Real Analysis Qual, Spring 2024

Problem 1. (Classic Technique), Solution Courtesty of Ethan Phan: Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n^t},\ t > 1$$

has a continuous derivative $f'(x)$ given by

$$f'(x) = \sum_{n=0}^{\infty} -\frac{\ln n}{n^t},\ t > 1.$$

Summations are nonsense at $n = 0$. The second summation is nonsense at $n = 1$. We assume these modifications.

Proof. We wish to show we can differentiate under the (discrete) integral. Let $f(x,t) = \frac{1}{x^t}$ for $t > 1, x > 0$. First, $f(x,t)$ is an $L^1(\mathbb{N})$ function, for $\sum_{n=1}^{\infty} f(n,t) = \sum_{n=1}^{\infty} 1/n^t$ is a p-series with $t > 1$. Now, we'll show f is differentiable at a fixed $t_0 > 1$. Let $\epsilon > 0$ such that $\epsilon < t_0 - 1$. Then for any $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $1 < c := t_0 - \epsilon < t$. Then:

$$\left|\frac{\partial f}{\partial t}(x,t)\right| = \left|-\frac{\ln(x)}{x^t}\right| = \frac{\ln(x)}{x^t} \leq \frac{\ln x}{x^c}$$

We claim this is in $L^1(\mathbb{N})$. Indeed, for $n \geq N$ large enough, $\ln(n) < n^{\left(\frac{c-1}{2}\right)+1}$ (where $\left(\frac{c-1}{2}\right) + 1 > 1$ still holds), so:

$$\frac{\ln(n)}{n^c} < \frac{n^{\frac{c-1}{2}+1}}{n^c} = \frac{1}{n^{\frac{c}{2}+\frac{1}{2}}}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+\frac{c}{2}}}$ converges because $\frac{1}{2} + \frac{c}{2} > 1$. So:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^c} = \sum_{n=1}^{N-1} \frac{\ln(n)}{n^c} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^c} \leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^c} + \sum_{n=N}^{\infty} \frac{1}{n^{\frac{1}{2}+\frac{c}{2}}} < \infty$$

Hence, $\frac{\ln(n)}{n^c} \in L^1(\mathbb{N})$. So we may interchange the (discrete) integral and derivative:

$$f'(t_0) = \left[\frac{\partial}{\partial t} \sum_{n=1}^{\infty} \frac{1}{n^t}\right](t_0) = \left[\sum_{n=1}^{\infty} \frac{\partial}{\partial t} \frac{1}{n^t}\right](t_0) = \sum_{n=1}^{\infty} -\frac{\ln(n)}{n^{t_0}}$$

and as $t_0 > 1$ is arbitrary, we have the result.

Next we wish to show $f'(t)$ is continuous. Take $t > 1$ and a sequence $t_n \to t$ (all $t_n > 1$). As $t > 1$, $\exists \epsilon$ s.t. $0 < \epsilon < t - 1$. As $t_n \to t$, only finitely many t_n's lie outside $(t - \epsilon, t + \epsilon)$, so let $c = \min\{t_n : t_n \notin (t - \epsilon, t + \epsilon)\}$, so that $1 < c < t_j$ for all j. We wish to show $\lim_{n\to\infty} f'(t_n) = f'(t)$. Note by the same argument above, $\left|-\frac{\ln(n)}{n^t}\right| \leq \frac{\ln(n)}{n^c} \in L^1(\mathbb{N})$. Hence by **DCT**:

$$\lim_{n\to\infty} f'(t_n) = \lim_{n\to\infty} \sum_{j=1}^{\infty} -\frac{\ln(j)}{j^{t_n}} = \sum_{j=1}^{\infty} \lim_{n\to\infty} \frac{-\ln(j)}{j^{t_n}} = \sum_{j=1}^{\infty} \frac{-\ln(j)}{j^t} = f'(t)$$

as $\frac{1}{j^t}$ is continuous in t for every fixed $j \geq 1$. $\square$

Problem 2. (Classic) Let $f : [1, \infty) \longrightarrow \mathbb{R}$ be a continuous, bounded function. Prove that

$$\lim_{n\to\infty} \int_1^\infty f(t)nt^{-n-1}\,\mathrm{d}t = f(1).$$

Proof. Say that $|f(t)| \leqslant M$. Then,

$$\int_1^\infty |f(t)nt^{-n-1}|\,\mathrm{d}t \leqslant Mn \int_1^\infty t^{-n-1}\,\mathrm{d}t \leqslant Mn \int_1^\infty 1/t^2\,\mathrm{d}t = Mn.$$

Therefore, $f(t)nt^{-n-1}$ is a Lebesgue integrable function. Hence, we may make the substitution $1/x = t$, given that $1/x$ is a $(0, 1) \longrightarrow (1, \infty)$ diffeomorphism. Note that $\mathrm{d}x = -t^{-2}\,\mathrm{d}t$, since $x = 1/t$. Hence, we $t^{-2}\,\mathrm{d}t$ with $\mathrm{d}x$. So,

$$\int_1^\infty f(t)nt^{-n-1}\,\mathrm{d}t = \int_0^1 f(1/x)nx^{n-1}\,\mathrm{d}x.$$

Now, $f(1/x) = g(x)$ is a bounded continuous function on $(0, 1]$ such that $g(1) = f(1)$. Therefore, we prove that

$$\lim \int_0^1 g(x)nx^{n-1}\,\mathrm{d}x = g(1).$$

Take the interval $(0, a]$ with $a < 1$. Suppose $g(x)$ is bounded by M. Then, for $x \in (0, a]$, we have $nx^{n-1} \leqslant na^{n-1}$. Given that $1/a > 1$, then $n/a^{n-1} \to 0$ as $n \to \infty$. Therefore, $na^{n-1} \to 0$ as $n \to \infty$. So, on $(0, a]$, $nx^{n-1} \to 0$ uniformly. Hence, taking N so that $n \geqslant N$ implies $nx^{n-1} < \epsilon$, we obtain

$$\int_0^a g(x)nx^{n-1}\,\mathrm{d}x \leqslant \int_0^a M\epsilon\,\mathrm{d}x \leqslant M\epsilon.$$

Taking $\epsilon \to 0$ shows that $\lim_{n\to\infty} \int_0^a g(x)nx^{n-1}\,\mathrm{d}x = 0$.

Now, on $[a, 1]$, define $M_a = \sup_{x\in[a,1]} g(x)$ and $N_a = \inf_{x\in[a,1]} g(x)$. Both M_a, N_a exist since g is bounded. Moreover, since $[a, 1]$ is compact and g is continuous, there are points $m_a, n_a \in [a, 1]$ such that $g(m_a) = M_a$ and $g(n_a) = N_a$. Since $a < 1$, in combination with the above we obtain

$$N_a = \lim N_a(1 - a^n) = \lim \int_a^1 N_a nx^{n-1}\,\mathrm{d}x + \lim \int_0^a g(x)nx^{n-1}\,\mathrm{d}x \leqslant \lim \int_0^1 g(x)nx^{n-1}\,\mathrm{d}x.$$

Likewise,

$$M_a = \lim N_a(1 - a^n) = \lim \int_a^1 N_a nx^{n-1}\,\mathrm{d}x + \lim \int_0^a g(x)nx^{n-1}\,\mathrm{d}x \geqslant \lim \int_0^1 g(x)nx^{n-1}\,\mathrm{d}x.$$

Therefore, for all a,

$$g(n_a) = N_a \leqslant \lim \int_0^1 g(x)nx^{n-1}\,\mathrm{d}x \leqslant M_a = g(m_a).$$

As $a \to 1$, we have $n_a, m_a \to 1$, and by continuity $g(n_a), g(m_a) \to 1$. Therefore,

$$g(1) = \lim \int_0^1 g(x) n x^{n-1} \, \mathrm{d}x.$$

Finally, $g(1) = f(1/1) = f(1)$, so

$$\lim_{n\to\infty} \int_1^\infty f(t) n t^{-n-1} \, \mathrm{d}t = f(1),$$

as required. □

Problem 3. Let ϕ be a continuously differentiable function on $\mathbb{R}$ such that $\phi(x) > 0$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| \geqslant 1$, and $\int_{\mathbb{R}} \phi(x) \, \mathrm{d}x = 1$. Put $K_n(x) = n\phi(nx)$. Then, (no proof required) $\int_{\mathbb{R}} K(x) \, \mathrm{d}x = 1$ and $K_n(x) = 0$ if $|x| \geqslant 1/n$. Recall that

$$f * K_n(x) = K_n * f(x) = \int_{\mathbb{R}} K_n(x-y) f(y) \, \mathrm{d}y.$$

Prove the following:

(i) **(Classic)** If $f \in L^1(\mathbb{R})$, then $f * K_n \in L^1(\mathbb{R})$ and has a continuous derivative for each n.

(ii) If $f \in L^1(\mathbb{R})$, $f * K_n$ converges to f in L^1 as $n \to \infty$.

Proof. We first prove (i). So, take $f \in L^1(\mathbb{R})$. Then,

$$\int |f * K_n(x)| \, \mathrm{d}x \leqslant \iint |K_n(x-y) f(y)| \, \mathrm{d}y \, \mathrm{d}x.$$

By Tonelli's

$$\begin{aligned}
\iint |K_n(x-y) f(y)| \, \mathrm{d}y \, \mathrm{d}x &= \iint |K_n(x-y) f(y)| \, \mathrm{d}x \, \mathrm{d}y \\
&= \int |f(y)| \int |K_n(x-y)| \, \mathrm{d}x \, \mathrm{d}y \\
&= \int |f(y)| \int |K_n(x)| \, \mathrm{d}x \, \mathrm{d}y \\
&= \left(\int |f(y)| \, \mathrm{d}y \right) \left(\int |K_n(x)| \, \mathrm{d}x \right).
\end{aligned}$$

Therefore, $||f * K_n||_1 \leqslant ||f||_1 \cdot ||K_n||_1 < \infty$. So, $f * K_n \in L^1(\mathbb{R})$.

Now, since ϕ is continuously differentiable, then K_n is continuously differentiable, for $K_n'(x) = n^2 \phi'(nx)$. The derivative of ϕ is continuous, and hence $K_n'(x)$ is a continuous function. Moreover, on $[-2, 2]$, for any point x, there is some open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \cap [-1, 1] = \emptyset$. Hence, $K_n|_{B_\epsilon(x)}$ is the 0 function, so $K_n'|_{B_\epsilon(x)}$ is the 0 function. Therefore, K_n' is supported on $[-2, 2]$, a compact interval. By continuity, we then obtain that K_n' is bounded, say by M. For each y fixed, $\frac{\partial}{\partial x}K_n(x-y) = K_n'(x-y)$, and thus has the same bound M. Therefore, $|\frac{\partial}{\partial x}K_n(x-y)f(y)| \leqslant M|f(y)|$, an integrable function. On the other hand, for each x, $|f * K_n(x)| \leqslant ||K_n||_\infty ||f||_1$. Since K_n is a compactly supported continuous function, $|f * K_n(x)|$ is therefore finite for all x. So, $K_n(x-y)f(y)$ is integrable for all x. Therefore, the criteria for differentiation under the integral sign are obtained. So,

$$f * K_n'(x) = \int \frac{\partial}{\partial x} K_n(x-y)f(y)\,\mathrm{d}y.$$

Since $K_n(x-y)f(y)$ are all dominated by an integrable function, then for (x_k) converging to x_0, thinking of $\frac{\partial}{\partial x}K_n(x_k-y)f(y)$ as a sequence of functions in k, we obtain by DCT

$$\lim f * K_n'(x_k) = \lim \int \frac{\partial}{\partial x} K_n(x_k-y)f(y)\,\mathrm{d}y = \int \frac{\partial}{\partial x} K_n(x_0-y)f(y)\,\mathrm{d}y = f * K_n'(x_0).$$

Hence, the derivative is continuous, proving (ii).

We prove (ii). So,

$$\int |K_n * f(x) - f(x)|\,\mathrm{d}x = \int \left| \int K_n(y)f(x-y)\,\mathrm{d}y - f(x) \right| \mathrm{d}x.$$

Since $\int K_n(y)\,\mathrm{d}y = 1$, then

$$\begin{aligned} &= \int \left| \int K_n(y)f(x-y)\,\mathrm{d}y - \int K_n(y)f(x) \right| \mathrm{d}y \\ &= \int \left| \int K_n(y)(f(x-y) - f(x))\,\mathrm{d}y \right| \mathrm{d}x \\ &\leqslant \iint |K_n(y)| \cdot |f(x-y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Since for $|y| > 1/n$ we have $K_n(y) = 0$, then

$$\iint |K_n(y)| \cdot |f(x-y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x = \int \int_{-1/n}^{1/n} |K_n(y)||f(x-y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x.$$

We apply Tonelli's

$$\begin{aligned} \int \int_{-1/n}^{1/n} |K_n(y)||f(x-y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x &= \int_{-1/n}^{1/n} \int |K_n(y)||f(x-y) - f(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-1/n}^{1/n} |K_n(y)| \int |f(x-y) - f(x)|\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

We have $y \to x$ as $n \to \infty$. Since translation is continuous, we choose N such that for all $n \geqslant N$, we have $|f(x-y) - f(x)| < \epsilon$. Therefore,

$$\int_{-1/n}^{1/n} |K_n(y)| \int |f(x-y) - f(x)|\,\mathrm{d}x\,\mathrm{d}y < \int_{-1/n}^{1/n} |K_n(y)|\epsilon\,\mathrm{d}y$$

$$
\begin{aligned}
&= \epsilon \int |K_n(y)| \, dy \\
&= \epsilon.
\end{aligned}
$$

Therefore, $K_n * f \to f$ in L^1. $\square$

Is there a way to prove the next problem without problem 3?

Problem 4. Let $f \in L^1(\mathbb{R})$. Define a linear transform T_f on $L^1(\mathbb{R})$ by $T_f(g) = f * g$. Show that

(i) $\sup_{||g||_1 \leqslant 1} ||T_f(g)||_1 = ||f||_1$, and

(ii) $T_f = 0$ if and only if $f = 0$ in $L^1(\mathbb{R})$.

Proof. First, by Tonelli

$$
||T_f(g)||_1 \leqslant \iint |f(x-y)g(y)| \, dy \, dx = \iint |f(x-y)g(y)| \, dx \, dy = \iint |f(x-y)| \, dx |g(y)| \, dy.
$$

If $||g||_1 \leqslant 1$, then by translation invariance,

$$
\iint |f(x-y)| \, dx |g(y)| \, dy = \left(\int |f(x)| \, dx \right) \left(\int |g(y)| \, dy \right) = ||f||_1 ||g||_1 \leqslant ||f||_1.
$$

Hence, $\sup_{||g||_1 \leqslant 1} ||T_f(g)||_1 \leqslant ||f||_1$. Recall from problem 3 that $K_n * f \to f$ in L^1 norm. By continuity of the norm, we obtain $||K_n * f||_1 \to ||f||_1$. Then,

$$
||T_f(K_n)||_1 = \int |K_n * f| \, dx = ||K_n * f||_1 \to ||f||_1.
$$

Since $||K_n||_1 = 1$, then we conclude that $||f||_1 \leqslant \sup_{||g||_1 \leqslant 1} ||T_f(g)||_1$, giving equality.

Now, suppose that $f = 0$. Then,

$$
T_f(g) = \int f(x-y) g \, dy = \int 0 g \, dy = 0.
$$

On the other hand, once again recall that $K_n * f \to f$ in L^1. Hence,

$$
0 = ||T_f(K_n)||_1 = ||K_n * f||_1 \to ||f||_1.
$$

Therefore, $||f||_1 = 0$, forcing $f = 0$. $\square$

Problem 5. Let f be a continuous complex valued function on $[0,1]$. Show that

$$
f([0,1]) = \{\lambda \in \mathbb{C} : m(f^{-1}(B_\epsilon(\lambda))) > 0 \text{ if } \epsilon > 0\},
$$

where $B_\epsilon(\lambda)$ is the open disc of radius ϵ centered at λ.

Proof. First suppose that $\lambda \in \mathbb{C}$ is such that $\lambda \in f([0,1])$. For $\epsilon > 0$, since f is continuous, then $f^{-1}(B_\epsilon(\lambda))$ is open. Hence, for some δ, we have a δ ball $B_\delta \subseteq f^{-1}(B_\epsilon(\lambda))$. Since $m(B_\delta) > 0$, then $m(f^{-1}(B_\epsilon(\lambda))) > 0$. Therefore,

$$\lambda \in \{\lambda \in \mathbb{C} : m(f^{-1}(B_\epsilon(\lambda))) > 0 \text{ if } \epsilon > 0\},$$

proving one direction.

Now, suppose that λ is such that for every ball $B_\epsilon(\lambda)$, the set $f^{-1}(B_\epsilon(\lambda))$ has positive measure. Choose a sequence (x_n) so that $x_n \in f^{-1}(B_{1/n}(\lambda))$. This is possible, given that each such set has positive measure, and therefore is nonempty. Since $[0,1]$ is compact, then there is a subsequence (x_{n_k}) which converges in $[0,1]$ to some point x_0. Given that $f^{-1}(B_{1/n}(\lambda)) \supseteq f^{-1}(B_{1/n+1}(\lambda))$, then we conclude that

$$x_0 \in \bigcap_{n=1}^{\infty} f^{-1}(B_{1/n}(\lambda)) = f^{-1}\left(\bigcap_{n=1}^{\infty} B_{1/n}(\lambda)\right).$$

Since $\bigcap_{n=1}^{\infty} B_{1/n}(\lambda) = \{\lambda\}$, then $x_0 \in f^{-1}(\{\lambda\})$, and hence $f(x_0) = \lambda$. Therefore, $\lambda \in f([0,1])$. $\square$

Problem 6.

(1) **(Classic)** Prove the following Riemann-Lebesgue identities for any $f \in L^1([-\pi,\pi])$:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t = 0, \quad \lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t = 0.$$

(2) Deduce the following: for any measurable set $E \subseteq [-\pi,\pi]$, and any sequence s_n of real numbers,

$$\lim_{n\to\infty} \int_E \cos^2(nt + s_n)\,\mathrm{d}t = m(E)/2.$$

Proof. First, take f to be the characteristic function of an interval $I = (a,b) \subseteq [-\pi,\pi]$. Note that

$$\int_{-\pi/n}^{\pi/n} \cos(nt)\,\mathrm{d}t = \frac{1}{n}\sin(nt)\Big|_{-\pi/n}^{\pi/n} = 0,$$

and

$$\int_{-\pi/n}^{\pi/n} \sin(nt)\,\mathrm{d}t = -\frac{1}{n}\cos(nt)\Big|_{-\pi/n}^{\pi/n} = 0.$$

From now on, we restrict to the case of $\cos(nt)$, for the arguments are identical. By translation, we conclude that each full period of $\cos(nt)$ completed in (a,b) contributes 0 to the integral $\int_a^b \cos(nt)\,\mathrm{d}t$. There are at most $\lfloor (b-a)\frac{n}{2\pi} \rfloor$ full periods of $\cos(nt)$ completed over (a,b). A partial period may be terminated by either the endpoints a, b, subtracting 2 from $\lfloor (b-a)\frac{n}{2\pi} \rfloor$. Hence,

$$(b-a)\frac{n}{2\pi} - 3 \leqslant \#\text{ of total periods} \leqslant \lfloor (b-a)\frac{n}{2\pi} \rfloor \leqslant (b-a)\frac{n}{2\pi}.$$

Multiplying the number of periods completed on (a,b) by $\frac{2\pi}{n}$ gives the total length of the subinterval on (a,b) which contributes 0 to the integral of $\cos(nt)$. Hence, if this subinterval is J_n, then

$$(b-a) - \frac{6\pi}{n} \leqslant m(J_n) \leqslant b-a.$$

Taking $n \to \infty$ we see that $m(J_n) \to b-a$. If $I = (a,b)$, then

$$-m(I\backslash J_n) = \int_{I\backslash J_n} -1 \,\mathrm{d}t \leqslant \int_{I\backslash J_n} \cos(nt)\,\mathrm{d}t \leqslant \int_{I\backslash J_n} 1 \,\mathrm{d} = m(I\backslash J_n).$$

Since $m(I\backslash J_n) = m(I) - m(J_n) \to 0$, then $\int_{I\backslash J_n} \cos(nt)\,\mathrm{d}t \to_{n\to\infty} 0$. Therefore,

$$\int_I \cos(nt)\,\mathrm{d}t = \int_{I\backslash J_n} \cos(nt)\,\mathrm{d}t + \int_{J_n} \cos(nt)\,\mathrm{d}t = \int_{I\backslash J_n} \cos(nt)\,\mathrm{d}t \to 0.$$

So, if $f(t)$ is the indicator function for an interval, then $\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t$.

Now, let $f(t) = \mathbb{1}_E$ be a simple function for $E \subseteq [-\pi,\pi]$. By one of the Littlewood principles, since $m(E) < \infty$, there is a finite collection of intervals $(I_k)_{k=1}^n$ such that $m(E \triangle \bigcup_{k=1}^n I_k) < \epsilon$. Set $A = \bigcup_{k=1}^n I_k$. Given that $E\triangle A = (E\backslash A) \cup (A\backslash E)$, we conclude that $m(E\backslash A) < \epsilon$. We may restrict the intervals I_k to be disjoint. Hence,

$$-\epsilon + \int_A \cos(nt)\,\mathrm{d}t \leqslant \int \mathbb{1}_E \cos(nt)\,\mathrm{d}t \leqslant \epsilon + \int_A \cos(nt)\,\mathrm{d}t.$$

Moreover, $\int_A \cos(nt)\,\mathrm{d}t = \sum_{k=1}^n \int_{I_k} \cos(nt)\,\mathrm{d}t \to 0$, by disjointness. Therefore, for all ϵ

$$\lim \left| \int \mathbb{1}_E \cos(nt)\,\mathrm{d}t \right| < \epsilon.$$

So, the limit is 0. By linearity, we therefore obtain $\lim \int_{-\pi}^{\pi} \phi(t)\cos(nt)\,\mathrm{d}t = 0$ for arbitrary simple functions.

Finally, let f be an arbitrary $L^1([-\pi,\pi])$ function. Since simple functions are dense in L^1, take ϕ within ϵ of f. Then, for all n,

$$\left| \int f\cos(nt)\,\mathrm{d}t - \int \phi\cos(nt)\,\mathrm{d}t \right| = \left| \int (f-\phi)\cos(nt)\,\mathrm{d}t \right| \leqslant \int |f-\phi|\,\mathrm{d}t < \epsilon.$$

On the other hand, taking $n \to \infty$, we have $\int \phi\cos(nt)\,\mathrm{d}t \to 0$. Therefore, by continuity of $|\cdot|$,

$$\left| \lim \int f\cos(nt)\,\mathrm{d}t \right| < \epsilon.$$

This holds for arbitrary ϵ, so $\lim \int f\cos(nt)\,\mathrm{d}t = 0$.

Now, to prove (2), we observe that

$$\cos^2(nt+s_n) = \frac{1+\cos(2nt+2s_n)}{2}.$$

Therefore,

$$\int_E \cos^2(nt + s_n)\,\mathrm{d}t = \int_E \frac{1 + \cos(2nt + 2s_n)}{2}\,\mathrm{d}t = m(E)/2 + \frac{1}{2}\int_E \cos(2nt + 2s_n)\,\mathrm{d}t.$$

Observe now that

$$\cos(2nt + 2s_n) = \cos(2nt)\cos(2s_n) - \sin(2nt)\sin(2s_n).$$

Moreover,

$$-\int_E \cos(2nt)\,\mathrm{d}t \leqslant \cos(2s_n)\int_E \cos(2nt)\,\mathrm{d}t \leqslant \int_E \cos(2nt)\,\mathrm{d}t.$$

Hence, as $n \to \infty$, the central term goes to 0 by (1). An identical argument holds for $\sin(2nt)\sin(2s_n)$. Therefore,

$$\int_E \cos^2(nt + s_n)\,\mathrm{d}t = m(E)/2 + \frac{1}{2}\int_E \cos(2nt + 2s_n)\,\mathrm{d}t \to m(E)/2,$$

completing the proof. □