Real Analysis Qual, Fall 2021

Problem 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and

$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{x_n + 1}{x_n + 2}.$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges, and find its limit.

Proof. We first show the sequence is bounded for $n \geqslant 2$. First, we prove that $x_n \geqslant 0$ for all n. At x_1, this holds. Suppose it holds at n. Then, $x_{n+1} = \frac{x_n+1}{x_n+2}$. Since $x_n \geqslant 0$, then this is a nonnegative fraction, so $x_{n+1} \geqslant 0$. Thus, $x_n \geqslant 0$ for all n. Hence, for all $n \geqslant 2$, we have

$$x_n = \frac{x_{n-1} + 1}{x_{n-1} + 2} \leqslant \frac{x_{n-1} + 2}{x_{n-1} + 2} \leqslant 1.$$

Consider the equation $x^2 + x - 1$. By the quadratic formula, this has two roots of the form $(\pm\sqrt{5} - 1)/2$. Set $L = (\sqrt{5} - 1)/2$. Observe that

$$L^2 + 2L - L - 1 = 0 \implies L = \frac{L+1}{L+2}.$$

So, L is a fixed point of $(x + 1)/(x + 2)$. Note also that $\frac{d}{dx}(x + 1)/(x + 2) = \frac{1}{(x+1)^2}$, which is positive for all $x \geqslant 0$. Therefore, $(x + 1)/(x + 2)$ is increasing in x. Observe also that $x^2 + x - 1$ is concave up, so the function is negative on the interval $[(-\sqrt{5}-1)/2, (\sqrt{5}-1)/2]$. Suppose first that $x_n \leqslant L$. Then, by the concavity of $x^2 + x - 1$, we have $x_n^2 + x_n - 1 \leqslant 0$. Therefore,

$$x_n^2 + 2x_n - x_n - 1 = x_n(x_n + 2) - (x_n + 1) \leqslant 0.$$

After some rearrangement, we obtain $x_n \leqslant \frac{x_n+1}{x_n+2} = x_{n+1}$. Moreover, since $(x + 1)/(x + 2)$ increases monotonically in x, then

$$x_{n+1} = \frac{x_n + 1}{x_n + 2} \leqslant \frac{L + 1}{L + 2} = L.$$

So, if $x_n \leqslant L$ for some n, then (x_n) increases monotonically on $[0, L]$, and is bounded above by L. So, by Monotone Convergence Theorem (x_n) converges.

On the other hand, suppose $x_n \geqslant L$ for some n. Then, again by the concavity of x^2+x-1, we obtain

$$x_n^2 + x - 1 \geqslant 0 \implies x_n \geqslant \frac{x_n + 1}{x_n + 2} = x_{n+1}.$$

Moreover, since $x_n \geqslant L$, since $(x + 1)/(x + 2)$ increases in x, then

$$\frac{x_n + 1}{x_n + 1} \geqslant \frac{L + 1}{L + 2} = L.$$

So, if $x_n \geqslant L$ for some n, then (x_n) decreases monotonically on the interval $[x_1, L]$. So, again, it is convergent by the Monotone Convergence Theorem. Finally, we have

$$c = \lim x_n = \lim x_{n+1} = \lim \frac{x_n + 1}{x_n + 2} = \frac{\lim x_n + 1}{\lim x_n + 2} = \frac{c + 1}{c + 2}.$$

So, $c(c+2) = c+1$ implying $c^2+c-1 = 0$. Since x_n is nonnegative for all n, we have $c \geqslant 0$. Therefore, c is the positive root of $x^2 + x - 1$, and this is L. $\square$

Problem 2.

(a) Let $F \subseteq \mathbb{R}$ be closed, and define

$$\delta_F(y) \coloneqq \inf_{x \in F} |x - y|.$$

For $y \notin F$, show that

$$\int_F |x-y|^{-2} \, \mathrm{d}x \leqslant \frac{2}{\delta_F(y)}.$$

(b) Let $F \subseteq \mathbb{R}$ be a closed set whose complement has finite measure. Define the function

$$I(x) \coloneqq \int_{\mathbb{R}} \frac{\delta_F(y)}{|x-y|^2} \, \mathrm{d}y.$$

Prove that $I(x) = \infty$ if $x \notin F$, however $I(x) < \infty$ for almost every $x \in F$.

Hint: Investigate $\int_F I(x)\, \mathrm{d}x$.

Proof. We prove (a). Note that $\delta_F(y) > 0$, otherwise we have $y \in F$. Observe that $F \cap (y, \infty) \subseteq (y + \delta_F(y), \infty)$. Therefore,

$$\int_{F \cap (\delta_F(y)+y, \infty)} \frac{1}{|x-y|^2} \, \mathrm{d}x \leqslant \int_{y+\delta_F(y)}^{\infty} \frac{1}{|x-y|^2} \, \mathrm{d}x.$$

Substituting $z = x - y$, we observe that the new bounds of integration are $(\delta_F(y), \infty)$. So, we have

$$\int_{F \cap (y, \infty)} \frac{1}{|x-y|^2} \, \mathrm{d}x \leqslant \int_{\delta_F(y)}^{\infty} \frac{1}{z^2} \, \mathrm{d}x = -\frac{1}{z}\bigg|_{\delta_F(y)}^{\infty} = \frac{1}{\delta_F(y)}.$$

Likewise, $F \cap (-\infty, y) \subseteq (-\infty, y - \delta_F(y)$. So,

$$\int_{F \cap (-\infty, y)} \frac{1}{|x-y|^2} \, \mathrm{d}x \leqslant \int_{-\infty}^{y-\delta(y)} \frac{1}{|x-y|^2} \, \mathrm{d}x.$$

Again, we perofrm the substitution $z = x - y$ to obtain

$$\int_{F \cap (-\infty, y)} \frac{1}{|x-y|^2} \, \mathrm{d}x \leqslant \int_{-\infty}^{-\delta(y)} \frac{1}{z^2} \, \mathrm{d}x = -\frac{1}{z}\bigg|_{-\infty}^{-\delta_F(y)} = \frac{1}{\delta_F(y)}.$$

Therefore,

$$\int_F \frac{1}{|x-y|^2} \, \mathrm{d}x = \int_{F \cap (y,\infty)} \frac{1}{|x-y|^2} \, \mathrm{d}x + \int_{F \cap (-\infty, y)} \frac{1}{|x-y|^2} \, \mathrm{d}x \leqslant \frac{2}{\delta_F(y)}.$$

We now prove (b). Observe that $I(x)$ is nonnegative. Moreover, the infimum over a set, and $|x-y|^{-2}$ are measurable functions, so $I(x)$, as the integral of their product, is a measurable function. Then, we may apply Tonelli's

$$\int_F I(x) \, \mathrm{d}x = \int_F \int \frac{\delta_F(y)}{|x-y|^2} \, \mathrm{d}y \, \mathrm{d}x$$

$$
\begin{aligned}
&= \int \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y + \int_F \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}
$$

Note that for $y \in F$, we have $\delta_F(y) = 0$. Thus,

$$
\begin{aligned}
\int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y + \int_F \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y &= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y + \int_F \int_F \frac{0}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}
$$

By (a), for each y, we have

$$
\begin{aligned}
\int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y &= \int_{F^c} \delta_F(y) \frac{2}{\delta_F(y)} \,\mathrm{d}y \\
&= 2\mu(F^c).
\end{aligned}
$$

By assumption, $2\mu(F^c)$ is finite. So, $\int_F I(x) \,\mathrm{d}x$ is finite, and thus $I(x) \neq \infty$ almost everywhere on F. Therefore, almost every $x \in F$ satisfies $I(x) < \infty$.

On the other hand, say that $x \notin F$. Then, since F is closed, there is some ϵ such that $(x-\epsilon, x+\epsilon) \subseteq F^c$. We may further choose ϵ so that $x-\epsilon, x+\epsilon \in F^c$. We have

$$
I(x) = \int \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y = \int_{F^c} \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y \geqslant \int_{x-\epsilon}^{x+\epsilon} \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y.
$$

Now, since $x-\epsilon, x+\epsilon \in F^c$, they have some minimal distance from F, say a, so that $\delta_F(y) \geqslant a$ for all $y \in (x-\epsilon, x+\epsilon)$. Therefore,

$$
\int_{x-\epsilon}^{x+\epsilon} \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y \geqslant \int_{x-\epsilon}^{x+\epsilon} \frac{a}{|x-y|^2} \,\mathrm{d}y.
$$

Finally, substitute $z = y - x$. Then,

$$
\int_{x-\epsilon}^{x+\epsilon} \frac{a}{|x-y|^2} \,\mathrm{d}y = \int_{-\epsilon}^{\epsilon} \frac{a}{z^2} \,\mathrm{d}z.
$$

Now, $\int_0^\epsilon 1/z^2 \,\mathrm{d}z = \infty$. Therefore, $I(x) = \infty$ when $x \notin F$. $\square$

Problem 3. Recall that a set $E \subseteq \mathbb{R}^d$ is measurable if for every $\epsilon > 0$ there is an open set $U \subseteq \mathbb{R}^d$ such that $m^*(U \backslash E) < \epsilon$.

(a) Prove that if E is measurable, then for all $\epsilon > 0$ there exists an elementary set F such that $m(E \triangle F) < \epsilon$. Here $m(E)$ denotes the Lebesgue measure of E, a set F is called elementary if it is a finite union of rectangles, and $E \triangle F$ denotes the symmetric difference of E and F.

(b) Let $E \subseteq \mathbb{R}$ be a measurable set such that $0 < m(E) < \infty$. Use part (a) to show that

$$\lim_{n\to\infty} \int_E \sin(nt)\,\mathrm{d}t = 0.$$

As stated, part (a) is false. One must also require that $m(E) < \infty$. So, we do the proof requiring that $m(E) < \infty$.

Proof. We prove (a). Take E measurable such that $m(E) < \infty$. Let V be the volume function on the collection of closed-open rectangles. Then, there is some collection $(R_m)_{m=1}^{\infty}$ of closed-open rectangles such that $E \subseteq \bigcup_{m=1}^{\infty} R_m$ and

$$m(E) \leqslant \sum_{m=1}^{\infty} V(R_m) \leqslant m(E) + \epsilon/2.$$

Now, set $A = \bigcup_{m=1}^{\infty} R_m$ and note that $m(A) \leqslant \sum_{m=1}^{\infty} V(R_m)$. Since A, E have finite measure, and $m(E) \leqslant m(A) \leqslant m(E) + \epsilon/2$, then $m(A \backslash E) \leqslant \epsilon/2$. Define $F_n \coloneqq \bigcup_{m=1}^{n} R_m$. Then,

$$m(E \triangle F_n) = m((E\backslash F_n) \cup (F_n \backslash E)) \leqslant m(E\backslash F_n) + m(F_n\backslash E) \leqslant m(E\backslash F_n) + m(A\backslash E).$$

Observe that $E\backslash F_n = E\backslash(E \cap F_n)$. Moreover, $E \cap F_n \leqslant E \cap F_{n+1}$ and $E \subseteq \bigcup_{m=1}^{\infty} R_m$. Therefore, $m(E) = \lim m(E\cap F_n)$. Choose n large enough such that $m(E) - m(E\cap F_n) < \epsilon/2$. Then, again applying finiteness, we have $m(E\backslash F_n) = m(E\backslash E \cap F_n) < \epsilon/2$. Hence,

$$m(E \triangle F_n) \leqslant m(E\backslash F_n) + m(A\backslash E) \leqslant \epsilon.$$

So, the result is proven.

We move on to (b). We first show that for an interval (a, b), we have

$$\lim_{n\to\infty} \int_a^b \sin(nt)\,\mathrm{d}t = 0.$$

We consider the integral over one full period of $\sin(nt)$. The period of $\sin(nt)$ has length $2\pi/n$. By translation invariance, we may compute the integral on the interval $[0, 2\pi/n]$. We have

$$\int_0^{2\pi/n} \sin(nt)\,\mathrm{d}t = \frac{-\cos(nt)}{n}\bigg|_0^{2\pi/n} = \frac{-\cos(2\pi) + \cos(0)}{n} = 0.$$

For a fixed n, $\sin(nt)$ completes at most $\lfloor (b-a)/(\frac{2\pi}{n}) \rfloor = \lfloor \frac{n(b-a)}{2\pi} \rfloor$ periods over (a, b). The endpoints a, b may interrupt a part of a period of $\sin(nt)$, so we can guarantee that $\sin(nt)$ comples at least $\lfloor \frac{n(b-a)}{2\pi} \rfloor - 2$ full periods on (a, b). Thus, there is subinterval I_n of (a, b) with length

$$\frac{2\pi}{n}\left(\left\lfloor \frac{n(b-a)}{2\pi} \right\rfloor - 2\right)$$

such that $\int_I \sin(nt)\,\mathrm{d}t = 0$, since $\sin(nt)$ has an integral of 0 over each period. Placing bounds on the floor function, we have

$$\frac{2\pi}{n}\left(\frac{n(b-a)}{2\pi} - 3\right) \leqslant \frac{2\pi}{n}\left(\left\lfloor \frac{n(b-a)}{2\pi} \right\rfloor - 2\right) \leqslant \frac{2\pi}{n}\left(\frac{n(b-a)}{2\pi} - 2\right).$$

The left hand side is $b-a-\frac{6\pi}{n}$, and the right hand side is $b-a-\frac{4\pi}{n}$. Therefore, taking $n\to\infty$, the length of I_n grows to $b-a$. So,

$$\int_{(a,b)\backslash I_n} -1\,\mathrm{d}t + \int_{I_n} \sin(nt)\,\mathrm{d}t \leqslant \int_a^b \sin(nt)\,\mathrm{d}t \leqslant \int_{(a,b)\backslash I_n} 1\,\mathrm{d}t + \int_{I_n} \sin(nt)\,\mathrm{d}t.$$

Choosing n large enough so that $m((a,b)\backslash I_n) < \epsilon$, we then have $-\epsilon \leqslant \int_a^b \sin(nt)\,\mathrm{d}t \leqslant \epsilon$. Thus, $\lim \int_a^b \sin(nt)\,\mathrm{d}t = 0$.

Consider now a measurable set E such that $m(E)<\infty$. By (a), there is a finite collection of rectangles $J_1,\dots,J_m$ such that $m(E\triangle\bigcup_{k=1}^m J_k) < \epsilon/2$. This requires in particular that $m(E\backslash\bigcup_{k=1}^m J_k) < \epsilon/2$. Set $B=\bigcup_{k=1}^m J_k$, and set $A=E\backslash B$. Then,

$$\left|\int_E \sin(nt)\,\mathrm{d}t\right| = \left|\int_A \sin(nt)\,\mathrm{d}t + \int_B \sin(nt)\,\mathrm{d}t\right| \leqslant \int_A 1\,\mathrm{d}t + \left|\int_B \sin(nt)\,\mathrm{d}t\right|.$$

Now,

$$\left|\int_B \sin(nt)\,\mathrm{d}t\right| \leqslant \sum_{k=1}^m \left|\int_{J_k} \sin(nt)\,\mathrm{d}t\right|.$$

Define N_k so that $|\int_{J_k} \sin(nt)\,\mathrm{d}t| < \epsilon/m$ for all $n\geqslant N_k$. This is possible by our argument on intervals above. Set $N=\max\{N_1,\dots,N_m\}$. Therefore, for all $n\geqslant N$,

$$\left|\int_B \sin(nt)\,\mathrm{d}t\right| \leqslant \sum_{k=1}^m \left|\int_{J_k} \sin(nt)\,\mathrm{d}t\right| \leqslant \epsilon/2.$$

So, for all $n\geqslant N$, we obtain

$$\left|\int_E \sin(nt)\,\mathrm{d}t\right| \leqslant \int_A 1\,\mathrm{d}t + \left|\int_B \sin(nt)\,\mathrm{d}t\right| \leqslant m(A)+\epsilon/2 = \epsilon.$$

Thus, $\lim \int_E \sin(nt)\,\mathrm{d}t = 0$. □

Problem 4. Let f be a measurable function on $\mathbb{R}$. Show that the graph of f has measure zero in $\mathbb{R}^2$.

Proof. Let $\Gamma_f = \{(x,f(x))\in\mathbb{R}^2 : x\in\mathbb{R}\}$. We first show that Γ_f is a measurable set. Define $h(x,y)=f(x)-y$. This is a difference of measurable functions, and thus measurable. Now, if $h(x,y)=0$, then $f(x)-y=0$, and so $y=f(x)$. Moreover, $h(x,f(x))=0$. Thus, $h^{-1}(\{0\}) = \{(x,f(x)) : x\in\mathbb{R}\}$. So, Γ_f is measurable.

Observe now that $\mathbb{1}_{\Gamma_f}$ is a nonnegative measurable function. Therefore, by Tonelli's Theorem,

$$m_2(\Gamma_f) = \int_{\mathbb{R}^2} \mathbb{1}_{\Gamma_f}\,\mathrm{d}m_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\Gamma_f}(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Since f is a function, then for each x, there is only one y such that $y=f(x)$. So, for x fixed, we have $\mathbb{1}_{\Gamma_f}(x,y) = \mathbb{1}_{\{f(x)\}}(y)$, an a.e. zero function. Hence,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\Gamma_f}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{f(x)\}}(y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}} 0\,\mathrm{d}x = 0.$$

Therefore, $m_2(\Gamma_f)=0$. □

Problem 5. Consider the Hilbert space $\mathcal{H} = L^2([0,1])$.

(a) Prove that if $E \subseteq \mathcal{H}$ is closed and convex then E contains an element of smallest norm.
Hint: Show that if $||f_n||_2 \to \inf\{f \in E : ||f||_2\}$ *then* $\{f_n\}$ *is a Cauchy sequence.*

(b) Construct a non-empty closed $E \subseteq \mathcal{H}$ which does not contain an element of smallest norm.

Proof. We prove (a). Set $\alpha = \inf\{f \in E : ||f||_2\}$. Applying the Parallelogram Law, we have

$$||f_n + f_m||_2^2 = 2||f_n||_2^2 + 2||f_m||_2^2 - ||f_n - f_m||_2^2.$$

By convexity, $\frac{1}{2}f_n - \frac{1}{2}f_m \in E$. Hence $||\frac{1}{2}f_n - \frac{1}{2}f_m||_2 \geqslant \alpha$. So, $||f_n - f_m||_2^2 \geqslant 4\alpha^2$. We have

$$2||f_n||_2^2 + 2||f_m||_2^2 - ||f_n - f_m||_2^2 \leqslant 2||f_n||_2^2 + 2||f_m||_2^2 - 4\alpha^2 \to_{n,m\to\infty} 0.$$

So, $||f_n - f_m||_2 \to_{n,m\to\infty} 0$. Therefore, the sequence is Cauchy. Since $\mathcal{H}$ is Hilbert, then it is complete, so $f_n \to f$ in $\mathcal{H}$. Since E is closed, then $f \in E$. By continuity of the norm, we have $\alpha = \lim ||f_n||_2 = ||f||$. Therefore, $f \in E$ has a minimal norm.

Define

$$f_n = \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(n+1).$$

Set $E = \{f_n\}_{n=1}^{\infty}$, and note that

$$\int_0^1 |f_n(x)|^2 \,\mathrm{d}x = \int_{\frac{1}{n+1}}^{\frac{1}{n}} (n+1)^2 \,\mathrm{d}x = (n+1)^2 \Big|_{\frac{1}{n+1}}^{\frac{1}{n}} = \frac{(n+1)^2}{n(n+1)} = 1 + \frac{1}{n}.$$

Hence, there is no minimal element in E. Moreover, each f_n has disjoint support. So,

$$||f_n + f_m||_2^2 = ||f_n||_2^2 + ||f_m||_2^2 \geqslant 2.$$

Therefore, there is no sequence $(f_{n_k})_{k=1}^{\infty}$ which is Cauchy. Thus, E has no limit points, so it is vacuously true that E contains all its limit points, and thus E is closed.

$\square$