Real Analysis Qual, Fall 2020

Problem 1. Show that if (x_n) is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, then

$$\lim nx_n = 0.$$

Proof. Since $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{k\to\infty} \sum_{n=k}^{\infty} x_n = 0$. Now, take $\epsilon > 0$. Choose k large enough that $\sum_{n=k}^{\infty} x_n < \epsilon/2$. Take $m \geqslant 2k$, and first suppose that m is even. Note that $x_m/2 \geqslant k$. Since the x_n are all nonnegative, we have

$$\frac{m}{2}x_m = \sum_{n=\frac{m}{2}}^{m} x_m \leqslant \sum_{n=\frac{m}{2}}^{m} x_n \leqslant \sum_{n=k}^{\infty} x_n < \epsilon/2.$$

So that $mx_m < \epsilon$ whenever m is even. On the other hand, if m is odd, then $m \geqslant 2k+1$ and hence $(m-1)/2 \geqslant k$. So,

$$\frac{m}{2}x_m \leqslant \frac{m+1}{2}x_m = \sum_{n=\frac{m-1}{2}}^{m} x_m \leqslant \sum_{n=\frac{m-1}{2}}^{m} x_n \leqslant \sum_{n=k}^{\infty} x_n < \epsilon/2.$$

Therefore, $mx_m < \epsilon$. We have covered both the even and odd case, so that for all $m \geqslant 2k$, we have $mx_m < \epsilon$. Therefore, $mx_m \to 0$. $\square$

Problem 2.

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Prove that

$$f(x) \leqslant \liminf_{y\to x} f(y) \text{ for each } x \in \mathbb{R} \Longleftrightarrow \{x \in \mathbb{R} : f(x) > a\} \text{ is open for all } a \in \mathbb{R}.$$

(b) A function f is called *lower semi-continuous* if it satisfies either condition in part (a) above. Prove that if $\mathcal{F}$ is any family of lower semi-continuous functions, then $g(x) = \sup\{f(x) : f \in \mathcal{F}\}$ is Borel measurable. *Note that $\mathcal{F}$ need not be a countable family.*

Proof. We first prove (a). Suppose that $f(x) \leqslant \liminf_{y\to x} f(y)$. Then, since $f(x) \in \inf\{|y - x| < \delta : f(y)\}$ for all x, we have $f(x) = \liminf_{y\to x} f(y)$. That is, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \inf_{|x-y|<\delta} f(y)| < \epsilon$. Now, take $x \in f^{-1}((a,\infty))$. Then, $f(x) \in (a,\infty)$. Pick ϵ so that the ϵ-ball $B_\epsilon(f(x))$ around $f(x)$ is contained in (a,∞). Choose δ such that $|f(x) - \inf_{|x-y|<\delta} f(y)| < \epsilon$. Take $y \in B_\delta(x)$. Then, $|x - y| < \delta$. If $f(y) \geqslant f(x)$, then $f(y) > a$, so $f(y) \in (a,\infty)$. Otherwise, if $f(y) \leqslant f(x)$, then $\inf_{|x-y|<\delta} f(y) \leqslant f(y) \leqslant f(x)$, and so $|f(y) - f(x)| < \epsilon$. Therefore, $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subseteq (a,\infty)$. Hence, for all $y \in B_\delta(x)$, we have $f(y) \in (a,\infty)$. In particular, $B_\delta(x) \subseteq f^{-1}((a,\infty))$, and so $f^{-1}((a,\infty)) = \{y \in \mathbb{R} : f(y) > a\}$ is open.

Now, suppose that $\{y \in \mathbb{R} : f(y) > a\} = f^{-1}((a,\infty))$ is open for all a. Then, for all $\epsilon > 0$, we have $f^{-1}((f(x) - \epsilon,\infty))$ is open. Therefore, there is some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}((f(x) - \epsilon,\infty))$. Thus, for all $y \in \mathbb{R}$ satisfying $|x - y| < \delta$, we have $f(y) \in (f(x) - \epsilon,\infty)$, and thus $f(y) > f(x) - \epsilon$. Since this holds for all y satisfying $|x - y| < \delta$, we obtain $\inf_{|x-y|<\delta} f(y) \geqslant f(x) - \epsilon$. By monotonicity of the infimum, for all $\delta' \leqslant \delta$, we achieve $\inf_{|x-y|<\delta'} f(y) \geqslant f(x) - \epsilon$. Therefore, $\liminf_{y\to x} f(y) = f(x)$, since we always have $\liminf_{y\to x} \leqslant f(x)$. Therefore, $\liminf_{y\to x} f(y) \geqslant f(x)$.

We prove (b). We show that each $g^{-1}((a,\infty))$ is measurable. This is sufficient, since the open rays to positive infinity generate the Borel σ-algebra. We prove $f^{-1}((a,\infty))$ is open. Take $x \in g^{-1}((a,\infty))$. We observe that $f(x) \nleqslant a$ for all $f \in \mathcal{F}$. Indeed, if $f(x) \leqslant a$ for all $f \in \mathcal{F}$, then $g(x) = \sup_{f\in\mathcal{F}} f(x) \leqslant a$, and thus $g(x) \notin (a,\infty)$ a contradiction. Therefore, there exists some f such that $f(x) > a$. So, there exists some δ-ball $B_\delta(x)$ such that $f(B_\delta(x)) \subseteq (a,\infty)$, by (a). So, for each $x \in B_\delta(x)$, we have $a < f(x) \leqslant g(x)$. Therefore, $g(B_\delta(x)) \subseteq (a,\infty)$. In particular, $B_\delta(x) \subseteq g^{-1}((a,\infty))$. So, $g^{-1}((a,\infty))$ is an open set as claimed, and thus Borel. So, g is Borel measurable. $\square$

Problem 3. **Don't understand how problem is well defined.** Let f be a nonnegative Lebesgue measurable function on $[1,\infty)$.

(a) Prove that
$$1 \leqslant \left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b \frac{1}{f(x)}\,\mathrm{d}x\right)$$
for any $1 \leqslant a < b < \infty$.

(b) Prove that if f satisfies $\int_1^t f(x)\,\mathrm{d}x \leqslant t^2 \log t$ for all $t \in [1,\infty)$, then $\int_1^\infty \frac{1}{f(x)}\,\mathrm{d}x = \infty$.

Hint: Write $\displaystyle\int_1^\infty \frac{1}{f(x)}\,\mathrm{d}x = \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \frac{1}{f(x)}\,\mathrm{d}x.$

Problem 4. Prove that if $xf(x) \in L^1(\mathbb{R})$, then
$$F(y) := \int f(x)\cos(yx)\,\mathrm{d}x$$
defines a C^1 function.

The statement in this problem is wrong. It is possible to have $xf(x) \in L^1(\mathbb{R})$, with $F(y)$ not continuous. Take
$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$
Observe that $xf(x) = \mathbb{1}_{(0,1)}$. In particular, $xf(x) \in L^1(\mathbb{R})$. Now, if F is continuous on $\mathbb{R}$, then F is continuous on $[0,1]$, and since $[0,1]$ is compact, then F is finite on $[0,1]$. So, $F(0) < \infty$. However,
$$F(0) = \int f(x)\cos(0x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x = \int_0^1 \frac{1}{x}\,\mathrm{d}x = \infty.$$
Thus, we obtain a contradiction. So, $F(y)$ is not continuous.

Problem 5. (Classic) Suppose $\phi \in L^1(\mathbb{R})$ with $\int \phi\,\mathrm{d}x = a$. For each $\delta > 0$ and $f \in L^1(\mathbb{R})$, define
$$A_\delta f(x) := \int f(x-y)\delta^{-1}\phi(\delta^{-1}y)\,\mathrm{d}y.$$

(a) Prove that $||A_\delta f||_1 \leqslant ||f||_1||\phi||_1$ for all $\delta > 0$.

(b) Prove that $A_\delta f \to af$ in $L^1(\mathbb{R})$ as $\delta \to 0^+$.

Hint: You may use, without proof, the fact that $\lim_{y\to 0}\int_{\mathbb{R}} |f(x-y) - f(x)|\,\mathrm{d}x = 0$ *for all* $f \in L^1(\mathbb{R})$.

Proof. We prove (a). Observe first that, upon applying the change of coordinates $x = \delta^{-1}y$, we have

$$\int |\delta^{-1}\phi(\delta^{-1}y)|\,\mathrm{d}y = \int |\phi(x)|\,\mathrm{d}x.$$

Now,

$$\begin{aligned}\int |A_\delta f(x)|\,\mathrm{d}x &= \int \left|\int f(x-y)\delta^{-1}\phi(\delta^{-1}y)\,\mathrm{d}y\right|\mathrm{d}x \\ &\leqslant \iint |f(x-y)\delta^{-1}\phi(\delta^{-1}y)|\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Applying Tonelli's Theorem, we reverse the order of integration to obtain

$$\begin{aligned}\iint |f(x-y)\delta^{-1}\phi(\delta^{-1}y)|\,\mathrm{d}y\,\mathrm{d}x &= \iint |f(x-y)\delta^{-1}\phi(\delta^{-1}y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int |\delta^{-1}\phi(\delta^{-1}y)| \int |f(x-y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int |\delta^{-1}\phi(\delta^{-1}y)| \int |f(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \left(\int |\delta^{-1}\phi(\delta^{-1}y)|\,\mathrm{d}y\right)\left(\int |f(x)|\,\mathrm{d}x\right) \\ &= \left(\int |\phi(x)|\,\mathrm{d}x\right)\left(\int |f(x)|\,\mathrm{d}x\right) \\ &= ||\phi||_1||f||_1.\end{aligned}$$

Thus, the claim is proven.

We now prove (b). There are two different possible proofs. One with compactly supported functions. We do the other. We have

$$A_\delta f(x) - af(x) = A_\delta f(x) - \left(\int \delta^{-1}\phi(\delta^{-1}y)\,\mathrm{d}y\right) f(x) = A_\delta f(x) - \left(\int \delta^{-1}\phi(\delta^{-1}y)f(x)\,\mathrm{d}y\right).$$

Hence,

$$A_\delta f(x) - af(x) = \int (f(x-y) - f(x))\delta^{-1}\phi(\delta^{-1}y)\,\mathrm{d}y.$$

Substitute $z = \delta^{-1}y$. Observe that $\delta z = y$. Then,

$$A_\delta f(x) - af(x) = \int (f(x-\delta z) - f(x))\phi(z)\,\mathrm{d}z.$$

Therefore, applying Tonelli's theorem as before, we have

$$||A_\delta f - af||_1 = \int \left| \int (f(x-\delta z) - f(x))\phi(z)\,\mathrm{d}z \right| \mathrm{d}x \leqslant \int \int |f(x-\delta z) - f(x)||\phi(z)|\,\mathrm{d}x\,\mathrm{d}z.$$

So,

$$||A_\delta f - af||_1 \leqslant \int \int |f(x-\delta z) - f(x)||\phi(z)|\,\mathrm{d}x\,\mathrm{d}z = \int ||f(\cdot - \delta z) - f||_1 |\phi(z)|\,\mathrm{d}z.$$

Now, $||f(\cdot - \delta z) - f||_1$ is a function in z. Pointwise, we have

$$\lim_{\delta \to 0^+} ||f(\cdot - \delta z) - f||_1 = \lim_{\delta \to 0^+} \int |f(x-\delta z) - f(x)|\,\mathrm{d}x = 0.$$

Moreover, for all z, $||f(\cdot - \delta z) - f||_1$ is bounded by $2||f||_1$. Observe, then, that $2||f||_1\phi(z)$ is an L^1 function which dominates $||f(\cdot - \delta z) - f||_1\phi(z)$, and which converges pointwise to 0. Therefore, by DCT, we have

$$\lim_{\delta \to 0^+} \int ||f(\cdot - \delta z) - f||_1 \phi(z)\,\mathrm{d}z = 0.$$

Thus, $||A_\delta f - af||_1 \to_{\delta \to 0^+} 0$, and so $A_\delta f \to af$ in L^1. $\square$