Real Analysis Qual, Fall 2021

Problem 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and

$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{x_n + 1}{x_n + 2}.$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges, and find its limit.

Proof. We first show the sequence is bounded for $n \geqslant 2$. First, we prove that $x_n \geqslant 0$ for all n. At x_1, this holds. Suppose it holds at n. Then, $x_{n+1} = \frac{x_n+1}{x_n+2}$. Since $x_n \geqslant 0$, then this is a nonnegative fraction, so $x_{n+1} \geqslant 0$. Thus, $x_n \geqslant 0$ for all n. Hence, for all $n \geqslant 2$, we have

$$x_n = \frac{x_{n-1} + 1}{x_{n-1} + 2} \leqslant \frac{x_{n-1} + 2}{x_{n-1} + 2} \leqslant 1.$$

Consider the equation $x^2 + x - 1$. By the quadratic formula, this has two roots of the form $(\pm\sqrt{5} - 1)/2$. Set $L = (\sqrt{5} - 1)/2$. Observe that

$$L^2 + 2L - L - 1 = 0 \implies L = \frac{L + 1}{L + 2}.$$

So, L is a fixed point of $(x + 1)/(x + 2)$. Note also that $\frac{d}{dx}(x + 1)/(x + 2) = \frac{1}{(x+1)^2}$, which is positive for all $x \geqslant 0$. Therefore, $(x + 1)/(x + 2)$ is increasing in x. Observe also that $x^2 + x - 1$ is concave up, so the function is negative on the interval $[(-\sqrt{5} - 1)/2, (\sqrt{5} - 1)/2]$. Suppose first that $x_n \leqslant L$. Then, by the concavity of $x^2 + x - 1$, we have $x_n^2 + x_n - 1 \leqslant 0$. Therefore,

$$x_n^2 + 2x_n - x_n - 1 = x_n(x_n + 2) - (x_n + 1) \leqslant 0.$$

After some rearrangement, we obtain $x_n \leqslant \frac{x_n+1}{x_n+2} = x_{n+1}$. Moreover, since $(x + 1)/(x + 2)$ increases monotonically in x, then

$$x_{n+1} = \frac{x_n + 1}{x_n + 2} \leqslant \frac{L + 1}{L + 2} = L.$$

So, if $x_n \leqslant L$ for some n, then (x_n) increases monotonically on $[0, L]$, and is bounded above by L. So, by Monotone Convergence Theorem (x_n) converges.

On the other hand, suppose $x_n \geqslant L$ for some n. Then, again by the concavity of $x^2 + x - 1$, we obtain

$$x_n^2 + x - 1 \geqslant 0 \implies x_n \geqslant \frac{x_n + 1}{x_n + 2} = x_{n+1}.$$

Moreover, since $x_n \geqslant L$, since $(x + 1)/(x + 2)$ increases in x, then

$$\frac{x_n + 1}{x_n + 1} \geqslant \frac{L + 1}{L + 2} = L.$$

So, if $x_n \geqslant L$ for some n, then (x_n) decreases monotonically on the interval $[1, L]$. So, again, it is convergent by the Monotone Convergence Theorem. Finally, we have

$$c = \lim x_n = \lim x_{n+1} = \lim \frac{x_n + 1}{x_n + 2} = \frac{\lim x_n + 1}{\lim x_n + 2} = \frac{c + 1}{c + 2}.$$

So, $c(c + 2) = c + 1$ implying $c^2 + c - 1 = 0$. Since x_n is nonnegative for all n, we have $c \geqslant 0$. Therefore, c is the positive root of $x^2 + x - 1$, and this is L. □

Problem 2.

(a) Let $F \subseteq \mathbb{R}$ be closed, and define

$$\delta_F(y) := \inf_{x \in F} |x - y|.$$

For $y \notin F$, show that

$$\int_F |x - y|^{-2} \, \mathrm{d}x \leqslant \frac{2}{\delta_F(y)}.$$

(b) Let $F \subseteq \mathbb{R}$ be a closed set whose complement has finite measure. Define the function

$$I(x) := \int_{\mathbb{R}} \frac{\delta_F(y)}{|x - y|^2} \, \mathrm{d}y.$$

Prove that $I(x) = \infty$ if $x \notin F$, however $I(x) < \infty$ for almost every $x \in F$.

Hint: Investigate $\int_F I(x) \, \mathrm{d}x$.

Proof. We prove (a). Note that $\delta_F(y) > 0$, otherwise we have $y \in F$. Observe that $F \cap (y, \infty) \subseteq (y + \delta_F(y), \infty)$. Therefore,

$$\int_{F \cap (\delta_F(y) + y, \infty)} \frac{1}{|x - y|^2} \, \mathrm{d}x \leqslant \int_{y + \delta_F(y)}^{\infty} \frac{1}{|x - y|^2} \, \mathrm{d}x.$$

Substituting $z = x - y$, we observe that the new bounds of integration are $(\delta_F(y), \infty)$. So, we have

$$\int_{F \cap (y, \infty)} \frac{1}{|x - y|^2} \, \mathrm{d}x \leqslant \int_{\delta_F(y)}^{\infty} \frac{1}{z^2} \, \mathrm{d}x = -\frac{1}{z}\bigg|_{\delta_F(y)}^{\infty} = \frac{1}{\delta_F(y)}.$$

Likewise, $F \cap (-\infty, y) \subseteq (-\infty, y - \delta_F(y)$. So,

$$\int_{F \cap (-\infty, y)} \frac{1}{|x - y|^2} \, \mathrm{d}x \leqslant \int_{-\infty}^{y - \delta(y)} \frac{1}{|x - y|^2} \, \mathrm{d}x.$$

Again, we perofrm the substitution $z = x - y$ to obtain

$$\int_{F \cap (-\infty, y)} \frac{1}{|x - y|^2} \, \mathrm{d}x \leqslant \int_{-\infty}^{-\delta(y)} \frac{1}{z^2} \, \mathrm{d}x = -\frac{1}{z}\bigg|_{-\infty}^{-\delta_F(y)} = \frac{1}{\delta_F(y)}.$$

Therefore,

$$\int_F \frac{1}{|x - y|^2} \, \mathrm{d}x = \int_{F \cap (y, \infty)} \frac{1}{|x - y|^2} \, \mathrm{d}x + \int_{F \cap (-\infty, y)} \frac{1}{|x - y|^2} \, \mathrm{d}x \leqslant \frac{2}{\delta_F(y)}.$$

We now prove (b). Observe that $I(x)$ is nonnegative. Moreover, the infimum over a set, and $|x - y|^{-2}$ are measurable functions, so $I(x)$, as the integral of their product, is a measurable function. Then, we may apply Tonelli's

$$\int_F I(x) \, \mathrm{d}x = \int_F \int \frac{\delta_F(y)}{|x - y|^2} \, \mathrm{d}y \, \mathrm{d}x$$

$$
\begin{aligned}
&= \int \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y + \int_F \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}
$$

Note that for $y \in F$, we have $\delta_F(y) = 0$. Thus,

$$
\begin{aligned}
\int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y + \int_F \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y &= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y + \int_F \int_F \frac{0}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}
$$

By (a), for each y, we have

$$
\begin{aligned}
\int_{F^c} \int_F \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y &= \int_{F^c} \delta_F(y) \frac{2}{\delta_F(y)} \,\mathrm{d}y \\
&= 2\mu(F^c).
\end{aligned}
$$

By assumption, $2\mu(F^c)$ is finite. So, $\int_F I(x) \,\mathrm{d}x$ is finite, and thus $I(x) \neq \infty$ almost everywhere on F. Therefore, almost every $x \in F$ satisfies $I(x) < \infty$.

On the other hand, say that $x \notin F$. Then, since F is closed, there is some ϵ such that $(x - \epsilon, x + \epsilon) \subseteq F^c$. We may further choose ϵ so that $x - \epsilon, x + \epsilon \in F^c$. We have

$$
I(x) = \int \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y = \int_{F^c} \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y \geqslant \int_{x-\epsilon}^{x+\epsilon} \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y.
$$

Now, since $x - \epsilon, x + \epsilon \in F^c$, they have some minimal distance from F, say a, so that $\delta_F(y) \geqslant a$ for all $y \in (x - \epsilon, x + \epsilon)$. Therefore,

$$
\int_{x-\epsilon}^{x+\epsilon} \frac{\delta_F(y)}{|x-y|^2} \,\mathrm{d}y \geqslant \int_{x-\epsilon}^{x+\epsilon} \frac{a}{|x-y|^2} \,\mathrm{d}y.
$$

Finally, substitute $z = y - x$. Then,

$$
\int_{x-\epsilon}^{x+\epsilon} \frac{a}{|x-y|^2} \,\mathrm{d}y = \int_{-\epsilon}^{\epsilon} \frac{a}{z^2} \,\mathrm{d}z.
$$

Now, $\int_0^\epsilon 1/z^2 \,\mathrm{d}z = \infty$. Therefore, $I(x) = \infty$ when $x \notin F$. □

Problem 4. Let f be a measurable function on $\mathbb{R}$. Show that the graph of f has measure zero in $\mathbb{R}^2$.

Proof. Let $\Gamma_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. We first show that Γ_f is a measurable set. Define $h(x, y) = f(x) - y$. This is a difference of measurable functions, and thus measurable. Now, if $h(x, y) = 0$, then $f(x) - y = 0$, and so $y = f(x)$. Moreover, $h(x, f(x)) = 0$. Thus, $h^{-1}(\{0\}) = \{(x, f(x)) : x \in \mathbb{R}\}$. So, Γ_f is measurable.

Observe now that $\mathbb{1}_{\Gamma_f}$ is a nonnegative measurable function. Therefore, by Tonelli's Theorem,

$$m_2(\Gamma_f) = \int_{\mathbb{R}^2} \mathbb{1}_{\Gamma_f} \, \mathrm{d}m_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\Gamma_f}(x, y) \, \mathrm{d}y \, \mathrm{d}x.$$

Since f is a function, then for each x, there is only one y such that $y = f(x)$. So, for x fixed, we have $\mathbb{1}_{\Gamma_f}(x, y) = \mathbb{1}_{\{f(x)\}}(y)$, an a.e. zero function. Hence,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\Gamma_f}(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{f(x)\}}(y) \, \mathrm{d}y \, \mathrm{d}x = \int_{\mathbb{R}} 0 \, \mathrm{d}x = 0.$$

Therefore, $m_2(\Gamma_f) = 0$. $\square$

Problem 5. Consider the Hilbert space $\mathcal{H} = L^2([0, 1])$.

(a) Prove that if $E \subseteq \mathcal{H}$ is closed and convex then E contains an element of smallest norm.

Hint: Show that if $||f_n||_2 \to \inf\{f \in E : ||f||_2\}$ *then* $\{f_n\}$ *is a Cauchy sequence.*

(b) Construct a non-empty closed $E \subseteq \mathcal{H}$ which does not contain an element of smallest norm.