Real Analysis Qual, Spring 2023

Problem 1.

(a) Demonstrate the existence of a positive function f that is both integrable and continuous on $\mathbb{R}$, but has the property that $\limsup_{x\to\infty} f(x) = \infty$.

(b) Prove that if f is both integrable and uniformly continuous on $\mathbb{R}$, then $\lim_{|x|\to\infty} f(x) = 0$.

Proof. We start with (a). Set $J_n = [2^n, 2^n + \frac{\pi}{4^n}]$. Define

$$f_n(x) := 2^n \mathbb{1}_{J_n}(x) \sin(4^n(x - 2^n)).$$

Define $F(x) := \sum_{n=1}^{\infty} f_n(x)$. Note that the J_n are pairwise disjoint, since $\pi < 4^n$, and $2^n + 1 < 2^{n+1}$. On each J_n, $F(x) = f_n(x)$, and f_n is continuous. So, F is continuous over the intervals J_n. Between consecutive J_n, J_{n+1}, is the interval $(2^n + \pi/4^n, 2^{n+1})$, on which F evaluates to 0. Moreover, F is 0 on $(-\infty, 0)$. So, F is piecewise continuous. The only possible points of discontinuity are the endpoints a_n, b_n of the intervals J_n. However, $F(a_n) = f_n(a_n) = f_n(b_n) = 0$, exactly the value of F over $(2^n + \pi/4^n, 2^{n+1})$. So, F is continuous.

Observe that the f_n are nonnegative, for if $x \notin J_n$, then $f_n(x) = 0$. Otherwise, $x \in J_n$, so $0 \leqslant 4^n(x - 2^n) \leqslant \pi$, and on this domain $\sin(t)$ is nonnegative. So, F is nonnegative. Moreover,

$$\int_{\mathbb{R}} f_n(x)\,\mathrm{d}x = \int_{2^n}^{2^n+\frac{\pi}{4^n}} 2^n \sin(4^n(x-2^n))\,\mathrm{d}x = \frac{-2^n\cos(4^n(x-2^n))}{4^n}\bigg|_{2^n}^{2^n+\pi/4^n} = \frac{2\cdot 2^n}{4^n} = \frac{2}{2^n}.$$

Define $g(n,x) = f_n(x)$. Under the counting measure, $F(x) = \int g(n,x)\,\mathrm{d}n$. Since F is nonnegative, then by Tonelli we have

$$\int F(x)\,\mathrm{d}x = \iint g(n,x)\,\mathrm{d}n\,\mathrm{d}x = \iint g(n,x)\,\mathrm{d}x\,\mathrm{d}n = \iint f_n(x)\,\mathrm{d}x\,\mathrm{d}n = \int \frac{2}{2^n}\,\mathrm{d}n = 2.$$

Given that F is nonnegative, then $F \in L^1(\mathbb{R})$ as needed.

Lastly, $\limsup_{x\to\infty} f(x) = \infty$. Consider the sequence of points $(x_n) = (2^n + \pi/(2\cdot 4^n))$. Then,

$$F\left(2^n + \frac{\pi}{2\cdot 4^n}\right) = f_n\left(2^n + \frac{\pi}{2\cdot 4^n}\right) = 2^n \sin\left(\frac{\pi}{2}\right) = 2^n.$$

Therefore, for all x, $\sup_{y\geqslant x} F(x) \geqslant F(x^n) = 2^n$. Hence, $\limsup_{x\to\infty} F(x) = \infty$.

Now we prove (b). Suppose that $\lim_{|x|\to\infty} f(x) \neq 0$. Then, for some $\epsilon > 0$, there is some infinite sequence of points (x_n) with $\lim_{n\to\infty} x_n = \infty$ such that $|f(x_n)| > \epsilon$. By uniform continuity we may choose δ such that for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon/2$. Therefore, for each δ-interval $B_\delta(x_n)$, for all $y \in B_\delta(x_n)$, we have $|f(x_n) - f(y)| < \epsilon/2$. Therefore, $|f(y)| > |f(x)| - \epsilon/2 > \epsilon/2$. Set $A = \bigcup_{n=1}^{\infty} B_\delta(x_n)$. Note that A has infinite measure. Then,

$$\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x \geqslant \int_A |f(x)|\,\mathrm{d}x > \int_A \epsilon/2\,\mathrm{d}x = m(A)\frac{\epsilon}{2} = \infty.$$

Therefore, f is not Lebesgue integrable. So, if $f \in L^1(\mathbb{R})$, then $\lim_{|x|\to\infty} f(x) = 0$. □

Problem 2.

(a) **(Classic Technique)** Let $G \subseteq \mathbb{R} \times \mathbb{R}$ be open and $f : G \longrightarrow \mathbb{R}$ be continuous. Prove that

$$F(x) \coloneqq \sup_{\{y \,:\, (x,y) \in G\}} f(x,y)$$

defines a Borel measurable function F on $\mathbb{R}$.

Hint: Recall that F is Borel measurable if $F^{-1}((a,\infty])$ is a Borel set for all $a \in \mathbb{R}$.

(The definition of F is not defined on all of $\mathbb{R}$, since there may be $x \in \mathbb{R}$, such that no $y \in \mathbb{R}$ satisfies $(x,y) \in G$. So, we assume that for every x, there exists such a y. This accords with the assumption in (b).)

(b) Prove that if g is a continuous function on $\mathbb{R}$, then the set of points where g is differentiable is a Borel measurable set, and that on this set g' is a Borel measurable function.

Hint: For each $n \in \mathbb{N}$ consider the functions

$$f_n(x,y) = \frac{g(x+y) - g(x)}{y}$$

restricted to the open sets $G_n = \{(x,y) : x \in \mathbb{R} \text{ and } 0 < |y| < 1/n\}$.

Proof. We prove (a). We show the set $A = F^{-1}((a,\infty)) = \{x \in \mathbb{R} : F(x) \in (a,\infty)\}$ is open. Take $x_0 \in A$. Then, $F(x_0) \in (a,\infty)$. Take ϵ such that $a < F(x_0) - \epsilon$. Then, by definition of sup, there is some y_0 such that $a < F(x_0) - \epsilon < f(x_0,y_0) \leqslant F(x_0)$. Since (a,∞) is open, we pick a new ϵ such that $B_\epsilon(f(x_0,y_0)) \subseteq (a,\infty)$. Since f is continuous, there is some δ such that $f(B_\delta(x_0,y_0)) \subseteq B_\epsilon(f(x_0,y_0))$. Hence, for all $(x,y_0) \in B_\delta(x_0,y_0)$, we have $f(x,y_0) > a$. So, for all $x \in (x_0 - \delta, x_0 + \delta)$, we then have $F(x) > f(x,y_0) > a$. Hence, $F(x) \in (a,\infty)$, so $x \in A$. Therefore, $(x_0 - \delta, x_0 + \delta) \subseteq A$. Since $x_0 \in A$ was chosen arbitrarily then A is open, and hence A is measurable.

For (b), take the f_n as defined. Define

$$F_n(x) \coloneqq \sup_{\{y \,:\, (x,y) \in G_n\}} f_n(x,y), \quad \text{and} \quad H_n(x) \coloneqq \inf_{\{y \,:\, (x,y) \in G_n\}} f_n(x,y).$$

By (a), F_n is a measurable function for each n. Moreover,

$$-H_n(x) = -\inf_{\{y \,:\, (x,y) \in G_n\}} f_n(x,y) = \sup_{\{y \,:\, (x,y) \in G_n\}} -f_n(x,y).$$

Since $-f_n$ are also continuous functions, then $-H_n$ is measurable for each n, again by (a). Thus, the H_n are measurable. Observe that $G_n \supseteq G_{n+1}$. Hence, for $x \in \mathbb{R}$, we have $F_n(x) \geqslant F_{n+1}(x)$ and $H_n(x) \leqslant H_{n+1}(x)$. In particular, these sequences are monotonic, so $F(x) \coloneqq \lim F_n(x)$ and $H(x) \coloneqq \lim H_n(x)$ are well defined functions over the extended reals. As limits of measurable functions, F and H are both measurable. Therefore, $A = \{x \in \mathbb{R} : F(x) = H(x)\}$ is a measurable set.

We first show that if $x_0 \in A$, then g is differentiable at x_0. So, observe that

$$H_n(x_0) \leqslant \mathbb{1}_{G_n}(x_0, y)\frac{g(x+y) - g(x)}{y} \leqslant F_n(x_0).$$

The middle term of the inequality is the difference quotient, and taking $n \to \infty$ sends $y \to 0$. Therefore, $n \to \infty$ gives $H(x_0) = g'(x_0) = F(x_0)$. On the other hand, say that g is differentiable at x_0. Suppose that $F(x_0) \neq g'(x_0)$. Then, there is some ϵ such that for all n sufficiently large, $|F_n(x_0) - g'(x_0)| > \epsilon$. Then, by definition of supremum, for each F_n we may find a corresponding y_n such that $f(x_0, y_n) - g'(x_0) > \epsilon/2$. As $n \to \infty$, then $y_n \to 0$. Howeover, by definition of f_n, we must have $f(x_0, y_n) \to g'(x_0)$. Hence, $F(x_0) = g'(x_0)$. An identical argument holds showing $H(x_0) = g'(x_0)$. Thus, $x_0 \in A$, since $H(x_0) = g'(x_0) = F(x_0)$. Therefore, A is the set of differentiable points of g. Finally, observe from our previous proofs that $F|_A = g'|_A$. Since F is a measurable function, and A is a measurable set, then $F|_A = g'|_A$ is measurable on A. □

Problem 3.

(a) Let $E \subseteq [0,1]$ be measurable with $m(E) = 0$. Prove that

$$m(\{y \in [0,1] : y^2 \in E\}) = 0.$$

Hint: First consider when $E \subseteq [a,1]$ for some $a > 0$.

(b) Prove that if f is a nonnegative measurable function on $[0,1]$, then

$$\int_{[0,1]} f(x)\,\mathrm{d}x = \int_{[0,1]} f(y^2)2y\,\mathrm{d}y.$$

Hint: Prove it for the characteristic function of an open interval, then the characteristic function of an open set, ..., and eventually for simple functions.

Using change of coordinates is not in the spirit of the problem. An alternative proof for 3a with these methods is given at the end of the document.

Proof. Suppose that $E \subseteq [\alpha, 1]$ with $\alpha > 0$. We first claim that $\sqrt{b} - \sqrt{a} < \frac{1}{\alpha}(b-a)$, for $b > a$ with $b, a \in [\alpha, 1]$. Indeed, since $a, b \in [0,1]$, then $\sqrt{a} \geqslant a$ and $\sqrt{b} \geqslant b$. Therefore, $\alpha \leqslant \sqrt{a} + \sqrt{b}$. So,

$$\alpha(\sqrt{b} - \sqrt{a}) \leqslant (\sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) = b - a.$$

Dividing both sides by α gives the result. Observe that $\sqrt{(a,b)} \subseteq (\sqrt{a}, \sqrt{b})$, since $\sqrt{\cdot}$ is monotonically increasing. So, if $I = (a,b)$, then $m(\sqrt{I}) \leqslant \frac{1}{\alpha}(b-a) = \frac{1}{\alpha}m((a,b))$. Now, since E is measurable, then there is some covering $(I_n)_{n=1}^\infty$ of E in open intervals such that $\sum_{n=1}^\infty m(I_n) < \epsilon$. Observe moreover that $\sqrt{E} \subseteq \bigcup_{n=1}^\infty \sqrt{I_n}$. Hence,

$$m(\sqrt{E}) \leqslant m\left(\bigcup_{n=1}^\infty \sqrt{I_n}\right) \leqslant \sum_{n=1}^\infty m(\sqrt{I_n}) \leqslant \sum_{n=1}^\infty \frac{1}{\alpha} m(I_n) = \frac{1}{\alpha}\epsilon.$$

Taking $\epsilon \to 0$ shows that $m(\sqrt{E}) = 0$.

We are ready to prove the full statement of (a). Take $E \subseteq [0,1]$ with $m(E) = 0$. Define $E_n = E \cap [1/n, 1]$. Note that $\bigcup_{n=1}^\infty E_n = E \backslash \{0\}$. Moreover, since $E_n \subseteq E$, then $m(E_n) = 0$. Therefore,

$$m\left(\bigcup_{n=1}^\infty \sqrt{E_n}\right) \leqslant \sum_{n=1}^\infty m(\sqrt{E_n}) = 0.$$

If $y \in \sqrt{E}$ with $y \neq 0$, then $y^2 \in E$, and so $y^2 \in E_n$ for some n. Therefore, $y \in \sqrt{E_n}$. Therefore, we conclude that $\sqrt{E}\backslash\{0\} \subseteq \bigcup_{n=1}^\infty \sqrt{E_n}$. The latter has measure 0, so $m(\sqrt{E}\backslash\{0\}) = 0$. Since $\{0\}$ has zero measure, then $m(\sqrt{E}) = 0$, completing the proof of (a).

We prove (b). Take an open interval (a,b). Then, for $f = \mathbb{1}_{(a,b)}$, we have $\int f \,\mathrm{d}x = m((a,b)) = b - a$. On the other hand, $\sqrt{(a,b)} = (\sqrt{a}, \sqrt{b})$. Therefore,

$$\int f(y^2) 2y \,\mathrm{d}y = \int \mathbb{1}_{(\sqrt{a},\sqrt{b})}(y) 2y \,\mathrm{d}y = \int_{\sqrt{a}}^{\sqrt{b}} 2y \,\mathrm{d}y = y^2 \Big|_{\sqrt{a}}^{\sqrt{b}} = b - a.$$

Now consider an open set U. We may write U as a disjoint union of open intervals $\bigcup I_\alpha$. We can guarantee this union is countable, since $\mathbb{Q}$ is dense in $\mathbb{R}$. Hence, $U = \bigcup_{n=1}^\infty I_n$, for I_n disjoint. Hence, $\mathbb{1}_U$ is the pointwise limit of $\sum_{n=1}^m \mathbb{1}_{I_n}$. Likewise, $\mathbb{1}_U(y^2)2y$ is the pointwise limit of $\sum_{n=1}^m \mathbb{1}_{I_n}(y^2)2y$. Since these are all nonnegative functions bounded by the integrable constant function 2, then by DCT,

$$\int_0^1 \mathbb{1}_U \,\mathrm{d}x = \lim_{m\to\infty} \int_0^1 \sum_{n=1}^m \mathbb{1}_{I_n} \,\mathrm{d}x = \lim_{m\to\infty} \sum_{n=1}^m \int_0^1 \mathbb{1}_{I_n} \,\mathrm{d}x = \lim_{m\to\infty} \sum_{n=1}^m \int_0^1 \mathbb{1}_{I_n}(y^2) 2y \,\mathrm{d}y,$$

and by DCT again,

$$\lim_{m\to\infty} \sum_{n=1}^m \int_0^1 \mathbb{1}_{I_n}(y^2)2y \,\mathrm{d}y = \lim_{m\to\infty} \int_0^1 \sum_{n=1}^m \mathbb{1}_{I_n}(y^2) 2y \,\mathrm{d}y = \int_0^1 \mathbb{1}_U(y^2) 2y \,\mathrm{d}y.$$

So, $\int_0^1 \mathbb{1}_U \,\mathrm{d}x = \int_0^1 \mathbb{1}_U(y^2)2y \,\mathrm{d}y$. Let U_n be a sequence of open sets such that $U_n \supseteq U_{n+1}$. Set $G = \bigcap_{n=1}^\infty U_n$. Then, $\mathbb{1}_G = \lim \mathbb{1}_{U_n}$, for if $x \in G$, then $x \in U_n$ for all n. Otherwise, if $x \notin G$, then there is some n such that $x \notin U_n$. By monotonicity, for all $m \geqslant n$, we then have $x \notin U_m$. Hence, $\mathbb{1}_G = \lim \mathbb{1}_{U_n}$ as claimed. For the same reason, $\mathbb{1}_G(y^2)2y = \lim \mathbb{1}_{U_n}(y^2)2y$. Again, these functions are all bounded by the integrable constant function 2. So, by DCT,

$$\int_0^1 \mathbb{1}_G \,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_{U_n} \,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_{U_n}(y^2) 2y \,\mathrm{d}y = \int_0^1 \mathbb{1}_G(y^2) 2y \,\mathrm{d}y.$$

Note that any G_δ set is an intersection of open sets $\bigcap_{n=1}^\infty U_n$. Moreover, since $\bigcap_{n=1}^\infty U_n = \bigcap_{n=1}^\infty \bigcap_{k=1}^n U_k$, so that the $\bigcap_{k=1}^n U_k$ decrease monotonically, then for any G_δ set G, we have $\int_0^1 \mathbb{1}_G \,\mathrm{d}x = \int_0^1 \mathbb{1}_G(y^2)2y \,\mathrm{d}y$.

Let E be an arbitrary measurable set. Then, by regularity, there is some G_δ set G such that $m(G\backslash E) = 0$ and $G \supseteq E$. We claim that $\sqrt{G\backslash E} \supseteq \sqrt{G}\backslash\sqrt{E}$. Indeed, if $y \in \sqrt{G}\backslash\sqrt{E}$, then $y^2 \in G$ but $y^2 \notin E$. Hence $y^2 \in G\backslash E$, so $y \in \sqrt{G\backslash E}$. By (a), $m(\sqrt{G\backslash E}) = 0$.

Therefore, $m(\sqrt{G}\backslash\sqrt{E}) = 0$. It follows that $\mathbb{1}_{\sqrt{G}} - \mathbb{1}_{\sqrt{E}}$ is an almost everywhere 0 function. So,

$$2y(\mathbb{1}_{\sqrt{G}}(y) - \mathbb{1}_{\sqrt{E(y)}}) = 2y(\mathbb{1}_G(y^2) - \mathbb{1}_E(y^2))$$

is an almost everywhere zero function. We conclude that $\int_0^1 2y\mathbb{1}_G(y^2)\,\mathrm{d}y = \int_0^1 2y\mathbb{1}_E(y^2)\,\mathrm{d}y$. Hence,

$$\int_0^1 \mathbb{1}_E\,\mathrm{d}x = \int_0^1 \mathbb{1}_G\,\mathrm{d}x = \int_0^1 \mathbb{1}_G(y^2)2y\,\mathrm{d}y = \int_0^1 \mathbb{1}_E(y^2)2y\,\mathrm{d}y.$$

By linearity of the integral, we may then obtain for arbitrary simple functions that $\int_0^1 \phi\,\mathrm{d}x = \int_0^1 \phi(y^2)2y\,\mathrm{d}y$.

Finally, let f be a measurable function with defined integral. We first suppose f is nonnegative. Take ϕ_n to be a sequence of nonnegative simple functions converging to f monotonically. Then, $2y\phi_n(y^2)$ converges pointwise to $2yf(y^2)$ monotonically. Therefore, by MCT, we obtain

$$\int_0^1 f\,\mathrm{d}x = \lim \int_0^1 \phi_n\,\mathrm{d}x = \lim \int_0^1 \phi_n(y^2)2y\,\mathrm{d}y = \int_0^1 f(y^2)2y\,\mathrm{d}y.$$

Finally, the result is obtained for f an arbitrary measurable function with defined integral by splitting f into negative and positive parts, and repeating the same argument on each part. $\square$

Problem 4. Let

$$F(t) = \int_0^\infty e^{-x^3\sin(t)}\,\mathrm{d}x.$$

(a) Prove that F is a well-defined real-valued differentiable function for all $t \in (0,\pi)$ with derivative

$$F'(t) = -\cos(t)\int_0^\infty x^3e^{-x^3\sin(t)}\,\mathrm{d}x.$$

(b) Prove that F has the further property that

$$\lim_{t\to 0^+} F(t) = \lim_{t\to\pi^-} F(t) = \infty.$$

Proof. Define $f(x,t) = e^{-x^3\sin(t)}$, and note that $\frac{\partial}{\partial t}f(x,t) = -\cos(t)x^3e^{-x^3\sin(t)}$. We claim that $f(x,t)$ is integrable in x for each fixed $t \in (0,\pi)$. Indeed, on $(0,1)$, $f(x,t)$ is bounded above by 1. On $(1,\infty)$, $e^{-x^3\sin(t)}$ is bounded by $x^3e^{-x^3\sin(t)}$, an integrable function. Hence, $f(x,t)$ is integrable. Choose $t_0 \in (0,\pi)$. Take ϵ small so that $B_\epsilon(t_0) \subseteq (0,\pi)$, and so that $t_0-\epsilon, t_0+\epsilon \in (0,\pi)$. Since $\sin(t)$ is continuous over the compact interval $[t_0-\epsilon, t_0+\epsilon]$, then it attains a minimal value α. Moreover, $\alpha \neq 1$, for this requires $t \in [t_0-\epsilon, t_0+\epsilon]$ such that $t = 0, \pi$. Therefore, given that $x \geqslant 0$, we have

$$|-\cos(t)x^3e^{-x^3\sin(t)}| \leqslant x^3e^{-x^3\sin(t)} \leqslant x^3e^{-x^3\alpha}.$$

Moreover,

$$\int_0^\infty x^3e^{-x^3\alpha}\,\mathrm{d}x = -\frac{1}{\alpha}e^{-x^3}|_0^\infty = \frac{1}{\alpha}.$$

Since $x^3e^{-x^3\alpha}$ is nonnegative, then it is Lebesgue integrable. Therefore, $|\frac{\partial}{\partial t}f(x,t)|$ is bounded by an integrable function for all $(x,t) \in (0,\infty) \times (t_0 - \epsilon, t_0 + \epsilon)$. Therefore, restricting F to the domain $(t_0 - \epsilon, t_0 + \epsilon)$, we may differentiate under the integral sign to obtain

$$F'(t_0) = \int_0^\infty \frac{\partial}{\partial t} f(x,t_0)\,\mathrm{d}x = \int_0^\infty -\cos(t_0)x^3e^{-x^3\sin(t_0)}\,\mathrm{d}x = -\cos(t_0)\int_0^\infty x^3e^{-x^3\sin(t_0)}\,\mathrm{d}x,$$

proving part (a).

Take a sequence (t_n) so that $t_n \to 0$. We enforce that this sequence by strictly monotonic, and we may further assume $(t_n) \subseteq (0, \pi/2)$. On $(0,\pi/2)$, $\sin(t)$ increases montonically, so $\sin(t_n)$ is monotonically decreasing to 0. Moreover, on $(0,\infty)$, e^{-x} decreases monotonically in x, so $e^{-x^3\sin(t_n)}$ increases monotonically as n increases, given that $\sin(t)$ is nonnegative on $(0,\pi)$. Therefore, $f_n(x) = f(x,t_n)$ is a sequence of monotonically increasing nonnegative functions. Furthermore, pointwise, given that $\sin(t_n) \to 0$, we have $f_n(x) \to 1$. Therefore, by MCT, we obtain

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty}\int_0^\infty f_n(x)\,\mathrm{d}x = \int_0^\infty 1\,\mathrm{d}x = \infty.$$

Hence, for $y \to 0^+$, we have $F(y) \to \infty$. We observe now that in the case of $y \to \pi^-$, we may proceed with an identical argument by taking $t_n \to \pi^-$, enforcing that the t_n increase monotonically, and that $(t_n) \subseteq (\pi/2, \pi)$. Then, as before, we obtain that $\sin(t_n)$ decreases montonically, so that $e^{-x^3\sin(t_n)}$ increases monotonically. We achieve a monotonically increasing sequence f_n, observe that $f_n \to 1$ pointwise, and apply MCT to once again obtain $\lim_{n\to\infty} F(t_n) = \infty$. □

Problem 5.

(a) Show, without appealing to methods from complex analysis, that

$$A := \int_0^\infty \frac{1}{(1+y)\sqrt{y}}\,\mathrm{d}y < \infty.$$

(b) Let $h(x) = \log(x)/x$ for all $x \in [0,\infty)$. Prove that $h \in L^2([0,\infty])$ with $||h||_2 \leqslant A$, by showing that

$$\left|\int_0^\infty \frac{\log(1+x)}{x} f(x)\,\mathrm{d}x\right| \leqslant A||f||_2$$

for all $f \in L^2([0,\infty))$.

Hint: Use the fact that $\log(1+x) = \displaystyle\int_0^x \frac{1}{1+y}\,\mathrm{d}y$.

Proof. We prove (a). Observe that $\frac{1}{(1+y)\sqrt{y}} \leqslant \frac{1}{\sqrt{y}}$. Moreover, $\frac{1}{\sqrt{y}}$ is Lebesgue integrable. Indeed, take $t_n \to 0$ monotonically. Then, $\mathbb{1}_{(t_n,1]}\frac{1}{\sqrt{y}} \to \frac{1}{\sqrt{y}}$ monotonically. Hence, by MCT,

$$\int_0^1 \frac{1}{\sqrt{y}}\,\mathrm{d}y = \lim \int_0^1 \mathbb{1}_{(t_n,1]}\frac{1}{\sqrt{y}}\,\mathrm{d}y = \lim\int_{t_n}^1 \frac{1}{\sqrt{y}}\,\mathrm{d}y = \lim 2\sqrt{y}\Big|_{t_n}^1 = 2 - \lim 2\sqrt{t_n} = 2.$$

On the other hand, $\frac{1}{(1+y)\sqrt{y}} \leqslant \frac{1}{y^{3/2}}$. We claim that $\frac{1}{y^{3/2}}$ is Lebesgue Integrable on $[1, \infty)$. Take $t_n \to \infty$ monotonically. Then, $\mathbb{1}_{[1,t_n]} \frac{1}{y^{3/2}} \to \frac{1}{y^{3/2}}$ monotonically. By MCT again,

$$\int_1^\infty \frac{1}{y^{3/2}} \, \mathrm{d}y = \lim \int \mathbb{1}_{[1,t_n]} \frac{1}{y^{3/2}} \, \mathrm{d}y = \lim \int_1^{t_n} \frac{1}{y^{3/2}} \, \mathrm{d}y = \lim -2y^{-1/2} \Big|_1^{t_n} = 2.$$

So, defining

$$f(y) := \begin{cases} \frac{1}{\sqrt{y}}, & \text{if } y \in (0, 1), \\ \frac{1}{y^{3/2}}, & \text{if } y \in [1, \infty), \end{cases}$$

we obtain $\frac{1}{(1+y)\sqrt{y}} \leqslant f(y)$, with $f(y)$ having finite integral. Since both of these functions are nonnegative, we conclude that A is finite.

Now, we move on to (b). So,

$$\begin{aligned} \left| \int_0^\infty \frac{\log(1+x)}{x} f(x) \, \mathrm{d}x \right| &= \left| \int_0^\infty \int_0^x \frac{1}{1+y} \, \mathrm{d}y \frac{f(x)}{x} \, \mathrm{d}x \right| \\ &= \left| \int_0^\infty \int_0^x \frac{1}{1+y} \left(\frac{f(x)}{x} \right) \mathrm{d}y \, \mathrm{d}x \right| \\ &\leqslant \int_0^\infty \int_0^x \left| \frac{1}{1+y} \right| \cdot \left| \frac{f(x)}{x} \right| \mathrm{d}y \, \mathrm{d}x. \end{aligned}$$

Let B be the region of integration for the above double integral. Observe that all points (x, y) satisfy $0 < x < \infty$ and $0 < y < x$. Equivalently, they all satisfy $0 < y < \infty$ and $y < x < \infty$. Hence, applying Tonelli's Theorem,

$$\begin{aligned} \int_0^\infty \int_0^x \left| \frac{1}{1+y} \right| \cdot \left| \frac{f(x)}{x} \right| \mathrm{d}y \, \mathrm{d}x &= \int_B \left| \frac{1}{1+y} \right| \cdot \left| \frac{f(x)}{x} \right| \mathrm{d}(x \times y) \\ &= \int_0^\infty \int_y^\infty \left| \frac{1}{1+y} \right| \cdot \left| \frac{f(x)}{x} \right| \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^\infty \left| \frac{1}{1+y} \right| \left(\int_y^\infty |f(x)| \cdot \left| \frac{1}{x} \right| \mathrm{d}x \right) \mathrm{d}y. \end{aligned}$$

On the domain (y, ∞), since $y > 0$, we have

$$\int_y^\infty \left| \frac{1}{x^2} \right| \mathrm{d}x = \int_y^\infty \frac{1}{x^2} \, \mathrm{d}x = -\frac{1}{x} \Big|_y^\infty = \frac{1}{y}.$$

Therefore, $||1/x||_2 = \frac{1}{\sqrt{y}}$. So, by Cauchy-Schwarz we obtain

$$\begin{aligned} \int_0^\infty \left| \frac{1}{1+y} \right| \left(\int_y^\infty |f(x)| \cdot \left| \frac{1}{x} \right| \mathrm{d}x \right) \mathrm{d}y &\leqslant \int_0^\infty \left| \frac{1}{1+y} \right| (||1/x||_2 \cdot ||f||_2) \, \mathrm{d}y \\ &= \int_0^\infty \frac{1}{(1+y)\sqrt{y}} ||f||_2 \, \mathrm{d}y \\ &= A||f||_2. \end{aligned}$$

So, the linear function $\Lambda(f) = \int_0^\infty (\log(1+x)/x) f(x)\,\mathrm{d}x$ satisfies $||\Lambda|| \leqslant A$. By duality, we observe then that $\log(x+1)/x \in L^2((0,\infty))$ and that $||(x+1)/x||_2 = ||\Lambda|| \leqslant A$. Finally, h is bounded by $\log(x+1)/x$ by monotonicity of the log function. Therefore, $h \in L^2((0,\infty))$ and $||h||_2 \leqslant A$. $\square$

Alternative Proof of 3(a)

Proof. Set $F = \{y \in [0,1] : y^2 \in E\}$. Since E is a measurable set, then $\mathbb{1}_E$ is a Lebesgue integrable function. Moreover, $\phi : [0,1] \longrightarrow [0,1]$ by $y \longmapsto y^2$ is a continuous and differentiable function, with invertible derivative. Hence, ϕ is a $[0,1] \longmapsto [0,1]$ diffeomorphism. Furthermore, $|\det D_\phi(y)| = |2y|$. So,

$$0 = m(E) = \int_0^1 \mathbb{1}_E(x)\,\mathrm{d}x = \int_0^1 \mathbb{1}_E(y^2)2y\,\mathrm{d}y.$$

Therefore, since $2y\mathbb{1}_E(y^2)$ is nonnegative, then it is almost everywhere 0. Since $2y = 0$ only at $y = 0$, we conclude that $\mathbb{1}_E(y^2)$ is almost everywhere 0. Moreover,

$$\{y \in [0,1] : \mathbb{1}_E(y^2) = 1\} = \{y \in [0,1] : y^2 \in E\} = F.$$

Therefore, $\mathbb{1}_E(y^2) = \mathbb{1}_F(y)$. Thus, we obtain

$$0 = \int_0^1 \mathbb{1}_E(y^2)\,\mathrm{d}y = \int_0^1 \mathbb{1}_F(y)\,\mathrm{d}y = m(F).$$

So, $m(F) = 0$ as needed. $\square$