Throwing Light on Black Holes

S.R. Manikandasriram — Sri Sankara Sen. Sec. School
Tanmay Shankar — Bala Vidya Mandir

Our Guru: Prof. Prasanta Kumar Tripathy

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Abstract

- In this project, we study Black Holes and their properties, by interpreting the solutions of the equations that govern them.

- What are Black Holes?
- Why are they attractive?
- How does one study them?

Coordinate Transformations

- Under coordinate transformations the line element remains unchanged.
- $F = \dfrac{d\vec{p}}{dt}$ (Valid for v<<c)
- This is invariant under Galilean transformations.
- When v is comparable to c, this breaks down, and Special relativity steps in.
- $F = \dfrac{d\vec{p}}{dt}$ where $\vec{p} = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$
- This is invariant under Lorentz transformation.

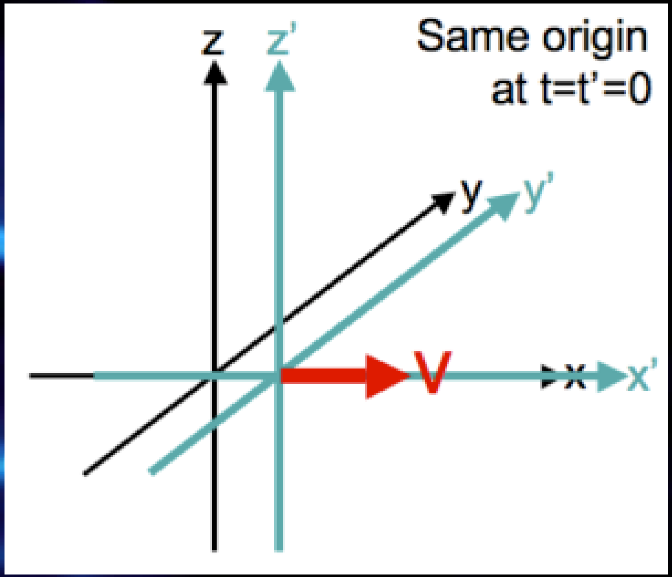
Lorentz Transformation

- The Line Element is:

$$s^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 .$$

- Standard Linear Transformation:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} .$$



- General Linear Transformations can also be performed.

Matrix Representation

- The necessary condition for a Lorentz Transformation is: $\eta_{\alpha\beta} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu}$
- It has a 6 parameter family of solutions.
- η → Flat Metric. Λ → Transformation Matrix

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Special Relativity

- The Principle of Relativity.
- The Principle of Invariant Light Speed.

General Relativity

- Incorporating gravity into Special Relativity, we get General Relativity.
- Equivalence Principle.
- Einstein’s Field Equations.

Einstein's Field Equations

- Requires:
  - Second order derivatives of the dynamic variable.
  - Invariant under general coordinate transformation.
- A Tensor of rank-2 is the perfect object that satisfies these conditions.
- This is an object that transforms as: $$T'_{ij} = \frac{\partial X_k}{\partial X'_i} \frac{\partial X_l}{\partial X'_j} T_{kl}$$
- It is invariant under linear coordinate transformations.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Here,
- $G_{\mu\nu}$ is Einstein's tensor.
- $g_{\mu\nu}$ is space-time metric.
- $T_{\mu\nu}$ is the Energy momentum tensor.
- Λ is the cosmological constant.

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

- R is the Ricci Scalar.
- R_{ab} is the Ricci Tensor.
- This is another expression for EFE.

Schwarzschild matrix

- This is the first non trivial solution of Einstein's field equation, and represents the space-time of a black hole.
- The solution for the line element is:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

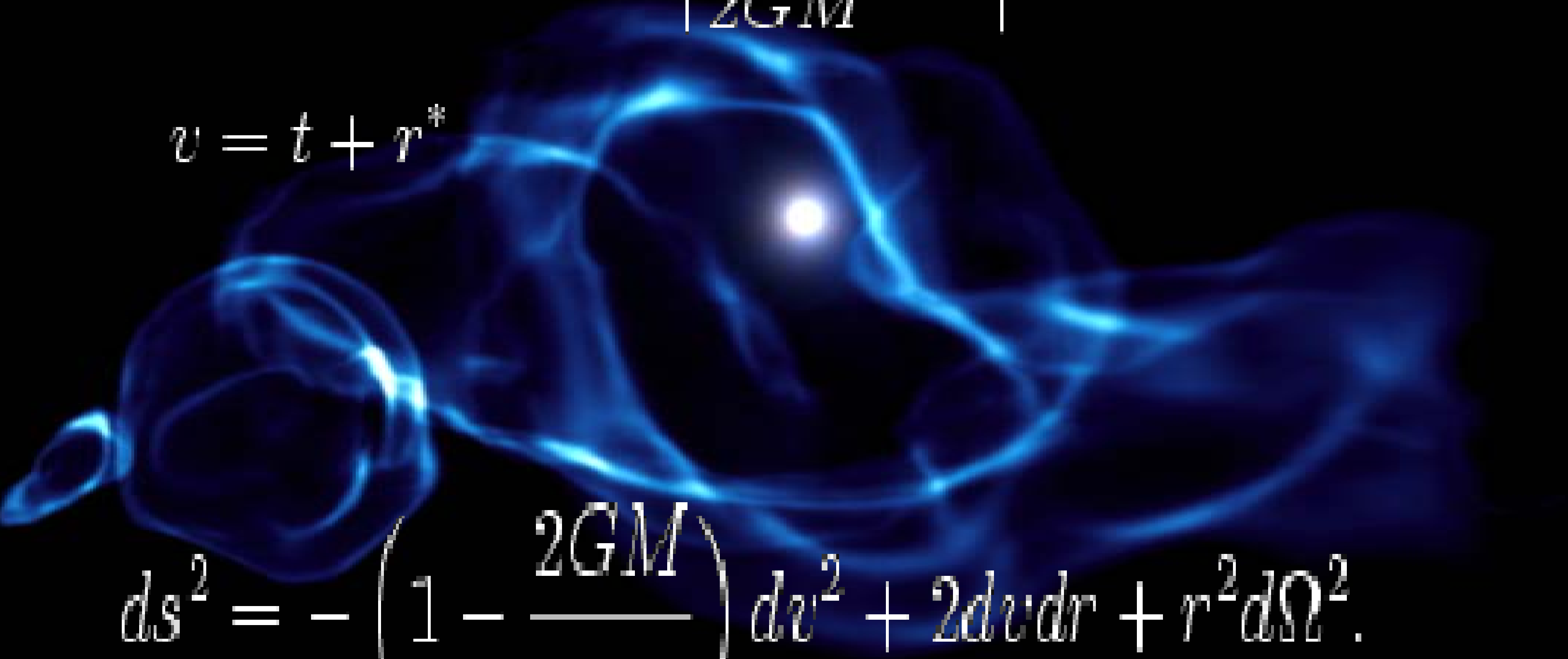
- Suggests apparent singularity, at r=2GM , but this can be resolved with proper coordinate system.

- A suitable set of co-ordinates, such as the Eddington-Finkelstein coordinates, the anomaly is removed, since the line element is well defined for r= 2GM.

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dV^2 + dU^2) + r^2 d\Omega^2,$$

$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|.$$

$$v = t + r^*$$



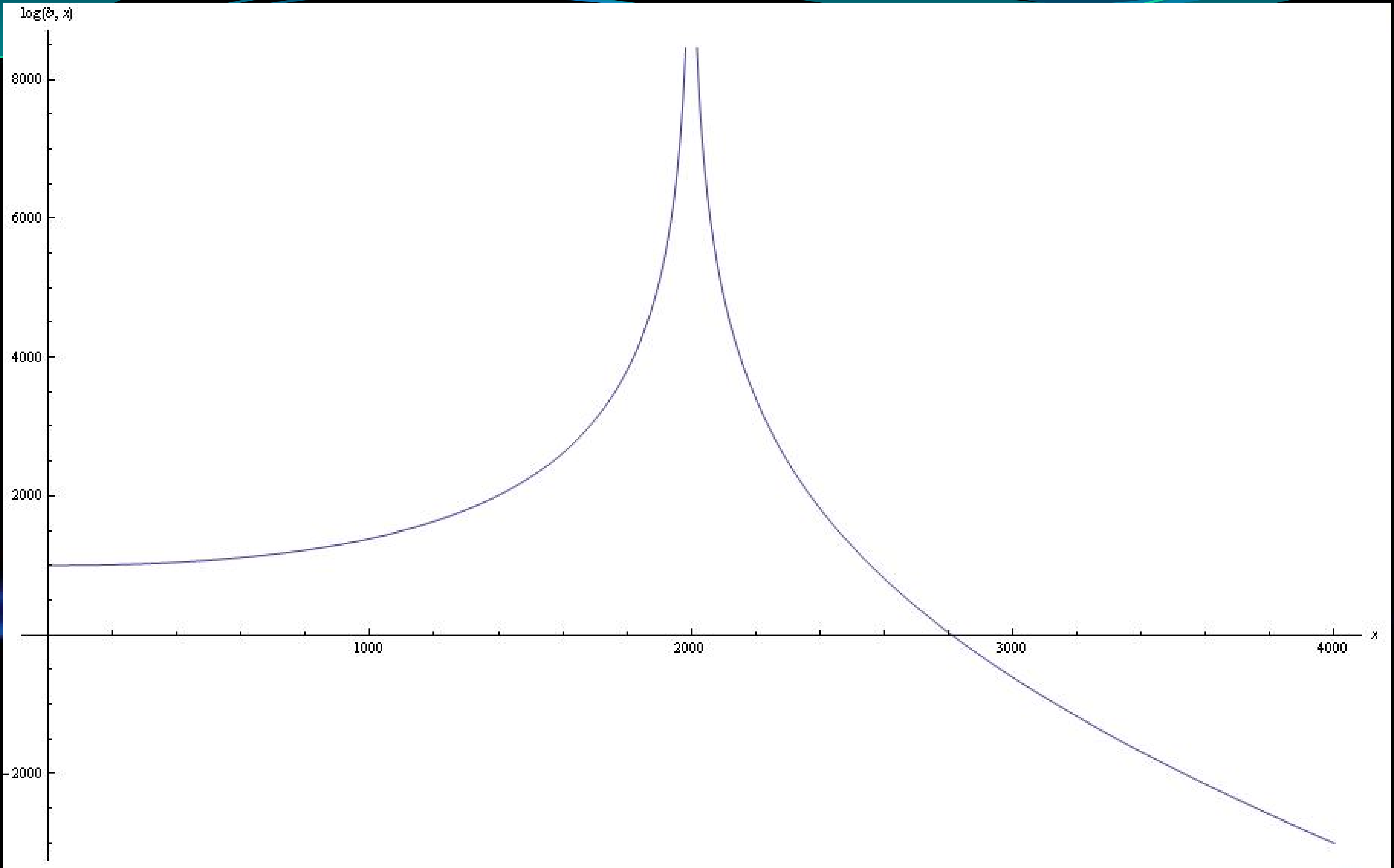
$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2.$$

•Light travels such that the line element ds = 0.
•Considering radial light rays, dΩ = 0.

$$-\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr = 0$$

•Trivially dv=0. Thus v is constant. Substituting in the Eddington-Finkelstein coordinates, we get:

$$v = t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right|.$$



T versus r graph.

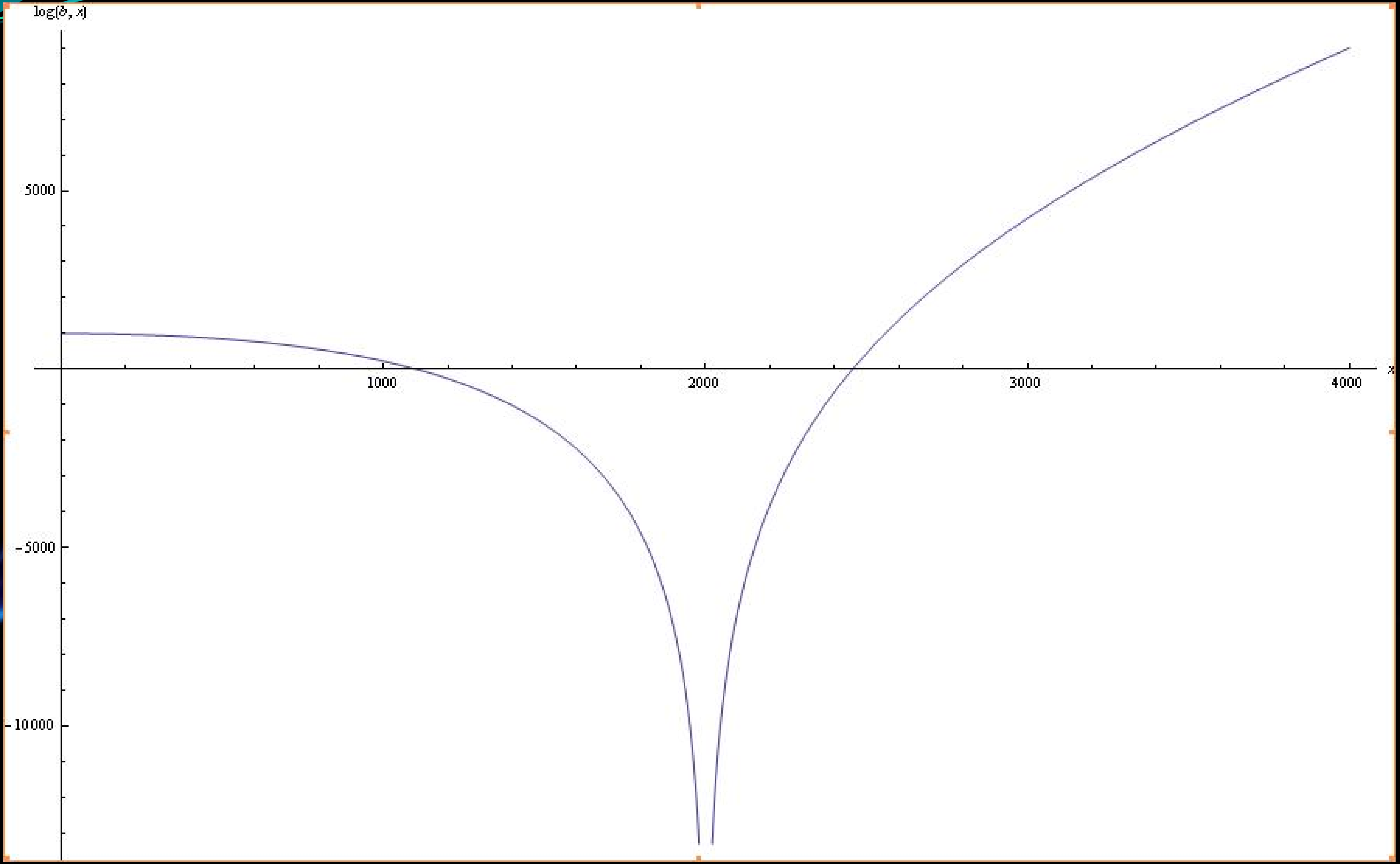
- The other solution is:

$$-\left(1-\frac{2M}{r}\right)dv+2dr=0$$

- Integrating, we get:

$$v-2\left(r+2M\log\left|\frac{r}{2M}-1\right|\right)=k$$

- Thus for r>2M, light rays are outgoing.
- For r<2M, light rays are ingoing.



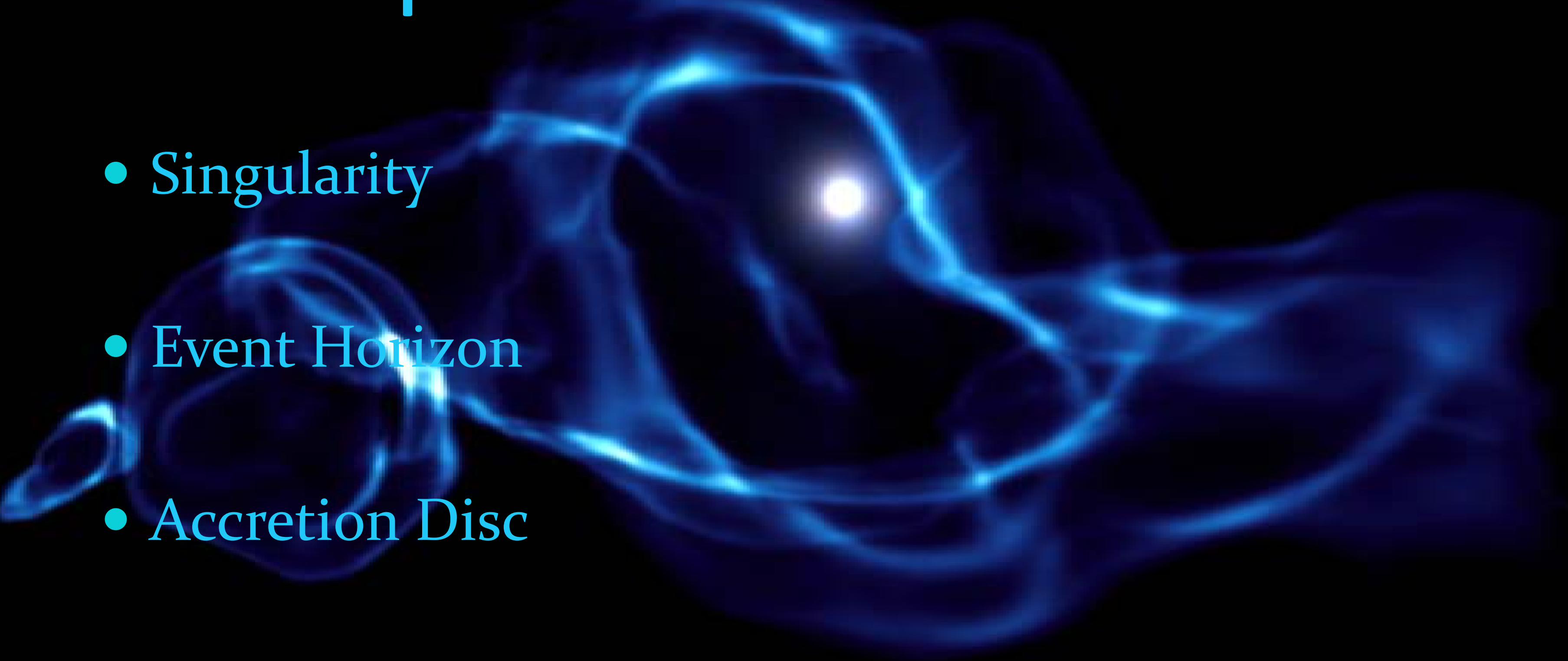
V versus r graph.

Properties of a Black Hole

- Nothing, not even light, can come out of a Black Hole.
- Mass.
- Charge.
- Angular momentum.

(Not applicable for simple black holes)

Components of a Black hole



- Singularity
- Event Horizon
- Accretion Disc
- Photon Sphere

References

- We used various sources of information, including:
  - Prof. Prasanta Kumar Tripathy.
  - Wikipedia
  - H. Jeffery's cartesian tensors
  - The Schwarzschild Black Hole
  - Black hole and entropy, by Atish Dhabolkar
  - Elementary Vector analysis (Prof. V. Balakrishnan)

THANK YOU!