

## Real Analysis Qual, Fall 2023

**Problem 1. (Classic Technique)** Let  $f_n(x) = \frac{nx^2}{n^3 + x^3}$ .

- (a) Prove that  $f_n$  converge uniformly to 0 on  $[0, M]$  for any  $M > 0$ , but does not converge uniformly to 0 on  $[0, \infty)$ .
- (b) Prove that the series  $\sum_{n=1}^{\infty} f_n(x)$  defines a continuous function on  $[0, \infty)$ .

*Proof.* We first prove (a). Over  $[0, M]$ , we have

$$0 \leq \frac{nx^2}{n^3 + x^3} \leq \frac{nx^2}{n^3} \leq \frac{x^2}{n^2} \leq \frac{M^2}{n^2} \rightarrow_{n \rightarrow \infty} 0.$$

Hence,  $f_n \rightarrow 0$  uniformly on  $[0, M]$ . On the other hand, set  $x_n = n$ . Then,

$$\frac{nx_n^2}{n^3 + x_n^3} = \frac{n^3}{2n^3} = \frac{1}{2}.$$

Then, on  $(0, \infty)$ , for every  $n$  we have  $f_n(x_n) = f_n(n) = \frac{1}{2}$ . Therefore, on  $(0, \infty)$ , we cannot have  $f_n \rightarrow 0$  uniformly, since for  $\epsilon < 1/2$ , there is no  $N$  such that for all  $n \geq N$ , we have  $|f_n(x)| < \epsilon$  for all  $x$ .

For (b), first write  $g(x, n) = f_n(x)$ . Set  $F = \sum_{n=1}^{\infty} f_n$ , and observe that  $F(x) = \int g(n, x) dn$ , taken with respect to the counting measure. Take  $x_0 \in (0, \infty)$ , and let  $(x_k)$  be a sequence converging to  $x_0$ . Then, fixing  $n$ , we have  $\lim_{k \rightarrow \infty} g(n, x_k) = g(n, x_0)$ , given that the  $g(n, x)$  is continuous in  $x$ . Therefore, viewing  $g(n, x_k)$  as a sequence of functions in  $k$ , we have  $g(n, x_k) \rightarrow_{k \rightarrow \infty} g(n, x_0)$  pointwise. Furthermore, since  $(x_k)$  is convergent, it is bounded, so we may write  $(x_k) \subseteq [0, M]$  for some  $M$ . By a computation above, given that  $n^2 \geq 1$ , we conclude that  $g(n, x_k) \leq M^2$  for all  $n, x_k$ . Since  $M^2$  is an integrable function on  $[0, M]$ , then by DCT we obtain

$$\lim_{k \rightarrow \infty} F(x_k) = \lim_{k \rightarrow \infty} \int g(n, x_k) dn = \int g(n, x_0) dn = F(x_0).$$

Therefore,  $F$  is continuous. □

**Problem 2.** Let  $(X, \mathcal{A})$  be a measure space, and let  $\mu$  be a nonnegative set function on  $\mathcal{A}$  that is finitely additive with  $\mu(\emptyset) = 0$ . Recall that a set function is said to be continuous from below if

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim \mu(A_j) \text{ whenever } A_j \text{ is an increasing sequence of sets in } \mathcal{A}.$$

Prove that

$$\mu \text{ is a measure} \iff \mu \text{ is continuous from below.}$$

*Proof.* First, say that  $\mu$  is continuous from below. Then, let  $E = \bigcup_{n=1}^{\infty} F_n$  for  $(F_n)_{n=1}^{\infty} \subseteq \mathcal{A}$  a countable pairwise disjoint collection of sets. Then, by continuity from below and finite additivity of  $\mu$ , we have

$$\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^n F_m\right) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{m=1}^n F_m\right) = \lim_{n \rightarrow \infty} \sum_{m=1}^n \mu(F_m) = \sum_{n=1}^{\infty} \mu(F_n).$$

Therefore,  $\mu$  is countably additive, and hence a measure.

Now, suppose that  $\mu$  is a measure. We must show that  $\mu$  is continuous from below. Say that  $(F_n)_{n=1}^{\infty}$  is an increasing sequence of sets in  $\mathcal{A}$ . Write  $E = \bigcup_{n=1}^{\infty} F_n$ . Also, set  $E_n = F_n \setminus F_{n-1}$ , except that  $E_1 = F_1$ . Note that the  $E_n$  are disjoint sets contained in  $E$ . Therefore, for all  $m$  we have

$$\sum_{n=1}^m \mu(E_n) \leq \mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

On the other hand,  $\bigcup_{n=1}^m E_n = F_1 \cup \bigcup_{n=1}^m (F_n \setminus F_{n-1}) = F_m$ . So,

$$\mu(F_m) = \sum_{n=1}^m \mu(E_n) \leq \mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

Taking  $m \rightarrow \infty$  therefore forces the equality  $\lim \mu(F_m) = \mu(E)$ . So,  $\mu$  is continuous from below.  $\square$

**Problem 3.** Prove that

$$1 - \frac{x^2}{2} \leq \cos(x) \leq e^{-x^2/2}$$

for all  $|x| \leq 1$ , and conclude from this that

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n}{2\pi}} \int_{|x| \leq 1} (\cos x)^n dx = 1.$$

*Hint:* You may use without proof that  $\int e^{-\pi x^2} dx = 1$ .

*Proof.* Recall that  $\cos(x)$  is analytic with Taylor series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ . Moreover,

$$e^{(-x^2/2)} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}.$$

For any alternating series  $S = \sum_{n=0}^{\infty} (-1)^n a_n$  for  $a_n \geq 0$  such that the  $a_n$  decrease monotonically, we have  $\sum_{n=0}^{2k-1} (-1)^n a_n \leq S \leq \sum_{n=0}^{2k} (-1)^n a_n$ . For  $x \in [-1, 1]$ , the terms  $x^{2n}/(2n)!$  and  $x^{2n}/(2^n (2n)!)$  do decrease monotonically. Hence,

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad \text{and} \quad 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} \leq e^{-x^2/2}.$$

With these bounds,

$$e^{-x^2/2} - \cos(x) \geq 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = \frac{4x^4 - x^6}{48}.$$

On  $x \in (-1, 1)$ ,  $x^4 \geq x^6$ , so  $e^{-x^2/2} - \cos(x) \geq 0$ . Therefore, we obtain  $1 - x^2/2 \leq \cos(x) \leq e^{-x^2/2}$  as needed.

Now,

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^1 (\cos(x))^n dx \leq \sqrt{\frac{n}{2\pi}} \int_{-1}^1 (e^{-x^2/2})^n dx = \sqrt{\frac{n}{2\pi}} \int_{-1}^1 e^{-((x\sqrt{n})/\sqrt{2})^2} dx.$$

We make the substitution  $y = \sqrt{n}x/\sqrt{2}$ . Then,  $(\sqrt{2}/\sqrt{n}) dy = dx$ . Moreover, define  $J_n = [-\sqrt{n}/\sqrt{2}, \sqrt{n}/\sqrt{2}]$ . So,

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^1 (e^{-x^2/2})^n dx = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{n}/\sqrt{2}}^{\sqrt{n}/\sqrt{2}} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) e^{-y^2} dy.$$

Now,  $\mathbb{1}_{J_n}(y) e^{-y^2}$  is a nonnegative and monotonically increasing sequence which converges pointwise to  $e^{-y^2}$ . Therefore,

$$\lim \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \int e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

On the other hand,

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^1 (\cos(x))^n dx \geq \sqrt{\frac{n}{2\pi}} \int_{-1}^1 \left(1 - \frac{x^2}{2}\right)^n dx.$$

Again, we substitute  $y = \sqrt{n}x/\sqrt{2}$  and obtain

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^1 \left(1 - \frac{x^2}{2}\right)^n dx = \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) \left(1 - \frac{y^2}{n}\right)^n dy.$$

Recall that  $\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$ . Substituting  $x = -y^2$ , we see that  $\mathbb{1}_{J_n}(y) \left(1 - \frac{y^2}{n}\right)^n$  converges pointwise to  $e^{-y^2}$ . On the other hand,

$$\left(1 - \frac{y^2}{n}\right)^n \leq \cos\left(\frac{y}{\sqrt{n}}\right)^n \leq \left(e^{-(y/\sqrt{n})^2/2}\right)^n = e^{-y^2/2}.$$

Now,  $e^{-y^2/2}$  is an integrable function. Hence, the functions  $\mathbb{1}_{J_n}(1 - (y^2/n))^n$  have an integrable dominant. By DCT, therefore, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) \left(1 - \frac{y^2}{n}\right)^n dy = \frac{1}{\sqrt{\pi}} \int e^{-y^2} dy = 1.$$

Therefore,  $\lim_{n \rightarrow \infty} \sqrt{n/2\pi} \int_{-1}^1 (\cos(x))^n dx = 1$ . □

**Problem 4.** Let  $a, b > 0$ . Prove that

$$\int_{[0,1] \times [0,1]} \frac{1}{x^a + y^b} dm_2(x, y) < \infty \iff \frac{1}{a} + \frac{1}{b} > 1$$

where  $m_2$  denotes the Lebesgue measure on  $\mathbb{R}^2$ .

*Hint:* One possible approach would be to consider separately the regions where  $x^a \leq y^b$  and  $x^a > y^b$ .

*Proof.* Set  $A = \{(x, y) \in [0, 1]^2 : x^a \geq y^b\}$ , and set  $B = \{(x, y) \in [0, 1]^2 : y^b \geq x^a\}$ . Note that

$$\int_A \frac{1}{2x^a} dm_2 \leq \int_A \frac{1}{x^a + y^b} dm_2 \leq \int_A \frac{1}{x^a} dm_2.$$

Likewise,

$$\int_B \frac{1}{2y^b} dm_2 \leq \int_B \frac{1}{y^b + x^a} dm_2 \leq \int_B \frac{1}{y^b} dm_2.$$

Hence, since all these functions are nonnegative on  $[0, 1]^2$ , we conclude that

$$\int \frac{1}{x^a + y^b} dm_2 < \infty \text{ if and only if } \int_A \frac{1}{x^a} dm_2, \int_B \frac{1}{y^b} dm_2 < \infty.$$

Observe that  $(x, y)$  is in  $A$  if and only if  $0 \leq x \leq 1$ ,  $0 \leq y \leq x^{a/b}$ . Therefore, by Tonelli's Theorem,

$$\int_A \frac{1}{x^a} dm_2 = \int_0^1 \int_0^{x^{a/b}} \frac{1}{x^a} dy dx = \int_0^1 x^{\frac{a}{b}-a} dx.$$

An analogous computation shows that

$$\int_B \frac{1}{y^b} dm_2 = \int_0^1 \int_0^{y^{a/b}} \frac{1}{y^b} dx dy = \int_0^1 x^{\frac{b}{a}-b} dy.$$

These integrals are finite if and only if  $\frac{a}{b} - a > 1$  and  $\frac{b}{a} - b > 1$ . These inequalities hold if and only if  $\frac{1}{a} + \frac{1}{b} > 1$ , proving the result.  $\square$

**Problem 5. (Classic Technique)** Let  $f_k \rightarrow f$  a.e. on  $\mathbb{R}$  with  $\sup_k \|f_k\|_{L^2(\mathbb{R})} < \infty$ . Prove that  $f \in L^2(\mathbb{R})$  and that

$$\lim_{k \rightarrow \infty} \int f_k g dx = \int f g dx$$

for all  $g \in L^2(\mathbb{R})$ .

*Hint:* First consider functions  $g$  supported on sets of finite measure and use Egorov's theorem.

*Proof.* First,  $|f_k|^2 \rightarrow |f|^2$  pointwise almost everywhere. Also, since  $\sup_k \|f_k\|_2 < \infty$ , we have  $\sup_k \|f_k\|_2^2 < \infty$ . Therefore, by Fatou's Lemma,

$$\int |f|^2 dx \leq \liminf \int |f_k|^2 dx \leq \sup_k \|f_k\|_2^2 < \infty.$$

Hence,  $f \in L^2(\mathbb{R})$ .

Suppose first that  $g$  is a compactly supported continuous function. Set  $A = \text{supp } g$ , and observe that  $g = \mathbb{1}_A g$ . By Cauchy-Schwarz,

$$\left| \int f g \, dx - \int f_k g \, dx \right| = \left| \int (f - f_k) \mathbb{1}_A \cdot g \, dx \right| \leq \| \mathbb{1}_A (f - f_k) \|_2 \cdot \| g \|_2.$$

Therefore, we prove that  $\| \mathbb{1}_A (f - f_k) \|_2 \rightarrow 0$ . Since  $A \subseteq \mathbb{R}$  is compact, then in particular it is bounded, so  $A$  has finite measure. Therefore, by Egorov's theorem, there exists  $E \subseteq A$  such that  $m(E) < \epsilon^2$  and  $f_k \rightarrow f$  uniformly on  $A \setminus E$ . Choose  $N$  such that for all  $k \geq N$  we have  $|f(x) - f_k(x)| < \epsilon$  for all  $x \in A \setminus E$ . Then,

$$\| \mathbb{1}_A (f - f_k) \|_2 \leq \| \mathbb{1}_E (f - f_k) \|_2 + \| \mathbb{1}_{A \setminus E} (f - f_k) \|_2 \leq \| \mathbb{1}_E f \|_2 + \| \mathbb{1}_E f_k \|_2 + \| \epsilon \mathbb{1}_{A \setminus E} \|_2.$$

Write  $M = \sup_k \| f_k \|_2$ . Then, by Cauchy-Schwarz,

$$\| \mathbb{1}_E f \|_2 \leq \sqrt{m(E)} \| f \|_2 < \epsilon \| f \|_2 \quad \text{and} \quad \| \mathbb{1}_E f_k \|_2 \leq \sqrt{m(E)} \| f_k \|_2 < \epsilon M.$$

Therefore,

$$\| \mathbb{1}_A (f - f_k) \|_2 \leq \epsilon (\| f \|_2 + M) + \epsilon \| \mathbb{1}_{A \setminus E} \|_2 < \infty.$$

Taking  $\epsilon \rightarrow 0$  gives the result. Therefore, when  $g$  is a compactly supported continuous function,

$$\lim_{k \rightarrow \infty} \int f_k g \, dx = \int f g \, dx.$$

Suppose now that  $g$  is an arbitrary  $L^2(\mathbb{R})$  function. Recall that compactly supported continuous functions are dense in  $L^2(\mathbb{R})$ . Take  $h$  within  $\epsilon$  of  $g$  in the  $L^2$  norm. Then, we have

$$\left| \int f g \, dx - \int f_k g \, dx \right| \leq \left| \int f (g - h) \, dx \right| + \left| \int f h \, dx - \int f_k h \, dx \right| + \left| \int f_k (h - g) \, dx \right|.$$

Furthermore,

$$\left| \int f (g - h) \, dx \right| \leq \| f \|_2 \| g - h \|_2 < \epsilon \| f \|_2,$$

and

$$\left| \int f_k (h - g) \, dx \right| \leq \| f_k \|_2 \| h - g \|_2 < \epsilon M.$$

Finally, choose  $N$  such that for all  $k \geq N$ , we have

$$\left| \int f h \, dx - \int f_k h \, dx \right| < \epsilon.$$

Then, for all  $k \geq N$ , we obtain

$$\left| \int f g \, dx - \int f_k g \, dx \right| \leq \epsilon (\| f \|_2 + M + 1).$$

Taking  $\epsilon \rightarrow 0$  gives

$$\lim_{k \rightarrow \infty} \int f_k g \, dx = \int f g \, dx$$

as claimed. □