Real Analysis Qual, Spring 2025

Problem 1. Let f be a continuous complex valued function on $[0,1]$. Show that

$$||f||_\infty = \sup\{|\lambda| : \lambda \in \mathbb{C}, m(f^{-1}(B_\epsilon(\lambda))) > 0, \text{ if } \epsilon > 0\}.$$

Proof. Let $A = \{|\lambda| : \lambda \in \mathbb{C}, m(f^{-1}(B_\epsilon(\lambda))) > 0, \text{ if } \epsilon > 0\}$. We first prove that if $a \in A$, then $a \leqslant ||f||_\infty$. So, take $a \in A$. Then, $a = |\lambda|$ such that for all $\epsilon > 0$, we have $m(f^{-1}(B_\epsilon(\lambda))) > 0$. Recall that

$$||f||_\infty = \sup\{C \in \mathbb{R} : m(\{x \in \mathbb{R} : |f(x)| > C\}) > 0\}.$$

Observe that

$$f^{-1}(B_\epsilon(\lambda)) = \{x \in \mathbb{R} : f(x) \in B_\epsilon(\lambda)\} = \{x \in \mathbb{R} : |f(x) - \lambda| < \epsilon\}.$$

Given that $|\lambda| - |f(x)| \leqslant ||\lambda| - |f(x)|| \leqslant |f(x) - \lambda| < \epsilon$, we have $|f(x)| > |\lambda| - \epsilon$. So,

$$f^{-1}(B_\epsilon(\lambda)) = \{x \in \mathbb{R} : |f(x) - \lambda| < \epsilon\} \subseteq \{x \in \mathbb{R} : |f(x)| > |\lambda| - \epsilon\}.$$

Since $f^{-1}(B_\epsilon(\lambda))$ has positive measure for all ϵ, then $\{x \in \mathbb{R} : |f(x)| > |\lambda| - \epsilon\}$ has positive measure for all ϵ. Therefore $a = |\lambda| \leqslant ||f||_\infty$.

Set $B = \{C \in \mathbb{R} : m(\{x \in \mathbb{R} : |f(x)| > C\}) > 0\})$. Observe that $|f|$ is a continuous function on a compact set, and hence it has a maximum value $|f(x_0)| = M$. We must have $\sup B \leqslant M$, for if $C > M$, then there is no x such that $|f(x)| > C$. On the other hand, we have $M \leqslant \sup B$, since $|f|^{-1}((M - \epsilon, \infty))$ is an open set containing x_0. Hence, it is a nonempty open set and thus must have positive measure. In particular, the set of elements x satisfying $|f(x)| > M - \epsilon$ is positive for all ϵ, therefore $M \leqslant \sup B$, forcing equality. Set $y = f(x_0)$, and note that $f^{-1}(B_\epsilon(y))$ is an open set containing x_0. So, it is a nonempty open set and thus has positive measure. Therefore, for all ϵ, $m(f^{-1}(B_\epsilon(y))) > 0$. It follows that $M = |y| \leqslant \sup A$. Therefore, $||f||_\infty = \sup A$. $\square$

Problem 2. Show that Lebesgue measurable functions f_n defined on $[a,b]$ converge in measure to f if and only if

$$\lim_{n\to\infty} \int_a^b \frac{|f_n - f|}{1 + |f_n - f|}\, \mathrm{d}x = 0.$$

Proof. First, observe that

$$\frac{d}{dx}\frac{x}{1+x} = \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2} > 0.$$

So, $\frac{x}{1+x}$ increases monotonically. Now suppose that

$$\lim_{n\to\infty} \int_a^b \frac{|f_n - f|}{1 + |f_n - f|}\, \mathrm{d}x = 0.$$

Say that f_n does not converge to f in measure. Then, for some ϵ, there is some δ and some subsequence f_{n_k} such that $m(\{x \in \mathbb{R} : |f(x) - f_{n_k}(x)| > \delta\}) > \epsilon$. Set $A_k = \{x \in \mathbb{R} :$

$|f(x) - f_{n_k}| > \delta\}$. Then, since $\frac{x}{1+x}$ is monotonically increasing and $\delta < |f_{n_k} - f|$ over A_k, we have

$$\int_a^b \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \,\mathrm{d}x > \int_{A_k} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \,\mathrm{d}x > \int_{A_k} \frac{\delta}{1+\delta} \,\mathrm{d}x > \epsilon \frac{\delta}{1+\delta},$$

contradicting the original convergence.

Now, suppose that $f_n \to f$ in measure. Define $E = \{x \in [a,b] : |f_n(x) - f(x)| \leqslant \epsilon\}$. Then,

$$\int_a^b \frac{|f_n - f|}{1 + |f_n - f|} \,\mathrm{d}x = \int_E \frac{|f_n - f|}{1 + |f_n - f|} \,\mathrm{d}x + \int_{E^c} \frac{|f_n - f|}{1 + |f_n - f|} \,\mathrm{d}x \leqslant \int_E \epsilon \,\mathrm{d}x + \int_{E^c} 1 \,\mathrm{d}x.$$

Moreover,

$$\int_E \epsilon \,\mathrm{d}x + \int_{E^c} 1 \,\mathrm{d}x \leqslant \int_a^b \epsilon \,\mathrm{d}x + m(E^c) = (b-a)\epsilon + m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}).$$

By convergence in measure, $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}) \to 0$ as $n \to \infty$. Choose N such that for all $n \geqslant N$ we have $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$. Then, we obtain for $n \geqslant N$ that

$$\int_a^b \frac{|f_n - f|}{1 + |f_n - f|} < (b - a + 1)\epsilon.$$

taking $\epsilon \to 0$ and $n \to \infty$ gives the result.

$\square$

Problem 3. Show that if f is Riemann integrable on $[a,b]$, and $f(x) = 0$ for $x \in \mathbb{Q} \cap [a,b]$ then $\int_a^b f(x) \,\mathrm{d}x = 0$.

Proof. Since f is Riemann integrable, then $\int_a^b f \,\mathrm{d}x = L(f, [a,b]) = U(f, [a,b])$, the lower and upper Riemann integrals respectively. Take a partition $\mathcal{P} = \{x_0 < \ldots < x_n\}$ of $[a,b]$. Then, since $[x_{j-1}, x_j] \cap \mathbb{Q} \neq \emptyset$, we have $\inf_{[x_{j-1},x_j]} f(x) \leqslant 0$. Hence,

$$L(f, \mathcal{P}, [a,b]) = \sum_{j=1}^n \inf_{[x_{j-1},x_j]} f(x)(x_j - x_{j-1}) \leqslant \sum_{j=1}^n 0 = 0.$$

Hence, $L(f, [a,b]) = \sup_{\mathcal{P}} L(f, \mathcal{P}, [a,b]) \leqslant 0$. Observe that $\sup_{[x_{j-1},x_j]} f(x) \geqslant 0$, since $\mathbb{Q} \cap [x_{j-1}, x_j] \neq \emptyset$. Therefore,

$$U(f, \mathcal{P}, [a,b]) = \sum_{j=1}^n \sup_{[x_{j-1},x_j]} f(x)(x_j - x_{j-1}) \geqslant \sum_{j=1}^n 0 = 0.$$

Since $U(f, [a,b]) = \inf_{\mathcal{P}} U(f, \mathcal{P}, [a,b])$, then $U(f, [a,b]) \geqslant 0$.

Therefore,

$$0 \leqslant U(f, [a,b]) = \int_a^b f \,\mathrm{d}x = L(f, [a,b]) \leqslant 0.$$

So, $\int_a^b f \,\mathrm{d}x = 0$. $\square$

Problem 4. Let $f_n \in L^2([0,1])$ be a sequence that is bounded in norm $||\cdot||_1$ and $f_n \to 0$ a.e.. Show that $\lim_{n\to\infty} ||f_n||_1 = 0$.
Hint: Use Egoroff and Cauchy-Schwarz.

Proof. By Egoroff's, choose $E \subseteq [0,1]$ such that $m(E) < \epsilon^2$ and $f_n \to 0$ uniformly on E^c. Take N so that for all $n \geqslant N$, $|f_n| < \epsilon$. Then,

$$\int_0^1 |f_n| \, \mathrm{d}x = \int_E |f_n| \, \mathrm{d}x + \int_{E^c} |f_n| \, \mathrm{d}x \leqslant \int \mathbb{1}_E |f_n| \, \mathrm{d}x + \int_{E^c} \epsilon \, \mathrm{d}x \leqslant ||\mathbb{1}_E||_2 ||f_n||_2 + \int_0^1 \epsilon \, \mathrm{d}x.$$

Now, $||\mathbb{1}_E||_2 = \sqrt{m(E)} < \epsilon$. Furthermore, $||f_n||_2$ is globally bounded by some constant, M. Therefore,

$$||\mathbb{1}_E||_2 ||f_n||_2 + \int_0^1 \epsilon \, \mathrm{d}x \leqslant \epsilon M + \epsilon = \epsilon(M+1).$$

Taking $\epsilon \to 0$ gives the result. □

Problem 5. Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of continuous functions g_n with compact supports such that $\lim_{n\to\infty} ||f - g_n||_1 = 0$.

The correct technique for this problem is to approximate interval indicator functions by trapezoids. Then, approximate general functions by linear sums of interval indicator functions. I am doing it with convolutions to improve my understanding of convolutions.

Proof. We first do this for integrable simple functions. Let J be some interval $[-\alpha, \alpha]$. Let E be a bounded measurable set. Set $g = \mathbb{1}_J / m(J)$. We claim that $\mathbb{1}_E * g$ is a compactly supported continuous function. First, we prove it has compact support. Since J, E are both bounded, then they are contained in some set $[-N, N]$. Take $x \notin [-2N, 2N]$. Then, $|x| > 2N$. For $y \in J$, $|x - y| > |x| - |y| \geqslant 2N - |y| \geqslant N$, given that $|y| \leqslant N$. Therefore, if $y \in J$, then $(x - y) \notin E, J$. So, $\mathbb{1}_E(x-y) g(y) = 0$. Hence,

$$\mathbb{1}_E * g(x) = \int \mathbb{1}_E(x-y) g(y) \, \mathrm{d}y = \int 0 \, \mathrm{d}y = 0.$$

Hence $\mathbb{1}_E * g$ is supported on the compact domain $[-2N, 2N]$. We claim that for any ϵ we may pick J so that $||\mathbb{1}_E * g - \mathbb{1}_E||_1 < \epsilon$. Now, observe that $\int g(y) \, \mathrm{d}y = 1$. So,

$$\begin{aligned}
||\mathbb{1}_E * g - \mathbb{1}_E||_1 &= \int \left| \int \mathbb{1}_E(x-y) g(y) \, \mathrm{d}y - \mathbb{1}_E(x) \right| \mathrm{d}x \\
&= \int \left| \int \mathbb{1}_E(x-y) g(y) \, \mathrm{d}y - \int g(y) \mathbb{1}_E(x) \, \mathrm{d}y \right| \mathrm{d}x \\
&= \int \left| \int \mathbb{1}_E(x-y) g(y) - g(y) \mathbb{1}_E(x) \, \mathrm{d}y \right| \mathrm{d}x \\
&\leqslant \int \int |(\mathbb{1}_E(x-y) - \mathbb{1}_E(x)) g(y)| \, \mathrm{d}y \, \mathrm{d}x.
\end{aligned}$$

Applying Tonelli's Theorem, we have

$$\begin{aligned}\int\int |(\mathbb{1}_E(x-y)-\mathbb{1}_E(x))g(y)|\,\mathrm{d}y\,\mathrm{d}x &= \int\int |(\mathbb{1}_E(x-y)-\mathbb{1}_E(x))g(y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int ||\mathbb{1}_E(\cdot-y)-\mathbb{1}_E||_1 g(y)\,\mathrm{d}y \\ &= \int_J \frac{||\mathbb{1}_E(\cdot-y)-\mathbb{1}_E||_1}{m(E)}\,\mathrm{d}y.\end{aligned}$$

The L^1 norm is continuous under translation. Taking $y\to 0$, we have $||\mathbb{1}_E(\cdot-y)-\mathbb{1}_E||_1\to 0$. Since $J=[-\alpha,\alpha]$ we choose α small enough that, given $y\in J$, then $||\mathbb{1}_E(\cdot-y)-\mathbb{1}_E||_1<\epsilon$. Therefore,

$$\int_J \frac{||\mathbb{1}_E(\cdot-y)-\mathbb{1}_E||_1}{m(E)}\,\mathrm{d}y \leqslant \int_J \frac{\epsilon}{m(J)}\,\mathrm{d}y=\epsilon.$$

Therefore, we may choose J such that $\mathbb{1}_E*(\mathbb{1}_J/m(J))$ is a compactly supported function within ϵ of $\mathbb{1}_E$ in the L^1 norm. Lastly, we prove that $\mathbb{1}_E*(\mathbb{1}_J/m(J))$ is a continuous function.

So, note that $\mathbb{1}_E(x-y)\mathbb{1}_J(y)/m(J)=\mathbb{1}_{(x-E)\cap J}(y)\leqslant \mathbb{1}_J(y)/m(J)$. Hence, for all x, $\mathbb{1}_E(x-y)\mathbb{1}_J(y)/m(J)$ is bounded by an integrable function in y. Now, take a sequence (x_n) with $x_n\to x_0$. Then, by DCT,

$$\begin{aligned}\lim \mathbb{1}_E*(\mathbb{1}_J/m(J))(x_n) &= \lim\int \mathbb{1}_E(x_n-y)\mathbb{1}_J(y)/m(J)\,\mathrm{d}y \\ &= \int \mathbb{1}_E(x_0-y)\mathbb{1}_J(y)/m(J)\,\mathrm{d}y \\ &= \mathbb{1}_E*(\mathbb{1}_J/m(J))(x_0).\end{aligned}$$

Hence, continuity is obtained. Therefore, measurable indicator functions may be approximated in the L^1 norm by a compactly supported continuous function f.

Now, let ϕ be an arbitrary integrable simple function. Then, ϕ is a linear sum of indicator functions of finite measurable sets. Hence, by triangle inequality of the L^1 norm, we can approximate it via a linear sum of compactly supported continuous functions. This function f is likewise continuous and compactly supported. Since simple functions are dense in L^1, and compactly supported continuous functions may be taken arbitrarily close to simple functions, then compactly supported continuous functions are dense in L^1, proving the result. □

Problem 6. Let $f\in L^1([0,1])$. Prove that

$$\lim_{n\to\infty}\int_0^1 f(x)|\sin(nx)|\,\mathrm{d}x=\frac{2}{\pi}\int_0^1 f(x)\,\mathrm{d}x.$$

Hint: Start with the easiest function f first.

Proof. Let f be the characteristic function of an interval. So, we show that

$$\lim_{n\to\infty}\int_a^b |\sin(nx)|\,\mathrm{d}x=\frac{2}{\pi}.$$

We first integrate a single arc of $|\sin(nx)|$ over $(0, \pi/n)$. So, making the change of coordinates $y = nx$, we observe that y ranges from 0 to π, and that $\frac{1}{n}\,\mathrm{d}y = \mathrm{d}x$.

$$\int_0^{\pi/n} |\sin(nx)| = \int_0^{\pi/n} \sin(nx)\,\mathrm{d}x = \frac{1}{n}\int_0^{\pi} \sin(y)\,\mathrm{d}y = \frac{1}{n}\left(-\cos(y)\Big|_0^{\pi}\right) = \frac{2}{n}.$$

Since each arc of $|\sin(nx)|$ is positive, and $|\sin(nx)|$ has a period of $(0, \pi/n)$, then each arc has the integral $\frac{2}{n}$ via translation. We count the minimum and maximum number of arcs of $|\sin(nx)|$ on (a, b) to estimate its integral. So, there are at most $\lfloor \frac{n}{\pi}(b-a) \rfloor$ complete arcs of $|\sin(nx)|$ on (a, b). Two arcs may be partially terminated at the point a or b. Hence, the total number of arcs is bounded below by $\lfloor \frac{n}{\pi}(b-a) \rfloor - 2$. So, we estimate

$$\frac{n}{\pi}(b-a) - 3 \leqslant \left\lfloor \frac{n}{\pi}(b-a) \right\rfloor - 2 \leqslant \#\text{ of arcs} \leqslant \left\lfloor \frac{n}{\pi}(b-a) \right\rfloor \leqslant \frac{n}{\pi}(b-a).$$

Say that J_n is the smallest interval on which all full arcs of $|\sin(nx)|$ are completed. The integral value contributed by each arc is $\frac{2}{n}$. So

$$\frac{2}{\pi}(b-a) - \frac{6}{n} \leqslant \int_{J_n} |\sin(nx)|\,\mathrm{d}x \leqslant \frac{2}{\pi}(b-a).$$

Each arc occures over a length π/n interval, so $(b-a) - \frac{3\pi}{n} \leqslant m(J_n) \leqslant (b-a)$. Then,

$$\frac{2}{\pi}(b-a) - \frac{6}{n} \leqslant \int_{J_n} |\sin(nx)|\,\mathrm{d}x \leqslant \int |\sin(nx)|\,\mathrm{d}x \leqslant \int_{I\setminus J_n} 1\,\mathrm{d}x + \int_{J_n} |\sin(nx)|\,\mathrm{d}x.$$

Therefore,

$$\frac{2}{\pi}(b-a) - \frac{6}{n} \leqslant \int_a^b |\sin(nx)|\,\mathrm{d}x \leqslant m(I) - m(J_n) + \frac{2}{\pi}(b-a).$$

Taking $n \to \infty$, we observe that $m(J_n) \to b - a$, so $m(I) - m(J_n) \to 0$. Moreover, $\frac{6}{n} \to 0$. Therefore,

$$\lim \int_a^b |\sin(nx)|\,\mathrm{d}x = \frac{2}{\pi}(b-a) = \frac{2}{\pi}\int_0^1 \mathbb{1}_{(a,b)}\,\mathrm{d}x.$$

By linearity of the integral, if ϕ is a step function, then we obtain

$$\lim \int_a^b |\sin(nx)|\phi\,\mathrm{d}x = \frac{2}{\pi}\int \phi\,\mathrm{d}x.$$

Now, let f be an arbitrary function. Since the Lebesgue measure is, in particular, a Lebesgue-Stieltjes measure, then we may approximate f in L^1 by step functions. So, take ϕ such that $||f - \phi||_1 < \epsilon$. Then,

$$\left|\int_0^1 f|\sin(nx)|\,\mathrm{d}x - \int_0^1 \phi|\sin(nx)|\,\mathrm{d}x\right| \leqslant \int_0^1 |f-\phi||\sin(nx)|\,\mathrm{d}x \leqslant \int_0^1 |f-\phi|\,\mathrm{d}x \leqslant \epsilon$$

for all n. Therefore,

$$\left|\int_0^1 f(x)|\sin(nx)|\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 f\,\mathrm{d}x\right| \leqslant \left|\int_0^1 (f-\phi)|\sin(nx)|\,\mathrm{d}x\right|$$

$$+\left|\int_0^1 \phi|\sin(nx)|\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 \phi\,\mathrm{d}x\right| + \left|\frac{2}{\pi}\int_0^1 \phi - f\,\mathrm{d}x\right|.$$

So,

$$\left|\int_0^1 f(x)|\sin(nx)|\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 f\,\mathrm{d}x\right| \leqslant \epsilon + \frac{2}{\pi}\epsilon + \left|\int_0^1 \phi|\sin(nx)|\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 \phi\,\mathrm{d}x\right|.$$

Taking $n \to \infty$ shows that we may take $\int_0^1 f|\sin(nx)|\,\mathrm{d}x$ to be within $2\epsilon + 2\frac{\pi}{\epsilon}$ of $\int_0^1 f\,\mathrm{d}x$ for all ϵ, giving the result. $\square$