

Real Analysis Qual, Fall 2021

Problem 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and

$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{x_n + 1}{x_n + 2}.$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges, and find its limit.

Proof. We first show the sequence is bounded for $n \geq 2$. First, we prove that $x_n \geq 0$ for all n . At x_1 , this holds. Suppose it holds at n . Then, $x_{n+1} = \frac{x_n + 1}{x_n + 2}$. Since $x_n \geq 0$, then this is a nonnegative fraction, so $x_{n+1} \geq 0$. Thus, $x_n \geq 0$ for all n . Hence, for all $n \geq 2$, we have

$$x_n = \frac{x_{n-1} + 1}{x_{n-1} + 2} \leq \frac{x_{n-1} + 2}{x_{n-1} + 2} \leq 1.$$

Consider the equation $x^2 + x - 1$. By the quadratic formula, this has two roots of the form $(\pm\sqrt{5} - 1)/2$. Set $L = (\sqrt{5} - 1)/2$. Observe that

$$L^2 + 2L - L - 1 = 0 \implies L = \frac{L + 1}{L + 2}.$$

So, L is a fixed point of $(x + 1)/(x + 2)$. Note also that $\frac{d}{dx}(x + 1)/(x + 2) = \frac{1}{(x+1)^2}$, which is positive for all $x \geq 0$. Therefore, $(x + 1)/(x + 2)$ is increasing in x . Observe also that $x^2 + x - 1$ is concave up, so the function is negative on the interval $[(-\sqrt{5} - 1)/2, (\sqrt{5} - 1)/2]$. Suppose first that $x_n \leq L$. Then, by the concavity of $x^2 + x - 1$, we have $x_n^2 + x_n - 1 \leq 0$. Therefore,

$$x_n^2 + 2x_n - x_n - 1 = x_n(x_n + 2) - (x_n + 1) \leq 0.$$

After some rearrangement, we obtain $x_n \leq \frac{x_n + 1}{x_n + 2} = x_{n+1}$. Moreover, since $(x + 1)/(x + 2)$ increases monotonically in x , then

$$x_{n+1} = \frac{x_n + 1}{x_n + 2} \leq \frac{L + 1}{L + 2} = L.$$

So, if $x_n \leq L$ for some n , then (x_n) increases monotonically on $[0, L]$, and is bounded above by L . So, by Monotone Convergence Theorem (x_n) converges.

On the other hand, suppose $x_n \geq L$ for some n . Then, again by the concavity of $x^2 + x - 1$, we obtain

$$x_n^2 + x_n - 1 \geq 0 \implies x_n \geq \frac{x_n + 1}{x_n + 2} = x_{n+1}.$$

Moreover, since $x_n \geq L$, since $(x + 1)/(x + 2)$ increases in x , then

$$\frac{x_n + 1}{x_n + 1} \geq \frac{L + 1}{L + 2} = L.$$

So, if $x_n \geq L$ for some n , then (x_n) decreases monotonically on the interval $[1, L]$. So, again, it is convergent by the Monotone Convergence Theorem. Finally, we have

$$c = \lim x_n = \lim x_{n+1} = \lim \frac{x_n + 1}{x_n + 2} = \frac{\lim x_n + 1}{\lim x_n + 2} = \frac{c + 1}{c + 2}.$$

So, $c(c + 2) = c + 1$ implying $c^2 + c - 1 = 0$. Since x_n is nonnegative for all n , we have $c \geq 0$. Therefore, c is the positive root of $x^2 + x - 1$, and this is L . \square

Problem 2.

(a) Let $F \subseteq \mathbb{R}$ be closed, and define

$$\delta_F(y) := \inf_{x \in F} |x - y|.$$

For $y \notin F$, show that

$$\int_F |x - y|^{-2} dx \leq \frac{2}{\delta_F(y)}.$$

(b) Let $F \subseteq \mathbb{R}$ be a closed set whose complement has finite measure. Define the function

$$I(x) := \int_{\mathbb{R}} \frac{\delta_F(y)}{|x - y|^2} dy.$$

Prove that $I(x) = \infty$ if $x \notin F$, however $I(x) < \infty$ for almost every $x \in F$.

Hint: Investigate $\int_F I(x) dx$.

Proof. We prove (a). Note that $\delta_F(y) > 0$, otherwise we have $y \in F$. Observe that $F \cap (y, \infty) \subseteq (y + \delta_F(y), \infty)$. Therefore,

$$\int_{F \cap (\delta_F(y) + y, \infty)} \frac{1}{|x - y|^2} dx \leq \int_{y + \delta_F(y)}^{\infty} \frac{1}{|x - y|^2} dx.$$

Substituting $z = x - y$, we observe that the new bounds of integration are $(\delta_F(y), \infty)$. So, we have

$$\int_{F \cap (y, \infty)} \frac{1}{|x - y|^2} dx \leq \int_{\delta_F(y)}^{\infty} \frac{1}{z^2} dz = -\frac{1}{z} \Big|_{\delta_F(y)}^{\infty} = \frac{1}{\delta_F(y)}.$$

Likewise, $F \cap (-\infty, y) \subseteq (-\infty, y - \delta_F(y))$. So,

$$\int_{F \cap (-\infty, y)} \frac{1}{|x - y|^2} dx \leq \int_{-\infty}^{y - \delta(y)} \frac{1}{|x - y|^2} dx.$$

Again, we perform the substitution $z = x - y$ to obtain

$$\int_{F \cap (-\infty, y)} \frac{1}{|x - y|^2} dx \leq \int_{-\infty}^{-\delta(y)} \frac{1}{z^2} dz = -\frac{1}{z} \Big|_{-\infty}^{-\delta(y)} = \frac{1}{\delta_F(y)}.$$

Therefore,

$$\int_F \frac{1}{|x - y|^2} dx = \int_{F \cap (y, \infty)} \frac{1}{|x - y|^2} dx + \int_{F \cap (-\infty, y)} \frac{1}{|x - y|^2} dx \leq \frac{2}{\delta_F(y)}.$$

We now prove (b). Observe that $I(x)$ is nonnegative. Moreover, the infimum over a set, and $|x - y|^{-2}$ are measurable functions, so $I(x)$, as the integral of their product, is a measurable function. Then, we may apply Tonelli's

$$\int_F I(x) dx = \int_F \int \frac{\delta_F(y)}{|x - y|^2} dy dx$$

$$\begin{aligned}
&= \int \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy \\
&= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy + \int_F \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy.
\end{aligned}$$

Note that for $y \in F$, we have $\delta_F(y) = 0$. Thus,

$$\begin{aligned}
\int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy + \int_F \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy &= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy + \int_F \int_F \frac{0}{|x-y|^2} dx dy \\
&= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy.
\end{aligned}$$

By (a), for each y , we have

$$\begin{aligned}
\int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} dx dy &= \int_{F^c} \delta_F(y) \frac{2}{\delta_F(y)} dy \\
&= 2\mu(F^c).
\end{aligned}$$

By assumption, $2\mu(F^c)$ is finite. So, $\int_F I(x) dx$ is finite, and thus $I(x) \neq \infty$ almost everywhere on F . Therefore, almost every $x \in F$ satisfies $I(x) < \infty$.

On the other hand, say that $x \notin F$. Then, since F is closed, there is some ϵ such that $(x - \epsilon, x + \epsilon) \subseteq F^c$. We may further choose ϵ so that $x - \epsilon, x + \epsilon \in F^c$. We have

$$I(x) = \int \frac{\delta_F(y)}{|x-y|^2} dy = \int_{F^c} \frac{\delta_F(y)}{|x-y|^2} dy \geq \int_{x-\epsilon}^{x+\epsilon} \frac{\delta_F(y)}{|x-y|^2} dy.$$

Now, since $x - \epsilon, x + \epsilon \in F^c$, they have some minimal distance from F , say a , so that $\delta_F(y) \geq a$ for all $y \in (x - \epsilon, x + \epsilon)$. Therefore,

$$\int_{x-\epsilon}^{x+\epsilon} \frac{\delta_F(y)}{|x-y|^2} dy \geq \int_{x-\epsilon}^{x+\epsilon} \frac{a}{|x-y|^2} dy.$$

Finally, substitute $z = y - x$. Then,

$$\int_{x-\epsilon}^{x+\epsilon} \frac{a}{|x-y|^2} dy = \int_{-\epsilon}^{\epsilon} \frac{a}{z^2} dz.$$

Now, $\int_0^\epsilon 1/z^2 dz = \infty$. Therefore, $I(x) = \infty$ when $x \notin F$. \square

Problem 3. Recall that a set $E \subseteq \mathbb{R}^d$ is measurable if for every $\epsilon > 0$ there is an open set $U \subseteq \mathbb{R}^d$ such that $m^*(U \setminus E) < \epsilon$.

- (a) Prove that if E is measurable, then for all $\epsilon > 0$ there exists an elementary set F such that $m(E \Delta F) < \epsilon$. Here $m(E)$ denotes the Lebesgue measure of E , a set F is called elementary if it is a finite union of rectangles, and $E \Delta F$ denotes the symmetric difference of E and F .

(b) Let $E \subseteq \mathbb{R}$ be a measurable set such that $0 < m(E) < \infty$. Use part (a) to show that

$$\lim_{n \rightarrow \infty} \int_E \sin(nt) dt = 0.$$

As stated, part (a) is false. One must also require that $m(E) < \infty$. So, we do the proof requiring that $m(E) < \infty$.

Proof. We prove (a). Take E measurable such that $m(E) < \infty$. Let V be the volume function on the collection of closed-open rectangles. Then, there is some collection $(R_m)_{m=1}^{\infty}$ of closed-open rectangles such that $E \subseteq \bigcup_{m=1}^{\infty} R_m$ and

$$m(E) \leq \sum_{m=1}^{\infty} V(R_m) \leq m(E) + \epsilon/2.$$

Now, set $A = \bigcup_{m=1}^{\infty} R_m$ and note that $m(A) \leq \sum_{m=1}^{\infty} V(R_m)$. Since A, E have finite measure, and $m(E) \leq m(A) \leq m(E) + \epsilon/2$, then $m(A \setminus E) \leq \epsilon/2$. Define $F_n := \bigcup_{m=1}^n R_m$. Then,

$$m(E \Delta F_n) = m((E \setminus F_n) \cup (F_n \setminus E)) \leq m(E \setminus F_n) + m(F_n \setminus E) \leq m(E \setminus F_n) + m(A \setminus E).$$

Observe that $E \setminus F_n = E \setminus (E \cap F_n)$. Moreover, $E \cap F_n \leq E \cap F_{n+1}$ and $E \subseteq \bigcup_{m=1}^{\infty} R_m$. Therefore, $m(E) = \lim m(E \cap F_n)$. Choose n large enough such that $m(E) - m(E \cap F_n) < \epsilon/2$. Then, again applying finiteness, we have $m(E \setminus F_n) = m(E \setminus (E \cap F_n)) < \epsilon/2$. Hence,

$$m(E \Delta F_n) \leq m(E \setminus F_n) + m(A \setminus E) \leq \epsilon.$$

So, the result is proven.

We move on to (b). We first show that for an interval (a, b) , we have

$$\lim_{n \rightarrow \infty} \int_a^b \sin(nt) dt = 0.$$

We consider the integral over one full period of $\sin(nt)$. The period of $\sin(nt)$ has length $2\pi/n$. By translation invariance, we may compute the integral on the interval $[0, 2\pi/n]$. We have

$$\int_0^{2\pi/n} \sin(nt) dt = \frac{-\cos(nt)}{n} \Big|_0^{2\pi/n} = \frac{-\cos(2\pi) + \cos(0)}{n} = 0.$$

For a fixed n , $\sin(nt)$ completes at most $\lfloor (b-a)/(\frac{2\pi}{n}) \rfloor = \lfloor \frac{n(b-a)}{2\pi} \rfloor$ periods over (a, b) . The endpoints a, b may interrupt a part of a period of $\sin(nt)$, so we can guarantee that $\sin(nt)$ completes at least $\lfloor \frac{n(b-a)}{2\pi} \rfloor - 2$ full periods on (a, b) . Thus, there is subinterval I_n of (a, b) with length

$$\frac{2\pi}{n} \left(\left\lfloor \frac{n(b-a)}{2\pi} \right\rfloor - 2 \right)$$

such that $\int_I \sin(nt) dt = 0$, since $\sin(nt)$ has an integral of 0 over each period. Placing bounds on the floor function, we have

$$\frac{2\pi}{n} \left(\frac{n(b-a)}{2\pi} - 3 \right) \leq \frac{2\pi}{n} \left(\left\lfloor \frac{n(b-a)}{2\pi} \right\rfloor - 2 \right) \leq \frac{2\pi}{n} \left(\frac{n(b-a)}{2\pi} - 2 \right).$$

The left hand side is $b - a - \frac{6\pi}{n}$, and the right hand side is $b - a - \frac{4\pi}{n}$. Therefore, taking $n \rightarrow \infty$, the length of I_n grows to $b - a$. So,

$$\int_{(a,b) \setminus I_n} -1 \, dt + \int_{I_n} \sin(nt) \, dt \leq \int_a^b \sin(nt) \, dt \leq \int_{(a,b) \setminus I_n} 1 \, dt + \int_{I_n} \sin(nt) \, dt.$$

Choosing n large enough so that $m((a,b) \setminus I_n) < \epsilon$, we then have $-\epsilon \leq \int_a^b \sin(nt) \, dt \leq \epsilon$. Thus, $\lim \int_a^b \sin(nt) \, dt = 0$.

Consider now a measurable set E such that $m(E) < \infty$. By (a), there is a finite collection of rectangles J_1, \dots, J_m such that $m(E \Delta \bigcup_{k=1}^m J_k) < \epsilon/2$. This requires in particular that $m(E \setminus \bigcup_{k=1}^m J_k) < \epsilon/2$. Set $B = \bigcup_{k=1}^m J_k$, and set $A = E \setminus B$. Then,

$$\left| \int_E \sin(nt) \, dt \right| = \left| \int_A \sin(nt) \, dt + \int_B \sin(nt) \, dt \right| \leq \int_A 1 \, dt + \left| \int_B \sin(nt) \, dt \right|.$$

Now,

$$\left| \int_B \sin(nt) \, dt \right| \leq \sum_{k=1}^m \left| \int_{J_k} \sin(nt) \, dt \right|.$$

Define N_k so that $|\int_{J_k} \sin(nt) \, dt| < \epsilon/m$ for all $n \geq N_k$. This is possible by our argument on intervals above. Set $N = \max\{N_1, \dots, N_m\}$. Therefore, for all $n \geq N$,

$$\left| \int_B \sin(nt) \, dt \right| \leq \sum_{k=1}^m \left| \int_{J_k} \sin(nt) \, dt \right| \leq \epsilon/2.$$

So, for all $n \geq N$, we obtain

$$\left| \int_E \sin(nt) \, dt \right| \leq \int_A 1 \, dt + \left| \int_B \sin(nt) \, dt \right| \leq m(A) + \epsilon/2 = \epsilon.$$

Thus, $\lim \int_E \sin(nt) \, dt = 0$. □

Problem 4. Let f be a measurable function on \mathbb{R} . Show that the graph of f has measure zero in \mathbb{R}^2 .

Proof. Let $\Gamma_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. We first show that Γ_f is a measurable set. Define $h(x, y) = f(x) - y$. This is a difference of measurable functions, and thus measurable. Now, if $h(x, y) = 0$, then $f(x) - y = 0$, and so $y = f(x)$. Moreover, $h(x, f(x)) = 0$. Thus, $h^{-1}(\{0\}) = \{(x, f(x)) : x \in \mathbb{R}\}$. So, Γ_f is measurable.

Observe now that $\mathbb{1}_{\Gamma_f}$ is a nonnegative measurable function. Therefore, by Tonelli's Theorem,

$$m_2(\Gamma_f) = \int_{\mathbb{R}^2} \mathbb{1}_{\Gamma_f} \, dm_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\Gamma_f}(x, y) \, dy \, dx.$$

Since f is a function, then for each x , there is only one y such that $y = f(x)$. So, for x fixed, we have $\mathbb{1}_{\Gamma_f}(x, y) = \mathbb{1}_{\{f(x)\}}(y)$, an a.e. zero function. Hence,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\Gamma_f}(x, y) \, dy \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{f(x)\}}(y) \, dy \, dx = \int_{\mathbb{R}} 0 \, dx = 0.$$

Therefore, $m_2(\Gamma_f) = 0$. □

Problem 5. Consider the Hilbert space $\mathcal{H} = L^2([0, 1])$.

- (a) Prove that if $E \subseteq \mathcal{H}$ is closed and convex then E contains an element of smallest norm.

Hint: Show that if $\|f_n\|_2 \rightarrow \inf\{f \in E : \|f\|_2\}$ then $\{f_n\}$ is a Cauchy sequence.

- (b) Construct a non-empty closed $E \subseteq \mathcal{H}$ which does not contain an element of smallest norm.

Proof. We prove (a). Set $\alpha = \inf\{f \in E : \|f\|_2\}$. Applying the Parallelogram Law, we have

$$\|f_n + f_m\|_2^2 = 2\|f_n\|_2^2 + 2\|f_m\|_2^2 - \|f_n - f_m\|_2^2.$$

By convexity, $\frac{1}{2}f_n - \frac{1}{2}f_m \in E$. Hence $\|\frac{1}{2}f_n - \frac{1}{2}f_m\|_2 \geq \alpha$. So, $\|f_n - f_m\|_2^2 \geq 4\alpha^2$. We have

$$2\|f_n\|_2^2 + 2\|f_m\|_2^2 - \|f_n - f_m\|_2^2 \leq 2\|f_n\|_2^2 + 2\|f_m\|_2^2 - 4\alpha^2 \rightarrow_{n,m \rightarrow \infty} 0.$$

So, $\|f_n - f_m\|_2 \rightarrow_{n,m \rightarrow \infty} 0$. Therefore, the sequence is Cauchy. Since \mathcal{H} is Hilbert, then it is complete, so $f_n \rightarrow f$ in \mathcal{H} . Since E is closed, then $f \in E$. By continuity of the norm, we have $\alpha = \lim \|f_n\|_2 = \|f\|$. Therefore, $f \in E$ has a minimal norm.

Define

$$f_n = \mathbb{1}_{[\frac{1}{n+1}, \frac{1}{n}]}(n+1).$$

Set $E = \{f_n\}_{n=1}^\infty$, and note that

$$\int_0^1 |f_n(x)|^2 dx = \int_{\frac{1}{n+1}}^{\frac{1}{n}} (n+1)^2 dx = (n+1)^2 \left| \frac{x}{n+1} \right|_{\frac{1}{n+1}}^{\frac{1}{n}} = \frac{(n+1)^2}{n(n+1)} = 1 + \frac{1}{n}.$$

Hence, there is no minimal element in E . Moreover, each f_n has disjoint support. So,

$$\|f_n + f_m\|_2^2 = \|f_n\|_2^2 + \|f_m\|_2^2 \geq 2.$$

Therefore, there is no sequence $(f_{n_k})_{k=1}^\infty$ which is Cauchy. Thus, E has no limit points, so it is vacuously true that E contains all its limit points, and thus E is closed. □