Real Analysis Qual, Fall 2024

Problem 1. Show that the function $f(x) = \frac{1}{1-e^{-x^2}}$ is uniformly continuous outside $(-\delta, \delta)$ for every δ, but it fails to be uniformly continuous on all of $\mathbb{R}$.

Now, f is not even defined on all of $\mathbb{R}$, but the sense of the problem is that even modification on a null set could not make it uniformly continuous.

Proof. First, observe that e^{x^2} grows monotonically in $|x|$. Hence e^{-x^2} decreases monotonically to 0 in $|x|$, so $(1-e^{-x^2})^{-1}$ increaases monotonically to 1 as $|x| \to \infty$. Take $\epsilon > 0$. Choose N such that if $|x| = N$, then $1-(1-e^{-x^2})^{-1} = |1-(1-e^{-x^2})^{-1}| < \epsilon/3$. By monotonicity, we have for all x satisfying $|x| \geqslant N$ that $|1-(1-e^{-x^2})^{-1}| < \epsilon$. On the other hand, $A = [-N, -\delta] \cup [\delta, N]$ is a closed and bounded set, and therefore compact. Since f is a continuous function on A, a compact set, then f is uniformly continuous on A. Therefore, we may choose c such that for all $x, y \in A$ satisfying $|x-y| < c$, we have $|f(x)-f(y)| < \epsilon/3$. We now claim that for all $x, y \in \mathbb{R}\backslash(-\delta, \delta)$, if $|x-y| < c$, then $|f(x)-f(y)| < \epsilon$. First, if $x, y \notin A$, then

$$|f(x)-f(y)| \leqslant |f(x)-1| + |1-f(y)| < 2\epsilon/3 < \epsilon.$$

If $x, y \in A$, then $|x-y| < c$ implies $|f(x)-f(y)| \leqslant \epsilon/3 < \epsilon$, and finally if $x \in A$ and $y \notin A$ such that $|x-y| < c$, we must then have $|x-N| < c$ if $x, y > 0$, or $|x+N| < c$ if $x, y < 0$, given that in the first case $x \leqslant N \leqslant y$ must hold, and in the latter case $y \leqslant -N \leqslant x$ must hold. Hence,

$$|f(x)-f(y)| \leqslant |f(x)-f(N)| + |f(N)-f(y)| < \epsilon/3 + |f(N)-1| + |1-f(y)| < \epsilon.$$

Therefore, f is uniformly continuous on $\mathbb{R}\backslash(-\delta, \delta)$.

Finally, f is not uniformly continuous on all of $\mathbb{R}$. Indeed, consider $x, x+h$. Then,

$$f(x) - f(x+h) = \frac{e^{-x^2} - e^{-(x+h)^2}}{(1-e^{-x^2})(1-e^{-(x+h)^2})} = \frac{e^{-x^2}(1-e^{-2xh-h^2})}{(1-e^{-x^2})(1-e^{-(x+h)^2})}.$$

For $x < 1$ with $x < h$, we have $x^2 \leqslant 2x^2 \leqslant 2xh + h^2$. Therefore, $e^{-x^2} \geqslant e^{-2xh-h^2}$, so $1-e^{-x^2} \leqslant 1-e^{-2xh-h^2}$. We obtain

$$1 \leqslant \frac{1-e^{-2xh-h^2}}{1-e^{-x^2}}.$$

Therefore,

$$\frac{e^{-x^2}(1-e^{-2xh-h^2})}{(1-e^{-x^2})(1-e^{-(x+h)^2})} \geqslant \frac{e^{-x^2}}{(1-e^{-(x+h)^2})}.$$

For x close enough to 0, this becomes

$$\frac{e^{-x^2}}{(1-e^{-(x+h)^2})} \geqslant \frac{1}{2(1-e^{-(x+h)^2})}.$$

Set $x = h/2$ so that $x < h$. Choose $h < c$. Then, $|(x+h)-x)| = h/2 < c$. Sending $h \to 0$,

$$\frac{1}{2(1-e^{-(x+h)^2})} = \frac{1}{2(1-e^{-(\frac{3}{2}h)^2})} \to \infty.$$

Therefore, for every ϵ, there is no c such that over all $\mathbb{R}$ if $|x-y| < c$ then $|f(x)-f(y)| < \epsilon$. $\square$

Problem 2. Give a sequence of measurable sets $A_1, A_2, \ldots$ in $[0,1]$, define

$$\limsup A_n := \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n, \quad \liminf A_n := \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n.$$

Show that

$$\liminf A_n \subseteq \limsup A_n$$

and that

$$\limsup m(A_n) \leqslant m(\limsup A_n), \quad m(\liminf A_n) \leqslant \liminf m(A_n).$$

Proof. Suppose that $x \in \liminf A_n$. Then, there is some k so that $x \in \bigcap_{n=k}^{\infty} A_n$. In particular, there is k so that for all $n \geqslant k$, we have $x \in A_n$. Therefore, $x \in \bigcup_{n=m}^{\infty} A_n$ for every m, so $x \in \limsup \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n$. It follows that $\liminf A_n \subseteq \limsup A_n$.

Set $E_k = \bigcup_{n=k}^{\infty} A_n$, so that $\limsup A_n = \bigcap_{k=1}^{\infty} E_k$. The E_k decrease monotonically and are contained in $[0,1]$. Hence, by continuity from above,

$$m(\limsup A_n) = \lim m(E_k).$$

For each fixed k, we have $m(E_k) \geqslant m(A_m)$ for all $m \geqslant k$. Hence, $m(E_k) \geqslant \sup_{m \geqslant k} m(A_m)$. So, the sequence $(m(E_k))$ bounds the sequence $(\sup_{m\geqslant k} m(A_m))$. Hence,

$$m(\limsup A_n) = \lim m(E_k) \geqslant \lim \sup_{m \geqslant k} m(A_m) = \limsup m(A_k).$$

Now set $F_k = \bigcap_{n=k}^{\infty} A_n$, and note that $\liminf A_n = \bigcup_{k=1}^{\infty} F_k$, with $F_k \subseteq F_{k+1}$. By continuity from above, we have

$$m(\liminf A_n) = \lim m(F_k).$$

Observe that $m(F_k) \leqslant m(A_m)$ for all $m \geqslant k$. So, $m(F_k) \leqslant \inf_{m \geqslant k} m(A_m)$. By the same argument with lim sup, we then have

$$m(\liminf A_n) = \lim m(F_k) \leqslant \lim \inf_{m \geqslant k} m(A_m) = \liminf m(A_k),$$

as needed. $\square$

Problem 3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable function. Recall that a point x is a critical point of f if $f'(x) = 0$, and a point y is a critical value of f if $y = f(x)$ for some critical point x. Prove that the set of all critical values of f has Lebesgue measure 0. *Hint: Consider the Mean Value Theorem.*

Proof. Let C be the set in $\mathbb{R}$ of critical points, and observe that $f(C)$ is the set of critical values. Set $I_n = (-n, n)$. Pick $\epsilon > 0$. Set $U = I_n \cap f'^{-1}(B_\epsilon(0))$. Since f' is continuous, then U is open. Observe that if $x \in C$, then $f'(x) = 0$, so $x \in f^{-1}(B_\epsilon(0))$. Therefore, $C \cap I_n \subseteq U$. We may write $U = \bigcup_{n=1}^{\infty} J_n$ for J_n pairwise disjoint open intervals. For any $x, y \in J_n$, then by the Mean Value Theorem, there is some intermediate c such that $f(x) - f(y) = f'(c)(x-y)$. Since c is intermediate, then $c \in J_n \subseteq U$, and hence $|f'(c)| < \epsilon$. Therefore,

$$|f(x) - f(y)| \leqslant |f'(c)||x-y| < \epsilon|x-y| < \epsilon m(J_n).$$

This holds for all $x, y \in J_n$. Note $f(J_n)$ is connected, since f is continuous and J_n is connected. Hence, $m(f(J_n)) = \sup_{x,y\in J_n} |f(x) - f(y)|$. Therefore, $m(f(J_n)) < \epsilon m(J_n)$. So,

$$m(f(U)) = m\left(f\left(\bigcup_{n=1}^{\infty} J_n\right)\right) = m\left(\bigcup_{n=1}^{\infty} f(J_n)\right) \leqslant \sum_{n=1}^{\infty} m(f(J_n)) < \epsilon \sum_{n=1}^{\infty} m(J_n) = \epsilon m(U).$$

Now, $U \subseteq I_n$, so $\epsilon m(U) \leqslant 2n\epsilon$. Therefore, $m(f(U)) \leqslant 2n\epsilon$. Recall that $C \cap I_n \subseteq U$, so $m(f(C \cap I_n)) \leqslant 2n\epsilon$. Taking $\epsilon \to 0$ gives $m(f(C \cap I_n)) = 0$ (note by completeness this shows also that $f(C \cap I_n)$, and hence $f(C)$ by countable unions, is measurable). Finally, note that $f(C \cap I_n)$ increases monotonically in n, since $C \cap I_n$ is monotonically increasing. So, we have

$$m(f(C)) = m\left(f\left(\bigcup_{n=1}^{\infty} I_n \cap C\right)\right) = m\left(\bigcup_{n=1}^{\infty} f(I_n \cap C)\right) = \lim m(f(I_n \cap C)) = 0.$$

Therefore, $m(f(C)) = 0$, with $f(C)$ the set of critical values of f. $\square$

Problem 4. Let E be a Lebesgue measurable set with $m(E) < \infty$. For each $x \in \mathbb{R}$, let $E + x = \{y + x : y \in E\}$, and define

$$f(x) = m(E \cap (E + x)).$$

Show that

(a) $f \in L^1(\mathbb{R})$, and

(b) $\lim_{|x|\to\infty} f(x) = 0$.

Proof. First, $\mathbb{1}_{E\cap(E+x)}(y)$ holds if and only if $y, y - x \in E$. Therefore, $\mathbb{1}_{E\cap(E+x)}(y) = \mathbb{1}_E(y)\mathbb{1}_E(y - x)$. So,

$$f(x) = m(E \cap (E + x)) = \int \mathbb{1}_{E\cap(E+x)}(y)\,\mathrm{d}y = \int \mathbb{1}_E(y)\mathbb{1}_E(y - x)\,\mathrm{d}y.$$

So, applying Tonelli's, we have

$$\int |f(x)|\,\mathrm{d}x = \iint \mathbb{1}_E(y)\mathbb{1}_E(y - x)\,\mathrm{d}y\,\mathrm{d}x = \int \int \mathbb{1}_E(y)\mathbb{1}_E(y - x)\,\mathrm{d}x\,\mathrm{d}y$$

We may pull $\mathbb{1}_E(y)$ to the outer integral, so that

$$\int \mathbb{1}_E(y) \int \mathbb{1}_E(y - x)\,\mathrm{d}x\,\mathrm{d}y = \int \mathbb{1}_E(y) \int \mathbb{1}_E(-x)\,\mathrm{d}x\,\mathrm{d}y = \left(\int \mathbb{1}_E(y)\,\mathrm{d}y\right)\left(\int \mathbb{1}_E(-x)\,\mathrm{d}x\right).$$

Since $\int \mathbb{1}_E(-x)\,\mathrm{d}x = \int \mathbb{1}_E(x)\,\mathrm{d}x$, we obtain $||f||_1 = m(E)^2 < \infty$, proving part (a).

Define $J_n \coloneqq [-n, n]$. Set $A_x = E \cap (E + x)$. Then,

$$m(A_x) = m((A_x\backslash J_n) \cup (A_x \cap J_n) \leqslant m(A_x\backslash J_n) + m(A_x \cap J_n) \leqslant m(E\backslash J_n) + m(A_x \cap J_n).$$

Observe that $E \cap J_n$ is monotonically increasing in n, and that $\bigcup_{n=1}^{\infty} E \cap J_n = E$. By continuity from below, we have $m(E) = \lim m(E \cap J_n)$. Since $m(E) < \infty$, then for $\epsilon > 0$ there is some n such that $m(E) - m(E \cap J_n) < \infty$. Observe that

$$A_x \cap J_n = (E \cap J_n) \cap ((E + x) \cap J_n)) \subseteq J_n \cap (J_n + x),$$

for, if $y \in E \cap J_n$ and $(E + x) \cap J_n$, then $y \in J_n$ and $y - x \in E \cap J_n \subseteq J_n$. Therefore, $y \in J_n \cap (J_n + x)$. Choose x so that $|x| > 2n$. Then, if $y \in J_n$, we have $|y| \leqslant n$, so $|y - x| \geqslant ||x| - |y|| > 2n - n = n$. In particular, $y - x \notin J_n$, since $|y - x| > n$. Therefore, $y \notin J_n \cap (J_n + x)$. So, $J_n \cap (J_n + x) = \emptyset$. It follows that for all $x \in \mathbb{R}$ satisfying $|x| > 2n$, we have

$$m(E \cap (E + x)) = m(A_x) \leqslant m(E \backslash J_n) + m(A_x \cap J_n) \leqslant \epsilon + m(J_n \cap (J_n + x)) = \epsilon.$$

We conclude that $m(E \cap (E + x)) \to 0$ as $|x| \to \infty$. □

Problem 5. For $t \in (0, \infty)$, define $f(t) := \displaystyle\int e^{-tx^2}\,\mathrm{d}x$. Show that

(a) $f'(t)$ exists, and

(b) $f'(t)$ is continuous.

Proof. Set $g(x, t) = e^{-tx^2}$. First, for all $t \in (0, \infty)$, $g(x, t)$ is a Lebesgue measurable function in x. Indeed, performing the substitution $y = \sqrt{t}x$,

$$\int e^{-tx^2}\,\mathrm{d}x = \int e^{-(\sqrt{t}x)^2}\,\mathrm{d}x = \frac{1}{\sqrt{t}} \int e^{-y^2}\,\mathrm{d}y = \frac{\sqrt{\pi}}{\sqrt{t}}.$$

Now, $|\frac{\partial}{\partial t} g(x, t)| = x^2 e^{-tx^2}$. Fix an open ray interval (a, ∞) with $a > 0$. Then, observe that $e^{-t} \leqslant e^{-a}$, hence $x^2 e^{-tx^2} \leqslant x^2 e^{-ax^2}$. There is some N sufficiently large that $x^2 \leqslant e^{ax^2/2}$ for all $|x| \geqslant N$. On the other hand, over $[-N, N]$, $x^2 e^{-ax^2}$ is a continuous function on a compact interval, and thus bounded, say by M. Therefore, by symmetry of the integral on $(-\infty, -N)$ and (N, ∞),

$$\int x^2 e^{-ax^2}\,\mathrm{d}x \leqslant \int_{-N}^{N} M\,\mathrm{d}x + 2 \int_{N}^{\infty} e^{ax^2/2} e^{-ax^2}\,\mathrm{d}x = 2NM + 2 \int e^{-(a/2)x^2}\,\mathrm{d}x.$$

Substituting $a/2$ for t in our integral computation above, we have $\int e^{-(a/2)x^2}\,\mathrm{d}x < \infty$. So, for all $t \in (a, \infty)$, $|\frac{\partial}{\partial t} g(x, t)|$ is bounded by the integrable function $x^2 e^{-ax^2}$. The criteria for differentiation under the integral are satisfied, so

$$f'(t) = \int \frac{\partial}{\partial t} g(x, t)\,\mathrm{d}x$$

for all $t \in (a, \infty)$. Since $a > 0$ was arbitrary, $f'(t)$ exists for each $t \in (0, \infty)$.

For our proof of continuity, take $t_n \to t_0$. Since $t_n, t_0 \in (0, \infty)$, then $\inf t_n > 0$, else there is an infinite sequence of t_n approaching 0, forcing $t_0 = 0$. Therefore, $\inf t_n = a > 0$. So, the sequence of functions $\frac{\partial}{\partial t} g(x, t_n)$ in x are bounded in absolute value by the integrable function

$|\frac{\partial}{\partial t} g(x,a)|$. Moreover, $\frac{\partial}{\partial t} g(x,t_n)$ are continuous in t, and hence $\lim \frac{\partial}{\partial t} g(x,t_n) = \frac{\partial}{\partial t} g(x,t_0)$, giving pointwise convergence in x. Therefore, by DCT,

$$\lim f'(t_n) = \lim \int \frac{\partial}{\partial t} g(x,t_n)\,\mathrm{d}x = \int \frac{\partial}{\partial t} g(x,t_0)\,\mathrm{d}x = f'(t_0).$$

So, $f'(t)$ is continuous. □

Problem 6. For the Lebesgue measure

(a) Define $L^\infty(\mathbb{R})$ and $||f||_\infty$ for $f \in L^\infty(\mathbb{R})$, and

(b) show that $L^\infty(\mathbb{R})$ is a Banach space.

Proof. The set $L^\infty(\mathbb{R})$ consists of all measurable functions f such that f is bounded on A and A^c is a null set. We define

$$||f||_\infty = \inf\{C \in \mathbb{R} : m(\{x \in \mathbb{C} : |f(x)| > C\}) > 0\}.$$

Now, suppose that (f_n) is a sequence of functions which is Cauchy in $L^\infty(\mathbb{R})$. We claim that $(f_n(x))$ converges for almost all $x \in \mathbb{R}$. So, $||f_m - f_n||_\infty < \epsilon$ implies that the set of x satisfying, $|f_n(x) - f_m(x)| > 2\epsilon$ is a null set. Let $A^{(k)}_{n,m}$ be the set of x such that $|f_n(x) - f_m(x)| > 2/k$. For each k, we may choose N_k such that $n, m \geqslant N_k$ implies $A^{(k)}_{n,m}$ is measure 0. Then, $A = \bigcup_{k=1}^\infty \bigcup_{n,m \geqslant N_k} A^{(k)}_{n,m}$ is a countable union of null sets and thus a null set. Every x not in A satisfies $|f_n(x) - f_m(x)| \leqslant ||f_n - f_m||_\infty < 2/k$ for all $n, m \geqslant N_k$. Hence, $(f_n(x))$ is a Cauchy sequence. By the completion of $\mathbb{R}$, it converges, so $f_n(x)$ converges pointwise a.e. to some function, say f.

We now prove that $|f|$ is almost everywhere bounded by some constant. Since (f_n) is Cauchy in L^∞, then $||f_n||_\infty$ is a bounded sequence. Suppose that it is bounded above by the constant M. Recall that for every $x \notin A$, we have $|f_n(x) - f_m(x)| \leqslant ||f_n - f_m||_\infty < 2/k$ for all $n, m \geqslant N_k$ sufficiently large. Hold k fixed. Set $B = \{x \in \mathbb{R} : |f_n(x)| > ||f_n||_\infty \text{ for some } n\}$. Observe that B is a union of null sets and hence a null set. Choose $x \notin A \cup B$. Then, $|f_n(x)| \leqslant |f_n(x) - f_m(x)| + |f_m(x)| \leqslant 2/k + M \leqslant 2M$, for all n sufficiently large. Moreover, for $x \notin A \cup B$, we have $\lim f_n(x) = f(x)$. Therefore, $|f(x)| \leqslant 2M$. So, outside of the null set $A \cup B$, f is bounded. So, $f \in L^\infty(\mathbb{R})$, hence $L^\infty(\mathbb{R})$ is Banach, for it is complete. □