Real Analysis Qual, Fall 2022

Problem 1. Let $\mathbb{1}_{[0,\infty)}$ denote the characteristic function of $[0,\infty)$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that $f(x) = \mathbb{1}_{[0,\infty)}(x)$ for almost every $x \in \mathbb{R}$ (with respect to the Lebesgue measure).

Proof. Let f be a function which equals $\mathbb{1}_{[0,\infty)}$ almost everywhere. Take $\epsilon > 0$. Observe that $0 \in f^{-1}(B_\epsilon(f(0)))$. If f is continuous, then there is some δ such that $B_\delta(0) \subseteq f^{-1}(B_\epsilon(f(0)))$. In particular, $f(B_\delta(0)) \subseteq B_\epsilon(f(0))$. We claim that $0, 1 \in f(B_\delta(0))$. Indeed, $(0,\delta) \subseteq B_\delta(0)$, and $\mathbb{1}_{[0,\infty)}((0,\delta)) = \{1\}$. Since $(0,\delta)$ does not have measure 0, then there must be some $y \in (0,\delta)$ such that $f(y) = \mathbb{1}_{[0,\infty)}(y) = 1$. An identical argument on $(-\delta, 0)$ shows that there is some $y \in (-\delta, 0)$ so that $f(y) = 0$. Hence, $0, 1 \in f(B_\delta(0))$ for all δ. Therefore, $0, 1 \in B_\epsilon(f(0))$. Taking $\epsilon \to 0$ gives a contradiction. Thus, f is not everywhere continuous. □

Problem 2. Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable family of Lebesgue measurable subsets of $\mathbb{R}^d$ with

$$\sum_{n=1}^{\infty} m(E_n) < \infty$$

where m denotes the Lebesgue measure on $\mathbb{R}^d$ and let

$$E = \{x \in X : x \in E_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

(a) **(Classic)** Show that $E = \bigcap_{N=1}^{\infty} \bigcup_{n \geqslant N} E_n$ and deduce that E is Lebesgue measurable with $m(E) = 0$.

(b) Show that
$$\chi_E(x) = \limsup_{n\to\infty} \chi_{E_n}(x)$$
for all $x \in \mathbb{R}^d$ where, for any subset $A \subseteq \mathbb{R}^d$, χ_A denotes the characteristic function of A.

Proof. Set $F = \bigcap_{N=1} \bigcup_{n\geqslant N} E_n$. Say that $x \in F$. Then, for all N, we have $x \in \bigcup_{n\geqslant N} E_n$. So, for all N, there is some $n \geqslant N$ such that $x \in E_n$. Hence, x is in infinitely many E_n, so $x \in E$, proving $F \subseteq E$. Now, take $x \in E$. Then, x is in infinitely many E_n. Therefore, for all N, there must be some $n \geqslant N$ such that $x \in E_n \subseteq \bigcup_{n\geqslant N} E_n$. Since this holds for all N, then $x \in \bigcap_{N=1} \bigcup_{n\geqslant N} E_n = F$, so $E \subseteq F$, giving equality. Note as well that

$$m\left(\bigcup_{n\geqslant N} E_n\right) \leqslant \sum_{n\geqslant N} m(E_n) \to_{N\to\infty} 0,$$

given that $\sum_{n=1}^{\infty} m(E_n)$ is convergent. Write $A_N = \bigcup_{n\geqslant N} E_n$, and note that $A_N \supseteq A_{N+1}$. Moreover, A_1 has finite measure, since $m(A_1) \leqslant \sum_{n=1}^{\infty} m(E_n) < \infty$. Therefore, by continuity from above,

$$m(E) = m\left(\bigcap_{N=1}^{\infty} A_n\right) = \lim_{N\to\infty} m(A_N) = 0.$$

So E is measure 0, proving part (a).

For (b), we have that

$$\limsup_{n\to\infty} \chi_{E_n}(x) = \lim_{n\to\infty} \sup_{k\geqslant n} \chi_{E_k}(x).$$

Now, either $\limsup_{n\to\infty} \chi_{E_n}(x) = 1$ or $\limsup_{n\to\infty} \chi_{E_n}(x) = 0$. In the case it equals 1, for all n, there exists some $k \geqslant n$ such that $x \in E_k$. Hence, x is in infinitely many E_k, so $x \in E$ and $\chi_E(x) = 1$. In the case it equals 0, then there is some n large enough such that for all $k \geqslant n$, we have $\chi_{E_k}(x) = 0$, and so x is in finitely many E_k. Thus, $x \notin E$, so $\chi_E(x) = 0$ as well. Since the output of $\limsup_{n\to\infty} \chi_{E_n}(x)$ and $\chi_E(x)$ can both only take on two values, then we have shown that $\limsup_{n\to\infty} \chi_{E_n}(x) = \chi_E(x)$. $\square$

Problem 3. **(Classic Technique)** Prove that if g is continuous with compact support on $\mathbb{R}^d$, then

$$\lim_{n\to\infty} \int |g(n^{1/n}x) - g(x)| \, \mathrm{d}x = 0$$

and deduce from this that if $f \in L^1(\mathbb{R}^d)$, then

$$\lim_{n\to\infty} \int |f(n^{1/n}x) - f(x)| \, \mathrm{d}x = 0.$$

Proof. We first prove that $n^{1/n} \to_{n\to\infty} 1$. Observe that $e^x \geqslant x$ for all $x \in (0, \infty)$, and hence

$$e = e^{x\cdot(1/x)} = (e^x)^{1/x} \geqslant x^{1/x}.$$

On the other hand, for all $x \geqslant 1$, we have $\sqrt[n]{x} \geqslant 1$ for all n, and hence $x^{1/x} \geqslant 1$. Therefore, the sequence $n^{1/n}$ is bounded by $[1, e]$. By Bolzano-Weierstrass, there is some convergent subsequence (n_k^{1/n_k}). Suppose that $n^{1/n} \not\to 1$. Then, there is some $\alpha > 1$ such that, for k sufficiently large, $n_k^{1/n_k} > \alpha$. In particular, $n_k > \alpha^{n_k}$ for all k sufficiently large. However, if $\alpha > 1$, then α^x is an exponential function, and hence there is some x large enough that for all $y > x$ we have $\alpha^y > y$. Picking $n_k > x$ gives a contradiction. So, we conclude that $n^{1/n}$ converges to 1.

Since supp g is compact, it is bounded. Say that $M = \sup_{x\in \operatorname{supp} g} |x|$. Choose N so that $N > M + 1$. Let B be the closed ball of radius $2N$. We claim that

$$\lim_{n\to\infty} \int |g(n^{1/n}x) - g(x)| \, \mathrm{d}x = \lim_{n\to\infty} \int_B |g(n^{1/n}x) - g(x)| \, \mathrm{d}x.$$

There is some m such that for all $n \geqslant m$, we have $|1 - n^{1/n}| > 1/2$. Therefore, for $x \notin B$, we have

$$|n^{1/n}x| = n^{1/n}|x| > \frac{1}{2}|x| > \frac{1}{2}(2N) = N.$$

So, $|n^{1/n}x| > M$, and hence $n^{1/n}x \notin \operatorname{supp} g$. Thus, for $n \geqslant m$, if $x \notin B$, then $x, n^{1/n}x \notin \operatorname{supp} g$. So,

$$\lim_{n\to\infty} \int |g(n^{1/n}x) - g(x)| \, \mathrm{d}x = \lim_{n\to\infty} \int_B |g(n^{1/n}x) - g(x)| \, \mathrm{d}x$$

as claimed.

Take $\epsilon > 0$. Since B is a closed ball of finite radius, it is a closed and bounded subset of $\mathbb{R}^d$. Therefore, by Heine-Borel, it is compact. Thus, given that g is a continuous function, it is uniformly continuous on B. Choose δ small enough that for all $x, y \in B$ satisfying $|x-y| < \delta$, we have $|g(x) - g(y)| < \epsilon$. Pick m such that for all $n \geqslant m$, we have $|1 - n^{1/n}| < \delta/2N$. Note that $|1 - n^{1/n}| < 1/2$. Moreover,

$$|n^{1/n}x - x| = |(1 - n^{1/n})x| = |1 - n^{1/n}| \cdot |x| < \frac{\delta}{2N} \cdot 2N < \delta.$$

Therefore, $|g(n^{1/n}x) - g(x)| < \epsilon$. Since $|1 - n^{1/n}| < 1/2$, by our preceding argument, we obtain for all $n \geqslant m$

$$\int |g(n^{1/n}x) - g(x)| \,\mathrm{d}x = \int_B |g(n^{1/n}x) - g(x)| \,\mathrm{d}x < \int_B \epsilon \,\mathrm{d}x = m(B)\epsilon.$$

Since B was chosen independently of ϵ, then taking $\epsilon \to 0$ gives the result for continuous functions with compact support.

Let $f \in L^1(\mathbb{R}^d)$. Continuous functions with compact support are dense in $L^1(\mathbb{R}^d)$, so choose g such that $||f - g||_1 < \epsilon$. Then, pick m such that for all $n \geqslant m$, we have $\int |g(n^{1/n}x) - g(x)| \,\mathrm{d}x < \epsilon$. We obtain

$$||f(n^{1/n}x) - f(x)||_1 \leqslant ||f(n^{1/n}x) - g(n^{1/n}x)||_1 + ||g(n^{1/n}x) - g(x)||_1 + ||g(x) - f(x)||_1.$$

By our choice, $||g(n^{1/n}x) - g(x)||_1, ||g(x) - f(x)||_1 < 2\epsilon$. On the other hand, substituting $y = n^{1/n}x$ and noting $\mathrm{d}y = n^{1/n} \,\mathrm{d}x$, we have

$$\int |f(n^{1/n}x) - g(n^{1/n}x)| \,\mathrm{d}x = \int |f(y) - g(y)| n^{1/n} \,\mathrm{d}x < n^{1/n}\epsilon \leqslant e \cdot \epsilon.$$

So, for all $n \geqslant m$, we have $||f(n^{1/n}x) - f(x)||_1 < 2/3\epsilon + e \cdot \epsilon$. Taking $\epsilon \to 0$ gives the result. $\square$

Problem 4. Let f be a function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a given enumeration $\{q_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x + q_n).$$

(a) Prove that F is a Lebesgue integrable function on $\mathbb{R}$ and hence that the series defining F converges for almost every $x \in \mathbb{R}$.

(b) Show, however, that this series is unbounded on every open interval, and in fact, any function G that agrees with F almost everywhere must be unbounded on every open interval.

Proof. Set $g(n,x) = \frac{1}{2^n} f(x+q_n)$. Since f is measurable, then g is measurable. Moreover, taken with respect to the counting measure on $\mathbb{N}$,

$$F(x) = \int g(n,x)\,\mathrm{d}x.$$

Observe that $g(n,x)$ is nonnegative for all n, x. Hence, by Tonelli's Theorem,

$$\int_{\mathbb{R}} F(x)\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{N}} g(n,x)\,\mathrm{d}n\,\mathrm{d}x = \int_{\mathbb{N}}\int_{\mathbb{R}} g(n,x)\,\mathrm{d}x\,\mathrm{d}n = \int_{\mathbb{N}}\int_{\mathbb{R}} \frac{1}{2^n} f(x+q_n)\,\mathrm{d}x\,\mathrm{d}r.$$

By invariance under shifts, we have

$$\int_{\mathbb{N}}\int_{\mathbb{R}} \frac{1}{2^n} f(x+q_n)\,\mathrm{d}x\,\mathrm{d}r = \int_{\mathbb{N}} \frac{1}{2^n} \int f(x)\,\mathrm{d}x\,\mathrm{d}r = \int_{\mathbb{N}} \frac{1}{2^n} \int_0^1 x^{-1/2}\,\mathrm{d}x\,\mathrm{d}n = \int_{\mathbb{N}} \frac{2}{2^n}\,\mathrm{d}n.$$

Putting it all together,

$$\int_{\mathbb{R}} |F(x)|\,\mathrm{d}x = \int_{\mathbb{R}} F(x)\,\mathrm{d}x = \int_{\mathbb{N}} \frac{2}{2^n}\,\mathrm{d}n = \sum_{n=1}^{\infty} \frac{2}{2^n} = 2.$$

Therefore, F is a Lebesgue integrable function, proving (a).

Take an open interval (a,b). By density of the rationals, there is some rational point $q_m \in (a,b)$. Let A be a null set. Note that $(q_m, q_m + 1/n) \not\subseteq A$ for any n, since $(q_m, q_m + 1/n)$ has nonzero measure. Therefore, we may take a sequence of points (x_n) satisfying $x_n \in (q_m, q_m + 1/n)\backslash A$. Observe that $x_n \to q_m$. For some k, $q_k = -q_m$. Therefore, $x_n + q_k \in (q_m + q_k, q_m + q_k + 1/n) = (0, 1/n) \subseteq (0,1)$. Since f is continuous on $(0,1)$ and $f(t) \to \infty$ as $t \to 0$, then $f(x_n + q_k) \to_{n\to\infty} \infty$. Furthermore, by nonnegativity of f,

$$\frac{1}{2^k} f(x_n + q_k) \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n + q_n) = F(x_n).$$

Therefore, $F(x_n) \to \infty$ as $n \to \infty$. Since $x_n \to q_m$, then the tail of (x_n) is contained in (a,b), so we conclude that F is unbounded on (a,b). Moreover, we chose the sequence (x_n) to avoid the arbitrary null set A. Therefore, if $G = F$ almost everywhere, then G is distinct from F on a null set A. Constructing (x_n) to avoid A shows that G is unbounded on every open interval. □

Problem 5. Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. Prove that the collection $\{u_{j,k}\}_{j,k=1}^{\infty}$ with

$$u_{j,k}(x,y) = u_j(x)u_k(y)$$

forms an orthonormal basis for $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. We show that $u_{j,k} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. So,

$$\int |u_{j,k}(x,y)u_{i,\ell}(x,y)|\,\mathrm{d}(x\times y) = \int |u_j(x)|^2 |u_k(y)|^2\,\mathrm{d}(x \times y).$$

The integrand is a nonnegative measurable function, so by Tonelli's theorem

$$\begin{aligned}\int |u_j(x)|^2|u_k(y)|^2\,\mathrm{d}(x\times y) &= \iint |u_j(x)|^2|u_k(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &= \left(\int |u_j(x)|^2\,\mathrm{d}x\right)\left(\int |u_k(y)|^2\,\mathrm{d}y\right)\\ &= 1\cdot 1.\end{aligned}$$

Therefore, $\int |u_{j,k}(x,y)|^2\,\mathrm{d}(x\times y) = 1$, so $u_{j,k}\in L^2(\mathbb{R}^d\mathbb{R}^d)$.

Hence $u_{j,k}\overline{u_{i,\ell}}\in L^2(\mathbb{R}^d\times\mathbb{R}^d)$, for

$$\int |u_{j,k}(x,y)||u_{i,\ell}(x,y)|\,\mathrm{d}(x\times y)\leqslant ||u_{j,k}||_2||u_{i,\ell}||_2 = 1\cdot 1 = 1$$

by Cauchy-Schwarz. Thus, by Fubini's Theorem,

$$\begin{aligned}\int u_{j,k}(x,y)\overline{u_{i,\ell}(x,y)}\,\mathrm{d}(x\times y) &= \iint u_j(x)\overline{u_i(x)}\cdot u_k(y)\overline{u_\ell(y)}\,\mathrm{d}x\,\mathrm{d}y\\ &= \left(\int u_j(x)\overline{u_i(x)}\,\mathrm{d}x\right)\left(\int u_k(y)\overline{u_\ell(y)}\,\mathrm{d}y\right).\end{aligned}$$

If either $j\neq i$ or $k\neq\ell$, then the result is 0, since in this case either u_i,u_j or u_k,u_ℓ are orthogonal. Otherwise, if $j=i$ and $k=\ell$, then the integrands are $|u_j|^2,|u_k|^2$, so, by the computation above, the result is 1. Therefore, $\{u_{j,k}\}_{j,k=1}^{\infty}$ is an orthonormal sequence in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$. Since it is an orthonormal sequence, $\{u_{j,k}\}_{j,k=1}^{\infty}$ is linearly independent. It remains to show that its spanning set is dense.

First, $g(x,y)\overline{u_{j,k}(x,y)}\in L^2(\mathbb{R}^d\times\mathbb{R}^d)$, for by Cauchy-Schwarz $||\,|g|\cdot|u_{j,k}|\,||_2<\infty$. Now, suppose that $\langle g,u_{j,k}\rangle = 0$ for all j,k. Then,

$$0=\int g\overline{u_{j,k}}\,\mathrm{d}(x\times y) = \iint g(x,y)\overline{u_j(x)u_k(y)}\,\mathrm{d}x\,\mathrm{d}y = \int \overline{u_k(y)}\left(\int g(x,y)\overline{u_j(x)}\,\mathrm{d}x\right)\mathrm{d}y.$$

Considering j as fixed, and regarding $\int g(x,y)\overline{u_j(x)}\,\mathrm{d}x$ as a function in y, we have for all k that

$$\left\langle \int g(x,y)\overline{u_j(x)}\,\mathrm{d}x, u_k\right\rangle = 0.$$

Since u_k is an orthonormal basis in $\mathbb{R}^d$, then $\int g(x,y)\overline{u_j(x)}\,\mathrm{d}x = 0$ for almost all $y\in\mathbb{R}^d$. For each j let A_j be the set of y such that $\int g(x,y)\overline{u_j(x)}\,\mathrm{d}x$ is nonzero. Then, $A=\bigcup_{j=1}^{\infty}A_j$ is still a null set. Therefore, for all $y\notin A$, and thus almost all $y\in\mathbb{R}^d$, $\langle g(x,y),u_j\rangle = 0$. Again, since u_j is an orthonormal basis in $\mathbb{R}^d$, then for almost all $y\in\mathbb{R}^d$, the function $g(x,y)$ is an almost everywhere 0 function. Let B be the set of x such that $g(x,y)$ is nonzero for $y\notin A$. Then, for all $(x,y)\notin A\times B$, we have $g(x,y)=0$. Since A,B are null sets, then $A\times B$ is a null set. So, $g(x,y)$ is a.e. the zero function. Therefore, $\{u_{j,k}\}_{j,k=1}^{\infty}$ has a dense span, so $\{u_{j,k}\}_{j,k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R}^d\times\mathbb{R}^d)$. □

Problem 6. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$. Prove that for any integrable function $f : X \longrightarrow \mathbb{C}$

$$\mu\left(\left\{x \in X : |f(x)| \geqslant \frac{1}{2}||f||_1\right\}\right) \geqslant \max\left\{\frac{||f||_1}{2||f||_\infty}, \frac{||f||_1^2}{4||f||_2^2}\right\}.$$

Proof. We observe that if either $||f||_\infty = \infty$ or $||f||_2 = \infty$, then at least one of the inequalities is clear. Hence, for our proofs, we may suppose that $f \in L^\infty(\mathbb{R})$ and $f \in L^2(\mathbb{R})$. Set $E = \{x \in X : |f(x)| \geqslant \frac{1}{2}||f||_1\}$. Since $m(X\backslash E) \leqslant 1$, we have

$$||f||_1 = \int_E |f| \,\mathrm{d}x + \int_{X\backslash E} |f| \,\mathrm{d}x \leqslant \int_E |f| \,\mathrm{d}x + \int_{X\backslash E} \frac{1}{2}||f||_1 \,\mathrm{d}x \leqslant \int_E |f| \,\mathrm{d}x + \frac{1}{2}||f||_1.$$

Therefore, after rearrangement, $\frac{1}{2}||f||_1 \leqslant \int_E |f| \,\mathrm{d}x$. We bound this further in two different ways. So,

$$\int_E |f| \,\mathrm{d}x \leqslant \int_E ||f||_\infty \,\mathrm{d}x = m(E)||f||_\infty.$$

Hence, we obtain

$$\frac{||f||_1}{2||f||_\infty} \leqslant m(E).$$

On the other hand, by Cauchy-Schwarz,

$$\int_E |f| \,\mathrm{d}x = \int |f|\mathbb{1}_E \,\mathrm{d}x \leqslant ||f||_2||\mathbb{1}_E||_2 = ||f_2||(m(E))^{1/2}.$$

Therefore,

$$\frac{||f||_1^2}{4||f||_2^2} \leqslant m(E).$$

Since $m(E)$ bounds both, then

$$\mu\left(\left\{x \in X : |f(x)| \geqslant \frac{1}{2}||f||_1\right\}\right) \geqslant \max\left\{\frac{||f||_1}{2||f||_\infty}, \frac{||f||_1^2}{4||f||_2^2}\right\},$$

proving the claim. $\square$