{(n-1)}$$

$$\therefore \quad \underline{\underline{|Adj(A)| = |A|^{n-1}}}$$

(3) Singular Matrix

$|A| = 0$ $\quad$ A^{-1} does not exist

$\parallel$gm area $= 0$.

(4) $|Adj(Adj(A))| = ?$

$|adj(A)| = |A|^{n-1}$

If $Adj(A) = B$

$|Adj(B)| = |B|^{n-1}$

$|B| = |Adj(A)|$

$= |A|^{n-1}$

$\therefore |Adj(B)| = |Adj(Adj(A))| = \left(|A|^{n-1}\right)^{n-1} \Rightarrow |A|^{(n-1)^2}$

Similarly $|Adj(Adj(Adj(A)))| = |A|^{(n-1)^3}$

(5) $|A \cdot B| = |A| \cdot |B|$.

$|A^k| = |A|^k$.

(6).

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(7) $Adj(0) = 0$.

(8) $adj(I) = I$

(9) $adj(cA) = c^{(n-1)}\, adj(A)$.

(10) $adj(A^T) = (adj(A))^T$

(11) $\text{adj}(A^{-1}) = (\text{adj}(A))^{-1}$

(12) $\text{Adj}(AB) = \text{Adj}(B) \cdot \text{Adj}(A)$.

(13) $(AB)^{-1} = B^{-1}A^{-1}$

(14) $\text{Adj}(A^{K}) = (\text{Adj}(A))^{K}$

(15) $(A^{T})^{-1} = (A^{-1})^{T}$

1.11 Solved Problems Elementary Row and Column Operations

① Let X be a square matrix. Consider the following two statements on X.

I. X is invertible

II. Determinant of X is non-zero.

Which of the following is TRUE?

(A) I implies II; II does not imply I

(B) II implies I; I does not imply II

(C) I does not imply II; II does not imply I.

(D) I and II are equivalent statements.

If A^{-1} exists it means $|A| \neq 0$ as $A^{-1} = \dfrac{Adj(A)}{|A|}$

$\therefore$ Both the statements are equivalent.

② Perform the following operations on the below matrix

$$\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 21 & 195 \end{bmatrix}$$

i. Add the third row to the second row.

ii. Subtract the third column from the first column.

The determinant of the resultant matrix is ______.

Elementary Row & Column Operations

① $R_i \leftrightarrow R_j$ (or) $C_i \leftrightarrow C_j$ - determinant will change in sign.

② $R_i \leftarrow cR_i$ (or) $C_j \leftarrow cC_j$ → Resultant determinant is multiplied by c

③ $R_i \leftarrow R_i + cR_j$ or $C_i \leftarrow C_i + cC_j$ → Resultant determinant will not change.

Coming back to the problem.

1. $R_2 \leftarrow R_2 + R_3$

II. $C_1 \leftarrow C_1 - C_3$

By these operations the determinant of the matrix remains unchanged.

If we observe $C_3 = 15 \times C_1$.

$\therefore \quad C_3 \leftarrow C_3 - 15C_1$.

$$\begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 0 \\ 13 & 2 & 0 \end{bmatrix}$$

$\therefore$ Determinant of the matrix $= 0$.

③ If the matrix A is such that

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{bmatrix}$$

$C_3 = 5 \times C_1$

$\therefore |A| = 0$.

(4)
$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}.$$

Which of the following is the matrix determinant NOT equal to

A
$$\begin{vmatrix} 1 & x(x+1) & (x+1) \\ 1 & y(y+1) & (y+1) \\ 1 & z(z+1) & (z+1) \end{vmatrix}$$

B
$$\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$$

C.
$$\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}.$$

D.
$$\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Soln.

Clearly by seeing option B if we perform $C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

we get back the original Matrix.

Similarly for Matrix of option C
if we perform $R_2 \leftarrow R_2 + R_3$
$R_1 \leftarrow R_1 + R_2$.

We get back the original matrix.

Also on option D. if we observe $C_2 \leftarrow C_2 - C_3$
$C_3 \leftarrow C_3 - C_1$

We get back the original matrix, the only remaining option is option 1.

(5) Let A, B, C, D be $n \times n$ matrices, each with non-zero determinant. If $ABCD = I$, then B^{-1} is .

A. $B^{-1}C^{-1}A^{-1}$

B. CDA

C. ADC.

D. Does not necessarily exist.

Soln

$$ABCD = I.$$

$$ABCD \times D^{-1} = I \cdot D^{-1}$$

$$ABC \times C^{-1} = D^{-1}C^{-1}$$

$$A^{-1} \cdot AB = A^{-1}D^{-1}C^{-1}$$

$$\Rightarrow B = A^{-1}D^{-1}C^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\therefore B^{-1} = CDA$$

option B.

1.12 Special types of Matrices and Properties

Orthogonal Matrix :-

- It is a square matrix
- $Q \cdot Q^T = I$ or $Q^T = Q^{-1}$

Intuitive Understanding

$$\underbrace{\begin{bmatrix} \longleftarrow r_1 \longrightarrow \\ \longleftarrow r_2 \longrightarrow \\ \longleftarrow r_3 \longrightarrow \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ r_1 & r_2 & r_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}}_{Q^T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r_1^T \cdot r_1 = 1$$

$$r_1 \cdot r_2 = r_1^T r_2 = 0.$$

$$r_2^T r_2 = 1$$

$$r_i \cdot r_j = 0 \text{ if } i \neq j$$

$$r_i \cdot r_j = 1 \text{ if } i = j$$

r_i's are unit vector

every r_i and r_j if $i \neq j$ are perpendicular vectors

and $i = j$ are parallel vectors.

- If we think of it from the perspective of a linear transform

$$Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \hat{i}_1 & \hat{j}_1 & \hat{k}_1 \end{bmatrix}_{3\times 3}$$

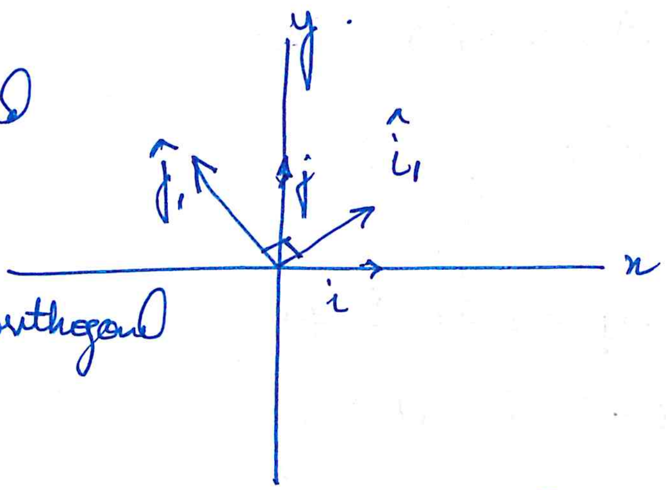
Initially $\hat{i} \perp \hat{j} \perp \hat{k}$

on transformation

$$\hat{i}_1 \perp \hat{j}_1 \perp \hat{k}_1$$

$$\|\hat{i}_1\| = \|\hat{j}_1\| = \|\hat{k}_1\| = 1$$

→ Rotation with no shear and no scaling is one example of an orthogonal matrix.



→ Mirror or Reflection is an example of an orthogonal matrix.

→ $|Q| = \pm 1$ volume of the parallelopiped $= \pm 1$.

volume does not change for an area only. orientation will change.

Preservation of Dot Product.

→ $U.V = U^T V = (Q.U).(QV)$

$$= (QU)^T (QV)$$

$$= U^T Q^T Q V$$

$$\Rightarrow \underline{\underline{U^T V}}.$$

- If the rows of an orthogonal matrix are interchanged it still remains an orthogonal matrix.

IDEMPOTENT MATRIX.

- It is a square Matrix such that $M \times M = M$

$$M = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$|M| = 0$ or $M = I$

- A is Idempotent $A^m = A$ for $n = 1, 2, \ldots$

Applications in ML & AI.

Involutary Matrix

$A^2 = I$ (Square Root Matrix of I).

or $\underline{\underline{A = A^{-1}}}$.

Geometrically :- If A is a linear transformation and its inverse is equal to itself then A is an involutary Matrix

ex
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad i = \sqrt{-1}$$

$A^2 = I$ & A is symmetric $\rightarrow$ A is orthogonal.

$A \cdot A = I$ & $A = A^T$ $\rightarrow A \cdot A^T = I$.

- An another way to look at it geometrically is an Involutary Matrix is like a mirror reflection, if applied twice we get back the original image/figure.

Properties of An Involutary Matrix

① $|A| = \pm 1$

② $\frac{1}{2}(A+I)$ is idempotent.

③ If A and B are Involutory then. AB is also involutary.

④ If A is involutary $A^n = \begin{cases} A & \text{if } n \text{ is odd.} \\ I & \text{if } n \text{ is even.} \end{cases}$

NILPOTENT MATRIX

If $N^k = 0$ for some k = +ve Integer

Smallest k is index of N.

ex $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Any triangular matrix with diagonal elements = 0 is a nilpotent matrix. with (index < n)

$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

Properties of Nilpotent Matrix

① $|I + N| = 1$

② $(I+N)^{-1} = \underbrace{I - N + N^2 - N^3 + \dots}\ + \underbrace{N^k + N^{k+1} + \dots}_{0}$

as $N^k = 0$

$N^{k+1} = 0$

$\vdots$

1.13 Trace Of A Matrix

Equality Of Matrices

$A_{m \times n} = B_{m \times n}$

- Orders should be same

and $A_{ij} = B_{ij} \ \forall i, j$.

Trace Of a Matrix

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$$

$$\text{trace}\left(\begin{bmatrix} -1 & 0 & 3 \\ 11 & 5 & 2 \\ 6 & 12 & -5 \end{bmatrix}\right) = -1 + 5 - 5 = -1$$

Geometric Intuition :- Related to Eigen values and Eigen Vectors

to be covered later.

→ $\text{Trace}(A+B) = \text{Trace}(A) + \text{Trace}(B)$.

→ $\text{Trace}(c.A) = c.\text{Trace}(A)$.

→ $\text{Trace}(A.B) = \text{Trace}(B.A)$ As long as AB and BA are defined.

→ $\text{Trace}(AB^T) = \text{Trace}(A^T B) = \text{Trace}(BA^T) = \text{Trace}(B^T A)$.

→ <u>Cyclic Property of Trace</u> :

$\text{Trace}(ABCD) = \text{Trace}(BCDA) = \text{Trace}(CDAB) = \text{Trace}(DABC)$

→ If A, B, C are symmetric

$\text{Trace}(ABC) = \text{Trace}(CBA) = \text{Trace}(ACB) = \text{Trace}(CAB)$.
$= \text{Trace}(BAC) = \text{Trace}(BCA)$.

→ * Common Mistake

$\text{trace}(AB) \neq \text{Trace}(A).\text{Trace}(B)$.

→ $\text{trace}(P^{-1}AP) = \text{trace}(P^{-1}(AP))$
$= \text{trace}(P^{-1}PA)$
$= \text{trace}(A)$.

- If A is a symmetric Matrix; B is a skew symmetric matrix

$$\text{trace}(AB) = 0.$$

1.14 Rank Of A Matrix

→ Very useful for solving a system of linear equations.

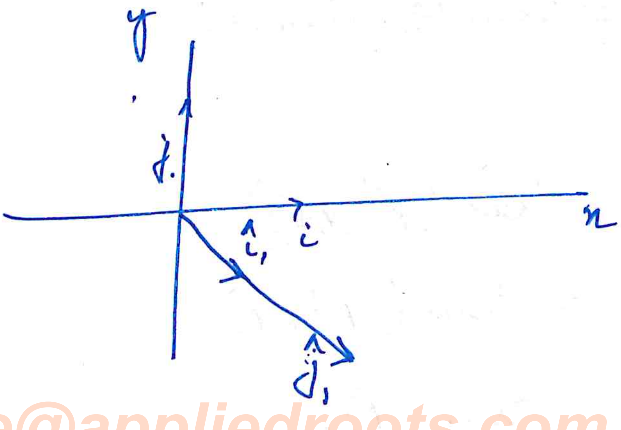
→ Rank is defined for both a square and rectangular matrix.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \qquad C_1 = \frac{C_2}{2}$$

$\hat{i}_1$ lands $\qquad \hat{j}_1$ lands.

If we perform transformation for any vector v.

$$\underset{2\times2}{A}\ \underset{2\times1}{V} = \underset{2\times1}{U}$$



- The transformed vector will lie on the same line as $\hat{i}_1$ and $\hat{j}_1$ lie on the same line

$|A| = 0$ area of the Parallelogram $= 0$. as $\hat{i}_1$ and $\hat{j}_1$ are linearly dependent.

Geometric Intuition of Rank(A).

- The Dimension of the vector space spanned by the column vectors of A span.

$$A = \begin{array}{c} \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \end{array}$$

Alternate way of thinking
Rank(A) = Maximum no of linearly independent columns of A.

If $C_1 \leftarrow aC_2 + bC_3$ C_1 is linearly dependent on C_2 and C_3.

Numerical way to think about Rank(A)

If A is a square matrix

if $|A| = 0$ Rank $\neq n$, Rank $< n$ } — Rank(A) = size of largest sub matrix with non zero determinant.

$|A| \neq 0$ Rank(A) $= n$

→ Column Rank is what we have discussed so far based on columns

→ Row Rank of a matrix is defined based on the rows of the matrix

- Max no of linearly independent rows of that matrix = Rank.

- Column Rank of the Matrix = Row rank of the matrix = Rank of the Matrix.

→ $Rank(A) = Rank(A^T)$

—

example

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \end{array}$$

$R_1 = -1 * R_2$.

Rank = 1.

→ If we have an m×n matrix A.

$\underline{Rank(A) \leq \min(m, n)}$.

<u>Compute Rank</u>

→ Elementary Row operations do not change the rank of a matrix.

1. Swap R_i and R_j

2. $R_i \leftarrow cR_i \quad c \neq 0$

3. $R_i \leftarrow R_i + kR_j$.

Gauss Elimination Method

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2 \quad \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

# of non zero rows = Rank of the Matrix. $= \underline{\underline{2}}$

The above form is the Echelon form of the Matrix.

Echelon Form

- The no of 0's before the first non zero number in any row is less than the no of such zeros in the next row.

- All zero rows must be below non zero rows.

1. $\text{rank}(A) \leq \min(m, n)$
2. Only zero matrix have rank $= 0$.
3. full row rank = rank $= m$.

full column rank = rank $= n$.

$A: n \times n$ full rank $\rightarrow$ rank $= n$

(4) $A_{n \times n}$ is invertible iff rank $(A) = n$.

if rank $(A) < n$ A is not invertible.

(5) $\text{Rank}(AB) \leq \text{Min}(\text{Rank}(A), \text{Rank}(B))$.

(6) $\text{Rank}(A+B) \leq \text{Rank}(A) + \text{Rank}(B)$

(7) Sylvester's Rank Inequality.

$$\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB).$$

(8) $\text{rank}(A^T A) = \text{rank}(A A^T) = \text{rank}(A) = \text{rank}(A^T)$

(8.1)

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$$\text{Rank}(P+Q) = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3}$$

$$\begin{bmatrix} 8 & 8 & 8 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 8 & 8 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 2$$

ex2

$$\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 21 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_3} \begin{bmatrix} 1 & 4 & 8 & 7 \\ 4 & 2 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 3 & 12 & 24 & 21 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 4R_1 \qquad R_3 \leftarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & -14 & -29 & 27 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\text{Rank} = 3}$$

1.15 Systems of Linear Equations - 1

- V. Important application of Linear Algebra.

$$3x + 2y - z = 1$$
$$2x - 2y + 4z = -2$$
$$-x + \frac{1}{2}y - z = 0.$$

$\downarrow$

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 4 \\ -1x & +1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Coeff Matrix A $\quad$ x $\quad$ b.

$$Ax = b.$$

Geometric Interpretation

1. $\underset{\downarrow}{A}\ \underset{\downarrow}{x} = \underset{\downarrow}{b}.$

   A: Lin transformation; x: i/p; b: o/p.

For the linear transformation represented by A and output b

what is the input x.

② Wikipedia https://en.wikipedia.org/wiki/System-of-linear-equations.

→ General form of eqn in 2D = $ax+by+c=0$.- st line.

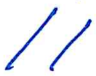
→ In 3D coordinate geometry → $ax+by+cz+d=0$ represents a plane in 3D.

→ In 4D space the equation $ax+by+cz+dt+e=0$ - represents a hyperplane in 4D.

→ Similarly a linear equation in 10 variables represents a hyperplane in 10D space.

→ 3 equations represents 3 planes in 3D space and the solution represented is the pt of intersection of the 3 planes.

Solution Set :- Set of all valid solutions to the linear system of equations

1. Infinitely many solutions →  l_1, l_2 overlapping lines
2. Single unique solution →  l_2 l_1
3. No Solution. →  l_1 l_2 parallel lines.

They will fall in one of the three categories.

Underdetermined and Overdetermined System of Equations

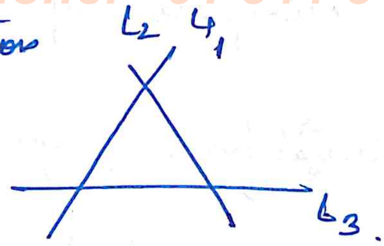
n - no of variables m - no of equations

① $m<n$ - Underdetermined system (cannot determine the soln)

② $m=n$ - Exactly determined system (1 soln, no soln, ∞ solutions)

(3) Over determined system of equations. $(m > n)$

We could have more than one solution



$n = 2 \quad m = 3$

<u>Independence of System of Linear Equations</u> :- A linear system of eqns are said to be independent if none of the equations can be arrived at from other equations algebraically.

ex1

(1) $3x + 6y = 6$.

(2) $6x + 4y = 12$

(2) $= 2 \times$ (1).

linearly dependent
System of Equations.

ex2

(1) $x - 2y = -1$

(2) $3x + 6y = 8$.

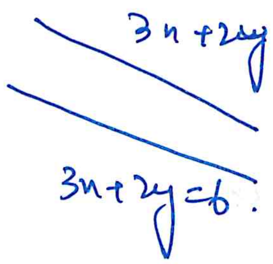
(3) $4x + 3y = 7$.

(3) $=$ (1) $+$ (2)

linearly dependent on system of Equation.

Inconsistent System of Equations

- If the system of equations is inconsistent :- No solution exists



|| lines.

- If the coefficient Matrix rows are linearly dependent and the constants do not satisfy the linear dependence then the given system of equations is inconsistent.

$$3x+2y=12$$
$$3x+2y=6.$$

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$R_1 = R_2 \times 1$

But in the constant matrix the same linear dependence does not hold.

<u>ex-2</u>

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$R_1 = \frac{1}{2} \times R_2$ and also in the constant matrix the same holds

It means that the two lines are overlapping and the system of equations has infinite no of solutions.

→ If the rows of the coeff. Matrix are linealy then the system of linear Equations is consistant.

Summary

Rows linearly Dependent

Yes. → Constant Matrix also linealy dependent with same factor as coeff Matrix

- Yes → Inconsistent S.of Eq. No Soln.
- No. → Consistent S.O.Eq. Infinite Solns.

No. → Consistent Sys. of Eq.

System Of Linear Equations -2.

Solving System of L. Equations

Elimination of Variables.

$x + 3y - 2z = 5$ — 1 ⟶ $x = 5 + 2z - 3y$.

$3x + 5y + 6z = 7$ — 2

$2x + 4y + 3z = 8$ — 3

substitution in eq 2 and 3

$-4y + 12z = -8 \Rightarrow y = 2 + 3z$ —— (4)

Substitute in eq 3.

We get $z = 2$

Substitute in eq (4)

We get $y = 8$

Substitute the values of y and z in any eq 1, 2, or 3 we get $x = -15$

GAUSS ELIMINATION METHOD.

The coefficients of the equation and the constants are written in a single matrix known as augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{A}$ $\underbrace{\quad}_{b}$

Need to reduce to 0.

$R_2 \longrightarrow R_2 - 3R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & -4 & +12 & -8 \\ 0 & -2 & 7 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{1}{2} R_2$.

$$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$$-2z = -4$$

$$z = 2$$

We can substitute in the 2nd row.

$$-4y + 12(z) = -8.$$

$$-4y + 24 = -8$$

$$y = 8.$$

Now we can substitute values of y and z in R_1

$$x + 3y - 2z = 5$$

$$x + 3(8) - 2(2) = 5.$$

$$x + 24 - 4 = 5$$

$$x = -15$$

→ Another way to solve is to proceed further and reduce the matrix in the form of an identity matrix, then the constants represent the values of x, y and z.

③ Cramers Rule

$$1x + 3y - 2z = 5$$
$$3x + 5y + 6z = 7$$
$$2x + 4y + 3z = 8.$$

$$Ax = b.$$

constants

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}$$

constants.

$$y = \frac{\begin{vmatrix} 1 & 5 & -2 \\ 3 & 7 & 6 \\ 2 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}$$

→ Not used much in practice as the calculation of determinants is not computationally efficient

However a geometric intuition of Cramers rule is available on a youtube video of 3Blue1Brown channel.

https://youtu.be/jBsC34PxzoM.

(4) Matrix Solution way to Solve System of Linear Equations.

$$Ax = b.$$

$$x = A^{-1}b.$$

→ If $|A| = 0$ — A is not invertable ⇒ ∞ no of solutions
or,
No solutions.

→ Easiest way amongall or most preffered is Gauss Elimination method.

Homogenous system of linear Equations.

$$Ax = b$$

if $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow Ax = 0$

$3x + 2y.$

$x + 2y = 0.$

$x = 0 \quad y = 0.$

(1) Zero/trivial soln = $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(2) If $|A| \neq 0$ then only zero solution exists.

(3) If $|A| = 0$ then ∞-many solution exists.

4. u & v are solution of $Ax=0$ then u+v is also a solution.

$$A(u+v) = Au + Av = 0$$

5. u is a solution → cu is also a solution

6. $Ax=b$ $Ax=0$.

↑ ↑

p is a soln v is a solution

Then p+v is also a soln to ~~Ax=b~~ $Ax=b$.

1.17 Solved Problems GATE 2017, 2014, 2015, 2005, 2004

Q Let $c_1, c_2, \ldots c_n$ be scalars not all zero, such that the following expression holds.

$$\sum_{i=1}^{n} c_i a_i = 0 .$$ where a_i is the column vector of $\mathbb{R}^n$

Consider the set of linear Equations

$$Ax = B$$

Where $A = [a_1 a_2 \ldots a_n]$ and $b = \sum_{i=1}^{n} a_i$

Then the set of Equations has.

A. A unique solution, has $x = j_n$ where j denotes n dimentional vector for all 1.

B. No solution.

C. Infintely many solutions

D. Fintely many solutions

Solution.

given $\sum_{i=1}^{n} c_i a_i = 0$.

c_i's are constants and a_i's represent the columns of the matrix

which mean in case of a 3x3 matrix

$$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0.$$

$$a_1 = -\frac{c_2}{c_1} a_2 - \frac{c_3}{c_1} a_3$$

a_1 is linearly dependent on a_2 and a_3.

Now this means that $|A| = 0$, the system of equations either have infintely many solutions or no solutions.

Also we are given that $\sum_{i=1}^{n} a_i = b$.

Which means that the sum of coffients = constant matrix.

i.e in place of x, y, z we put $(1,1,1)$ we will get the sum of the coffients, in other words $(1,1,1)$ is a solution to the system of linear equations, we cannot say that no soln exist, and therefore the system has infinite no of solns. option C.

→ Q Consider a system, each consisting of m linear equations in n variables.

I. If $m<n$ then all such systems have a solution.

II. If $m>n$, then none of these systems has a solution.

III. If $m=n$ then there exist a system which has a solution

Which of the following is correct?

(A) I, II and III are true

(B) Only II and III are true

(C) Only III is true

(D) None of them is true.

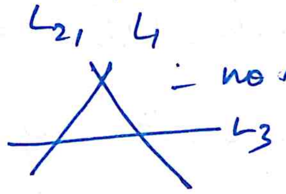
Soln

Statement I. lets take $m=1$ $n=2$.

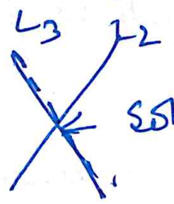
This is an underdetermined system, all such systems do not have a soln it just represents one line.

Statement II $m=3$, $n=2$

3 lines



no soln.



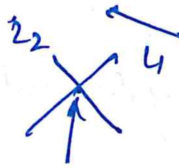
Soln exists.

Soln may exist, Statement II is false.

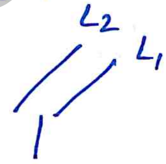
Statement III. If $m=n$

$m=2$ $n=2$

2 lines 2 variables.



soln exists



Solutions ~~need~~ does not exist.

∴ Statement 3 is true.

option c is correct.

Q → If the following system has a non trivial solution

$$px + qy + rz = 0$$

$$qx + \cancel{p}\overset{r}{}y + pz = 0$$

$$rx + py + qz = 0.$$

Which of the following options is True?

(A) $p-q+r=0$ or $p=q-r$.

(B) $p+q-r=0$ or $p=-q=r$.

(C) $p+q+r=0$ or $p=q=r$.

(D) $p-q+r=0$ or $p=-q=-r$.

Soln If a homogeneous sys of linear Equations then $|A|=0$

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0.$$

If we ~~opt~~ observe the 3rd option $p=q=r$ $|A|=0$ option 3 is correct

Q → Consider the following system of equations in three real numbers. x_1, x_2, x_3.

$$2x_1 - x_2 + 3x_3 = 1$$

$$3x_1 - 2x_2 + 5x_3 = 2$$

$$-x_1 + 4x_2 + x_3 = 3.$$

The system of equations have

(A) No Solution

(B) A Unique Solution

(C) More than one but finite no of solutions

(D) An infinite no of solutions.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & -2 & 5 \\ -1 & 4 & 1 \end{bmatrix}$$

$|A| = -6 \neq 0$ ∴ The system has a unique son option B.

Q → What values of x, y, z would satisfy the following system of linear Equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

A) $x=6, y=3, z=2$

B) $x=12$ $y=3$ $z=-4$

C) $x=6$ $y=6$ $z=-4$

D) $x=12$ $y=-3$ $z=0$

Soln on Substituting value we can come to a solution

option C is correct.

1.18 Eigen Values And Eigen Vectors

Geometric Intuition

$A_{n \times n} \longrightarrow$ Square Matrix {Transformation in nD space}

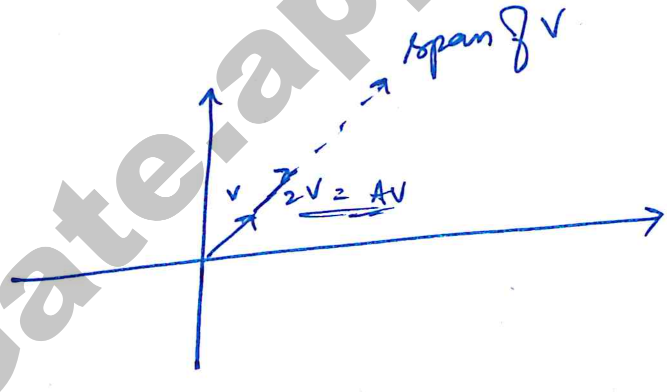
$$Av = \lambda v$$

$v \in \mathbb{R}^d$: real valued d-dim vector.

v is the eigen vector of A

λ is the eigen value of A.

→ V is that special vector of A if we apply linear transformation of A it is equivalent to scaling it by a factor of λ.



say $\lambda = 2$.

$|\lambda| > 1 \Rightarrow$ stretched

$|\lambda| < 1 \Rightarrow$ shrunk.

$|\lambda| < 0$ reversed.

Animations from 3Blue1Brown youtube channel for Eigen values and Eigen vectors :- http://youtu.be/PFDu9oVAE-g.

Algebraic and Numeric way

$$AV = \lambda V$$

$$\Rightarrow AV = \lambda I V$$

$$\Rightarrow AV - \lambda I V = 0.$$

$$\Rightarrow (A - \lambda I) V = 0.$$

~~If v is~~ $V \neq 0$.

$|A - \lambda I| = 0$ → Characteristic Eqn.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0.$$

$$\lambda^2 - 4\lambda - 3 = 0.$$

$$\lambda_1 = 3, \lambda_2 = 1$$

$(A-\lambda I)v_1 = 0.$

$\lambda_1 = 3.$

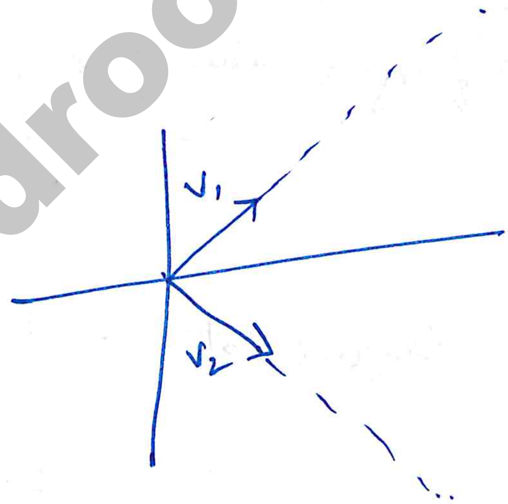
$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0.$$

$-v_{11} + v_{12} = 0$

$v_{11} - v_{12} = 0.$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \longrightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$



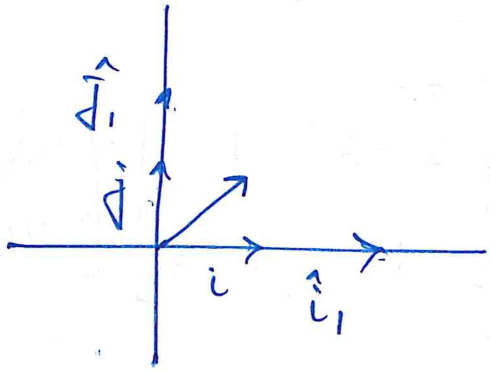
Properties of E.V. & E.Vectors.

① It is possible that A has no eigen value and eigen vectors

ex Rotation Matrix $A_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = R_{90°}.$

$\therefore$ Such a matrix does not have any eigen values

(2) $A = S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



$|A - \lambda I| = 0$

$(2-\lambda)(2-\lambda) = 0.$

$\lambda = 2$ single value.

$\boxed{\text{No of eigen values} <= n}$ for a $n \times n$ matrix.

(3) A = Diagonal Matrix = $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\lambda = 2, 3.$

$(2-\lambda)(3-\lambda) = 0.$

→ Diagonal Matrix are the E. Values. and basis vectors are the eigen vectors.

(4) $\text{Trace}(A) = \sum_{i=1}^{n} \lambda_i$

(5) $|A| = \prod_{i=1}^{n} \lambda_i$

(8) If λ is the eigen value of A then 1. λ^n is is eigen value of A^n.

2. $K\lambda$ is eigen value of (KA)
K is a scalar $\neq 0$

3. $\lambda + K$ = Eigen value of $\underline{(A+KI)}$

$\rightarrow$ Eigen value of the matrix represented by $\underline{\underline{a_0 I + a_1 A + a_2 A^2}}$ is

$\underline{\underline{a_0 + a_1\lambda + a_2\lambda^2}}$.

(9) If λ is eigen value of A and $|A| \neq 0$ then $\frac{1}{\lambda}$ is eigen value of A^{-1}.

(10) Eigen Value of $(Adj(A)) = \frac{|A|}{\lambda}$.

(11) In case of a triangular matrix the eigen values are the diagonal elements itself.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix} \quad (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 2, 3$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 7 \end{bmatrix} \rightarrow \lambda = 2, 6, 7.$$

True in case of upper and lower triangular matrix.

12) Eigen Value (A) = Eigen Value (A^T)

→ If A is an orthogonal matrix then $AA^T = I$ $A^T = A^{-1}$

λ is an eigen value of A then $\frac{1}{\lambda}$ is also an eigen value of A.

13) If we have a real and symmetric matrix, eigen values are real valued.

→ If $a + ib$ is an eigen value of A then $a - ib$ is also an eigen value of A.

14) If For a skew symmetric matrix A which is real valued, the eigen values are zero or purely imaginary.

15) Eigen Value (AB) = Eigen Value (BA).

16) If $\lambda_1, \lambda_2 \dots \lambda_n$ are non repeating eigen values of A then the eigen vectors of A $V_1, V_2 \dots V_n$ are linearly independent.

1.19 Cayley- Hamilton Theorem And Diagnolization

- If we have a matrix A, its characteristic polynomial is given by.

$P(\lambda) = |A-\lambda I| = 0$.

For example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $P(\lambda) = |A-\lambda I| = \lambda^2 - 5\lambda - 2 = 0$.

* → The matrix itself satisfies the characteristic Equation. Cayley Hamilton theorem

$A^2 - 5A - 2I = 0$.

Applications

→ 1. We can represent higher powers of A in terms of simple operations on matrix A. we can do this very fast this way.

$A^2 = 5A + 2I$

$A^3 = 5A^2 + 2A$

$= 5(5A + 2I) + 2A$

$= 27A + 10I$

Similarly we can compute $A^4, A^5 \ldots$ any power of A.

2. Compute Inverse.

$A^2 = 5A + 2I$

Multiply by A^{-1} throughout. $\Rightarrow$ $A = 5I + 2A^{-1}$

$\Rightarrow A^{-1} = \frac{A}{2} - \frac{5}{2}I$

Also higher -ve exponents can be calculated by Cayley Hamilton theorem.

Diagonalization : A matrix "A" is said to be diagonalizable if there exists a matrix P such that $P^{-1}AP$ = diagonal matrix. $\begin{bmatrix} a_1 & \cdots & 0 \\ 0 & a_2 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots 0 & a_n \end{bmatrix}$

If we write the equation

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots \\ 0 & \lambda_2 & 0 & \cdots & \cdots \\ 0 & 0 & \lambda_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & \lambda_n \end{bmatrix}$$

Multiplying both sides with P.

$$AP = P\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots \\ 0 & \lambda_2 & 0 & & \\ 0 & 0 & \lambda_3 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \lambda_n \end{bmatrix}$$

If $P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ V_1 & V_2 & V_3 & \cdots & V_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}_{n \times n}$.

$$A \cdot P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ AV_1 & AV_2 & \cdots & AV_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ V_1\lambda_1 & V_2\lambda_2 & V_3\lambda_3 & \cdots & V_n\lambda_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$\therefore$

$$\left.\begin{aligned} AV_1 &= V_1\lambda_1 \\ AV_2 &= V_2\lambda_2 \\ &\vdots \\ AV_n &= V_n\lambda_n . \end{aligned}\right\} \text{ This is the defination of Eigen Values, Vectors.}$$

Given a matrix A we can easily find the diagonalization matrix by using its eigen vectors.

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ V_1 & V_2 & \cdots & V_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \cdot & \cdot \\ \cdots & \cdots & \lambda_n \end{bmatrix} = D$$

* For the matrix A to be diagonalizable the matrix P should be invertable i.e in other words the Eigen Vectors should be linearly Independent.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}$$ on calculating E.V we get $\lambda_1 = 3$ $\lambda_2 = 2$ $\lambda_3 = 1$.

$$V_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

We get.

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applications of diagonalization

→ Compute Higher Exponents of any matrix.

$$P^{-1}AP = D.$$

$$\cancel{P P^{-1}} A \cancel{P P^{-1}} = P D P^{-1}$$

$$A^2 = P D \cancel{P^{-1}} \cancel{P} D P^{-1}$$

$$\Rightarrow P D^2 P^{-1}$$

Similarly $\underline{\underline{A^k = PD^kP^{-1}}}$

$$A^{100} = PD^{100}P^{-1}$$

Instead of 100 Matrix Multiplications we can do just 2 Matrix multiplication. As it is very easy to compute exponent of diagonal Matrices.

1.20 LU DECOMPOSITION

- Very Similar to Gauss Elimination

- If we have a system of linear Equations $\underline{Ax=b}$ and we have to find a solution to it i.e. we need to find x.

- We try to decompose A as $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

For example lets consider $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ if we are trying to decompose A as LU

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

On multiplying L and U we get.

$$LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & (L_{21}U_{12}+U_{22}) & (L_{21}U_{13}+U_{23}) \\ L_{31}U_{11} & (L_{31}U_{12}+L_{32}U_{22}) & (L_{31}U_{13}+L_{32}U_{23}+U_{33}) \end{bmatrix}$$

On comparing the above matrix with A and equating the corresponding terms

$\underline{U_{11}=1}$

$\underline{U_{12}=2}$

$\underline{U_{13}=4}$.

$L_{21}U_{11}=3$

$L_{21}(1)=3 \Rightarrow \underline{L_{21}=3}$

$L_{21}U_{12}+U_{22}=8$

$3(2)+U_{22}=8$

$\underline{U_{22}=2}$.

$L_{21}U_{13}+U_{23}=14$

$3(4)+U_{23}=14$

$\underline{U_{23}=2}$.

$L_{31}U_{11}=2$

$\underline{L_{31}=2}$

$L_{31}U_{12}+L_{32}U_{22}=6$

$2(2)+L_{32}(2)=6$.

$\underline{L_{32}=1}$

$L_{31}U_{13}+L_{32}U_{23}+U_{33}=13$

$2(4)+1(2)+U_{33}=13$

$\underline{U_{33}=3}$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

We can use LU Decomposition to solve system of linear Equations.

Step 1 :- Decompose the coeff Matrix A into two lower and upper triangular matrix.

Step 2 :-

$$L\underbrace{Ux}_{y} = b$$

$$Ly = b.$$

solve for y.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

Step 3 Now solve for x.

$$Ux = y.$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

→ This method is very easy to solve using the LU decomposition as we have upper and lower triangular matrices and they are already in reduced form.

→ Does Every matrix have a LU decomposition? - No

We need to check the determinant of the leading matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$A_1 = 1$

$|A_1| = 1 \neq 0$

$A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

$|A_2| = 8 - 6 = 2 \neq 0$

$A_3 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

$|A_3| = 6 \neq 0$.

→ If the determinant of all the leading submatrices $\neq 0$ then the LU decomposition of that matrix exists otherwise the LU-decomposition of that matrix does not exist.

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e.g

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

$A_1 = 1$

$|A_1| = 1$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad |A_2| = 0 \Rightarrow \cancel{0}.$$

As a hack we can swap R_2 and R_3 then in this case ...

$$A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad |A_2| \neq 0.$$

If $|A_3|$ was $= 0$ then we cannot apply any hack.

→ LU Decomposition is used as a part of many computations tools in different systems.

→ LU decomposition is not applicable if A is not invertable or $|A_3| = 0$.

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