ME237 Project
Nonlinear Control of a Cart Pendulum System

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December 16, 2011

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1 Introduction

The inverted pendulum on a moving cart has been an ubiquitous example of both linear and nonlinear systems analysis and design. It was originally used to demonstrate using linear control to stabilize an unstable nonlinear system via Jacobian linearization. In this report, some nonlinear system analyses and control techniques are investigated for this particular system. Namely, we are looking for nonlinear control laws that regulate the pendulum to the upright position and the cart at its desired position.

The report is organized as follows. Three nonlinear controllers are discussed in the order of complexity. It will be seen shortly that with increased complexity, the controller gets better performance. Three control strategies are *1*) input to pendulum angle feedback linearization plus LQR, *2*) input to cart displacement feedback linearization plus energy regulation, and *3*) Energy, cart displacement and velocity regulation.

2 Dynamics

In this section, the dynamic model of the cart pendulum system is derived. Since there are constraint forces existing in the system, it is easier to use Lagrangian mechanics than Newtonian. Figure 1 shows a schematic diagram of the system.

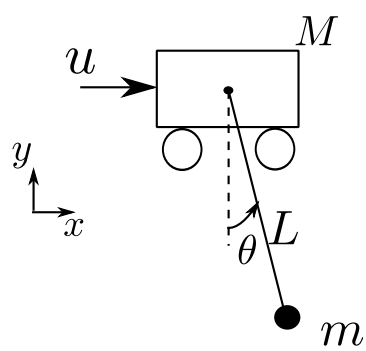

Figure 1: The schematic diagram of a inverted pendulum on a moving cart

To apply Lagranges equations, we determine expressions for the kinetic and the potential energies as the cart and the pendulum undergo translational and rotational

motions respectively. The horizontal displacement of the cart from the predefined zero position is denoted as x and the rotational displacement of the pendulum from the pendent position as θ. The only actuation in the system is the force u exerted on the cart. From the geometry, the kinetic energy can be expressed as,

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}M\dot{x}_{cart}^2 + \frac{1}{2}m(\dot{x}_{pen}^2 + \dot{y}_{pen}^2) \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\cos(\theta)\dot{\theta}\dot{x})\end{aligned} \tag{1}$$

and the potential energy can be expressed as,

$$\begin{aligned}\mathcal{V} &= mgy_{pen} \\ &= -mgL\cos(\theta)\end{aligned} \tag{2}$$

The Lagrangian is thus,

$$\begin{aligned}\mathcal{L} &= \mathcal{T} - \mathcal{V} \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\cos(\theta)\dot{\theta}\dot{x}) + mgL\cos(\theta)\end{aligned} \tag{3}$$

We choose x, θ as two generalized coordinates, the Lagrange's equations become,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = u \tag{4}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0 \tag{5}$$

Through some algebraic manipulations, we obtain two governing *equations of motion* as follows,

$$(M+m)\ddot{x} + mL\cos(\theta)\ddot{\theta} - mL\sin(\theta)\dot{\theta}^2 = u \tag{6}$$

$$L\ddot{\theta} + \cos(\theta)\ddot{x} + g\sin(\theta) = 0 \tag{7}$$

or equivalently the following standard form,

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = f \tag{8}$$

where

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}, \quad M(q) = \begin{bmatrix} M+m & mL\cos(\theta) \\ mL\cos(\theta) & mL^2 \end{bmatrix}, \quad C(q,\dot{q}) = \begin{bmatrix} 0 & -mL\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ mgL\sin(\theta) \end{bmatrix}, \qquad f = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

The system is difficult to control because it is *underactuated*, meaning that we have control only in one channel of the actuation vector f. Otherwise, the system can be easily controlled. For example, one can use the general MIMO sliding mode control proposed by Slotine *et.al.*. Note also that $\dot{M} - 2C$ is skew-symmetric, which will be used in the third controller design.

For the purpose of control, we obtain the state space equations and define four state variables $(x_1, x_2, x_3, x_4)^T = (x, \dot{x}, \theta, \dot{\theta})^T$,

$$\dot{x}_1 = x_2 \tag{9}$$

$$\dot{x}_2 = \frac{1}{m\sin^2(x_3) + M}\left[mL\sin(x_3)x_4^2 + mg\cos(x_3)\sin(x_3) + u\right] \tag{10}$$

$$\dot{x}_3 = x_4 \tag{11}$$

$$\dot{x}_4 = \frac{1}{(-L)(m\sin^2(x_3) + M)}\left[mL\sin(x_3)\cos(x_3)x_4^2 + (M+m)g\sin(x_3) + \cos(x_3)u\right] \tag{12}$$

3 Equilibrium and Stability Analysis

The equilibria of the cart-pendulum system are obtained to be,

$$\begin{cases} x_{1e} = \alpha, \alpha \in \mathbb{R} \\ x_{2e} = 0 \\ x_{3e} = k\pi, k \in \mathbb{Z} \\ x_{4e} = 0 \end{cases} \tag{13}$$

Mathematically, there are infinitely many equilibria in this system. Physically, the equilibria are the upright and the pendent positions of the pendulum with arbitrary cart displacement. Their stabilities are of interest.

The Jacobian linearized model around the equilibrium $[0, 0, \pi, 0]^T$ can be obtained directly from the state space equations (Equation 9 through Equation 12), which yields,

$$\dot{x} = Ax + Bu$$

where

$$A_{upright} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \qquad B_{upright} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} \tag{14}$$

It has two zero eigenvalues and two real eigenvalues with opposite signs. Therefore, this equilibrium in the original nonlinear system is *unstable*.

Similarly, the Jacobian linearization around the equilibrium $[0, 0, 0, 0]^T$ yields the following A, B matrices,

$$A_{pendent} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-(M+m)g}{ML} & 0 \end{bmatrix} \qquad B_{pendent} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix} \tag{15}$$

It also has two zero eigenvalues and a pair of pure complex eigenvalues.

From Lyapunov first theorem, the stability of the pendulum's pendent position can not be concluded from the Jacobian linearization because of the presence of eigenvalues on the imaginary axis. However, for this particular system, the stability can be directly inferred based on the physical intuition and the "δ-ϵ" type of definition of stability.

The pendulum's pendent position is stable in the sense of Lyapunov, because for any given ϵ-ball around that position, we can find some perturbation δ which is small enough such that the cart and the pendulum oscillate within the given ϵ-ball.

4 Input to Pendulum Angle Feedback Linearization

This section shows an input-output feedback linearizion control law where the original nonlinear plant is considered as a single-input-single-output system. The pendulum angular displacement θ is considered to be the output. Our control objective is to regulate the pendulum to the upright position. It should be easy to identify that this particular output has relative degree two. Thus there are internal dynamics existing

in the closed-loop system, which correspond to the cart motion. The control input shows up in the second derivative of the pendulum angle,

$$\dot{y} = \dot{\theta} \tag{16}$$

$$\begin{aligned} \ddot{y} &= \ddot{\theta} \\ &= \frac{1}{(-L)(m\sin^2(\theta)+M)}\Big[mL\sin(\theta)\cos(\theta)\dot{\theta}^2 \\ &+ (M+m)g\sin(\theta) + \cos(\theta)u\Big] \\ &\triangleq v \end{aligned} \tag{17}$$

The following synthetic control input $v = -c_1\dot{\theta} - c_2(\theta - \theta_d)$ makes the feedback linearized system become a second-order stable system. The parameters c_1 and c_2 can be chosen so that the poles are placed at the desired location.

$$\frac{\theta(s)}{\theta_d(s)} = \frac{c_2}{s^2 + c_1 s + c2} \tag{18}$$

The real input u can then be computed from the synthetic input v, provided $\cos(\theta) \neq 0$,

$$\tau = \frac{1}{\cos(\theta)}\Big[(-L)(m\sin^2(\theta)+M)v - mL\sin(\theta)\cos(\theta)\dot{\theta}^2 - (M+m)g\sin(\theta)\Big] \tag{19}$$

One drawback of the feedback linearization is the possibility of unstable internal dynamics. This is usually hard to analyze, instead one usually analyzes the stability of the zero dynamics. In this application, the zero dynamics, associated with the cart motion, can be derived to be,

$$\ddot{x} = 0 \tag{20}$$

Therefore, the cart motion is both unobservable and unstable, meaning that although the pendulum angle can be controlled perfectly with any linear system control technique, the cart displacement has no way to be controlled and will approach infinity eventually.

Simulation:
The plant parameters are chosen as, $M = 3kg, m = 1kg, L = 0.5m$. The poles of the linearized system are placed at $p_{1,2} = -2.2 \pm 3.3j$ so that the system has 10% overshoot and 1.7s settling time, by choosing $c_1 = 4.5, c_2 = 16$. The lower part of

Figure 2 shows exactly the expected response of the pendulum angular displacement. The pendulum is regulated to the upright position and kept there. However, as a consequence of being unobservable in the system, the cart horizontal displacement blows up to infinity, which leads to an unstable system.

Equation 19 indicates that the control input approaches infinity as the pendulum approaches $\theta = \pi/2$. In this simulation, the input saturation is not considered. However, we should expect difficulties in the physical implementation of this control law.

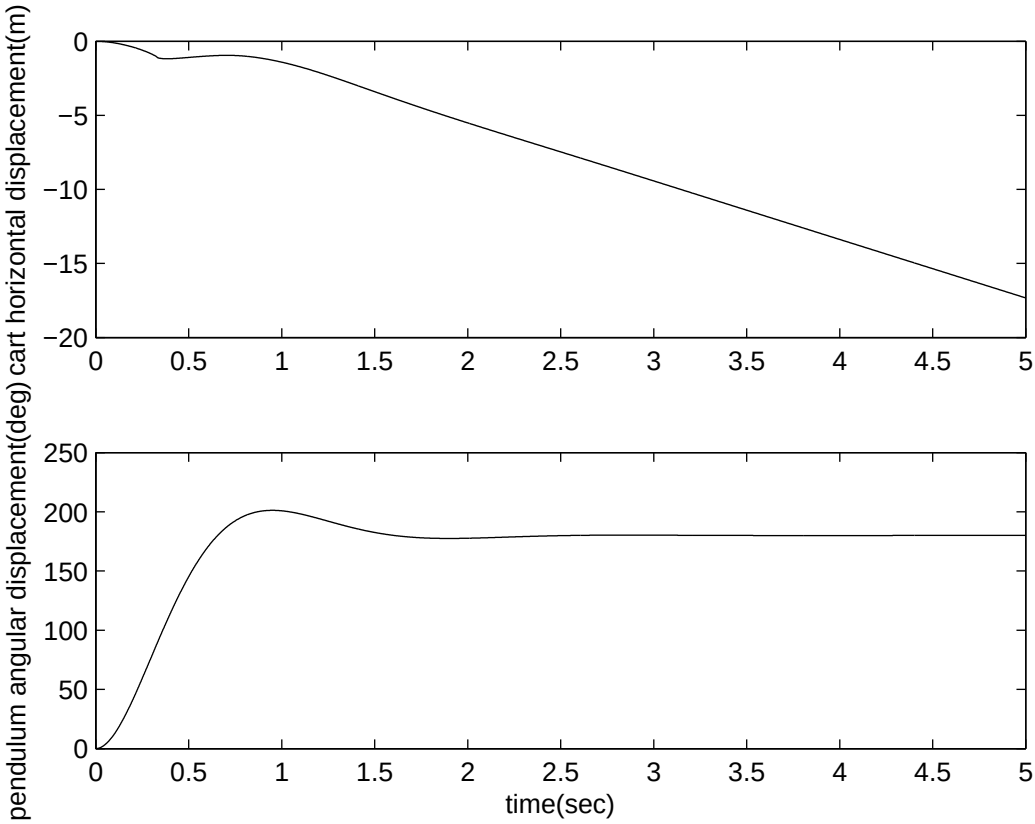

Figure 2: Cart acceleration to pendulum angle feedback linearization control

In this preliminary example, we have shown a simple input-output feedback linearization controller that successfully brings the pendulum to the desired position

but sacrifices the stability of the system. We also notice that the cart displacement is kept small when the pendulum first reaches the upright position. In the following section, we propose a linear quadratic regulator which regulates the system to its unstable equilibrium when the state variables are near the unstable equilibrium.

4.1 Full state feedback control using Jacobian linearization

Although the feedback linearization controller is not able to stabilize the cart position, we observed in the previous simulation (Figure 2) that it swung up the pendulum close to the upright position and kept the cart near where it started. This needs careful tuning of the parameters c_1 and c_2 in the Equation 24. Therefore, a full state feedback controller can be used to stabilize the Jacobian linearized system model.

From the Jacobian linearized model around the equilibrium $[0, 0, \pi, 0]^T$ in Equation 14, the controllability matrix is derived,

$$\mathcal{C} = \begin{bmatrix} \frac{mg}{M^2L} & 0 & \frac{1}{M} & 0 \\ 0 & \frac{mg}{M^2L} & 0 & \frac{1}{M} \\ \frac{(M+m)g}{M^2L^2} & 0 & \frac{1}{ML} & 0 \\ 0 & \frac{(M+m)g}{M^2L^2} & 0 & \frac{1}{ML} \end{bmatrix} \tag{21}$$

The system is controllable, provided $M \neq 0$, which is always true. If a full-state feedback controller is used, the poles of the closed-loop system, i.e. the eigenvalues of $(A - BK)$, can be arbitrarily placed by the proper choice of the feedback gain matrix K. Furthermore, there exists a unique asymptotically stabilizing infinite-horizon LQR because of the Jacobian linearized system is both controllable and observable (the system is observable since we assume we have all the measurements).

Simulation:
The same parameter numerical values are used in this simulation. The infinite-horizon LQR is designed to minimize the following cost function,

$$J = \int \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}^T Q \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + Ru \tag{22}$$

$$Q = \text{diag}(20, 1, 10, 1) \qquad R = 1$$

The feedback linearization control law is switched to the LQR control law when θ enters $[0.9\pi, 1.1\pi]$.

The system response shown in Figure 3 indicates that the combination of two control laws successfully regulates all state variables to the desired positions.

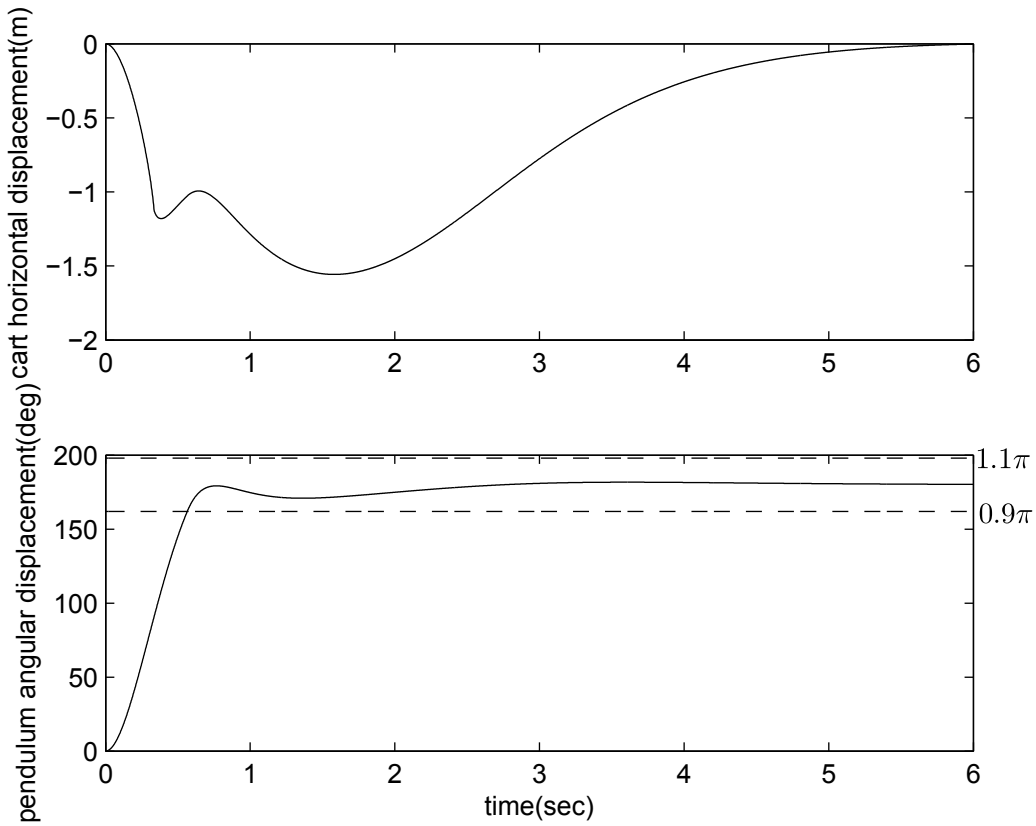

Figure 3: System response using both feedback linearization swing-up and state feedback regulating control

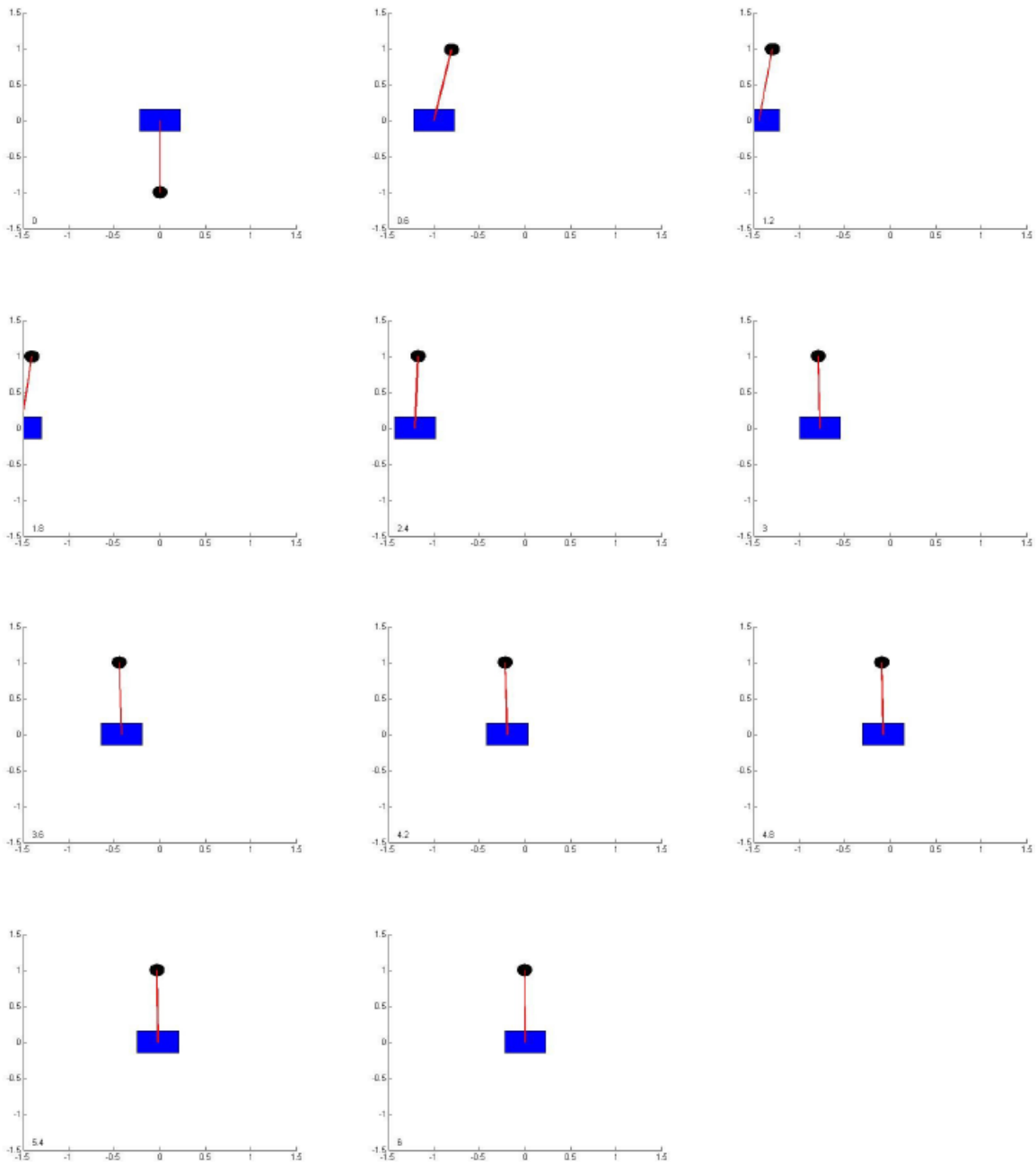

Figure 4: Animation of system response

4.2 Effect of model uncertainties

The greatest drawback of the input-output feedback linearization is that the model uncertainty cannot be taken into account in the controller design. Also in this application, the cart displacement is unobservable, meaning that we don't really control it but just hope for the best. In the last section, we have already shown that with *1*) perfect knowledge of the model, *2*) knowledge of the initial condition, *3*) careful tunning of the controller parameters , we are able to regulate both the cart and the pendulum to the desired positions. However, this approach is not robust to model uncertainties.

Simulation:
This simulation demonstrates the effect of model uncertainties. The actual cart mass is set to $2.9kg$(3.3%error) and the actual pendulum mass is set to $1.1kg$(10%error).

From Figure 5, we see that model errors result in bad response of the cart and the pendulum displacements. Namely, the pendulum is stuck on the "switching line $\theta = 0.9\pi$" and the cart drifts far away from 0. Although both of them are eventually regulated to the desired positions, the stability of the open-loop unstable equilibrium is not guaranteed.

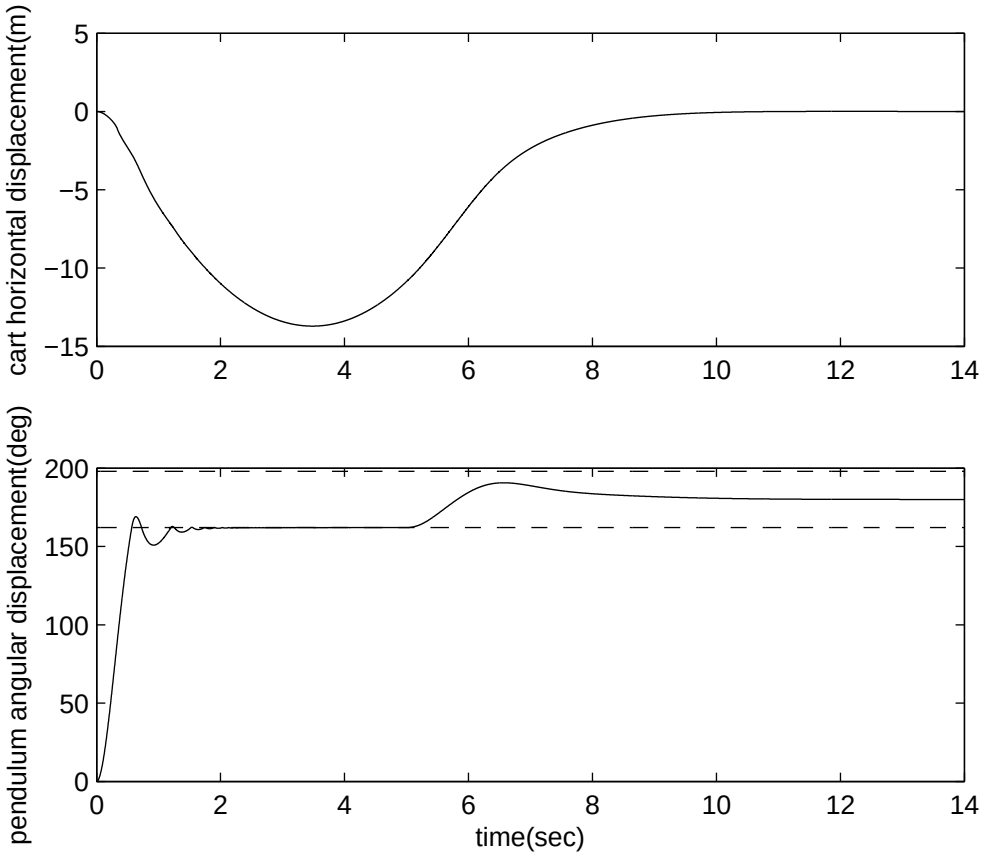

Figure 5: Effect of existence of model uncertainties

5 Input to Cart Displacement Feedback Linearization

It has been seen that the input to pendulum angle feedback linearization makes the cart displacement and velocity unobservable, thus the resulting closed-loop system is not stable. In this section, the input to cart displacement feedback linearization is utilized instead. Although the desired cart displacement profile which results in the desired pendulum angle is still not clear at this stage (meaning that the synthetic control is not easy to be determined), the state space equations involving the syn-

thetic control input are much simpler compared with the ones with real input.

It has been mentioned that the system is underactuated. For this class of physical systems, Spong *et.al.* proposed to partially linearize the system using the output which has actuation [1]. This suggests we have the cart displacement as the output. From Equation 10, the relationship between the real input u and the synthetic input τ is easy to obtain,

$$\ddot{x} \triangleq \tau = \frac{1}{m\sin^2(\theta) + M}\left[mL\sin(\theta)\dot{\theta}^2 + mg\cos(\theta)\sin(\theta) + u\right] \tag{23}$$

$$u = (m\sin^2(\theta) + M)\tau - mL\sin(\theta)\dot{\theta}^2 - mg\cos(\theta)\sin(\theta) \tag{24}$$

The linearized system has also relative degree two. The new state space equations follow easily from here,

$$\dot{x} = v \tag{25}$$

$$\dot{v} = \tau \tag{26}$$

$$\dot{\theta} = \omega \tag{27}$$

$$\dot{\omega} = -\frac{g}{L}\sin\theta - \frac{1}{L}\cos\theta\tau \tag{28}$$

More descriptive notations are used since we will design the controller from physical intuition. v and ω here represent the cart velocity and the pendulum angular velocity respectively. We will see shortly that we can design the synthetic controller to regulate the total energy of the system to zero, i.e. both the pendulum energy and the cart velocity.

5.1 Energy regulation

The normalized total energy of the pendulum is,

$$E_p = -\cos\theta - 1 + \frac{1}{2}\frac{L}{g}\omega^2 \tag{29}$$

Zero pendulum energy is desirable because it corresponds to a homoclinic orbit, which will be discussed in detail in Section 6.1. The rate of change of E_p can be derived to be,

$$\begin{aligned}\dot{E}_p &= \sin\theta\omega + \omega\frac{L}{g}(-\frac{g}{L}\sin\theta - \frac{1}{L}\cos\theta\tau) \\ &= -\frac{1}{L}\cos\theta\omega\tau\end{aligned} \tag{30}$$

Consider a Lyapunov function candidate V,

$$V = \frac{1}{2}E_p^2 + \frac{1}{2}v^2 \tag{31}$$

The Lie derivative gives,

$$\dot{V} = (-\frac{E_p}{L}\cos\theta\omega + v)\tau \tag{32}$$

V is made non-increasing by the following synthetic input,

$$\tau \triangleq \frac{E_p}{L}\cos\theta\omega - v \tag{33}$$

$$\Longrightarrow \dot{V} \leq -(\frac{E_p}{L}\cos\theta\omega - v)^2 \leq 0 \tag{34}$$

The proposed controller is not able to steer the states to the desired trajectory if the pendulum starts at the stable equilibrium position. In order to avoid that, we exclude that particular equilibrium in the domain of attraction $\Omega = \{v, \theta, \omega | V < V(\text{stable eq}) = 1\}$. To apply LaSalle's invariant principle, we need to compute the largest invariant set M in $P = \{v, \theta, \omega \in \Omega | \dot{V} = 0\}$.

Theorem 1. *The largest invariant set M in P is zero pendulum energy, zero cart velocity (i.e. zero total energy of the entire system). Therefore, any solution starting in the domain of attraction converges to M as $t \to \infty$. The cart displacement is not regulated, but is guaranteed to be bounded.*

Proof. For every element in P

$$\dot{V} = \tau = \frac{E_p}{L}\cos\theta\omega - v = 0 \tag{35}$$

which gives,

$$\tau = 0 \Rightarrow \begin{cases} \dot{v} = 0 \\ \dot{\omega} = -\frac{g}{L}\sin\theta \end{cases} \tag{36}$$

Also,

$$V = \frac{1}{2}E_p^2 + \frac{1}{2}v^2 = \frac{1}{2}(1 + \frac{1}{L^2}\cos^2\theta\omega^2)E_p^2 = \text{constant} \tag{37}$$

$E_p = v = 0$ obviously satisfies all the conditions. We are going to show E_p cannot be nonzero by contradiction.

Since both V and v are constants, E_p must also be a constant. For a nonzero constant E_p, we have $\cos\theta\omega =$ constant. Its derivative gives,

$$\sin\theta(\omega^2 + \frac{g}{L}\cos\theta) = 0 \tag{38}$$

$\sin\theta = 0$ corresponds to cases where the pendulum is either in its stable or unstable equilibrium. The former case is excluded by predefined domain of attraction Ω. The latter case is excluded by assuming nonzero E_p. Hence,

$$\omega^2 + \frac{g}{L}\cos\theta = 0 \tag{39}$$

Combining this and the pendulum energy expression in Equation 29 gives constant ω, which again leads to $\sin\theta = 0$.

Therefore, the largest invariant set M in P is $M = \{v, \theta, \omega \in \Omega | E_p = 0, v = 0\}$. □

The following theorem concludes the derivation in this section.

Theorem 2. *For the cart pendulum system, where the pendulum is not initially resting at the pendent position, the following control law (combining Equations 24 and 33)*

$$u = (m\sin^2(\theta) + M)(\frac{E_p}{L}\cos\theta\omega - v) - mL\sin(\theta)\dot{\theta}^2 - mg\cos(\theta)\sin(\theta) \tag{40}$$

regulates the system to its zero energy motion, in which the pendulum periodically comes arbitrarily close to the upright position and the cart has zero velocity. A linear controller can be used to regulate the system to an unstable equilibrium, as shown in Section 4.1 . The cart displacement is guaranteed to be bounded.

Simulation:
The same plant parameters are used as before. The initial pendulum displacement is set to be $0.1rad$ and the other are all zeros. Thus the system is initially in the domain of attraction. It can be seen from Figure 6 that the pendulum's motion converges to the homoclinic orbit (zero energy motion), and the cart velocity converges to zero so that the displacement is bounded. A linear controller can be used when the pendulum is at those "plateaus" in the lower figure.

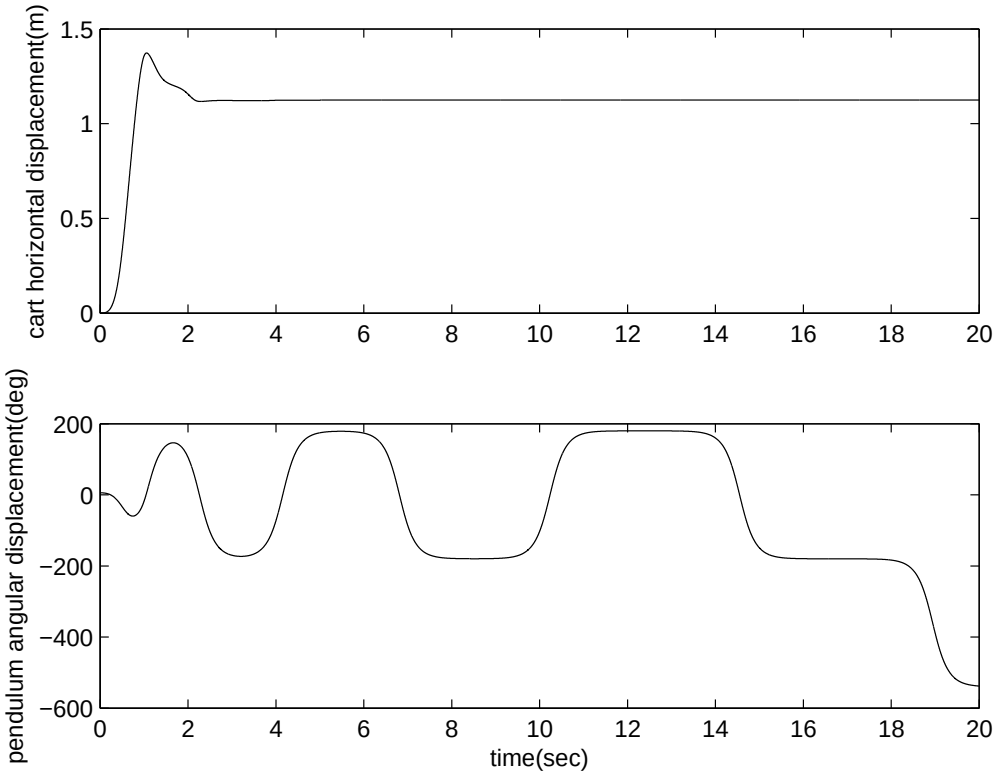

Figure 6: Input to cart displacement feedback linearization plus energy regulation control

6 Energy and Cart Displacement Regulation

This section presents a nonlinear control law that regulates both the cart displacement and the total energy of the system. Compared with two methods introduced in the previous sections where the cart displacement is not regulated during the pendulum swing-up, this controller guarantees that all state variables converge to the region near the unstable equilibrium so that we can then switch to a linear controller in order to regulate the linearized system.

This method is not originally my work. It was proposed by Fantoni *et.al.*[2]. The controller design is based on Lyapunov stability theorem. The original local asymp-

totic stability proof is done using LaSalle's invariant principle, and is simplified in this report. Therefore, it serves as a very good example demonstrating the power of stability theorems in nonlinear controller designs.

6.1 Energy regulation and swing-up of the pendulum

The total mechanical energy E of the system is obtained to be,

$$E = \frac{1}{2}\dot{q}^T M(q)\dot{q} + mgL(-\cos(\theta) - 1) \tag{41}$$

where q and M have already been defined in Equation 8. Two terms correspond to the kinetic and potential energies respectively. The zero potential energy level is chosen such that the unstable equilibrium has exactly zero energy.

We are going to regulate the energy of the system to zero because it corresponds to a homoclinic orbit if $\dot{x}$ is kept zero at the same time,

$$\frac{1}{2}mL^2\dot{\theta}^2 = mgL(\cos(\theta) + 1) \tag{42}$$

As the energy converges to zero, the pendulum converges to the swinging motion in the interval $(-\pi, \pi)$. The system will periodically come arbitrarily near the unstable equilibrium. A linear controller can then be used near the equilibrium. This motivates us to look for a nonlinear control law that regulates the energy, the cart displacement and the velocity all to zero.

The rate of change of the total energy is just the product of the cart's velocity and the external force, which will simplify the controller design.

$$\begin{aligned}
\dot{E} &= \dot{q}^T(M(q)\ddot{q} + \frac{1}{2}\dot{M}(q)\dot{q} + G(q)) \\
&= \dot{q}^T(-C(q,\dot{q})\dot{q} - G(q) + f + \frac{1}{2}\dot{M}(q)\dot{q} + G(q)) \\
&= \dot{q}^T f \\
&= \dot{x}u
\end{aligned} \tag{43}$$

In the derivation above, we use the skew-symmetric property of $\frac{1}{2}\dot{M} + C$.

6.2 Controller design using Lyapunov stability theorem

We will utilize Lyapunov stability theorem to obtain the control law. Considering the three aforementioned objectives, an intuitive Lyapunov function candidate is,

$$V(q,\dot{q}) = \frac{k_E}{2}E^2 + \frac{k_x}{2}x^2 + \frac{k_v}{2}\dot{x}^2 \tag{44}$$

It is certainly a *positive semi-definite* function, provided all the constant coefficients k_E, k_x, k_v are positive. The positive definiteness is not necessary because we are going to use LaSalle's invariant principle. The Lie derivative along system trajectories yields,

$$\begin{aligned} \dot{V} &= k_E E\dot{E} + k_x x\dot{x} + k_v \dot{x}\ddot{x} \\ &= \dot{x}(k_E E u + k_x x + k_v \ddot{x}) \end{aligned} \tag{45}$$

The following control input will make $\dot{V}$ *negative semi-definite*,

$$k_E E u + k_x x + k_v \ddot{x} \triangleq -k\dot{x}, \quad k > 0 \tag{46}$$

Plugging in the expression of $\ddot{x}$ in Equation 12,

$$(k_E E + \frac{k_v}{m\sin^2(\theta) + M})u + \frac{k_v(mL\sin^2(\theta)\dot{\theta} + mg\cos(\theta)\sin(\theta))}{m\sin^2(\theta) + M} + k_x x = -k\dot{x} \tag{47}$$

Controller parameters k_E and k_v should be selected properly so that the coefficient in front of the control input is non-zero. Note that E has a minimum value $-2mgL$ corresponding to the pendulum resting at its stable equilibrium. $\sin^2(\theta)$ is bounded above by 1. Therefore, the following condition should be satisfied,

$$\frac{k_v}{k_E} > 2mgL(m + M) \tag{48}$$

The control input u can be rewritten explicitly as,

$$u \triangleq -\frac{k_v(mL\sin^2(\theta)\dot{\theta} + mg\cos(\theta)\sin(\theta)) + (m\sin^2(\theta) + M)(k_x x + k\dot{x})}{(m\sin^2(\theta) + M)k_E E + k_v} \tag{49}$$

So far, we have established a *continuously differentiable and non-increasing* function V. In the next section, the local asymptotic stability of zero energy, zero cart displacement and cart velocity will be shown using LaSalle's invariance principle.

6.3 Asymptotic stability proof using LaSalle's invariance principle

The proposed controller is not able to steer the states to the desired trajectory if the pendulum starts at the stable equilibrium position. In order to avoid that, we exclude that particular equilibrium in the domain of attraction $\Omega = \{x, \dot{x}, \theta, \dot{\theta} | V < V(\text{stable eq}) = 2k_E m^2 g^2 L^2\}$. Ω is a compact set with the property that every system trajectory that starts in Ω remains in it for all future time. To apply LaSalle's theorem, we need to compute the largest invariant set M in $P = \{x, \dot{x}, \theta, \dot{\theta} \in \Omega | \dot{V} = 0\}$.

Theorem 3. *The largest invariant set M in P is zero total energy, zero cart displacement and velocity. Therefore, any solution starting in the domain of attraction converges to M as $t \to \infty$*

Proof. From $\dot{V} = -k\dot{x}^2 = 0$, we have $V =$ constant, $x =$ constant. The total energy of the system is also constant by its definition. Equation 46 gives that Eu is a constant. $E = 0$ is obviously a particular solution, which consequently leads to $x = 0$ by Equation 46.

We will show by contradiction that nonzero energy is not possible. Suppose $E \neq 0$, the system dynamic Equations 6 and 7 with $\ddot{x} = 0$ yields,

$$\sin\theta(g\cos\theta + L\dot{\theta}^2) = C_1 \tag{50}$$

(C with a subscript represents a constant number in the following analysis)
The energy expression gives,

$$L\dot{\theta}^2 = C_2 + 2g(\cos\theta + 1) \tag{51}$$

Combining these two,

$$\sin\theta(3g\cos\theta + C_3) = C_4 \tag{52}$$

Taking the derivative of this equation, we will obtain,

$$\dot{\theta}(3g(\cos^2\theta - \sin^2\theta) + C_3\cos\theta) = 0 \tag{53}$$

$$\implies 3g(\cos^2\theta - \sin^2\theta) + C_3\cos\theta = 0 \tag{54}$$

$$\text{or} \qquad \dot{\theta} = 0 \tag{55}$$

With some algebraic manipulation, we can show that either case leads to $\dot{\theta} = 0$. Thus $\ddot{\theta} = 0$. From Equation 7, $\sin\theta = 0$, which corresponds to the pendulum

pendent position. However, it has been excluded from the definition of the domain of attraction Ω. We reach a contradiction! $\square$

The following theorem concludes the derivation shown in this section.

Theorem 4. *For the cart pendulum system, where the pendulum is not initially resting at the pendent position, the controller law in Equation 49 with positive k_E, k_x, k_x, k satisfying Equation 48 is able to asymptotically regulates the system to zero energy, zero cart displacement and velocity. All the states are guaranteed to periodically come arbitrarily close to the unstable equilibrium. A linear controller can then be used to further stabilize the system to the open-loop unstable equilibrium.*

Simulation:
The same plant parameters are used as before. The initial pendulum displacement is set to $0.1rad$ so that the system is initially in the domain of attraction. The initial cart displacement is set to $0.5m$. The controller parameters are set as follows,

$$k_E = 1, \quad k_v = 50, \quad k_x = 100, \quad k = 300 \tag{56}$$

They are tuned in order to get fast convergence rate. However, the analytic relationship between them is undetermined. It can be seen from Figure 7 that the pendulum's motion converges to the homoclinic orbit (zero energy motion), and the cart displacement converges to zero. A linear controller can be switched to after around 40 seconds.

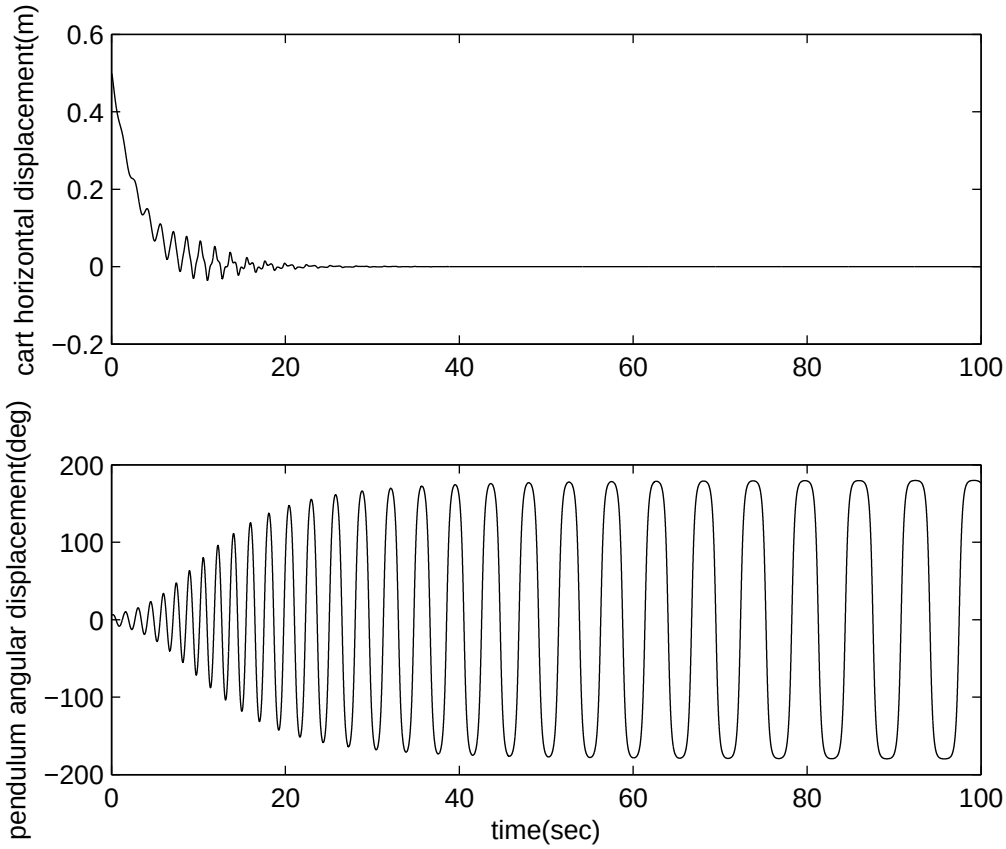

Figure 7: The energy and cart displacement regulation

7 Summary

Three control laws are presented in the report. The controller design involves input-output feedback linearization, Lyapunov 2nd theorem and LaSalle's invariant principle. With increased complexity of the control law, the performance of the closed-loop system gets better.

References

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