

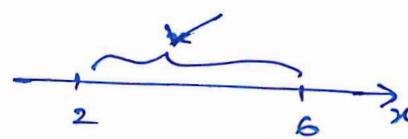
* Why study logic?

We will learn about Propositional logic and Predicate logic.

Suppose person P₁ and P₂ write the following code

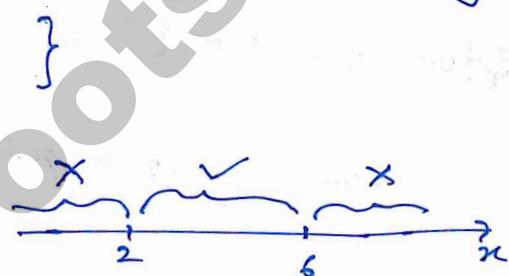
(P₁)

```
if ((x > 2) && (x < 6))  
{  
    == do something  
}
```



(P₂)

```
if (! ((x ≤ 2) || (x ≥ 6)))  
{  
    == do something  
}
```



As a software engineer, people do code reviews and software testers would test if the code is behaving in the expected manner. So verifying and validating the code is a part of software engineering and in general a part of programming.

Logic gives us a bunch of tools which consists of symbols, notations, properties and laws to verify or solve the problem if given two programs are equivalent or not. When we have large OS like Windows O.S or Linux O.S, these tools can be utilized to verify and validate the code.

Suppose $n > 2$ and n is an integer. Given 2 statements:

S₁: if n is prime then n is odd

S₂: if n is not odd then n is not prime

With the help of logic we can also verify, if two given statements are equivalent or not.

For $n > 2$

$\left\{ \begin{array}{l} S_1: \text{for all } x \in \{2, 3, \dots, n-1\} \text{ if } \frac{x \text{ does not divide } n}{x | n \leftarrow} \\ \text{then } n \text{ is prime.} \end{array} \right.$

S₂: If there exists an element $x \in \{2, 3, \dots, n-1\}$ such that $\underbrace{x | n}_{x \text{ divides } n}$, then n is not prime.

$x \text{ divides } n$	
2	1 6
2	X 5
3	1 6
3	X 8

There is an area in computer science and engineering called formal verification.

Formal verification is an extension of

logic (that we will learn). Formal

verification is used by electronic engineers

to verify circuit designs. It is also used

in software code verification. It is also used

in computer networks design to verify the behaviour
of the network.

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* Propositional Logic - I :

Propositional Logic / calculus / 0th order logic :

Proposition : It is a declarative (declares a fact) statement that is either true or false but not both.

Propositions are often represented as

p, q, r, s, \dots

E.g. p : New Delhi is the Capital of India — True

q : $1+2=3$ — True

r : $4+6=9$ — False

s : What's the time? — Not a proposition.

p_1 : $x+1=2 \rightarrow \begin{cases} T & \text{if } x=1 \\ F & \text{if } x \neq 1 \end{cases}$ } Not a proposition

p_2 : There exists $x \in \mathbb{R}$, such that $x+1=2$

↳ True

p_3 : He is tall — Not a proposition
He is tall
pronoun

p_4 : This sentence is false — Not a proposition
This sentence is false
This statement is called a Liar Paradox in Logic

Note:

Most of these concepts that we learn in propositional logic have been first developed formally by Aristotle around 2300 years ago.

④ Logical operators on propositions :

① Negation (unary operation) : Negation is denoted [operation on 1 proposition] by $\{\neg P, \tilde{P}, P', \bar{P}\}$

P: Today is Friday

$\neg P$: Today is not Friday

It's not Friday today

It's not the case that today is Friday

Truth Table

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

$\neg(\neg P) = P$

② Conjunction: $\rightarrow P \wedge q, P \cdot q, pq$

P: Today is Friday

q: It is raining today

$P \wedge q$: Today is Friday and it's raining today

\wedge is a logical AND. In C programming, logical AND is represented as `&&`. In Digital circuit design, we have learnt about AND GATE, which operates the same way.

Conjunction is a binary operator as it operates on 2 propositions.

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P	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

③ Disjunction: $p \vee q$, $p+q$, logical OR

In C programming \rightarrow ||

In DLC \rightarrow OR GATE

$p \vee q$: Today is Friday

or it's raining today.

p

q

P	q	$P \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

④ Exclusive OR: \oplus

In DLC \rightarrow XOR Gate

P	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$P \oplus q = (P \wedge \neg q) \vee (\neg P \wedge q) \leftarrow \text{using } \neg, \vee, \wedge$$

$$= (P \vee q) \wedge (\neg P \vee \neg q)$$

Proof:

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P	$q \vee \neg q$	$p \oplus q$	$(p \wedge \neg q)$	$(\neg p \wedge q)$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	$T \wedge F = F$	$\neg T = F$	F
T	F	T	$T \wedge T = T$	$\neg F = F$	T
F	T	T	$F \wedge F = F$	$T \wedge T = T$	T
F	F	F	$F \wedge T = F$	$T \wedge F = F$	F

⑤ Conditional operation / Implication:

$p \rightarrow q$

hypothesis conclusion
 premise consequence
 antecedent

if p then q
 if p is true, then q is true
 p implies q
 p only if q
 q if p
 q when p
 q follows from p
 q whenever p
 q unless $\neg p$
 p sufficient condition
 for q is p .
 p necessary condition
 for p is q
 q is necessary for p .
 if p, q

E.g. if you get 100 marks, then you will get A grade

$\text{you get 100 marks} \rightarrow \text{you get A grade}$

	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$$p \rightarrow q = \neg p \vee q$$

E.g. p : x learns discrete maths

q : x gets a good job.

$p \rightarrow q$: if x learns discrete maths, then x gets a good job.

$(p \rightarrow q)$ and $(q \rightarrow p)$ are not equivalent.

$p \rightarrow q$: x gets a good job unless x does not learn discrete maths.

Converse: $q \rightarrow p$

Contrapositive: $\neg q \rightarrow \neg p$

Inverse: $\neg p \rightarrow \neg q$

E.g. I will stay only if you go.

I will stay \rightarrow you go.

Converse: $q \rightarrow p$

If you go then I will stay.

⑥ Biconditional / iff :

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \rightarrow q$	p	q	$\neg(p \oplus q)$
T	T	T	F		T
T	F	F	T		F
F	T	F	T	F	
F	F	T	F		T

$$p \leftrightarrow q = \neg(p \oplus q)$$

Eg: you can fly iff you have a ticket.

$p \leftrightarrow q$

$p \leftrightarrow q :$

- p if and if q
- p is necessary & sufficient for q
- if p then q and conversely
- p iff q
- if p then q and if q then p
- q unless $\neg p$ and p unless $\neg q$

Note:

Logical equivalence: \Leftrightarrow, \equiv

$p \Leftrightarrow q$ means that $p \leftrightarrow q$ is a tautology.

Logical implication: \Rightarrow

$p \Rightarrow q$ means that $p \rightarrow q$ is a tautology.

$p \Rightarrow q$ means that $p \rightarrow q$ is a tautology soon.

Note: we will see the definition of tautology soon.

⑦ NAND & NOR $\rightarrow (P \downarrow q) = \neg(P \vee q)$

$$G(P \uparrow q) = \neg(P \wedge q)$$

P	q	$(P \uparrow q)$	$(P \downarrow q)$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

⑧ The operator precedence in logic:

\neg
 \wedge
 \vee
 \rightarrow
 \leftrightarrow

E.g., $\neg P \vee q$
 $= (\neg P) \vee q$

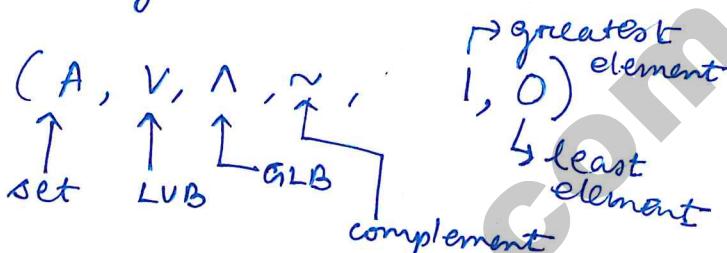
E.g., $\neg P \wedge q \vee P$
 $= (\neg P) \wedge q \vee P$

E.g., $\neg P \vee q \rightarrow S$
 $((\neg P) \vee q) \rightarrow S$

④ Logic and Algebraic structures:

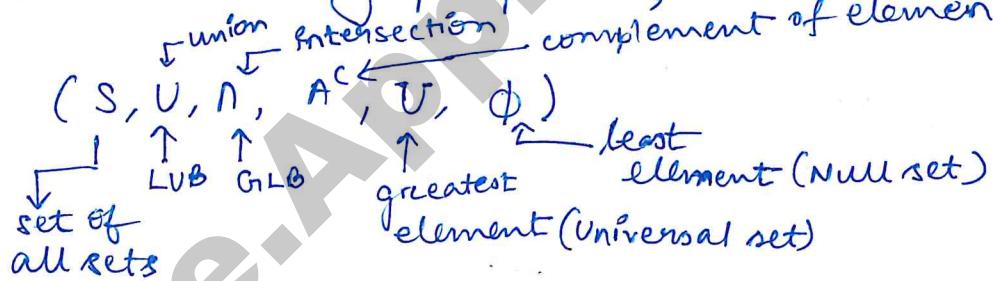
We have seen Lattice in algebraic structures,

which is a 6-tuple



A lattice where every pair of element has an LUB, GLB, every element has a complement and if the greatest and the least elements exists, such a lattice is called a Boolean Algebra.

From set theory perspective,



From logic perspective, V is OR

\wedge is AND

1 is True

0 is False

True \leftarrow False

The smallest Boolean Algebra, $A_2 = \{0, 1\}$,

$(A, V, \wedge, \sim, 1, 0)$, it contains only 2 elements the greatest and the least element.

A propositional logic is nothing but the boolean algebra that we have studied in the algebraic structures.

∴ Propositional logic is distributed, complemented and bounded lattice.

Now we will study the properties of propositional logic using the properties of Boolean Algebra.

Properties:

① Closure: $\forall x, y \in A, (x \vee y) \in A$

$$x = T$$

$$y = F$$

$$x \vee y = T$$

Note: The smallest Boolean Algebra,

$$A = \{F, T\}$$

Again, $x \wedge y \in A$

$$\bar{x} = \neg x \in A$$

These are propositional logic perspective.

Now, from set theory perspective,

$$\forall A, B \in \mathcal{S},$$

$$A \cup B \in \mathcal{S}$$

$$A \cap B \in \mathcal{S}$$

$$A^c = \bar{A} \in \mathcal{S}$$

Note: Algebraic structures connect the concepts in sets and propositional logic via boolean algebra (which is a

special type of lattice.

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② Commutative:

$$\begin{aligned} x \vee y &= y \vee x \\ x \wedge y &= y \wedge x \end{aligned} \quad \left. \begin{array}{l} \text{propositional} \\ \text{logic perspective} \end{array} \right\}$$

$$\begin{aligned} A \cap B &= B \cap A \\ A \cup B &= B \cup A \end{aligned} \quad \left. \begin{array}{l} \text{set theoretic view} \end{array} \right\}$$

③ Distributive property:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

④ Identity:

$$x \vee 0 = x = x \vee F$$

$$x \wedge 1 = x = x \wedge T$$

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

⑤ Complement

$$(\exists \neg x \vee x)$$

There exists complement of x for all x .

$$x \vee \top x = T$$

$$x \wedge \top x = F$$

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \emptyset$$

⑥ Idempotent law:

$$x \wedge x = x$$

$$x \vee x = x$$

$$A \cup A = A$$

$$A \cap A = A$$

Note: In Algebraic structures, ($x * x = x$) where * is any operator.

⑦ Absorption Law:

$$x \wedge (x \vee y) = x$$

$$x \vee (x \wedge y) = x$$

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

⑧ De Morgan's Law:

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$\neg(p \vee q) = \neg p \wedge \neg q$$

$$\neg(p \wedge q) = \neg p \vee \neg q$$

⑨ Law of double complement:

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$$\neg(\neg p) = p$$

$$(A')' = A$$

⑩ Domination Law:

$$p \vee 1 = 1$$

$$p \wedge 0 = 0$$

$$A \cup U = U \quad , \text{where } U \text{ is the universal set.}$$

$$A \cap \emptyset = \emptyset$$

✳ Tautology, Contradiction, Contingency & Commutative and Associative Operators

Tautology: Any proposition (simple or complex) is a tautology if it always True.

In case of complex proposition, if its Truth value is True for all possible values, then the given complex proposition is a tautology.

E.g.,

P	T	P V T
T	T	T
F	T	T

We need to prove if the given complex proposition(s) is a tautology or not.

p	q	r	s
T	T	F	F

But, to find if all values in s are True for all possibilities of p, q and r, is a very cumbersome process.

Instead let us simplify the given complex proposition (S).

$$\begin{aligned}
 S &: ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow ((p \rightarrow r)) \\
 &\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) \\
 &\xrightarrow{\text{Apply DeMorgan's Law}} \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r) \\
 &\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r) \\
 &\equiv \neg p \vee (p \wedge \neg q) \vee r \vee (q \wedge \neg r) \\
 &\equiv (\neg p \vee p) \wedge (\neg p \vee \neg q) \vee (r \vee q) \wedge (r \vee \neg r) \\
 &\equiv (\neg p \vee p) \wedge (\neg p \vee \neg q) \wedge T \\
 &\equiv T \wedge (\neg p \vee \neg q) \wedge T \\
 &\equiv T \wedge (T \vee \neg r) \wedge T \\
 &\equiv T \wedge T \wedge T \\
 &\equiv T
 \end{aligned}$$

\therefore complex proposition S is always True.
Therefore S is a tautology.

Note:

If we have n -variables, P_1, P_2, \dots, P_n .

Each of these variables can be True/False, i.e.,
two possibilities.

Therefore the total number of rows that
we can have is $2 * 2 * 2 * 2 * \dots * n$ times

$$= 2^n.$$

If we have 2 variables, number of

$$\text{rows} = 2^2 = 4$$

4 variables — 16 rows

3 variables — 8 rows.

	P_1	P_2	\dots	P_n	f
1.					True/False
2.					T/F
3.					True/False
\vdots					\vdots
2^n					True/False

The number of proposition function f

possible is $2 * 2 * 2 * 2 * \dots * 2^n$ times
 $= 2^{2^n}$

2^{2^n} is superexponential.

For $n=3$, $2^n = 8$, $2^{2^n} = 2^8 = 256$ Phone: +91 844-844-0102

Therefore, the truth table approach may not be always feasible, as when n increases, number of rows increases exponentially.

It is usually better to use the properties and laws of propositional logic to prove the propositions.

Contradiction / fallacy / invalid :

If a proposition p is always false, then we call the proposition as contradiction / fallacy / invalid. In other words, if the truth table results in False for all combinations of propositional variables, then it is called a contradiction.

Contingency :

If a proposition is sometimes true and sometimes false.

Example: $(p \vee q)$ is a contingency.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

E.g. $\beta: ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (r \rightarrow p)$

$$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg r \vee p)$$

$$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg r \vee p)$$

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg r \vee p)$$

$$\equiv p \vee (p \vee \neg q) \vee \neg r \vee (q \vee \neg r)$$

$$\equiv p \vee \neg r \quad (\text{using absorption rule})$$

Therefore β is sometimes true and sometimes false.

Note: If an expression (propositional statement) is a tautology or a contingency, it is said to be satisfiable.

If an expression is a contradiction, it is said to be non-satisfiable as no combination of truth values (T/F) assignments to the propositional variables, will result in a True for the whole expression.

Example:

$$p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$$

$$\text{RHS. } (p \Rightarrow q) \wedge (p \Rightarrow r)$$

$$\equiv (\neg p \vee q) \wedge (\neg p \vee r)$$

$$\equiv \neg p \vee (q \wedge r) = \text{LHS}$$

The given propositional statement is a tautology.

$$(Q) P \Rightarrow (q \vee r) \equiv (P \Rightarrow q) \vee (P \Rightarrow r)$$

LHS.

$$\neg P \vee (q \vee r)$$

$$(\neg P \vee q \vee r)$$

RHS.

$$(\neg P \vee q) \vee (\neg P \vee r)$$

$$(\neg P \vee q \vee r)$$

$$\therefore LHS = RHS$$

\therefore The given propositional statement is also a tautology.

$$(Q) (P \Leftrightarrow q) \wedge (P \Leftrightarrow r)$$

$$\equiv (\neg P \vee q) \wedge (\neg q \vee P) \wedge (P \vee r) \wedge (\neg r \vee \neg P)$$

$$\equiv (\neg P \vee (q \wedge \neg q)) \wedge (P \vee (r \wedge \neg r))$$

$$\equiv (\neg P) \wedge (P)$$

$$\equiv F$$

which means the propositional statement is a contradiction.

- ① $\wedge, \vee, \text{XOR}, \Leftrightarrow$ — All these operators are commutative and associative.

$$a \vee b \equiv b \vee a \quad \text{— commutative}$$

$$a \vee (b \vee c) \equiv (a \vee b) \vee c \quad \text{— associative.}$$

- ② \Rightarrow — not commutative, not associative

$$(P \Rightarrow q) \not\equiv (q \Rightarrow P)$$

$$\therefore (\neg p \vee q) \not\equiv (\neg q \vee p)$$

∴ Not commutative.

- ③ NAND (\uparrow) NOR (\downarrow) } Commutative but not associative.

- * Functionally complete set, Normal Forms (PDNF, PCNF)

Functional completeness (FC):

A set F of boolean functions is functionally complete if we can express any boolean function using functions in set F.

E.g.: $F: \{\wedge, \vee, \neg\}$: FC set

Any complex boolean function using **XOR, NAND, NOR, ...**
can be represented using an FC set.

Q. Why is the concept of functional completeness important?

If we are designing digital logic circuits, we will have logic gates, like AND gate, OR gate, NOT gate, which are actual physical ^{electronic} circuits. what functional completeness tells us is that using AND, OR and NOT gate we can represent any boolean function we need.

E.g : $\{\wedge, \vee, \neg, \rightarrow, \Leftrightarrow, \uparrow, \downarrow\}$: FC set.

$\{\wedge, \vee\}$: Not FC set because we can't represent $\neg p$ given proposition p.

$\{\wedge, \neg\}$
 $\{\vee, \neg\}$
 $\{\rightarrow, \neg\}$
 $\{\uparrow\}$
 $\{\downarrow\}$

: FC and also minimally FC.

Note: Minimally FC set : These are the FC set for which there is no subset of F is FC.

E.g: F: $\{\wedge, \vee, \neg\}$ is not minimally FC because the subset $\{\wedge, \neg\}$ or $\{\vee, \neg\}$ are FC.

Note: $\{\leftrightarrow, \top\}$ is not a FC set. **Phone: +91 844-844-0102**



In Digital Logic circuit perspective, using NAND (\uparrow) / NOR (\downarrow) gates, we can construct any boolean function.

E.g. Let's show that $\{\uparrow\}$ is FC. Given proposition P ,

$$P \uparrow P = \uparrow(P \wedge P) = \uparrow P$$

Given propositions p and q ,

$$(P \uparrow P) \uparrow (q \uparrow q)$$

$$\equiv \uparrow(P \wedge P) \uparrow \uparrow(q \wedge q)$$

$$\equiv \uparrow(P) \uparrow \uparrow(q)$$

$$\equiv \uparrow(\uparrow P \wedge \uparrow q)$$

$$\equiv P \vee q \quad (\text{using DeMorgan's law})$$

Similarly,

$$(P \uparrow q) \uparrow (P \uparrow q)$$

$$\equiv P \wedge q$$

Therefore using $\{\uparrow\}$ we are able to simulate $\{\neg, \wedge, \vee\}$. Therefore $\{\uparrow\}$ is also F.C.

E.g. Let's see $\{\downarrow\}$ OR NOR: is 9FC 844044-0102

Using the similar approach that we used for $\{\uparrow\}$, we can show that,

$$P \downarrow P = \neg(P \vee P) = \neg P$$

$$P \vee Q = (P \downarrow Q) \downarrow (P \downarrow Q)$$

$$P \wedge Q = (P \downarrow P) \downarrow (Q \downarrow Q)$$

Therefore, FC sets are a extremely interesting concept in Digital Logic Circuits that help us understand how many types of circuits that we need to be able to simulate / generate any boolean function.

Normal Forms:

To explain the Normal Forms, let's take an example to understand it.

E.g. $P \leftrightarrow (Q \rightarrow R')$ where R' or \bar{R} or $\neg R$ are same.

Let 0 represent False and 1 represent True.

P			q			r			$q \rightarrow r'$		$P \leftrightarrow (Q \rightarrow R')$		min term	$\begin{matrix} \wedge \\ 0:P' \\ 1:P \end{matrix}$	max term	$\begin{matrix} \vee \\ 0:P \\ 1:P' \end{matrix}$
3	0	0	0		1					0			$m_0 = .$		$M_0 = P \vee q \vee r$	
	0	0	1		1					0			$m_1 = .$		$M_1 = P \vee q \vee \bar{r}$	
	0	1	0		1					0			$m_2 = .$		$M_2 = P \vee \bar{q} \vee r$	
	0	1	1		0					1			$m_3 = P' \wedge q \wedge \bar{r}$		$M_3 = .$	
	1	0	0		1					1			$m_4 = P \wedge q' \wedge \bar{r}'$		$M_4 = .$	
	1	0	1		1					1			$m_5 = P \wedge q' \wedge \bar{r}$		$M_5 = .$	
	1	1	0		1					1			$m_6 = P \wedge q \wedge \bar{r}'$		$M_6 = .$	
	1	1	1		0					0			$m_7 = .$		$M_7 = P' \wedge q' \wedge \bar{r}'$	

$$\begin{array}{l} \text{APPLIED} \\ \text{ROOTS} \\ p \leftrightarrow (q \rightarrow r') \end{array}$$

$$= \neg q \vee \neg r$$

$$= q' \vee r'$$

We know,

$$p \leftrightarrow q$$

$$= (p \rightarrow q) \wedge (q \rightarrow p)$$

$$\text{Therefore, } p \leftrightarrow (q \rightarrow r') \equiv (m_3 \vee m_4 \vee m_5 \vee m_6)$$

Each m is built using $\{p, q, r, \text{negation}, \wedge\}$

And $(m_3 \vee m_4 \vee m_5 \vee m_6)$ is in the standard form.

This standard form is called principal disjunctive normal Form (PDNF)

Also,

$$p \leftrightarrow (q \rightarrow r') \equiv M_0 \wedge M_1 \wedge M_2 \wedge M_7$$

Each M is built using $\{p, q, r, \text{negation}, \vee\}$

This standard form is called Principal conjunctive Normal Form (PCNF).

This is extremely useful in digital logic circuit design. We can represent any functions using $\{\neg, \vee, \wedge\}$, i.e., OR, NOT and AND gates.

Note: 1. Given any expression, PCNF & PDNF are unique.
 2. Given $x = y$,
 we can check for :

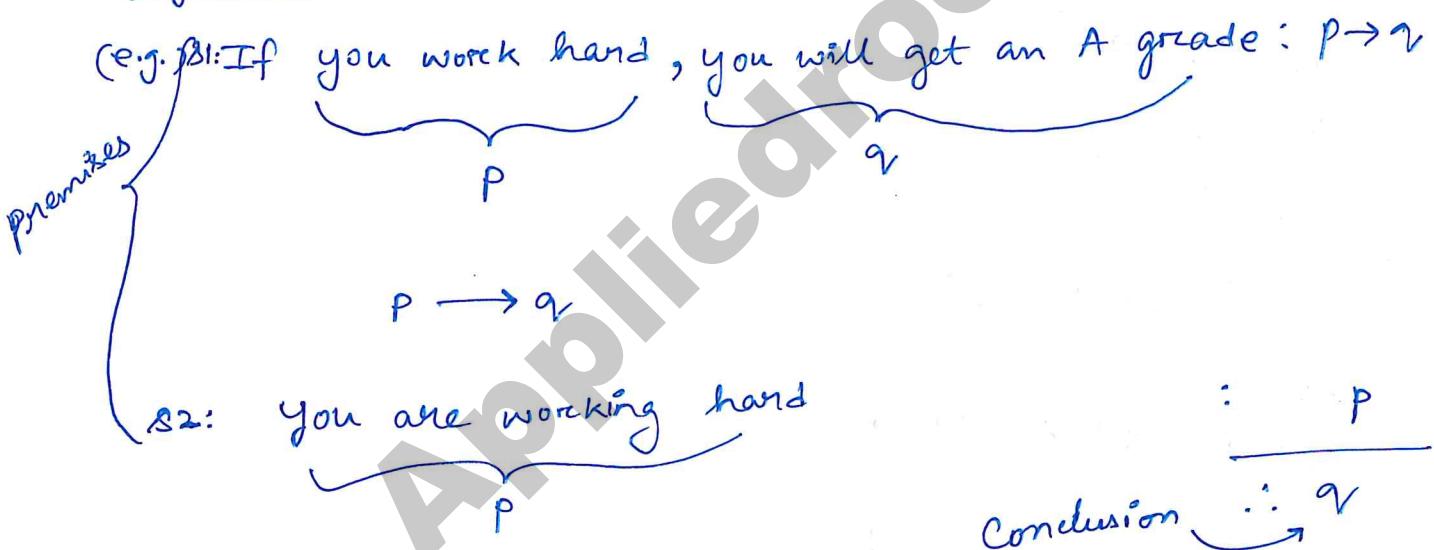
$$PDNF(x) = PDNF(y)$$

$$PCNF(x) = PCNF(y)$$

$$3. (\# \text{ terms in PCNF}) + (\# \text{ terms in PCNF}) = 2^n$$

Arguments and Rules of Inference:

Arguments



Arguments are basically a set of premises followed by a logically valid conclusion.

Let,

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{array} \left. \begin{array}{c} \} \\ \} \end{array} \right. \begin{array}{l} \text{premises} \\ \text{conclusion} \end{array}$$

In a world where all of $P_1, P_2, P_3, \dots, P_n$ are True, q should logically follow (always).

$((P_1 \wedge P_2 \wedge P_3 \wedge P_4 \dots \wedge P_n) \rightarrow q)$ is tautology

$$\begin{aligned} &\equiv T \\ &\equiv 1 \end{aligned} \quad \left. \right\} \text{always}$$

\therefore To conclude q from $(P_1 \wedge P_2)$ we need to prove that $((P_1 \wedge P_2) \rightarrow q)$ is True always or is a tautology.

$$P_1 : p \rightarrow q$$

$$\frac{P_2 : p}{C : q}$$

$$(P_1 \wedge P_2 \rightarrow q)$$

$$\equiv ((p \rightarrow q) \wedge p) \rightarrow q$$

$$\equiv ((\neg p \vee q) \wedge p) \rightarrow q$$

$$\equiv \neg ((\neg p \vee q) \wedge p) \vee q$$

$$\equiv (\neg(\neg p \vee p)) \wedge (\neg p \vee \neg q) \vee q$$

$$\equiv T \wedge (\neg p \vee \neg q \vee q)$$

$$\equiv T \wedge (\neg p \vee T)$$

$$\equiv T \wedge T$$

$$\equiv T \equiv 1 \quad (\text{RHS})$$

Because $(CP, \wedge p_2) \rightarrow q$ is True, we can conclude q given premises p_1 and p_2 .
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If we have to make conclusions from premises, this method is too time taking. Therefore we will learn to use some simple rules called the the rules of inference and see how these simple rules can actually simplify the whole process for us. We will also see how to use this rules to make conclusions with the given premises.

Rules of Inference:

① Addition :
$$\frac{P}{\therefore P \vee q}$$

In a world where p is true, would p and q be also true?

Yes, because if p is true, we don't have to think about q .

② Simplification :
$$\frac{P \wedge q}{\therefore p}$$

$$\frac{P \wedge q}{\therefore q}$$

In a world where both p and q are True, is P True?

Yes, ofcourse.

③ Conjunction:

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

In a world where p and q are both True, $(p \wedge q)$ is also True.

④ Modus Ponens: $p \rightarrow q$

$$\begin{array}{c} p \\ \hline \therefore q \end{array}$$

It is also called Rule of Detachment.

⑤ Modus Tollens: $p \rightarrow q$

$$\begin{array}{c} \neg q \\ \hline \therefore \neg p \end{array}$$

We can re-write it:

$$\underbrace{((p \rightarrow q) \wedge \neg q) \rightarrow \neg p}_{\text{Is this a Tautology?}} \equiv 1$$

Is this a Tautology?

It is also called proof by contradiction.

⑥ Hypothetical syllogism:

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

This is transitive property of implication.

⑦ Constructive dilemma: $P \rightarrow Q$

$$\begin{array}{c} r \rightarrow s \\ P \vee r \\ \hline \therefore (Q \vee s) \end{array}$$

⑧ Destructive dilemma: $P \rightarrow Q$

$$\begin{array}{c} r \rightarrow s \\ \neg Q \vee \neg s \\ \hline \therefore \neg P \vee \neg r \end{array}$$

⑨ Disjunctive syllogism: $P \vee Q$

$$\begin{array}{c} \neg P \\ \hline \therefore Q \end{array}$$

This is also called proof by elimination.

In a world where $(P \vee Q)$ is true and $\neg P$ is true,
i.e., P is not true, therefore Q is true.

E.g. $P \rightarrow (S \oplus r)$

$$\left. \begin{array}{c} P \\ r \\ \hline \therefore ? \end{array} \right\}$$

using modus ponens:
 $\rightarrow S \oplus r \equiv (S \wedge \neg r) \vee (\neg S \wedge r)$

$$\begin{array}{c} r \\ \hline \therefore ? \end{array}$$

If r is True, $\neg r$ is False. $\therefore (S \wedge \neg r)$ is False.

$$\text{Now, } F \vee (\neg S \wedge r) \equiv (\neg S \wedge r)$$

$$\text{since } r \text{ is True, } \neg S \wedge r \equiv r$$

$$\therefore r \wedge r \equiv r \equiv T$$

E.g.,

$$\begin{array}{c}
 P \rightarrow (S \vee R) \\
 P \\
 \hline
 \therefore ?
 \end{array}
 \left\{
 \begin{array}{c}
 \text{modus} \\
 \text{ponens}
 \end{array}
 \right\}
 \frac{\quad}{S \vee R}
 \left\{
 \begin{array}{c}
 F \\
 \neg R \\
 \hline
 T
 \end{array}
 \right\}
 \left\{
 \begin{array}{c}
 \text{Proof by} \\
 \text{elimination}
 \end{array}
 \right\}
 \frac{\quad}{\therefore S}$$

E.g.,

$$\begin{array}{c}
 P \rightarrow (S \oplus R) \\
 P \\
 \hline
 \therefore ?
 \end{array}
 \left\{
 \begin{array}{c}
 \text{modus} \\
 \text{ponens}
 \end{array}
 \right\}
 \frac{\quad}{S \oplus R = (S \wedge \neg R) \vee (\neg S \wedge R)}
 \left\{
 \begin{array}{c}
 T \\
 \neg \{ \neg R \\
 \hline
 \therefore S
 \end{array}
 \right\}$$

Given $\neg R$ is True,
means $R = \text{False}$.

$$\neg S \wedge F \equiv F$$

$$\text{and } (S \wedge \neg R) \equiv (S \wedge T) \equiv S$$

$\therefore S$ is the conclusion.

(*) Some more Translations from English to logical statements:

More English Translations (This is extremely important for non-native english speakers).

① It's raining but it's sunny.
 $\underbrace{P}_{\text{It's raining}}$ but $\underbrace{q}_{\text{it's sunny}}$

$$P \text{ but } q$$

$$P \wedge q$$

② You will not get an A grade unless you work hard.

$$\neg p \vee q \equiv p \rightarrow q$$

Note: p unless $q \equiv p \vee q$

③ I worked hard also I got a good grade

$$p \text{ also } q \equiv p \wedge q$$

④ I worked hard as well I got a good grade

$$p \text{ as well } q \equiv p \text{ as well as } q \equiv p \wedge q$$

⑤ p or q is true but not both
Exactly one of p or q (is True)

$$\left. \begin{array}{l} \\ \end{array} \right\} \rightarrow p \oplus q \quad (\text{XOR})$$

⑥ Atleast one of p or q
Either p or q

$$\left. \begin{array}{l} \\ \end{array} \right\} \rightarrow p \vee q$$

⑦ Neither p nor q } $\rightarrow p \downarrow q \equiv \neg p \wedge \neg q \equiv \neg(p \vee q)$

⑧ Atleast one of p or q is false
Either p is false or q is false

$$\left. \begin{array}{l} \\ \end{array} \right\} \begin{aligned} p \uparrow q &\equiv \neg p \vee \neg q \\ &\equiv \neg(p \wedge q) \end{aligned} \quad (\text{NAND})$$

⑨ p is necessary for q $\equiv q \rightarrow p$ Phone: +91 844-844-0102

$p \text{ is required for } q$
 $p \text{ is a must for } q$

⑩ p is sufficient for q $\equiv p \rightarrow q$

p is enough for q

E.g., A degree is required to apply for this job.

p q

$\equiv q \rightarrow p$ where q: you can apply for
this job

p: you have a degree.

A degree is enough to apply for this job

p q

$\equiv p \rightarrow q$

⑪ You cannot ride a rollercoaster if you
are under 3 feet tall unless you are older
than 10 years.

p: you can ride a roller coaster

q: you are under 3 feet tall

r: you are older than 10 years.

p unless $q \equiv p \vee q$

q if $p \equiv p \rightarrow q$

p if $q \equiv q \rightarrow p$

$$\neg p \text{ if } q \text{ unless } r$$

$$\neg q \rightarrow \neg p \vee r$$

$$= (q \rightarrow \neg p) \vee r$$

④ Predicate Logic : An Introduction

Predicate logic / Predicate calculus / 1st order logic :

While studying propositional logic, we have seen statements like :

$x+2 = 5$ and we said that it is not a proposition, because for some values of x , the statement holds true, while in other cases the statement is false, i.e., the same statement is both True and False. \therefore It is not a proposition.

Let $x \in \mathbb{R}$

$x+2 = 5$ is True if $x=3$
False if $x \neq 3$

We can say,

there exists $x \in \mathbb{R}$ such that $x+2 = 5$.

This is a proposition.

$\exists x \ x+2 = 5$

There exists x such that

We have to define the domain of x clearly, i.e., whether x is an integer, or real number or complex number, or x is an integer greater than 5, etc.

Domain of x means all the possible values of x .

If Domain of $x = \mathbb{R} - \{3\}$,

then $\exists x (x+2=5)$ is False.

If Domain of $x = \{3\}$,

then $\exists x (x+2=5)$ is True.

\exists is called an existential quantifier as it is quantifying what x would be.

It is often written as :

$\exists x P(x)$ where $P(x) : x+2=5$

In English :

There exists x such that $P(x)$

There is an x such that $P(x)$

There is atleast one x such that $P(x)$.

For all $x \in \mathbb{R}, (x+1) > x$

$\forall x \ p(x)$; Domain of $x = \mathbb{R}$
 Universal quantifier

English: for all $x, p(x)$

for every $x, p(x)$

for each $x, p(x)$

for any $x, p(x)$

given any $x, p(x)$

All of $x, p(x)$

for arbitrary $x, p(x)$

$\forall x \ p(x) \quad x \in \{x_1, x_2, x_3, \dots\} = \text{Domain of } x$

$$\textcircled{1} \quad (p(x_1) \wedge p(x_2) \wedge p(x_3) \wedge p(x_4) \wedge \dots) \Rightarrow \forall x \ p(x)$$

$$(x_1+1 > x_1) \wedge (x_2+1 > x_2) \wedge (x_3+1 > x_3) \wedge \dots \Rightarrow \forall x \ p(x)$$

$$\textcircled{2} \quad \exists x \ p(x) \quad x \in \{x_1, x_2, x_3, \dots\}$$

$$(p(x_1) \vee p(x_2) \vee p(x_3) \vee \dots) \Rightarrow \exists x \ p(x)$$

E.g. $x \in \mathbb{Z}^+$

$p(x) : \sqrt{x}$ is real

$\neg p(x) : \sqrt{x}$ is not real

Note: $\sqrt{\mathbb{Z}^+}$ = real but $\sqrt{2^-}$ = complex number.

$$\forall x P(x) \equiv T$$

: For all values of x ,
 \sqrt{x} is real. It is true.

$$\exists x P(x) \equiv T$$

↓

There exists a value of x for which \sqrt{x} is real. It is true.

Similarly,

$$\forall x \neg P(x) \equiv F$$

$$\exists x \neg P(x) \equiv F$$

E.g. $x \in \mathbb{R}$

$$P(x) : x+2=5$$

$$\forall x P(x) \equiv \text{False}$$

$$\exists x P(x) \equiv \text{True}$$

Uniqueness Quantifier! (Not widely used)

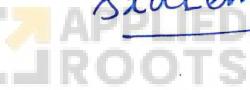
$$\exists ! x P(x) \quad \text{or} \quad \exists_1 x P(x)$$

There exists a unique x s.t. $P(x)$
such that

There is one and only x such that $P(x)$

$$\exists_1 x (x+2=5) \quad \text{s.t. } x \in \mathbb{R}$$

Statements and Quantifiers on 2 variables



(e.g) $x, y \in \mathbb{R}$

$$P(x, y) : x + y = 5$$

① $\forall x \forall y P(x, y) \equiv \forall_x \forall_y (x + y = 5) \equiv \text{False}$

\equiv For all x and for all y

$$(x + y) = 5$$

② $\exists x \exists y P(x, y) \equiv \exists_x \exists_y (x + y = 5) \equiv \text{True}$

③ $\forall x \exists y P(x, y) \equiv \forall x \exists y (x + y = 5)$

$$x + y = 5$$

$$\Rightarrow y = 5 - x$$

$$\therefore \forall x \exists y \underset{x}{(x + y = 5)} \equiv \text{True}$$

④ $\exists x \forall y P(x, y) \equiv \exists x \forall y (x + y = 5) \equiv \text{False}$

For a fixed value of x , y also becomes fixed since $y = 5 - x$. Therefore y does not have all possible values in the real numbers range.

⑤ $\forall y \exists x (x + y = 5) \equiv \text{True}$

For any value y , we can find a value of x such that $(x + y) = 5$.

(e.g) $x, y \in \mathbb{Z}^+$
 $P(x, y) = x + y = 5$

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$\forall x \forall y P(x, y) \equiv F$
 $\begin{matrix} \forall x \\ \forall y \end{matrix}$

$\exists x \exists y P(x, y) \equiv T$
 $\begin{matrix} \exists x \\ \exists y \end{matrix}$

$\exists x \forall y P(x, y) \equiv F$

$\forall y \exists x P(x, y) \equiv F$
 $\begin{matrix} \forall y \\ \exists x \end{matrix}$

$$(6 + x) = 5$$

$$x \in \mathbb{Z}^+$$

(e.g) $P(x, y) : x + y^2 = 10 \quad x, y \in \mathbb{R}$

Ⓐ $\forall x \forall y x + y^2 = 10 \equiv \text{False}$
 $\begin{matrix} \forall x \\ \forall y \end{matrix}$

Ⓑ $\exists x \exists y x + y^2 = 10 \equiv \text{True}$
 $\begin{matrix} \exists x \\ \exists y \end{matrix}$

Ⓒ $\exists x \forall y (x + y^2 = 10) \equiv F \quad ;: y = \sqrt{10 - x}$
 $\begin{matrix} \exists x \\ \forall y \end{matrix}$

$\therefore x$ and y are dependent here, selecting
 x , makes $y = \sqrt{10 - x}$. Therefore we
can't take any values of y . Therefore
the statement is false.

Ⓓ $\exists y \forall x ((x + y^2 = 10) \equiv F \quad ;: 10 - y^2 = x$
 $\begin{matrix} \exists y \\ \forall x \end{matrix}$

② $\forall x \exists y (x+y^2=10)$ **Phone: +91 844-844-0102**
Applied Roots

$$\begin{aligned}y &= \sqrt{10-x} \\&= \sqrt{10-11} \\&= \sqrt{-1} \notin \mathbb{R}\end{aligned}$$

③ $\forall y \exists x (x+y^2=10) \equiv T$
 \downarrow
 $x = 10 - y^2$

If y is a real number, y^2 is also a real number, $(10-y^2)$ is also a real number.

\therefore The statement is True.

Statements and quantifiers on 3 variables :

Let $x, y, z \in \mathbb{R}$

$$P(x, y, z) : x + y + z = 10$$

④ $\forall x \forall y \forall z P(x, y, z) \equiv \text{False}$

For all x , for all y , for all z , the statement is not True.

⑤ $\forall n \forall y \exists z P(x, y, z) \equiv \text{True}$

Take any values (real numbers) of x and y ,
 $z = (10 - x - y)$ is also a real number. \therefore True.

$$\forall x \exists z \forall y P(x, y, z) \equiv \text{False}$$

$$\begin{aligned}y &= 10 - z - x \\&= 10 - z_1 - 11 \quad (\text{say } z = z_1) \\&= -1 - z_1\end{aligned}$$

The moment we set $z = z_1$, y is also fixed as y is dependent on z_1 .

$\therefore y$ cannot take any value that it wants.
 \therefore It fails. \therefore False.

Note: The sequence of assignment is always left to right, i.e., first assign x , then assign z and then y .

$$\textcircled{2} \exists z \forall n \forall y (x+y+z=10) \equiv \text{False.}$$

z
 z_1

$$x+y=10-z_1$$

If we set $z = z_1$, for all values of x and for all values of y , the statement need not be True.

⑥ $\exists z \exists y \forall x (x+y+z=10) \equiv \text{False}$

$$\begin{array}{ccc} \downarrow & \downarrow \\ z_1 & y_1 \end{array}$$

If we fix z and y as z_1 and y_1 , this statement will not hold for all values of x .

⑦ $\exists x \exists y \exists z \forall w (x+y+z=10) \equiv \text{True}$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow \\ w_1 & w_2 & w_3 & w_4 \end{array}$$

⑧ $\forall x \exists y \exists z (x+y+z=10) \equiv \text{True}$

$$y+z = \underbrace{10-x}_{\text{IR}}$$

We can allocate y and z , values which are real numbers and satisfy the statement.

⑨ Properties of Predicate Logic

① $\forall x P(x) \Rightarrow \exists x P(x)$

$$P \rightarrow Q$$

		$P \rightarrow Q$	$P \rightarrow Q \equiv \neg P \vee Q$
		P	Q
✓	T	T	✓
T	F	F	
{F	F	T	✓
F	T	T	

$P \rightarrow Q$ is 'if P then Q '.

It means, if P is True, then Q has to be True. If P is False, we don't care what Q is, $P \rightarrow Q$ will always be True.

$$\forall x P(x) \equiv (P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \dots)$$

$$x \in \{x_1, x_2, x_3, \dots\}$$

To make notations simpler,

$$P(x_i) = P_i$$

$$Q(x_j) = Q_j$$

Let's try to prove

$$\forall x P(x) \Rightarrow \exists x P(x)$$

LHS.

$$\begin{aligned} \forall x P(x) &\equiv (P(x_1) \wedge P(x_2) \wedge \dots) \\ &\equiv P_1 \wedge P_2 \wedge P_3 \wedge \dots \end{aligned}$$

RHS.

$$\exists x P(x) \equiv P_1 \vee P_2 \vee P_3 \vee \dots$$

From modus ponens :-

$$\frac{\begin{array}{c} P \rightarrow Q \\ P \end{array}}{\therefore Q}$$

Given P is True, then for $P \rightarrow Q$ to be

True, Q has to be True.

$$\therefore (P_1 \wedge P_2 \wedge P_3 \wedge \dots) \Rightarrow (P_1 \vee P_2 \vee P_3 \vee \dots)$$

T T T

\therefore LHS implies RHS.

Let's now try to see:

$$\exists x P(x) \stackrel{?}{\Rightarrow} \forall x P(x)$$

$$P_1 \vee P_2 \vee P_3 \dots \Rightarrow P_1 \wedge P_2 \wedge P_3 \wedge \dots$$

For this to be true if only P_1 is True and other propositions are False, will hold.
 \therefore LHS is True.

But for the same truth value (True/False) assignment of the propositions in RHS, RHS will result in False.

$$\therefore T \stackrel{?}{\Rightarrow} F$$

F
It is not a tautology.

$$\therefore \exists x P(x) \not\Rightarrow \forall x P(x)$$

② De'Morgan's Laws:

$$\textcircled{a} \quad \neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\textcircled{b} \quad \neg(\exists x P(x)) \equiv \forall x (\neg P(x))$$

Let's try to prove \textcircled{a} :-

$$\neg(\forall x P(x)) \Leftrightarrow \exists x \neg P(x)$$

First, we try to prove:

$$\neg(\forall x P(x)) \Rightarrow \exists x (\neg P(x))$$

$$\text{LHS: } \neg(P_1 \wedge P_2 \wedge P_3 \dots) \quad \text{RHS: } (\neg P_1 \vee \neg P_2 \vee \neg P_3 \dots)$$

Using De Morgan's, we can prove $LHS \rightarrow RHS$.
Now, let's try to prove $RHS \rightarrow LHS$.

$$\exists x \neg P(x) \Rightarrow \neg (\forall x P(x))$$

$$(\neg P_1 \vee \neg P_2 \vee \neg P_3 \dots) \Rightarrow \neg (P_1 \wedge P_2 \wedge P_3 \dots)$$

Again, we use Demorgan's and prove that
 $RHS \rightarrow LHS$.

$$\therefore \neg (\forall x P(x)) \Leftrightarrow (\exists x \neg P(x))$$

③ Nested Quantifiers:

$$\exists x \forall y P(x, y) \Rightarrow \forall y \exists x P(x, y)$$

LHS:

$$\exists x \forall y P(x, y)$$

$$\equiv \exists x (\forall y P(x, y)) \wedge P(x, y_1) \wedge P(x, y_2) \wedge P(x, y_3) \wedge \dots$$

$$\equiv \exists x (P(x, y_1) \wedge P(x, y_2) \wedge P(x, y_3) \wedge \dots)$$

$$\equiv (P(x_1, y_1) \wedge P(x_1, y_2) \wedge P(x_1, y_3) \wedge \dots)$$

$$\vee (P(x_2, y_1) \wedge P(x_2, y_2) \wedge P(x_2, y_3) \wedge \dots)$$

$$\vee (P(x_3, y_1) \wedge P(x_3, y_2) \wedge P(x_3, y_3) \wedge \dots)$$

$$\equiv (P_{11} \wedge P_{12} \wedge P_{13} \wedge \dots) \vee$$

$$(P_{21} \wedge P_{22} \wedge P_{23} \wedge \dots) \vee$$

$$(P_{31} \wedge P_{32} \wedge P_{33} \wedge \dots)$$

Note: While expanding \forall gives \wedge and
 \exists gives \vee .

But,

$$\forall x \exists y P(x, y) \Rightarrow \exists y \forall x P(x, y)$$

shortcut to remember:

$$\forall x (\exists y P(x, y)) \Rightarrow \exists y \forall x P(x, y)$$

We can pullout $\forall y$ ahead of $\exists x$

$$\forall x (\exists y P(x, y)) \Rightarrow \exists y \forall x P(x, y)$$

\times we cannot pullout $\exists y$ ahead of $\forall x$.

$$\text{E.g., } \forall x \exists y (x+y=0) \Rightarrow \exists y \forall x (x+y=0)$$

$x, y \in \mathbb{R}$

The RHS is False as if we fix y , $\forall x$ does not hold. $\therefore \exists y \forall x (x+y=0)$ is False.

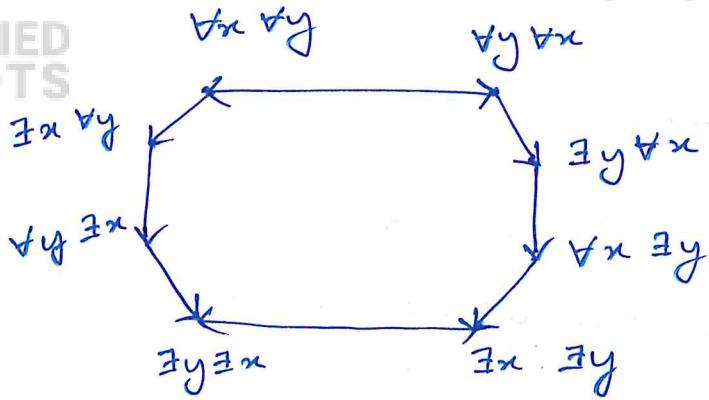
But LHS is True.

$$\therefore \underbrace{T \rightarrow F}_F$$

$$\therefore \text{LHS} \not\Rightarrow \text{RHS}.$$

This Theorem is called One-way Theorem,
As $\text{LHS} \Rightarrow \text{RHS}$ but $\text{RHS} \not\Rightarrow \text{LHS}$.

$$\begin{aligned} ④ \quad & \forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y) \\ & \exists x \exists y P(x, y) \Leftrightarrow \exists y \exists x P(x, y) \end{aligned}$$



$$\forall x \forall y P(x,y) \rightarrow \exists x \forall y P(x,y) : \text{True}$$

$$\forall y \forall x P(x,y) \rightarrow \forall x \exists y P(x,y) : \text{True}$$

$$\forall x \exists y P(x,y) \rightarrow \exists y \forall x P(x,y) : \text{False}$$

⑤ Distributive property:

$$(A) \quad \forall x (P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$$

$$\equiv (P_1 \wedge Q_1) \wedge (P_2 \wedge Q_2) \wedge \dots \equiv (P_1 \wedge P_2 \wedge P_3 \wedge \dots) \wedge \\ (Q_1 \wedge Q_2 \wedge Q_3 \wedge \dots)$$

Hence, LHS is equivalent to RHS.

$$(B) \quad \exists x (P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$$

$$\equiv (P_1 \vee Q_1) \vee (P_2 \vee Q_2) \vee \dots \equiv (P_1 \vee P_2 \vee P_3 \vee \dots) \\ \vee (Q_1 \vee Q_2 \vee Q_3 \vee Q_4 \vee \dots)$$

Hence, LHS is equivalent to RHS.

③ $\forall x (P(x) \vee Q(x)) \stackrel{?}{\Rightarrow} (\forall x (P(x)) \vee (\forall x Q(x)))$ Phone: 91844-844-0102

$$\equiv (P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge (P_3 \vee Q_3) \wedge \dots \quad \left| \begin{array}{l} = (P_1 \wedge P_2 \wedge P_3 \wedge \dots) \\ \vee (Q_1 \wedge Q_2 \wedge Q_3 \wedge \dots) \end{array} \right.$$

We try to find out if an assignment of the truth values result in

$$T \rightarrow F$$

Then it would mean that $(T \rightarrow F)$ is false.

$$\begin{aligned} \text{LHS.} \\ &= (P_1^T \vee Q_1^F) \wedge (P_2^F \vee Q_2^T) \\ &\quad \wedge (P_3^T \vee Q_3^F) \wedge \dots \\ &= T \end{aligned}$$

$$\begin{aligned} \text{RHS.} \\ &= (P_1^T \wedge P_2^F \wedge P_3^T \wedge \dots) \\ &\quad \vee (Q_1^F \wedge Q_2^T \wedge Q_3^F \wedge \dots) \\ &= F \vee F \\ &= F \end{aligned}$$

$\therefore \text{LHS} \rightarrow \text{RHS}$

$$\begin{aligned} &= T \rightarrow F \\ &= F \end{aligned}$$

$\therefore \forall x (P(x) \vee Q(x)) \not\Rightarrow (\forall x P(x)) \vee (\forall x Q(x))$

$$\begin{aligned} \text{④ } (\forall x P(x)) \vee (\forall x Q(x)) &\stackrel{?}{\Rightarrow} \forall x (P(x) \vee Q(x)) \\ &\equiv (P_1 \wedge P_2 \wedge P_3 \wedge \dots) \vee (Q_1 \wedge Q_2 \wedge Q_3 \wedge \dots) \quad \left| \begin{array}{l} = (P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge \\ (P_3 \vee Q_3) \wedge \dots \end{array} \right. \end{aligned}$$

For LHS to be True, either each of the propositions P_1, P_2, P_3, \dots has to be True or,

$q_1, q_2, q_3 \dots$ has to be True or both
has to be False.

Given the Truth values, RHS will have:

$$(i) \frac{(P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge \dots}{T \quad F}$$

$$(ii) \frac{(P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge \dots}{F \quad T}$$

$$(iii) \frac{(p_1 \vee q_1)}{T} \wedge \frac{(p_2 \vee q_2)}{T} \wedge \dots$$

In all the above possibilities R+IS will be true.

$$\therefore T \xrightarrow{\hspace{1cm}} T \\ (\text{LHS}) \qquad (\text{RHS})$$

$$\therefore (\forall x P(x) \wedge \forall x Q(x)) \Rightarrow ((\forall x P(x) \vee \forall x Q(x))$$

$$(E) \exists x (P(x) \wedge Q(x)) \Rightarrow (\exists x P(x)) \wedge (\exists x Q(x))$$

$$\text{E: } \exists x (P(x)) \rightarrow (\exists x P(x)) \Rightarrow \exists x (P(x) \wedge Q(x))$$

$$(F) (\exists x \varphi(x)) \wedge (\neg \exists x \psi(x))$$

$$\vdash \neg p \wedge q \equiv (\forall x P(x)) \vee q$$

$$\textcircled{G} \quad \forall n (P(n) \wedge Q) = \cdots \\ \qquad \qquad \qquad \vdots \vdots \vdots (P_0 \wedge Q) \wedge \cdots$$

$$\equiv (P_1 \wedge Q) \wedge (P_2 \wedge Q) \wedge (P_3 \wedge Q)$$

$$\vdash (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \dots) \wedge \vartheta$$

$$\exists x \forall n P(n) \wedge Q \equiv RHB.$$

Note: Q is not a statement which contains x.

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$$\textcircled{H} \quad \exists x (P(x) \wedge Q) \equiv (\exists x P(x)) \wedge Q$$

$$\textcircled{I} \quad \forall x (P(x) \vee Q) \equiv (\forall x P(x)) \vee Q$$

$$\textcircled{J} \quad \exists x (P(x) \vee Q) \equiv (\exists x P(x)) \vee Q$$

E.g. $\forall x (P(x) \vee Q) \Leftrightarrow (\exists x P(x)) \vee Q$

Let's check for LHS \Rightarrow RHS first.

$$\equiv (\forall x P(x)) \vee Q \Rightarrow (\exists x P(x)) \vee Q$$

\therefore LHS \Rightarrow RHS

Let's check for RHS \Rightarrow LHS now.

$$\exists x P(x) \not\Rightarrow \forall x P(x)$$

\therefore RHS $\not\Rightarrow$ LHS.

E.g. $\forall x (P(x) \rightarrow Q(x)) \Rightarrow (\forall x P(x) \rightarrow (\forall x Q(x)))$

$$\equiv (\neg P_1 \vee Q_1) \wedge (\neg P_2 \vee Q_2) \wedge (\neg P_3 \vee Q_3) \wedge \dots \quad \left| \begin{array}{l} \equiv \neg(P_1 \wedge P_2 \wedge P_3 \dots) \\ \vee (Q_1 \wedge Q_2 \wedge Q_3 \wedge \dots) \end{array} \right.$$
$$\equiv (\neg P_1 \vee \neg P_2 \vee \neg P_3 \dots) \\ \vee (Q_1 \wedge Q_2 \wedge Q_3 \wedge \dots)$$

Whenever LHS is True, RHS is also True.

$\top \rightarrow \top$ holds

\therefore LHS \Rightarrow RHS.

Note: To expand we use the property:

$$P \rightarrow Q \equiv \neg P \vee Q.$$

E.g. $(\forall x P(x)) \rightarrow (\forall x Q(x)) \not\Rightarrow \forall x (P(x) \rightarrow Q(x))$

$$\begin{aligned} & \neg P_1 \vee \neg P_2 \vee \neg P_3 \dots \\ & \vee (\underset{F}{Q_1} \wedge \underset{F}{Q_2} \wedge \underset{F}{Q_3} \wedge \dots) \\ & \equiv T \end{aligned}$$

$$\left. \begin{aligned} & (\neg P_1 \vee \neg Q_1) \wedge (\neg P_2 \vee \neg Q_2) \\ & \wedge (\underset{F}{\neg P_3} \vee \underset{F}{\neg Q_3}) \wedge \dots \\ & = (T \vee F) \wedge (F \vee F) \\ & \wedge (F \vee F) \wedge \dots \\ & = T \wedge F \wedge F \wedge \dots \\ & = F \end{aligned} \right\}$$

$$\therefore T \rightarrow F$$

(LHS) (RHS)

\therefore we show that,

$$(\forall x P(x)) \rightarrow (\forall x Q(x)) \not\Rightarrow \forall x (P(x) \rightarrow Q(x))$$

E.g. $\exists x (P(x) \rightarrow Q(x)) \not\Rightarrow (\exists x P(x)) \rightarrow (\exists x Q(x))$
 It can be proved by finding
 $T \rightarrow F$.

E.g. $(\exists x P(x)) \rightarrow (\exists x Q(x)) \Rightarrow \exists x (P(x) \rightarrow Q(x))$

Here, $T \rightarrow T$ holds.

 * Translation from English to Predicate Logic
and vice-versa.

Precedence: \forall, \exists have the highest priority

$$\forall x P(x) \rightarrow Q(x)$$

$$\equiv (\forall x P(x)) \rightarrow Q(x)$$

$$\forall x P(x) \rightarrow \exists x Q(x)$$

$$(\forall x P(x)) \rightarrow (\exists x Q(x))$$

Translations:

Predicate statements \leftrightarrow English.

Basic Rules:

① \exists : some
at least one

\forall : all
any
every

Eg. Every student in the class has studied DM.

$x \in \underbrace{\text{students in the class}}$
Domain / Universe

x is a variable that represents any element
as long as the element belongs to the set
(students in the class).

$\forall x S(x)$, x has studied DM.

Note:
In case where domain is not clearly defined, we need to define domain as broadly as possible.

$x \in \underbrace{\text{set of persons}}_{\text{Domain}}$

$c(x)$: x is a student in the class.

$\forall x (c(x) \rightarrow S(x))$

For all x , if x is a student in the class,

then x have studied DM.

And if x is not a student in the class,

then we don't care about the person,

as we don't have that information.

The following statement is incorrect:

$\forall x (c(x) \wedge S(x))$ — common mistake

It means for every person in the universe, they should belong to my class and they should have studied DM.

And there may be some person x , who

does not belong to the class

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This is not what has been said in the given statement. Therefore it is an error/mistake while translation that we should be aware of.

Note: \forall with \rightarrow is correct.

\forall with \wedge could lead to incorrect translation.

E.g. Some students in the class have studied DM.
 $x \in$ set of students in class.

Case 1: Domain is clearly stated.

$\exists x \ S(x)$

Case 2: Domain is not clearly stated.

$x \in$ set of people } we define a broad domain.

$\exists x (C(x) \wedge S(x))$

The following is a common mistake:

$\exists x (C(x) \rightarrow S(x))$

What if there are no students in the class, then $S(x)$ may have any truth value (T/F)

and $(C(x) \rightarrow S(x))$ will always be True.

$\therefore x$ will exist.

\therefore It is a wrong translation.

E.g. who is } AND ←
 which is }
 There is someone who is taller than 7 feet

$P(x)$: x is a person

$x \in$ all objects } broad domain

$T(x)$: x is taller than 7 feet.

$$\therefore \exists x (P(x) \wedge T(x))$$

(AND)

E.g. Humming Bird is small.

$x \in$ all objects } broad domain

$H(x)$: x is a Humming bird

$s(x)$: x is small

$$\therefore \exists x (H(x) \wedge s(x))$$

'All Humming birds are small' will be

$\forall x (H(x) \rightarrow s(x)) \equiv$ Humming birds are small

$B(x)$: x is a boy.

E.g. Every tall boy in the class plays BasketBall.

$x \in$ Students in the class

$T(x)$ = x is Tall

$B(x)$: x is a boy.

$BB(x)$: x plays basketball.

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$$\forall x (\{B(x) \wedge T(x)\} \rightarrow BB(x))$$

E.g. No & Not.

No hardworking person is poor.

$x \in$ set of persons (in the universe)

$P(x)$: x is poor.

$H(x)$: x is hardworking

Let, $\neg P(x) = R(x)$: x is rich.

No hardworking person is poor

$\equiv \neg(\text{a hardworking person is poor})$

$\equiv \neg(\exists x H(x) \wedge P(x))$

$\equiv \forall x (\neg H(x) \vee \neg P(x))$

$\equiv \forall x (H(x) \rightarrow \neg P(x))$

$\equiv \forall x (H(x) \rightarrow R(x))$

\equiv Every hardworking person is rich.

E.g. Not all hardworking people are poor.

$\equiv \neg(\text{all hardworking people are poor})$

$\equiv \neg(\forall x (H(x) \rightarrow P(x)))$ De Morgan's law

$\equiv \exists x (H(x) \wedge \neg P(x))$

$\equiv \exists x (H(x) \wedge R(x))$

= There are some people who are both rich and hardworking.

E.g. Pearls and Rubies are precious.

$x \in$ all objects

$P(x)$: x is a pearl

$R(x)$: x is a Ruby

$Prec(x)$: x is precious.

$$\forall x(P(x) \vee R(x)) \rightarrow Prec(x)$$

④ More Translations and examples

E.g. Lions and Tigers attack when hungry or threatened.

x : any object

$$S1: \forall x(L(x) \vee T(x)) \Rightarrow [(H(x) \vee Th(x)) \Rightarrow A(x)]$$

Given,
 $L(x)$: x is a Lion

$T(x)$: x is a Tiger

$H(x)$: x is Hungry

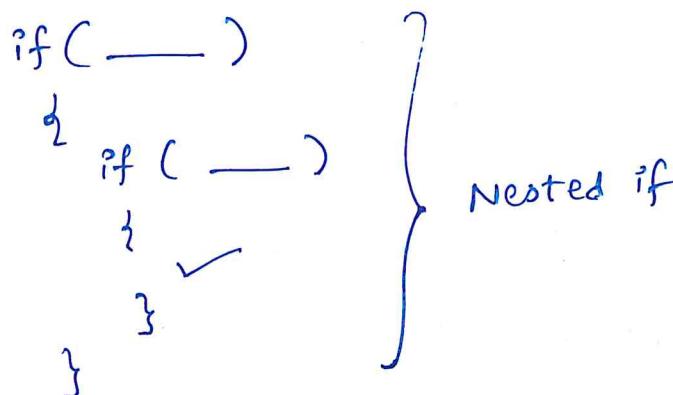
$Th(x)$: x is Threatened

$A(x)$: x attacks.

Let's check if it is a correct translation.

$\forall x (L(x) \vee T(x)) \Rightarrow [R(x) \vee Th(x)] \Rightarrow A(x)$

$\forall x (\text{if } \checkmark \text{ then } \{ \text{ if } \checkmark \text{ then } \checkmark \})$



It can be very complicated sometimes to read using nested if.

Therefore let's simplify it.

$$\begin{aligned} P &= L(x) \vee T(x) \\ Q &= R(x) \vee Th(x) \\ R &= A(x) \end{aligned}$$

\therefore we have:- (we are ignoring $\forall x$ for now).

$$\begin{aligned} P &\Rightarrow (Q \Rightarrow R) \\ &\equiv \neg P \vee (\neg Q \vee R) \\ &\equiv \neg P \vee \neg Q \vee R \\ &\equiv \neg(P \wedge \neg Q) \vee R \\ &\equiv (P \wedge \neg Q) \rightarrow R \end{aligned}$$

\therefore we have,

$$\beta_2: \forall x [\{L(x) \vee T(x)\} \wedge \{R(x) \vee Th(x)\}] \Rightarrow A(x)$$

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For all x , if x is a lion or a tiger, and, if x is hungry or threatened, then it will attack.

\therefore It is a correct translation.

Note: $\forall x (L(x) \wedge T(x))$ is incorrect because an animal cannot be both lion and a tiger.

And,

$$S_3: \forall x (L(x) \wedge T(x)) \Rightarrow [(H(x) \vee Th(x)) \Rightarrow A(x)]$$

F. T/F
 It will always be True

\therefore It is a wrong translation.

$$S_4: \forall x [\underbrace{\{H(x) \vee Th(x)\}}_P \rightarrow \{ \underbrace{(L(x) \vee T(x)) \rightarrow A(x)}_{q} \}]$$

$$\begin{aligned}
 p \rightarrow (q \rightarrow r) &\equiv \neg p \vee \neg q \vee r \\
 &\equiv \neg(p \wedge q) \vee r \\
 &\equiv (p \wedge q) \rightarrow r \\
 &\equiv S_2
 \end{aligned}$$

$\therefore S_4$ is also a correct translation.

e.g. All purple mushrooms are poisonous.



$$\delta_1: \forall x [(P(x) \wedge M(x)) \Rightarrow P_o(x)]$$

It says:

For all x , if x is a mushroom and x is purple, then it is poisonous.

It is a correct translation as the statement is only talking about purple mushrooms.

$$\delta_2: \forall x [P(x) \rightarrow \{M(x) \rightarrow P_o(x)\}]$$

$$P \rightarrow \{M \rightarrow P_o\} \equiv \neg(P \wedge M) \vee P_o \equiv (P \wedge \neg M) \rightarrow P_o \equiv \delta_1.$$

$$\delta_3: \forall x [M(x) \rightarrow \{P(x) \rightarrow P_o(x)\}] \equiv \delta_1.$$

$$\delta_4: \forall x [\{P(x) \rightarrow M(x)\} \rightarrow P_o(x)] \equiv \neg(\neg P \vee M) \vee P_o$$

$$\equiv (P \wedge \neg M) \vee P_o \equiv \neg(\neg P \vee M) \vee P_o \equiv (\neg P \vee M) \rightarrow P_o$$

$\neq \delta_1$

$\therefore \delta_4$ is an incorrect translation.

Note: Try to always write a simple statement with one or minimum possible implications. Then compare other statements with the simple statement.

(e.g.) No one can fool everyone all the time



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$x, y \in$ set of people.

$F(x, y) \equiv x$ can fool y all the time.

$$\neg (\underbrace{\exists x \forall y F(x, y)})$$

There exists x such that x can fool
all the y all the time.

$$\equiv \forall x \exists y \neg F(x, y)$$

For all x , there exists a y such that
 x cannot fool y all the time.

(e.g.) $x \in$ set of students in a college

$F(x, y) : x \& y$ are friends

$\forall x \forall y F(x, y) :$ All students in the college
are friendly with each other

$\forall x \exists y F(x, y) :$ Every student is friendly
with someone in a college.

$\exists y \forall x F(x, y) :$ Some students are friendly
with everyone in a college.

$\exists y \exists x F(x, y) :$ There is atleast one pair of
friends in a college.

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(e.g.) Some boys are taller than all the girls in the class.

Let, $x \in$ students in the class.

$$\exists x [B(x) \wedge \{ \forall y G(y) \rightarrow T(x, y) \}]$$

Mistake: $\exists x [B(x) \wedge \{ \forall y G(y) \underline{\wedge} T(x, y) \}]$

(e.g.)
 $x \in$ all painters
 $y \in$ all paintings.

$P(x, y)$: x has painted y

$\exists x \forall y P(x, y)$: some painters have painted all paintings.

$\forall y \exists x P(x, y)$: some paintings are painted by every painter.

$\exists x \forall y P(x, y)$: For every painting, there exists someone who has painted it.

(e.g.) $\forall x \{(x \neq 0) \rightarrow \exists y (xy = 1)\}$

$y, x \in \mathbb{R}$

= Every non-zero real number has an inverse (multiplicative inverse).

= Every non zero real number x , there exists another real number y such that $x \cdot y = 1$.

(e.g.) $\neg(x=1) \wedge \forall y [\exists z (x=y*z) \Rightarrow \{(y=x) \vee (y=1)\}]$

$x, y, z \in \mathbb{R}$

If x is not equal to 1 and (if y is a divisor of x , then y is either x or y is 1).

The statement says :

$x \neq 1$

y and z are divisors of x .

For every y there exists a z .

If the above is true then $y=x$ or $y=1$

E.g.:

$$x=6$$

$$x=6 = 2 * 3$$

$\downarrow \quad \downarrow$
 $y \quad z$

E.g. $x=5 = 5 * 1$

$$x=5 = 1 * 5$$

$\downarrow \quad \downarrow$
 $y \quad z$

$x \neq 1$ but divisors of x are x and 1.

Such a number is called a prime number.

Let's try to simplify the statement:-

$$\forall y [\exists z (x=y*z) \Rightarrow \{(y=z) \vee (y=1)\}]$$

$$\equiv \forall y [\neg (\exists z x=y*z) \vee (y=x) \vee (y=1)]$$

$$\equiv \forall y [\forall z (x \neq y*z) \vee (y=x) \vee (y=1)]$$

For the statement to be true one of the above has

to be true. But in this case, the simplification
does not seem to be intuitive.

E.g. There is exactly one apple on the table.
 $y, x \in$ all objects on the table.

$$S_1: \exists x (A(x) \wedge \forall y \{y \neq x \rightarrow \neg A(y)\})$$

$A(x)$: x is an Apple.

The statement S_1 means:

There exists an Apple x and for all y , if object y is not same as object x (Apple), then y is not an apple.

$$\equiv \exists x (A(x) \wedge \forall y \{A(y) \rightarrow (y=x)\})$$

$$\equiv \exists_1 x A(x)$$

$$\equiv \underbrace{\exists x A(x)}_{\geq 1} \wedge \underbrace{\forall x \forall y (A(x) \wedge A(y) \rightarrow (x=y))}_{\leq 1}$$

$= 1$
exactly one apple.

Atleast 2 apples :

$$\exists x [A(x) \wedge \exists y (A(y) \wedge (x \neq y))]$$

We have one item (Apple x) and another item (Apple y) and they both are not the same. That is, there are atleast 2 apples.

Exactly 2 apples :

$$\underbrace{\exists x [A(x) \wedge \exists y (A(y) \wedge (x \neq y))]}_{\text{atleast}} \wedge$$

$$\forall z \left\{ A(z) \Rightarrow \left\{ \begin{array}{l} (z=x) \vee (z=y) \end{array} \right\} \right\} \underbrace{\quad}_{\text{atmost}}$$

$$\equiv \exists x \exists y [(x \neq y) \wedge A(x) \wedge A(y) \wedge \forall z (A(z) \Leftrightarrow (z=x \vee z=y))]$$

Atmost 2 :

$$\forall x \forall y \forall z (\{ A(x) \wedge A(y) \wedge A(z) \} \Rightarrow \begin{cases} z=x \\ \text{or} \\ z=y \\ \text{or} \\ x=y \end{cases})$$

If we have three apples, then two of them are the same.