

Real Analysis Qual, Spring 2025

Problem 1. Let f be a continuous complex valued function on $[0, 1]$. Show that

$$\|f\|_\infty = \sup\{|\lambda| : \lambda \in \mathbb{C}, m(f^{-1}(B_\epsilon(\lambda))) > 0, \text{ if } \epsilon > 0\}.$$

Proof. Let $A = \{|\lambda| : \lambda \in \mathbb{C}, m(f^{-1}(B_\epsilon(\lambda))) > 0, \text{ if } \epsilon > 0\}$. We first prove that if $a \in A$, then $a \leq \|f\|_\infty$. So, take $a \in A$. Then, $a = |\lambda|$ such that for all $\epsilon > 0$, we have $m(f^{-1}(B_\epsilon(\lambda))) > 0$. Recall that

$$\|f\|_\infty = \sup\{C \in \mathbb{R} : m(\{x \in \mathbb{R} : |f(x)| > C\}) > 0\}.$$

Observe that

$$f^{-1}(B_\epsilon(\lambda)) = \{x \in \mathbb{R} : f(x) \in B_\epsilon(\lambda)\} = \{x \in \mathbb{R} : |f(x) - \lambda| < \epsilon\}.$$

Given that $|\lambda| - |f(x)| \leq ||\lambda| - |f(x)|| \leq |f(x) - \lambda| < \epsilon$, we have $|f(x)| > |\lambda| - \epsilon$. So,

$$f^{-1}(B_\epsilon(\lambda)) = \{x \in \mathbb{R} : |f(x) - \lambda| < \epsilon\} \subseteq \{x \in \mathbb{R} : |f(x)| > |\lambda| - \epsilon\}.$$

Since $f^{-1}(B_\epsilon(\lambda))$ has positive measure for all ϵ , then $\{x \in \mathbb{R} : |f(x)| > |\lambda| - \epsilon\}$ has positive measure for all ϵ . Therefore $a = |\lambda| \leq \|f\|_\infty$.

Set $B = \{C \in \mathbb{R} : m(\{x \in \mathbb{R} : |f(x)| > C\}) > 0\}$. Observe that $|f|$ is a continuous function on a compact set, and hence it has a maximum value $|f(x_0)| = M$. We must have $\sup B \leq M$, for if $C > M$, then there is no x such that $|f(x)| > C$. On the other hand, we have $M \leq \sup B$, since $|f|^{-1}((M - \epsilon, \infty))$ is an open set containing x_0 . Hence, it is a nonempty open set and thus must have positive measure. In particular, the set of elements x satisfying $|f(x)| > M - \epsilon$ is positive for all ϵ , therefore $M \leq \sup B$, forcing equality. Set $y = f(x_0)$, and note that $f^{-1}(B_\epsilon(y))$ is an open set containing x_0 . So, it is a nonempty open set and thus has positive measure. Therefore, for all ϵ , $m(f^{-1}(B_\epsilon(y))) > 0$. It follows that $M = |y| \leq \sup B$. Therefore, $\|f\|_\infty = \sup B$. \square

Problem 2. Show that Lebesgue measurable functions f_n defined on $[a, b]$ converge in measure to f if and only if

$$\lim_{n \rightarrow \infty} \int_a^b \frac{|f_n - f|}{1 + |f_n - f|} dx = 0.$$

Proof. First, observe that

$$\frac{d}{dx} \frac{x}{1+x} = \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2} > 0.$$

So, $\frac{x}{1+x}$ increases monotonically. Now suppose that

$$\lim_{n \rightarrow \infty} \int_a^b \frac{|f_n - f|}{1 + |f_n - f|} dx = 0.$$

Say that f_n does not converge to f in measure. Then, for some ϵ , there is some δ and some subsequence f_{n_k} such that $m(\{x \in \mathbb{R} : |f(x) - f_{n_k}(x)| > \delta\}) > \epsilon$. Set $A_k = \{x \in \mathbb{R} :$

$|f(x) - f_{n_k}| > \delta\}$. Then, since $\frac{x}{1+x}$ is monotonically increasing and $\delta < |f_{n_k} - f|$ over A_k , we have

$$\int_a^b \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} dx > \int_{A_k} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} dx > \int_{A_k} \frac{\delta}{1 + \delta} dx > \epsilon \frac{\delta}{1 + \delta},$$

contradicting the original convergence.

Now, suppose that $f_n \rightarrow f$ in measure. Define $E = \{x \in [a, b] : |f_n(x) - f(x)| \leq \epsilon\}$. Then,

$$\int_a^b \frac{|f_n - f|}{1 + |f_n - f|} dx = \int_E \frac{|f_n - f|}{1 + |f_n - f|} dx + \int_{E^c} \frac{|f_n - f|}{1 + |f_n - f|} dx \leq \int_E \epsilon dx + \int_{E^c} 1 dx.$$

Moreover,

$$\int_E \epsilon dx + \int_{E^c} 1 dx \leq \int_a^b \epsilon dx + m(E^c) = (b - a)\epsilon + m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}).$$

By convergence in measure, $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}) \rightarrow 0$ as $n \rightarrow \infty$. Choose N such that for all $n \geq N$ we have $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$. Then, we obtain for $n \geq N$ that

$$\int_a^b \frac{|f_n - f|}{1 + |f_n - f|} dx < (b - a + 1)\epsilon.$$

taking $\epsilon \rightarrow 0$ and $n \rightarrow \infty$ gives the result. □

Problem 3. Show that if f is Riemann integrable on $[a, b]$, and $f(x) = 0$ for $x \in \mathbb{Q} \cap [a, b]$ then $\int_a^b f(x) dx = 0$.

Proof. Since f is Riemann integrable, then $\int_a^b f dx = L(f, [a, b]) = U(f, [a, b])$, the lower and upper Riemann integrals respectively. Take a partition $\mathcal{P} = \{x_0 < \dots < x_n\}$ of $[a, b]$. Then, since $[x_{j-1}, x_j] \cap \mathbb{Q} \neq \emptyset$, we have $\inf_{[x_{j-1}, x_j]} f(x) \leq 0$. Hence,

$$L(f, \mathcal{P}, [a, b]) = \sum_{j=1}^n \inf_{[x_{j-1}, x_j]} f(x) (x_j - x_{j-1}) \leq \sum_{j=1}^n 0 = 0.$$

Hence, $L(f, [a, b]) = \sup_{\mathcal{P}} L(f, \mathcal{P}, [a, b]) \leq 0$. Observe that $\sup_{[x_{j-1}, x_j]} f(x) \geq 0$, since $\mathbb{Q} \cap [x_{j-1}, x_j] \neq \emptyset$. Therefore,

$$U(f, \mathcal{P}, [a, b]) = \sum_{j=1}^n \sup_{[x_{j-1}, x_j]} f(x) (x_j - x_{j-1}) \geq \sum_{j=1}^n 0 = 0.$$

Since $U(f, [a, b]) = \inf_{\mathcal{P}} U(f, \mathcal{P}, [a, b])$, then $U(f, [a, b]) \geq 0$.

Therefore,

$$0 \leq U(f, [a, b]) = \int_a^b f dx = L(f, [a, b]) \leq 0.$$

So, $\int_a^b f dx = 0$. □

Problem 4. Let $f_n \in L^2([0, 1])$ be a sequence that is bounded in norm $\|\cdot\|_1$ and $f_n \rightarrow 0$ a.e.. Show that $\lim_{n \rightarrow \infty} \|f_n\|_1 = 0$.

Hint: Use Egoroff and Cauchy-Schwarz.

Proof. By Egoroff's, choose $E \subseteq [0, 1]$ such that $m(E) < \epsilon^2$ and $f_n \rightarrow 0$ uniformly on E^c . Take N so that for all $n \geq N$, $|f_n| < \epsilon$. Then,

$$\int_0^1 |f_n| dx = \int_E |f_n| dx + \int_{E^c} |f_n| dx \leq \int \mathbb{1}_E |f_n| dx + \int_{E^c} \epsilon dx \leq \|\mathbb{1}_E\|_2 \|f_n\|_2 + \int_0^1 \epsilon dx.$$

Now, $\|\mathbb{1}_E\|_2 = \sqrt{m(E)} < \epsilon$. Furthermore, $\|f_n\|_2$ is globally bounded by some constant, M . Therefore,

$$\|\mathbb{1}_E\|_2 \|f_n\|_2 + \int_0^1 \epsilon dx \leq \epsilon M + \epsilon = \epsilon(M + 1).$$

Taking $\epsilon \rightarrow 0$ gives the result. \square

Problem 5. Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of continuous functions g_n with compact supports such that $\lim_{n \rightarrow \infty} \|f - g_n\|_1 = 0$.

The correct technique for this problem is to approximate interval indicator functions by trapezoids. Then, approximate general functions by linear sums of interval indicator functions. I am doing it with convolutions to improve my understanding of convolutions.

Proof. We first do this for integrable simple functions. Let J be some interval $[-\alpha, \alpha]$. Let E be a bounded measurable set. Set $g = \mathbb{1}_J/m(J)$. We claim that $\mathbb{1}_E * g$ is a compactly supported continuous function. First, we prove it has compact support. Since J, E are both bounded, then they are contained in some set $[-N, N]$. Take $x \notin [-2N, 2N]$. Then, $|x| > 2N$. For $y \in J$, $|x - y| > |x| - |y| \geq 2N - |y| \geq N$, given that $|y| \leq N$. Therefore, if $y \in J$, then $(x - y) \notin E$. So, $\mathbb{1}_E(x - y)g(y) = 0$. Hence,

$$\mathbb{1}_E * g(x) = \int \mathbb{1}_E(x - y)g(y) dy = \int 0 dy = 0.$$

Hence $\mathbb{1}_E * g$ is supported on the compact domain $[-2N, 2N]$. We claim that for any ϵ we may pick J so that $\|\mathbb{1}_E * g - \mathbb{1}_E\|_1 < \epsilon$. Now, observe that $\int g(y) dy = 1$. So,

$$\begin{aligned} \|\mathbb{1}_E * g - \mathbb{1}_E\|_1 &= \int \left| \int \mathbb{1}_E(x - y)g(y) dy - \mathbb{1}_E(x) \right| dx \\ &= \int \left| \int \mathbb{1}_E(x - y)g(y) dy - \int g(y) \mathbb{1}_E(x) dy \right| dx \\ &= \int \left| \int \mathbb{1}_E(x - y)g(y) - g(y) \mathbb{1}_E(x) dy \right| dx \\ &\leq \int \int |(\mathbb{1}_E(x - y) - \mathbb{1}_E(x))g(y)| dy dx. \end{aligned}$$

Applying Tonelli's Theorem, we have

$$\begin{aligned} \int \int |(\mathbb{1}_E(x-y) - \mathbb{1}_E(x))g(y)| dy dx &= \int \int |(\mathbb{1}_E(x-y) - \mathbb{1}_E(x))g(y)| dx dy \\ &= \int \|\mathbb{1}_E(\cdot - y) - \mathbb{1}_E\|_1 g(y) dy \\ &= \int_J \frac{\|\mathbb{1}_E(\cdot - y) - \mathbb{1}_E\|_1}{m(E)} dy. \end{aligned}$$

The L^1 norm is continuous under translation. Taking $y \rightarrow 0$, we have $\|\mathbb{1}_E(\cdot - y) - \mathbb{1}_E\|_1 \rightarrow 0$. Since $J = [-\alpha, \alpha]$ we choose α small enough that, given $y \in J$, then $\|\mathbb{1}_E(\cdot - y) - \mathbb{1}_E\|_1 < \epsilon$. Therefore,

$$\int_J \frac{\|\mathbb{1}_E(\cdot - y) - \mathbb{1}_E\|_1}{m(E)} dy \leq \int_J \frac{\epsilon}{m(J)} dy = \epsilon.$$

Therefore, we may choose J such that $\mathbb{1}_E * (\mathbb{1}_J/m(J))$ is a compactly supported function within ϵ of $\mathbb{1}_E$ in the L^1 norm. Lastly, we prove that $\mathbb{1}_E * (\mathbb{1}_J/m(J))$ is a continuous function.

So, note that $\mathbb{1}_E(x-y)\mathbb{1}_J(y)/m(J) = \mathbb{1}_{(x-E) \cap J}(y) \leq \mathbb{1}_J(y)/m(J)$. Hence, for all x , $\mathbb{1}_E(x-y)\mathbb{1}_J(y)/m(J)$ is bounded by an integrable function in y . Now, take a sequence (x_n) with $x_n \rightarrow x_0$. Then, by DCT,

$$\begin{aligned} \lim \mathbb{1}_E * (\mathbb{1}_J/m(J))(x_n) &= \lim \int \mathbb{1}_E(x_n - y)\mathbb{1}_J(y)/m(J) dy \\ &= \int \mathbb{1}_E(x_0 - y)\mathbb{1}_J(y)/m(J) dy \\ &= \mathbb{1}_E * (\mathbb{1}_J/m(J))(x_0). \end{aligned}$$

Hence, continuity is obtained. Therefore, measurable indicator functions may be approximated in the L^1 norm by a compactly supported continuous function f .

Now, let ϕ be an arbitrary integrable simple function. Then, ϕ is a linear sum of indicator functions of finite measurable sets. Hence, by triangle inequality of the L^1 norm, we can approximate it via a linear sum of compactly supported continuous functions. This function f is likewise continuous and compactly supported. Since simple functions are dense in L^1 , and compactly supported continuous functions may be taken arbitrarily close to simple functions, then compactly supported continuous functions are dense in L^1 , proving the result. \square

Problem 6. (Classic) Let $f \in L^1([0, 1])$. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin(nx)| dx = \frac{2}{\pi} \int_0^1 f(x) dx.$$

Hint: Start with the easiest function f first.

Proof. Let f be the characteristic function of an interval. So, we show that

$$\lim_{n \rightarrow \infty} \int_a^b |\sin(nx)| dx = \frac{2}{\pi}.$$

We first integrate a single arc of $|\sin(nx)|$ over $(0, \pi/n)$. So, making the change of coordinates $y = nx$, we observe that y ranges from 0 to π , and that $\frac{1}{n} dy = dx$.

$$\int_0^{\pi/n} |\sin(nx)| = \int_0^{\pi/n} \sin(nx) dx = \frac{1}{n} \int_0^\pi \sin(y) dy = \frac{1}{n} \left(-\cos(y) \Big|_0^\pi \right) = \frac{2}{n}.$$

Since each arc of $|\sin(nx)|$ is positive, and $|\sin(nx)|$ has a period of $(0, \pi/n)$, then each arc has the integral $\frac{2}{n}$ via translation. We count the minimum and maximum number of arcs of $|\sin(nx)|$ on (a, b) to estimate its integral. So, there are at most $\lfloor \frac{n}{\pi}(b-a) \rfloor$ complete arcs of $|\sin(nx)|$ on (a, b) . Two arcs may be partially terminated at the point a or b . Hence, the total number of arcs is bounded below by $\lfloor \frac{n}{\pi}(b-a) \rfloor - 2$. So, we estimate

$$\frac{n}{\pi}(b-a) - 3 \leq \left\lfloor \frac{n}{\pi}(b-a) \right\rfloor - 2 \leq \# \text{ of arcs} \leq \left\lfloor \frac{n}{\pi}(b-a) \right\rfloor \leq \frac{n}{\pi}(b-a).$$

Say that J_n is the smallest interval on which all full arcs of $|\sin(nx)|$ are completed. The integral value contributed by each arc is $\frac{2}{n}$. So

$$\frac{2}{\pi}(b-a) - \frac{6}{n} \leq \int_{J_n} |\sin(nx)| dx \leq \frac{2}{\pi}(b-a).$$

Each arc occurs over a length π/n interval, so $(b-a) - \frac{3\pi}{n} \leq m(J_n) \leq (b-a)$. Then,

$$\frac{2}{\pi}(b-a) - \frac{6}{n} \leq \int_{J_n} |\sin(nx)| dx \leq \int |\sin(nx)| dx \leq \int_{I \setminus J_n} 1 dx + \int_{J_n} |\sin(nx)| dx.$$

Therefore,

$$\frac{2}{\pi}(b-a) - \frac{6}{n} \leq \int_a^b |\sin(nx)| dx \leq m(I) - m(J_n) + \frac{2}{\pi}(b-a).$$

Taking $n \rightarrow \infty$, we observe that $m(J_n) \rightarrow b-a$, so $m(I) - m(J_n) \rightarrow 0$. Moreover, $\frac{6}{n} \rightarrow 0$. Therefore,

$$\lim \int_a^b |\sin(nx)| dx = \frac{2}{\pi}(b-a) = \frac{2}{\pi} \int_0^1 \mathbb{1}_{(a,b)} dx.$$

By linearity of the integral, if ϕ is a step function, then we obtain

$$\lim \int_a^b |\sin(nx)| \phi dx = \frac{2}{\pi} \int \phi dx.$$

Now, let f be an arbitrary function. Since the Lebesgue measure is, in particular, a Lebesgue-Stieltjes measure, then we may approximate f in L^1 by step functions. So, take ϕ such that $\|f - \phi\|_1 < \epsilon$. Then,

$$\left| \int_0^1 f |\sin(nx)| dx - \int_0^1 \phi |\sin(nx)| dx \right| \leq \int_0^1 |f - \phi| |\sin(nx)| dx \leq \int_0^1 |f - \phi| dx \leq \epsilon$$

for all n . Therefore,

$$\left| \int_0^1 f(x) |\sin(nx)| dx - \frac{2}{\pi} \int_0^1 f dx \right| \leq \left| \int_0^1 (f - \phi) |\sin(nx)| dx \right|$$

$$+ \left| \int_0^1 \phi |\sin(nx)| dx - \frac{2}{\pi} \int_0^1 \phi dx \right| + \left| \frac{2}{\pi} \int_0^1 \phi - f dx \right|.$$

So,

$$\left| \int_0^1 f(x) |\sin(nx)| dx - \frac{2}{\pi} \int_0^1 f dx \right| \leq \epsilon + \frac{2}{\pi} \epsilon + \left| \int_0^1 \phi |\sin(nx)| dx - \frac{2}{\pi} \int_0^1 \phi dx \right|.$$

Taking $n \rightarrow \infty$ shows that we may take $\int_0^1 f |\sin(nx)| dx$ to be within $2\epsilon + 2\frac{\pi}{\epsilon}$ of $\int_0^1 f dx$ for all ϵ , giving the result. \square