Real Analysis Qual, Spring 2020

Problem 1. Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k\to\infty} \int_0^1 kx^{k-1} f(x)\,\mathrm{d}x = f(1).$$

Proof. First, for $0 \leqslant x < 1$, we claim $kx^{k-1} \to 0$ pointwise. If $x = 0$, then the claim is obvious. Otherwise, x^{-k+1} tends to infinity in k. Therefore, by L'Hopital's Rule, we have

$$\lim_{k\to\infty} kx^{k-1} = \lim_{k\to\infty} \frac{k}{x^{-k+1}} = \lim_{k\to\infty} \frac{1}{(-k+1)x^{-k}} = 0.$$

Moreover, kx^{k-1} is monotone. So, on the interval $[0, a]$ for $a < 1$, kx^{k-1} is bounded by ka^{k-1}. Therefore, $kx^{k-1} \to 0$ uniformly on $[0, a]$.

Now, f is a continuous function on the compact interval $[0, 1]$, and so f is bounded by some constant M. Then, for all $a \in [0, 1)$,

$$\int_0^a kx^{k-1}(-M)\,\mathrm{d}x \leqslant \int_0^a kx^{k-1} f(x)\,\mathrm{d}x \leqslant \int_0^a kx^{k-1} M\,\mathrm{d}x.$$

Since $kx^{k-1} \to 0$ uniformly on $[0, a]$, we obtain $\lim \int_0^a kx^{k-1} f(x)\,\mathrm{d}x = 0$ for each a. So, for all $a \in [0, 1)$, we have

$$\lim \int_0^1 kx^{k-1} f(x)\,\mathrm{d}x = \lim \int_a^1 kx^{k-1} f(x)\,\mathrm{d}x.$$

Define N_a to be the lower bound of $f(x)$ and M_a to be the upper bound of $f(x)$ over the compact interval $[a, 1]$. Then,

$$N_a - a^k N_a = \int_a^1 kx^{k-1} N_a\,\mathrm{d}x \leqslant \int_a^1 kx^{k-1} f(x)\,\mathrm{d}x \leqslant \int_a^1 kx^{k-1} M_a\,\mathrm{d}x = M_a - a^k M_a.$$

Since $a^k \to 0$, then taking $k \to \infty$, we have for all $a \in [0, 1)$,

$$N_a \leqslant \lim_{k\to\infty} \int_a^1 kx^{k-1} f(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_0^1 kx^{k-1} f(x)\,\mathrm{d}x \leqslant M_a.$$

Finally, by continuity of f, as $a \to 1$, we have $N_a \to f(1)$ and $M_a \to f(1)$. Therefore,

$$f(1) \leqslant \lim \int_0^1 kx^{k-1} f(x)\,\mathrm{d}x \leqslant f(1).$$

So, we obtain equality, as needed. □

Problem 2. Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

(a) Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E with the property that

$$m_*(B) = m_*(E).$$

(b) Prove that if $E \subseteq \mathbb{R}$ has the property that $m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$ for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \backslash N$ with $m_*(N) = 0$.

Proof. We begin with (a). By definition, if $E \subseteq \mathbb{R}$, then

$$m_*(E) = \inf \left\{ \sum_{n=1}^{\infty} V(I_n) : (I_n)_{n=1}^{\infty} \text{ are open-closed intervals, } E \subseteq \bigcup_{n=1}^{\infty} I_n \right\},$$

where $V((a, b]) = b - a = m_*(I_n)$. From definition of infimum, there exists a sequence $(I_n^{(k)})_{n=1}^{\infty}$ such that $E \subseteq \bigcup_{n=1}^{\infty} I_n^{(k)}$ and $\sum_{n=1}^{\infty} m_*(I_n^{(k)}) \leqslant m_*(E) + 1/k$. Define $B_k = \bigcup_{n=1}^{\infty} I_n^{(k)}$. Then, B_k is a countable union of open-closed intervals, and thus Borel. Set $B = \bigcap_{k=1}^{\infty} B_k$ and note that B is Borel. For each k, we have $B_k \supseteq E$, and so $B \supseteq E$. Moreover, for all k, we have $m_*(E) \leqslant m_*(B) \leqslant m_*(B_k) \leqslant m_*(E) + 1/k$. Thus, $m_*(E) = m_*(B)$ for B Borel.

For (b), take E as defined. Set $I_n = (-n, n)$. Note $\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$. For each $E \cap I_n$, by (a), choose B_n a Borel set so that $m_*(E \cap I_n) = m_*(B_n)$ and $E \cap I_n \subseteq B_n$. Note that $E \cap I_n \subseteq I_n \cap B_n \subseteq B_n$, so that $m_*(I_n \cap B_n) = m_*(E \cap I_n)$. Since I_n is Borel, then $I_n \cap B_n$ is also Borel. Hence, we may assume that B_n is contained in I_n. Therefore,

$$m_*(B_n) = m_*(B_n \cap E) + m_*(B_n \cap E^c) = m_*(B_n \cap (I_n \cap E)) + m_*(B_n \cap E^c).$$

Since $M_*(B_n) = m_*(I_n \cap E) = m_*(B_n \cap (I_n \cap E))$, we may conclude that $m_*(B_n \cap E^c) = 0$, and that $m_*(B_n \backslash E) \leqslant m_*(B_n \backslash (I_n \cap E)) = 0$. Define $B = \bigcup_{n=1}^{\infty} B_n$. Now, B_n is a countable union of Borel sets, and hence Borel. Set $N = B \backslash E$, and note now that $E = B \backslash N$. We have

$$m_*(N) = m_* \left(\left(\bigcup_{n=1}^{\infty} B_n \right) \backslash E \right) \leqslant m_* \left(\bigcup_{n=1}^{\infty} (B_n \backslash E) \right) \leqslant \sum_{n=1}^{\infty} m_*(B_n \backslash E) = 0.$$

Therefore, there exists B a Borel set such that $E = B \backslash N$ for $m_*(N) = 0$. $\square$

Problem 3.

(a) Prove that if $f \in L^1(\mathbb{R})$, then

$$\lim_{N \to \infty} \int_{|x| \geqslant N} |f(x)| \, \mathrm{d}x = 0$$

and demonstrate that it is not necessarily the case that $f(x) \to 0$ as $N \to \infty$.

(b) Prove that if $f \in L^1([1, \infty))$ and decreasing, then $\lim_{x \to \infty} f(x) = 0$ and in fact $\lim_{x \to \infty} x f(x) = 0$.

(c) If $f : [1, \infty) \to [0, \infty)$ is decreasing with $\lim_{x \to \infty} x f(x) = 0$, does this ensure $f \in L^1([1, \infty))$?

Proof. We start with (a). First, suppose f is a compactly supported continuous function. Then, supp f is bounded, say by M. Therefore, for all $|x| \geqslant M$, we have $f(x) = 0$. So, for all $N \geqslant M$, we obtain

$$\int_{|x| \geqslant N} |f(x)| \, \mathrm{d}x \geqslant \int_{|x| \geqslant M} |f(x)| \, \mathrm{d}x = \int_{|x| \geqslant M} 0 \, \mathrm{d}x = 0.$$

Therefore,

$$\lim_{N\to\infty}\int_{|x|\geqslant N}|f(x)|\,\mathrm{d}x=0.$$

Now, say that $f\in L^1(\mathbb{R})$. Continuous functions with compact support are dense in $L^1(\mathbb{R})$. Let $\epsilon>0$. Choose g to be a continuous function with compact support such that $||f-g||_1<\epsilon$. Then, take N large enough so that $\int_{|x|\geqslant N}|g|\,\mathrm{d}x=0$. We have

$$\int_{|x|\geqslant N}|f(x)|\,\mathrm{d}x\leqslant\int_{|x|\geqslant N}|f(x)-g(x)|\,\mathrm{d}x+\int_{|x|\geqslant N}|g(x)|\,\mathrm{d}x\leqslant\int_{\mathbb{R}}|f(x)-g(x)|\,\mathrm{d}x<\epsilon.$$

So,

$$\lim_{N\to\infty}\int_{|x|\geqslant N}|f(x)|\,\mathrm{d}x=0.$$

Now, define $f_N=\sum_{n=1}^{N}\mathbb{1}_{[2^n,2^n+1/2^n]}$. Then, $f_N\to f_\infty$ pointwise. Moreover, $f_N\leqslant f_{N+1}$, so the f_N are monotone. Finally, the f_N are all nonnegative. Thus, by MCT, we have

$$1=\sum_{n=1}^{\infty}\frac{1}{2^n}=\lim\sum_{n=1}^{N}\frac{1}{2^n}=\lim\sum_{n=1}^{N}\int\mathbb{1}_{[2^n,2^n+1/2^n]}\,\mathrm{d}x=\lim\int f_N\,\mathrm{d}x=\int f_\infty\,\mathrm{d}x.$$

Since $|f_\infty|=f_\infty$, then $f_\infty\in L^1(\mathbb{R})$. But, $\lim_{x\to\infty}f_\infty(x)\neq 0$, since for all n we have $f_\infty(2^n)=1$.

Now we prove (b). Say that $f(x)$ is monotone in x. Say there is some $x_0\in[1,\infty)$ such that $f(x_0)<0$. Then, for all $x\geqslant x_0$, we have $f(x)\leqslant f(x_0)$. Therefore,

$$\int_{x_0}^{\infty}f(x)\,\mathrm{d}x\geqslant\int_{x_0}^{\infty}f(x_0)\,\mathrm{d}x=-\infty.$$

So, $\int_1^\infty|f|\,\mathrm{d}x=\infty$, a contradiction. Thus, $f(x)\geqslant 0$ for all x. Therefore, $f(x)$ decreases monotonically in x and is bounded below, so $\lim f(x)=L$ for some L. Say that $L>0$. Then, there exist constants M,N such that for all $x\geqslant N$ we have $f(x)\geqslant M>0$. Then,

$$\int_N^\infty|f(x)|\,\mathrm{d}x=\int_N^\infty f(x)\,\mathrm{d}x\geqslant\int_N^\infty M\,\mathrm{d}x=\infty,$$

again a contradiction. Thus, $L=0$. Furthermore, fix x. Then,

$$\int_{x/2}^{x}f(t)\,\mathrm{d}t\geqslant\int_{x/2}^{x}f(x)\,\mathrm{d}t=f(x)\Big|_{\frac{x}{2}}^{x}xf(x)-\frac{x}{2}f(x)=\frac{xf(x)}{2}.$$

Now, since the tail of the integral of an L^1 function goes to 0, then $\lim_{x\to\infty}\int_{x/2}^{x}f(t)\,\mathrm{d}t=0$. Therefore, $\lim_{x\to\infty}\frac{xf(x)}{2}=0$, so $\lim_{x\to\infty}xf(x)=0$.

Finally, we prove (c). Consider

$$f(x)=\frac{1}{(x+1)\ln(x+1)}$$

which is defined on all $[1,\infty)$. Then, $xf(x) = \frac{x}{(x+1)\ln(x+1)} \to_{x\to\infty} 0$. Moreover, $f(x)$ is decreasing, since $(x+1)\ln(x+1)$ increases monotonically on $[1,\infty)$. Finally, the antiderivative of $f(x)$ on $[1,\infty)$ is $\ln(\ln(x+1))$. Since $\ln(\ln(x+1)) \to \infty$ as $x \to \infty$, and $\ln(\ln(2))$ is a constant, then

$$\infty = \int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty |f(x)|\,\mathrm{d}x.$$

So $f(x) \notin L^1([1,\infty])$. $\square$

Problem 4. Let $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Argue that $H(x,y) = f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this that

$$f * g(x) = \int f(y)g(x-y)\,\mathrm{d}y$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$||f * g||_1 \leqslant ||f||_1||g||_1.$$

Proof. Since f, g are measurable functions, and $(x,y) \longmapsto xy$ is a continuous function, then $H(x,y)$ is a measurable function. By Tonelli's Theorem, we obtain

$$\begin{aligned}
\int |H(x,y)|\,\mathrm{d}m_2 &= \int\int |f(y)g(x-y)|\,\mathrm{d}x\,\mathrm{d}y \\
&= \int |f(y)| \int |g(x-y)|\,\mathrm{d}x\,\mathrm{d}y \\
&= \int |f(y)| \int |g(x)|\,\mathrm{d}x\,\mathrm{d}y \\
&= \left(\int |f(y)|\,\mathrm{d}y\right)\left(\int |g(x)|\,\mathrm{d}x\right).
\end{aligned}$$

Therefore, $H(x,y) \in L^1(\mathbb{R}^2)$. Furthermore,

$$\int |f * g(x)|\,\mathrm{d}x = \int \left|\int f(y)g(x-y)\,\mathrm{d}y\right| \mathrm{d}x \leqslant \int\int |f(y)g(x-y)|\,\mathrm{d}y\,\mathrm{d}x = ||H||_1.$$

So, $f * g \in L^1(\mathbb{R})$, and

$$||f * g||_1 \leqslant ||H_1|| = \int |H(x,y)|\,\mathrm{d}m_2 = \left(\int |f(y)|\,\mathrm{d}y\right)\left(\int |g(x)|\,\mathrm{d}x\right) = ||f||_1||g||_1,$$

completing the proof. $\square$

Problem 5. Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-n+1} \mathrm{d}x.$$

Proof. Recall that $\lim_{n\to\infty}(1+\frac{y}{n})^n \to e^y$ pointwise. Likewise, $1+\frac{x^2}{n} \to 1$ pointwise. Thus, for x fixed we have

$$\lim_{n\to\infty} \mathbb{1}_{[0,n]} \left(1+\frac{x^2}{n}\right)^{-n+1} = \lim_{n\to\infty} \frac{1+\frac{x^2}{n}}{\left(1+\frac{x^2}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{x^2}{n}\right)^n} = e^{-x^2}.$$

Moreover, for each n, applying the Binomial Theorem and observing that $x \geqslant 0$, we have the inequality

$$\left(1+\frac{x^2}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x^2}{n}\right)^k \geqslant \binom{n}{2}\left(\frac{x^2}{n}\right)^2 = \frac{n-1}{2n}x^4 \geqslant \frac{1}{4}x^4.$$

Therefore,

$$\left(1+\frac{x^2}{n}\right)^{-n+1} = \frac{1+\frac{x^2}{n}}{\left(1+\frac{x^2}{n}\right)^n} \leqslant \frac{4\left(1+\frac{x^2}{n}\right)}{x^4} \leqslant \frac{4+4x^2}{x^4}.$$

Furthermore, on the interval $[0,1]$, since $1+x^2/n \geqslant 1$ for all $x \in [0,1]$, we have

$$\left(1+\frac{x^2}{n}\right)^{-n+1} = \frac{1}{\left(1+\frac{x^2}{n}\right)^{n-1}} \leqslant 1.$$

Define

$$f(x) := \begin{cases} 1, \text{ if } x \in [0,1], \\ \frac{4+4x^2}{x^4}, \text{ if } x \in (1,\infty). \end{cases}$$

By our previous arguments, $f(x)$ globally bounds $(1+\frac{x^2}{n})^{-n+1}$. Moreover,

$$\int_0^\infty |f(x)|\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x + \int_1^\infty f(x)\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x + \int_1^\infty \frac{4+4x^2}{x^4}\,\mathrm{d}x < \infty.$$

So, $(1+\frac{x^2}{n})^{-n+1}$ has the integrable dominant f. Therefore, by DCT, we obtain

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x^2}{n}\right)^{-n+1} \mathrm{d}x = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{3},$$

given that $\int_0^\infty e^{-x^2}\,\mathrm{d}x$ is the standard Gaussian integral. $\square$

Problem 6.

(a) Show that $L^2([0,1]) \subseteq L^1([0,1])$ and that $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

(b) For $f \in L^1([0,1])$ define

$$\hat{f}(n) := \int_0^1 f(x)e^{-2\pi i n x}\,\mathrm{d}x.$$

Prove that if $f \in L^1([0,1])$ and $\{\hat{f}(n)\} \subseteq \ell^1(\mathbb{Z})$, then

$$S_N f(x) = \sum_{|n|\leqslant N} \hat{f}(n)e^{2\pi i n x}$$

converges uniformly on $[0,1]$ to a continuous function g that equals f almost everywhere.

Hint: One possible approach is to argue that if $f \in L^1([0,1])$ with $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$, then $f \in L^2([0,1])$.

Proof. We begin with (a). Let f be an $L^2([0,1])$ function. Then, by Hölder's Inequality, we have

$$||f||_1 \leqslant ||f||_2||1||_2 = ||f||_2 < \infty.$$

So, $f \in L^1([0,1])$. Now, say that $(a_n) \in \ell^1(\mathbb{Z})$. Since $\mathbb{Z}, \mathbb{N}$ are both countable, we may simple assume this series is indexed in $\mathbb{N}$. Then, $a_n \to 0$. So, pick N such that for all $n \geqslant N$, we have $|a_n| < 1$. Then, $|a_n|^2 \leqslant |a_n|$, so we obtain

$$\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{N} |a_n|^2 + \sum_{n=N}^{\infty} |a_n|^2 \leqslant \sum_{n=1}^{N} |a_n|^2 + \sum_{n=1}^{\infty} |a_n| < \infty.$$

Therefore, $(a_n) \in \ell^2(\mathbb{Z})$.

We now prove (b). Take $M \geqslant N$. Then,

$$||S_N f - S_M f||_1 = \left\| \sum_{N \leqslant |n| \leqslant M} \widehat{f}(n) e^{-2\pi i n x} \right\| \leqslant \sum_{N \leqslant |n| \leqslant M} \int \left| \widehat{f}(n) e^{-2\pi i n x} \right| \mathrm{d}x = \sum_{N \leqslant |n| \leqslant M} |\widehat{f}(n)|.$$

Since $(\widehat{f}(n)) \in \ell^1(\mathbb{Z})$, then we conclude that $S_N f$ is Cauchy. Since $L^1([0,1])$ is complete, then $S_N f$ converges to some $g \in L^1([0,1])$. Moreover, observe that $|\widehat{f}(n) e^{-2\pi i n x}| \leqslant |\widehat{f}(n)|$ for all x. Then, since $\sum_{n \in \mathbb{Z}} |\widehat{f}(n)| < \infty$, by the Weierstrass M-test, we have a uniform convergence $S_N f \to g$. Since each $S_N f$ is continuous, then g is continuous. Now, since $S_N f \to g$ uniformly, then we may pass the limit through the integral. That is,

$$\widehat{g}(n) = \int_0^1 g(x) e^{-2\pi i n x} \, \mathrm{d}x = \lim \int_0^1 S_N f(x) e^{-2\pi i n x} \, \mathrm{d}x = \lim \int_0^1 \widehat{f}(n) \, \mathrm{d}x = \widehat{f}(n),$$

by the orthogonality of $(e^{2\pi i n x})_{n \in \mathbb{Z}})$ over $L^2([0,1])$. Therefore, f and g have the same Fourier coefficients. By uniqueness of fourier coefficients, we conclude that $f = g$ a.e.. $\square$