Real Analysis Qual, Spring 2021

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $E_n \in \mathcal{M}$ be a measurable set for $n \geqslant 1$. Let $f_n = \chi_{E_n}$ be the indicator function of the set E_n. Prove that

(a) $f_n \to 1$ uniformly if and only if there exists $N \in \mathbb{N}$ such that $E_n = X$ for all $n \geqslant N$.

(b) $f_n(x) \to 1$ for almost every x if and only if

$$\mu\left(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X\backslash E_k)\right) = 0.$$

Proof. We prove (a). First suppose that $f_n \to 1$ uniformly. Then, there is some N such that for all $n \geqslant N$, we have $|1 - \chi_{E_n}(x)| < 1/2$ for all x. If there is some x such that $x \notin E_n$, then $|1 - \chi_{E_n}(x)| = |1 - 0| = 1 \not< 1/2$. Therefore, we have $E_n = X$ for all $n \geqslant N$. For the reverse direction, suppose there is some n such that $E_n = X$ for all $n \geqslant N$. Then, take $\epsilon > 0$ to be arbitrary. For all $n \geqslant N$, we have $|1 - \chi_{E_n}(x)| = |1 - \chi_X(x)| = 0 < \epsilon$. So, $f_n \to 1$ uniformly.

Now we prove (b). Suppose that $f_n(x) \to 1$ for a.e. x. Observe $x \in \bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X\backslash E_k)$ if for all $n \geqslant 0$, there exists some $k \geqslant n$, such that $x \in X\backslash E_k$. In particular, for all $n \geqslant 0$, there exists $k \geqslant 0$ such that $x \notin E_k$, so that $f_k(x) = 0$. Therefore, $f_k(x) \not\to 1$ as $k \to \infty$. Thus, x belongs to the set of points A such that $f_n(x) \not\to 1$. Since $f_n(x) \to 1$ for a.e. x, then A has measure 0, so

$$\mu\left(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X\backslash E_k)\right) \leqslant \mu(A) = 0,$$

as needed.

Suppose alternatively that $\mu(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X\backslash E_k)) = 0$. Consider the set of points A such that $f_n(x) \not\to 1$ as $n \to \infty$. Suppose that $x \in A$. Then, for all $\epsilon > 0$, there is no $n \geqslant 0$, such that for all $k \geqslant n$ we have $|1 - f_n(x)| < \epsilon$. In particular, choosing $\epsilon = 1/2$, for all $n \geqslant 0$, there is some $k \geqslant n$ such that$|1 - f_k(x)| > 1/2$. Then, $f_n(x) \neq 1$, so $f_k(x) = 0$, and thus $x \in X\backslash E_k$. So, for all $n \geqslant 0$, there is some $k \geqslant n$ such that $x \in X\backslash E_k$. So, for all $n \geqslant 0$, we have $x \in \bigcup_{k\geqslant n} X\backslash E_k$, and thus $x \in \bigcap_{n\geqslant 0}\bigcup_{x\geqslant n} X\backslash E_k$. This is a 0 measure set, and it contains A, so A is a 0 measure set. $\square$

Problem 2. (Classic) Calculate the limit

$$L := \lim_{n\to\infty}\int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)}\,\mathrm{d}x.$$

Proof. Pointwise $\mathbb{1}_{[0,n]}\cos(x/n) \to 1$, since $x/n \to 0$ pointwise and $\cos(0) = 1$. Therefore, for all $x > 0$ we have

$$\lim_{n\to\infty}\mathbb{1}_{[0,n]}\frac{\cos(x/n)}{x^2 + \cos(x/n)} = \frac{1}{x^2 + 1}.$$

Since $3/2 < \pi/2$, then on the interval $[0, 3/2]$, $\cos(x/n)$ decreases monotonically for all n. Moreover, since $3/2 \leqslant 3/2n$ for all n, then we have $\cos(3/2n) \geqslant \cos(3/2)$ for all n. Thus, for all n, on the interval $[0, 3/2]$,

$$\left|\frac{\cos(x/n)}{x^2 + \cos(x/n)}\right| \leqslant \frac{1}{x^2 + \cos(x/n)} \leqslant \frac{1}{x^2 + \cos(3/2)}.$$

Moreover, for all n we have $\cos(x/n) \geqslant -1$. Therefore, for $x \in [3/2, \infty)$ we have

$$\left|\frac{\cos(x/n)}{x^2 + \cos(x/n)}\right| \leqslant \frac{1}{x^2 + \cos(x/n)} \leqslant \frac{1}{x^2 - 1}.$$

Define

$$f(x) = \begin{cases} \frac{1}{x^2 + \cos(3/2)}, & \text{if } x \in [0, 3/2], \\ \frac{1}{x^2 - 1}, & \text{if } x \in (3/2, \infty). \end{cases}$$

Note that $|f| = f$, and that, by the foregoing, we have

$$\left|\frac{\cos(x/n)}{x^2 + \cos(x/n)}\right| \leqslant f(x)$$

for all x. Moreover, f is Lebesgue integrable on $[0, \infty]$. Indeed,

$$\int_0^\infty |f(x)| \, \mathrm{d}x = \int_0^{3/2} f(x) \, \mathrm{d}x + \int_{3/2}^\infty f(x) \, \mathrm{d}x = \int_0^{3/2} \frac{1}{x^2 + \cos(3/2)} \, \mathrm{d}x + \int_{3/2}^\infty \frac{1}{x^2 - 1} \, \mathrm{d}x.$$

Both these integrals are finite, so f is Lebesgue integrable as claimed. Therefore, by DCT,

$$\lim_{n \to \infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)} \, \mathrm{d}x = \int_0^\infty \frac{1}{x^2 + 1} \, \mathrm{d}x.$$

Now, observe that $\tan : [0, 2\pi) \longrightarrow [0, \infty)$ is a diffeomorphism. So, we perform the substitution $x = \tan(\theta)$. Note that $\mathrm{d}x = \sec^2(\theta) \, \mathrm{d}\theta$. Therefore, since $\tan^2(\theta) + 1 = \sec^2(\theta)$, we have

$$\lim_{n \to \infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)} \, \mathrm{d}x = \int_0^\infty \frac{1}{x^2 + 1} \, \mathrm{d}x = \int_0^{2\pi} \frac{1}{\tan^2(\theta) + 1} \sec^2(\theta) \, \mathrm{d}\theta = \int_0^{2\pi} 1 \, \mathrm{d}x = 2\pi.$$

□

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $(f_n)_{n=1}^\infty \subseteq L^1(X, \mu)$ and $f \in L^1(X, \mu)$ such that $f_n(x) \to x$ as $n \to \infty$ for almost every $x \in X$. Prove that for every $\epsilon > 0$ there exists $M > 0$, and a set $E \subseteq X$, such that $\mu(E) < \epsilon$ and $|f_n(x)| \leqslant M$ for all $x \in X \backslash E$ and $n \in \mathbb{N}$.

Proof. Define $A_m = \{x \in X : \exists n \in \mathbb{N}, |f_n(x)| > m\}$. Observe that the A_m are monotone, since if $x \in A_{m+1}$, there is some n such that $|f_n(x)| > m + 1 \geqslant m$, and so $x \in A_m$. For $x \in \bigcap_{m=1}^\infty A_m$, the sequence $(f_n(x))$ does not converge, since for every M, there exists some n such that $|f_n(x)| \geqslant M$. Since f_n converges pointwise a.e., then $\bigcap_{m=1}^\infty A_m$ must be a null set. Since X is a finite measure space, and so A_1 in particular is finite, then by continuity from below

$$0 = \mu\left(\bigcap_{m=1}^\infty A_m\right) = \lim \mu(A_m).$$

So, let $\epsilon > 0$ be arbitray. Pick M so that $\mu(A_M) < \epsilon$. Then, for $x \in X \backslash A_M$, we must have $|f(x)| \leqslant M$. Thus, $A_M = E$ satisfies the requirements of our set, proving the claim. □

Problem 4. **(Classic Technique)** Let f and g be Lebesgue Integrable on $\mathbb{R}$. Let $g_n(x) = g(x-n)$. Prove that

$$\lim_{n\to\infty} ||f - g_n||_1 = ||f||_1 + ||g||_1.$$

Proof. We first suppose that f, g are continuous functions with compact support. Since $\operatorname{supp} f, \operatorname{supp} g$ compact, then they are bounded. So, say that $\operatorname{supp} f$ is supported on $[-N, N]$ and that $\operatorname{supp} g$ is supported on $[-M, M]$. Observe that if $g_n(x) \neq 0$, then $g(x-n) \neq 0$, and this happens if and only if $x - n \in [-M, M]$ which occurs if and only if $x \in [n-M, n+M]$. So, $g_n(x)$ is supported on $[n-M, n+M]$. Take $n \geqslant N+M$. Then, $[n-M, n+M] \cap [-N, N] = \emptyset$, and so $f(x) \neq 0$ if and only if $g_n(x) = 0$, and vice versa. Therefore,

$$\int |f - g_n| \, \mathrm{d}x = \int_{-N}^{N} |f - g_n| \, \mathrm{d}x + \int_{n-M}^{n+M} |f - g_n| \, \mathrm{d}x = \int_{-N}^{N} |f| \, \mathrm{d}x + \int_{n-M}^{n+M} |g_n| \, \mathrm{d}x.$$

Now, applying the change of coordinates $y = x - n$, we have

$$\int_{n-M}^{n+M} |g_n| \, \mathrm{d}x = \int \mathbb{1}_{[-M,M]}(x-n)|g(x-n)| \, \mathrm{d}x = \int \mathbb{1}_{[-M,M]}(y)|g(y)| \, \mathrm{d}y = \int_{-M}^{M} |g| \, \mathrm{d}x.$$

So,

$$\int |f - g_n| \, \mathrm{d}x = \int_{-N}^{N} |f| \, \mathrm{d}x + \int_{-M}^{M} |g| \, \mathrm{d}x = \int |f| \, \mathrm{d}x + \int |g| \, \mathrm{d}x$$

for all n sufficiently large.

Now, take f, g to be arbitrary L^1 functions. Take $\epsilon > 0$. Let ϕ be within $\epsilon/4$ of f, and let ψ be within $\epsilon/4$ of g in the L^1-norm. Then, choose n sufficiently large that $||\phi - \psi_n|| = ||\phi||_1 + ||\psi||_1$, which is possible by the above proof. Observe that using the proper change of coordinates, we have $||g_n - \psi_n|| = ||g - \psi||$. So,

$$\Big| ||f - g_n|| - ||\phi - \psi_n|| \Big| \leqslant ||f - g_n - (\phi - \psi_n)|| \leqslant ||f - \phi|| + ||g_n - \psi_n|| < \epsilon/2.$$

We have

$$\Big| ||f - g_n|| - ||\phi|| - ||\psi|| \Big| \leqslant \Big| ||f - g_n|| - ||\phi - \psi_n|| \Big| + \Big| ||\phi - \psi_n|| - ||\phi|| - ||\psi|| \Big| = \epsilon/2.$$

So, for all n sufficiently large,

$$\Big| ||f - g_n|| - ||f|| - ||g|| \Big| \leqslant \Big| ||f - g_n|| - ||\phi|| - ||\psi|| \Big| + \Big| ||\phi|| + ||\psi|| - ||f|| - ||g|| \Big| \leqslant \epsilon/2 + \epsilon/2 = \epsilon,$$

completing the proof. □

Problem 5. Let $f_n \in L^2([0, 1])$ for $n \in \mathbb{N}$. Assume that

(a) $||f_n||_2 \leqslant n^{-51/100}$, for all $n \in \mathbb{N}$, and

(b) $\hat{f}_n$ is supported in the interval $[2^n, 2^{n+1}]$, that is

$$\hat{f}_n(k) = \int_0^1 f(x) e^{-2\pi i k x} \, \mathrm{d}x = 0, \text{ for } k \notin [2^n, 2^{n+1}].$$

Prove that $\sum_{n=1}^{\infty} f_n$ converges in the Hilbert space $L^2([0,1])$.

Proof. We prove that (f_n) is an orthogonal sequence. Say that $m \neq n$. Then,

$$\langle f_n, f_m \rangle = \sum_{k=1}^{\infty} \langle f_n e^{-2\pi ikx}, f_m e^{-2\pi ikx} \rangle \leqslant \sum_{k=1}^{\infty} ||f_n e^{-2\pi ikx}||_2 ||f_m e^{-2\pi ikx}||_2.$$

Observe that $||f_n e^{-2\pi ikx}|| = |\hat{f}_n(k)|$. Moreover, $\hat{f}_n(k)\hat{f}_m(k) \neq 0$ if and only if $k \in [2^n, 2^{n+1}]$ and $k \in [2^m, 2^{m+1}]$. If $m \neq n$, then these two intervals are disjoint, and so

$$\langle f_n, f_m \rangle = \sum_{k=1}^{\infty} ||f_n e^{-2\pi ikx}||_2 ||f_m e^{-2\pi ikx}||_2 \sum_{k=1}^{\infty} = \sum_{k=1}^{\infty} |\hat{f}_n(k)\hat{f}_m(k)| = \sum_{k=1}^{\infty} 0 = 0.$$

So, the f_n are orthogonal as claimed. Therefore, by the Pythagorean Theorem

$$\left\| \sum_{n=k}^{N} f_n \right\|_2^2 = \sum_{n=k}^{N} ||f_n||_2^2 \leqslant \sum_{n=k}^{N} n^{-102/100}.$$

Taking $N \to \infty$, we have

$$\left\| \sum_{n=k}^{\infty} f_n \right\|_2^2 \leqslant \sum_{n=k}^{\infty} n^{-102/100} < \infty.$$

For all k, this is a p-series, and thus convergent. Moreover, we have $\sum_{n=k}^{\infty} n^{-102/100} \to_{k\to\infty} 0$. Therefore, $S_N = \sum_{n=1}^{N} f_n$ is a Cauchy sequence. Since Hilbert spaces are complete, then the S_N converge. $\square$

Problem 6. Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function, and for $x \in \mathbb{R}$ define the set

$$E_x := \{y \in \mathbb{R} : m(\{x \in \mathbb{R} : f(x,z) = f(x,y)\}) > 0\}.$$

Show that

$$E := \bigcup_{x \in \mathbb{R}} \{x\} \cup E_x$$

is a measurable subset of $\mathbb{R} \times \mathbb{R}$.
Hint: Consider the measurable function $h(x,y,z) := f(x,y) - f(x,z)$.

There is some lore behind this problem. It is a known hard problem. It went unsolved during the qual, and apparently the official solution was considered dubious by some of the professors. Neil Lyall has explicitly told some graduate students not to study this problem for future quals. I have not tried to think of a solution to this problem, since I consider my time better spent on answering other questions.