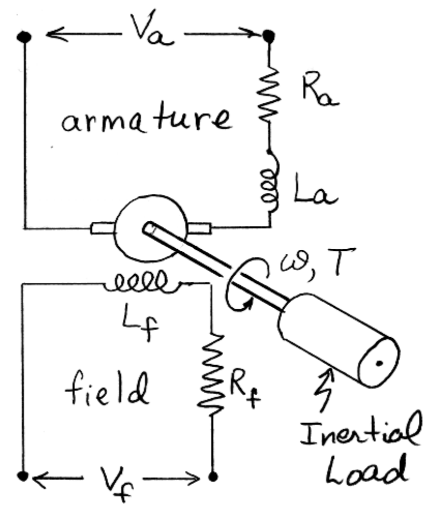
DC Motor Transfer Functions

(Reference: Dorf and Bishop, Modern Control Systems, 9th Ed., Prentice-Hall, Inc. 2001)

The figure at the right represents a DC motor attached to an inertial load. The voltages applied to the field and armature sides of the motor are represented by V_f and V_a. The resistances and inductances of the field and armature sides of the motor are represented by R_f, L_f, R_a, and L_a. The torque generated by the motor is proportional to i_f and i_a the currents in the field and armature sides of the motor.

$$\boxed{T_m = K\, i_f\, i_a} \tag{1.1}$$



DC Motor

Field-Current Controlled:

In a field-current controlled motor, the armature current i_a is held constant, and the field current is controlled through the field voltage V_f. In this case, the motor torque increases linearly with the field current. We write

$$T_m = K_{mf}\, i_f$$

By taking Laplace transforms of both sides of this equation gives the transfer function from the input current to the resulting torque.

$$\boxed{\frac{T_m(s)}{I_f(s)} = K_{mf}} \tag{1.2}$$

For the field side of the motor the voltage/current relationship is

$$\begin{aligned} V_f &= V_R + V_L \\ &= R_f i_f + L_f \left(di_f / dt \right) \end{aligned}$$

The transfer function from the input voltage to the resulting current is found by taking Laplace transforms of both sides of this equation.

$$\boxed{\frac{I_f(s)}{V_f(s)}=\frac{\left(1/L_f\right)}{s+\left(R_f/L_f\right)}}$$

(1st order system) (1.3)

The transfer function from the input voltage to the resulting motor torque is found by combining equations (1.2) and (1.3).

$$\boxed{\frac{T_m(s)}{V_f(s)}=\frac{T_m(s)}{I_f(s)}\frac{I_f(s)}{V_f(s)}=\frac{\left(K_{mf}/L_f\right)}{s+\left(R_f/L_f\right)}}$$

(1st order system) (1.4)

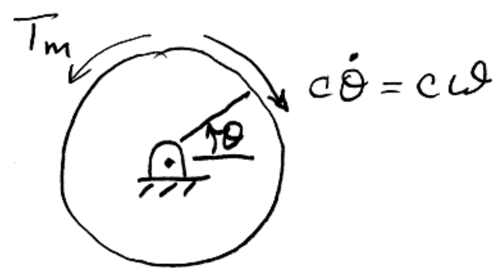
So, a step input in field voltage results in an exponential rise in the motor torque.

An equation that describes the rotational motion of the inertial load is found by summing moments

$$\sum M = T_m - cw = J\dot{w} \qquad \text{(counterclockwise positive)}$$

or

$$\boxed{J\dot{w}+cw=T_m}$$



Free Body Diagram
of the Inertial Load

Thus, the transfer function from the input motor torque to rotational speed changes is

$$\boxed{\frac{w(s)}{T_m(s)}=\frac{\left(1/J\right)}{s+\left(c/J\right)}}$$

(1st order system) (1.5)

Combining equations (1.4) and (1.5) gives the transfer function from the input field voltage to the resulting speed change

$$\boxed{\frac{w(s)}{V_f(s)}=\frac{w(s)}{T_m(s)}\frac{T_m(s)}{V_f(s)}=\frac{\left(K_{mf}/L_fJ\right)}{\left(s+c/J\right)\left(s+R_f/L_f\right)}}$$

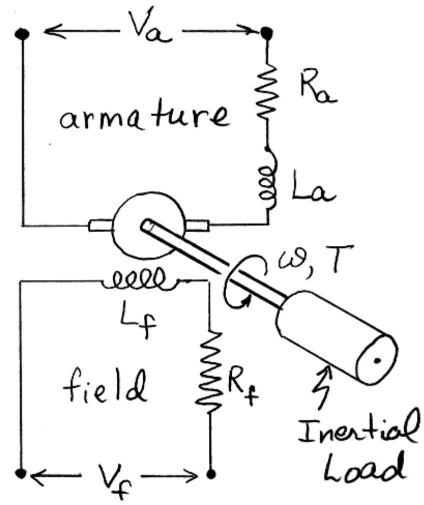
(2nd order system) (1.6)

Finally, since $w = dq/dt$, the transfer function from input field voltage to the resulting rotational position change is

$$\boxed{\frac{q(s)}{V_f(s)}=\frac{q(s)}{w(s)}\frac{w(s)}{V_f(s)}=\frac{\left(K_{mf}/L_fJ\right)}{s\left(s+c/J\right)\left(s+R_f/L_f\right)}}$$

(3rd order system) (1.7)

Armature-Current Controlled:



DC Motor

In a armature-current controlled motor, the field current i_f is held constant, and the armature current is controlled through the armature voltage V_a. In this case, the motor torque increases linearly with the armature current. We write

$$T_m = K_{ma}\, i_a$$

The transfer function from the input armature current to the resulting motor torque is

$$\boxed{\frac{T_m(s)}{I_a(s)} = K_{ma}} \tag{1.8}$$

The voltage/current relationship for the armature side of the motor is

$$V_a = V_R + V_L + V_b \tag{1.9}$$

where V_b represents the "back EMF" induced by the rotation of the armature windings in a magnetic field. The back EMF V_b is proportional to the speed w, i.e. $V_b(s) = K_b w(s)$. Taking Laplace transforms of Equation (1.9) gives

$$\boxed{V_a(s) - V_b(s) = \left(R_a + L_a s\right) I_a(s)} \tag{1.10}$$

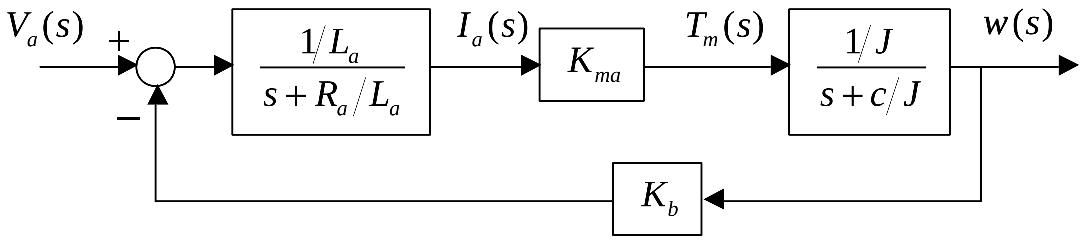
or

$$\boxed{V_a(s) - K_b w(s) = \left(R_a + L_a s\right) I_a(s)} \tag{1.11}$$

As before, the transfer function from the input motor torque to rotational speed changes is

$$\boxed{\frac{w(s)}{T_m(s)} = \frac{\left(1/J\right)}{s + \left(c/J\right)}} \qquad (1^{st} \text{ order system}) \tag{1.12}$$

Equations (1.8), (1.11) and (1.12) together can be represented by the closed loop block diagram shown below.



Block diagram reduction gives the transfer function from the input armature voltage to the resulting speed change.

$$\boxed{\frac{w(s)}{V_a(s)} = \frac{\left(K_{ma}/L_a J\right)}{\left(s+R_a/L_a\right)\left(s+c/J\right)+\left(K_b K_{ma}/L_a J\right)}} \qquad \text{(2}^{\text{nd}}\text{ order system)} \qquad (1.13)$$

The transfer function from the input armature voltage to the resulting angular position change is found by multiplying Equation (1.13) by $1/s$.