Real Analysis Qual, Fall 2025

Problem 1. Let f be an $\mathbb{R}$-valued measurable function on $[0,1]$. Put $A = f^{-1}(\mathbb{Z})$, and for $n \in \mathbb{N}$, $f_n(x) := [\cos(\pi f(x))]^{2n}$. Show that A is measurable, and that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = m(A).$$

Proof. First, the singletons $\{n\}$ for $n \in \mathbb{Z}$ are measurable. So, $\mathbb{Z} = \bigcup_{n\in\mathbb{Z}}\{n\}$ is a countable union of measurable sets, and hence measurable. Since f is a measurable function, then $f^{-1}(\mathbb{Z}) = A$ is measurable. Set $E = \{x \in [0,1] : |\cos(\pi f(x))| < 1\}$, so that $E^c = \{x \in [0,1] : |\cos(\pi f(x))| = 1\}$. Observe that

$$\mathbb{1}_E(x)\cos(\pi f(x))^{2n} = \mathbb{1}_E(x)|\cos(\pi f(x))|^{2n}$$

converges pointwise to 0, since if $x \in E^c$, then the above function is 0, and otherwise $|\cos(\pi f(x))| < 0$, so that $|\cos(\pi f(x))|^{2n} \to 0$. Moreover, $|\cos(\pi f(x))| \leqslant 1$ over all $[0,1]$. Therefore, by DCT,

$$\lim \int_0^1 \mathbb{1}_E(x) f_n(x)\,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_E(x)\cos(\pi f(x))^{2n}\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0.$$

Hence,

$$\lim \int_0^1 f_n\,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_E f_n\,\mathrm{d}x + \lim \int_0^1 \mathbb{1}_{E^c} f_n\,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_{E^c}(x)|\cos(\pi f(x))|^{2n}\,\mathrm{d}x.$$

Over E^c, $|\cos(\pi f(x))| = 1$. So,

$$\lim \int_0^1 \mathbb{1}_{E^c}(x)|\cos(\pi f(x))|^{2n}\,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_{E^c}\,\mathrm{d}x = \int_0^1 \mathbb{1}_{E^c}\,\mathrm{d}x.$$

Finally, we observe that $|\cos(\pi f(x))| = 1$ if and only if $f(x)$ is an integer. Therefore, $E^c = \{x \in [0,1] : f(x) \in \mathbb{Z}\} = A$. So,

$$\lim \int_0^1 f_n\,\mathrm{d}x = \lim \int_0^1 \mathbb{1}_E f_n\,\mathrm{d}x + \lim \int_0^1 \mathbb{1}_{E^c} f_n\,\mathrm{d}x = \int_0^1 \mathbb{1}_{E^c}\,\mathrm{d}x = \int_0^1 \mathbb{1}_A\,\mathrm{d}x = m(A),$$

completing the proof. $\square$

Problem 2. (Classic) For $x \neq 0$, define $f(x)$ by the series

$$f(x) := \sum_{n=0}^{\infty} e^{-n|x|}.$$

(i) Let $d > 0$. Show that for $x \in (-\infty, -d) \cup (d, \infty)$ the series converges uniformly, and $f(x)$ is uniformly continuous.

(ii) Show that for $x \in (-\infty, 0) \cup (0, \infty)$ the series is not uniformly convergent, and $f(x)$ is not uniformly continuous.

Proof. Define $s_N(x) = \sum_{n=0}^{N} e^{-n|x|}$. We show that for every ϵ there is some M such that for all $N \geqslant M$ and for all x satisfying $|x| \in (d, \infty)$, we have $|f(x) - s_N(x)| < \epsilon$

$$|f(x) - s_N(x)| = f(x) - s_N(x) = \sum_{n=N}^{\infty} e^{-n|x|}.$$

Observe that $e^{-n|x|} \leqslant e^{-nd}$ given that $d < |x|$. Therefore,

$$\sum_{n=N}^{\infty} e^{-n|x|} \leqslant \sum_{n=N}^{\infty} e^{-nd} = \sum_{n=N}^{\infty} (e^{-d})^n.$$

Since $e^d > 1$ for all $d > 0$, then $e^{-d} < 1$, and so $\sum_{n=0}^{\infty}(e^{-d})^n$ is a convergent power series of nonnegative terms. Therefore, choose M large enough that for all $N \geqslant M$, we have $\sum_{n=N}^{\infty}(e^{-d})^n < \epsilon$. Then, for all $N \geqslant M$, and for all x with $|x| \in (d, \infty)$, we obtain $|f(x) - s_N(x)| < \epsilon$. Therefore, $s_N \to f$ uniformly.

Observe that for all $|x| > d/2$, we have $e^{-|x|} < 1$.

$$\sum_{n=0}^{\infty} e^{-n|x|} = \sum_{n=0}^{\infty} (e^{-|x|})^n = \frac{1}{1 - e^{-|x|}}.$$

So, $f(x)$ is a continuous on all $\mathbb{R} \backslash [-d/2, d/2]$. Moreover, $\lim_{|x| \to \infty} f(x) = 1$, and since $-e^{-|x|}$ increases monotonically as $|x| \to \infty$, then $f(x)$ decreases monotonically in $|x|$. Hence, choose N so large that for all x with $|x| \geqslant N$, we have $1 \leqslant f(x) \leqslant 1 + \epsilon/3$. Observe moreover that f is continuous on the compact set $A = [-N, -d] \cup [d, N]$, and hence uniformly continuous on this set. Choose δ small enough that for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon/3$. We claim now that for all x, y with $|x|, |y| > d$, that $|f(x) - f(y)| < \epsilon$. If $x, y \in A$, we have already shown this holds. If $x, y \notin A$, then $|x|, |y| \geqslant N$, and so

$$|f(x) - f(y)| \leqslant |f(x) - 1| + |1 - f(y)| \leqslant \epsilon/3 + \epsilon/3 < \epsilon.$$

Finally, suppose that $x \in A$ and $y \notin A$ are such that $|x - y| < \delta$. Suppose $y > N$, and note the proof is identical if $y < -N$. Then, $x \leqslant N \leqslant y$, so $|x - N| < \delta$. Therefore,

$$|f(x) - f(y)| \leqslant |f(x) - f(N)| + |f(N) - f(y)| < \epsilon/3 + 2\epsilon/3 < \epsilon.$$

Therefore, f is uniformly continuous on $(-\infty, -d) \cup (d, \infty)$.

Finally, we show that s_N does not converge uniformly to f, and that f is not uniformly continuous. For uniform convergence, note that since $|x| > 0$, $e^{-|x|} < 1$. Therefore, the following is a power series, so

$$|f(x) - s_N(x)| = \sum_{n=N}^{\infty} e^{-n|x|} = e^{-N|x|} \sum_{n=0}^{\infty} e^{-nx} = \frac{e^{-N|x|}}{1 - e^{-|x|}}.$$

As $|x| \to 0$, $e^{-|x|}, e^{-N|x|} \to 1$. Thus, $\frac{e^{-N|x|}}{1-e^{-|x|}} \to \infty$ as $|x| \to 0$. Therefore, there is no N large enough that $|f(x) - s_N(x)| < \epsilon$ for all x. For uniform continuity, set $x = h$ and $y = 2h$. Choose $h > 0$. Since $h < 2h$, $f(h) > f(2h)$. Therefore, by some algebraic manipulations,

$$|f(h) - f(2h)| = f(h) - f(2h) = \frac{e^{-h} - e^{-2h}}{(1 - e^{-h})(1 - e^{-2h})} = \frac{e^{-h}}{1 - e^{-2h}}.$$

Sending $h \to 0$ has $e^{-h} \to 1$ and $1 - e^{-2h} \to 0$. So, $|f(h) - f(2h)| \to \infty$. For any δ, then $|h - 2h| = h < \delta$ eventually as $h \to 0$. So, f is not uniformly continuous on $\mathbb{R}\backslash\{0\}$. □

Problem 3. Let f be a bounded real valued function on $[a, b]$. Assume that

$$\sup\left\{\int_{[a,b]} \phi \, \mathrm{d}m : \phi \text{ is simple }, \phi \leqslant f\right\} = \inf\left\{\int_{[a,b]} \phi \, \mathrm{d}m : \phi \text{ is simple }, \phi \geqslant f\right\}.$$

Show that f is Lebesgue measurable.

Proof. Let the value of the inf/sup be α. By definition of supremum, take a sequence ψ_n of simple functions with $\psi_n \leqslant f$ such that $|\int_a^b \psi_n \, \mathrm{d}m - \alpha| < 1/n$. By definition of infimum, take a sequence ϕ_n of simple functions with $\phi_n \geqslant f$ such that $|\int_a^b \phi_n \, \mathrm{d}m - \alpha| < 1/n$. We may enforce that the ϕ_n, ψ_n are monotonic by redefining ϕ_n to be the infimum of the previous n simple functions, and ψ_n to be the supremum of the previous n simple functions. The redefined ϕ_n, ψ_n are still simple functions with the property $\phi_n \geqslant f \geqslant \psi_n$. Moreover, if ϕ_n' was the original simple function before redefinition, then $\alpha \leqslant \int_a^b \phi_n \, \mathrm{d}m \leqslant \int_a^b \phi_n' \, \mathrm{d}m$, so $|\int_a^b \phi_n \, \mathrm{d}m - \alpha| < 1/n$ stilll holds. An equivalent argument on ψ_n holds as well.

We first claim that $\phi_n - \psi_n$ converges in the L^1 norm to 0. Indeed

$$\left|\int |\phi_n - \psi_n| \, \mathrm{d}m\right| = \left|\int \phi_n - \psi_n \, \mathrm{d}m\right| \leqslant \left|\int \phi_n \, \mathrm{d}m - \alpha\right| + \left|\alpha - \int \psi_m n \, \mathrm{d}m\right| < 2/n.$$

On the other hand, set ϕ to be the pointwise limit of the ϕ_n, and let ψ be the pointwise limit of the ψ_n. These pointwise limits exist, for $\phi_n(x)$ decreases monotonically and is bounded below by $f(x)$, and $\psi_n(x)$ increases monotonically and is bounded above by $f(x)$. Therefore, $\phi_n - \psi_n$ converges pointwise to $\phi - \psi$. Since ϕ_n decreases monitically and ψ_n increases monotonically, then $\phi_n - \psi_n$ is a monotonically decreasing sequence, so in particular it is bounded above by the integrable function $\phi_1 - \psi_1$. Hence, by DCT and the fact that $\phi \geqslant f \geqslant \psi$, we have

$$0 = \lim \int \phi_n - \psi_n \, \mathrm{d}m = \int \phi - \psi \, \mathrm{d}m = \int |\phi - \psi| \, \mathrm{d}m.$$

Therefore, $\phi = \psi$ almost everywhere. Since $\psi \leqslant f \leqslant \phi$, then $f = \phi$ almost everywhere. Therefore, f is a Lebesgue measurable function up to redefinition on a null set, and hence a Lebesgue measurable function. □

Problem 4. Let f be Lebesgue integrable on $(0, a)$, and let

$$g(x) = \int_x^a \frac{f(t)}{t} \, \mathrm{d}t.$$

Show that g is measurable and integrable on $(0, a)$, and

$$\int_0^a g(x) \, \mathrm{d}x = \int_0^a f(x) \, \mathrm{d}x.$$

Proof. First, let $R = \{(x,t) \in (0,a)^2 : t \geqslant x\}$, and note that R is a measurable set on $(0,a)^2$. Furthermore,

$$g(x) = \int_x^a \frac{f(t)}{t}\,\mathrm{d}t = \int_0^a \mathbb{1}_{(x,a)}(t)\frac{f(t)}{t}\,\mathrm{d}t = \int_0^a \mathbb{1}_R(x,t)\frac{f(t)}{t}\,\mathrm{d}t.$$

Note that $b(x,t) = \frac{f(t)}{t}$ is a $(0,a)^2$ measurable function, since

$$b^{-1}(A) = \left\{(x,t) \in (0,a)^2 : \frac{f(t)}{t} \in A\right\} = \left(\left(\frac{f(t)}{t}\right)^{-1}(A)\right) \times (0,a),$$

where $(\frac{f(t)}{t})^{-1}(A)$ is a measurable set in $(0,a)$, since $\frac{f(t)}{t}$ is a measurable function. Set $h(x,t) = \mathbb{1}_R(x,t)b(x,t)$. Then, $h(x,t)$ is a product of measurable functions and hence measurable. We claim that $h(x,t)$ is Lebesgue integrable. So, by Tonelli's Theorem,

$$\int_{(0,a)^2} |h(x,t)|\,\mathrm{d}m_2 = \int_{(0,a)^2} \left|\mathbb{1}_R(x,t)\frac{f(t)}{t}\right|\mathrm{d}m_2 = \int_0^a \left|\frac{f(t)}{t}\right| \int_0^a \mathbb{1}_R(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Now,

$$\int_0^a \mathbb{1}_R(x,t)\,\mathrm{d}x = \int_0^a \mathbb{1}_{x\leqslant t}\,\mathrm{d}t = \int_0^t 1\,\mathrm{d}t = t = |t|.$$

Therefore,

$$\int_{(0,a)^2} |h(x,t)|\,\mathrm{d}m_2 = \int_0^a \left|\frac{f(t)}{t}\right| |t|\,\mathrm{d}t = \int_0^a |f(t)|\,\mathrm{d}t < \infty.$$

So, $h(x,t)$ is Lebesgue integrable, as claimed. Therefore, by Fubini's Theorem,

$$\int_0^a h_x(t)\,\mathrm{d}t = \int_0^a h(x,t)\,\mathrm{d}t = \int_0^a \mathbb{1}_R(x,t)\frac{f(t)}{t}\,\mathrm{d}t = g(x)$$

is a Lebesgue integrable function. Therefore, applying Fubini's Theorem again, we have

$$\int_0^a g(x)\,\mathrm{d}x = \int_0^a \int_0^a \mathbb{1}_R(x,t)\frac{f(t)}{t}\,\mathrm{d}t\,\mathrm{d}x = \int_0^a \int_0^a \mathbb{1}_R(x,t)\frac{f(t)}{t}\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, by the same argument as above,

$$\int_0^a \int_0^a \mathbb{1}_R(x,t)\frac{f(t)}{t}\,\mathrm{d}x\,\mathrm{d}t \int_0^a \frac{f(t)}{t} \int_0^a \mathbb{1}_R(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^a f(t)\,\mathrm{d}t.$$

Therefore, $\int_0^a g(x)\,\mathrm{d}x = \int_0^a f(t)\,\mathrm{d}t$. □

Problem 5. Let $f \in L^1(\mathbb{R})$. Define

$$F(x,r) \coloneqq \frac{1}{r}\int_{[x-r,x+r]} f(y)\,\mathrm{d}y,\ \ (x,r) \in \mathbb{R} \times (0,\infty).$$

Show that F is continuous in (x,r).

Proof. Let (x_n, r_n) converge to (x_0, r_0) in $\mathbb{R} \times (0, \infty)$. Set $g_n = \mathbb{1}_{[x_n - r_n, x_n + r_n]}$. We claim that $g_n \to g_0$ pointwise, except on the boundary points $x_0 - r_0, x_0 + r_0$. Otherwise, suppose that $g_0(y) = 1$. Then, $|y - x_0| < r_0$. Choose ϵ so that that $|y - x_0| + \epsilon < r_0$. Note that $x_n \to x_0$ and $r_n \to r_0$. Therefore, pick N so that for all $n \geqslant N$ we have $|x_0 - x_n| < \epsilon/2$ and $|r_0 - r_n| < \epsilon/2$. Then, $|y - x_n| \leqslant |y - x_0| + |x_0 - x_n| < |y - x_0| + \epsilon/2$. On the other hand, $|r_n - r_0| < \epsilon/2$, so $r_n > r_0 - \epsilon/2$. Since $|y - x_0| + \epsilon < r_0$, then $|y - x_0| + \epsilon/2 < r_0 - \epsilon/2 < r_n$. Therefore, $|y - x_n| < r_n$, and so $g_n(y) = 1$. So, if $g_0(y) = 1$, then $g_n(y) \to 1$. Now, suppose that $g_0(y) = 0$. Then, $|y - x_0| > r_0$. Choose ϵ so that $|y - x_0| > r_0 + \epsilon$. Pick N large enough such that for all $n \geqslant N$ we have $|x_0 - x_n| < \epsilon/2$ and $|r_0 - r_n| < \epsilon/2$. Therefore,

$$r_n + \epsilon/2 < r_0 + \epsilon < |y - x_0| - |x_0 - x_n| + \epsilon/2 \leqslant ||y - x_0| - |x_0 - x_n|| + \epsilon/2 \leqslant |y - x_n| + \epsilon/2.$$

Subtracting $\epsilon/2$ from both sides, we have $r_n < |y - x_n|$. Therefore, for all $n \geqslant N$, we have $g_n(y) = 0$. Since g_0 only outputs 0 or 1, we see that $g_n \to g_0$ pointwise.

Observe moreover that $1/x$ is a continuous function on $(0, \infty)$. Therefore, $1/r_n \to 1/r_0$. So,

$$\lim \frac{g_n(y)}{r_n} f(y) = \frac{g_0(y)}{r_0} f(y)$$

for all $y \in \mathbb{R}$. Set $a = \inf r_n$. Observe that $a > 0$, for if $a = 0$, then there is an infinite sequence of r_n approaching 0. Since r_n is a convergent series, then $r_n \to 0$. But, $r_n \to r_0 > 0$, hence we must have $a > 0$. Therefore, since g_n is an indicator function,

$$\left| \frac{g_n(y)}{r_n} f(y) \right| = \left| \frac{f(y)}{r_n} \right| \leqslant \left| \frac{f(y)}{a} \right|.$$

Now, $\frac{1}{a} f(y)$ is a constant multiple of a Lebesgue integrable function, and therefore Lebesgue integrable. So, the sequence of functions $\frac{g_n(y)}{r_n} f(y)$ has an integrable dominant. Therefore, by DCT, we obtain

$$\lim F(x_n, r_n) = \lim \frac{1}{r_n} \int_{[x_n - r_n, x_n + r_n]} f(y) \, \mathrm{d}y = \lim \int \frac{g_n(y)}{r_n} f(y) \, \mathrm{d}y = \int \frac{g_0(y)}{r_0} f(y) \, \mathrm{d}y.$$

Finally,

$$\int \frac{g_0(y)}{r_0} f(y) \, \mathrm{d}y = \frac{1}{r_0} \int_{[x_0 - r_0, x_0 + r_0]} f(y) \, \mathrm{d}y = F(x_0, r_0).$$

Thus, F is continuous on $\mathbb{R} \times (0, \infty)$. □

Problem 6.

(i) **(Classic)** For $f \in L^(\mathbb{R})$, $t \in \mathbb{R}$, define $\tau_t(f)(x) = f(x - t)$. Show that $t \longmapsto \tau_t(f)$ is a continuous map from $\mathbb{R}$ to $L^1(\mathbb{R})$.

(ii) Let $f \in L^1(\mathbb{R})$, and let g be a bounded measurable function. Show that $h = f * g$ is uniformly continuous, where $f * g(y) = \int_{\mathbb{R}} f(y - x) g(x) \, \mathrm{d}x$.

Proof. Take $f \in L^1(\mathbb{R})$. Let $t_n \to t_0$. First suppose that f is a compactly supported continuous function. Define $f_n(x) = f(x - t_n)$, and observe that by continuity we have $f_n \to f$ pointwise. Hence, $|f - f_n| \to 0$ pointwise. Say that f is compactly supported on $[-N, N]$. Since $t_n \to t_0$, then the t_n are contained in some set $[-M, M]$. Hence, if $f_n(x) \neq 0$, then

$$|x| - M \leqslant |x| - |t_n| \leqslant ||t_n| - |x|| \leqslant |x - t_n| < N.$$

So, $x \in [-N - M, M + N]$, and thus the f_n are all supported on the common compact set $[-M - N, M + N] = A$. Since $|f|$ is a continuous function supported on a compact set, then it has some upper bound C. Since the $|f_n|$ are shifted versions of $|f|$, then $|f_n| < C$ for all n. Since $\operatorname{supp} f_n \subseteq A$, then $C\mathbb{1}_A$ bounds f_n for all n. Moreover, $C\mathbb{1}_A$ is an integrable function, given that A is a bounded set. It follows that $|f - f_n|$ is bounded by the integrable dominant $2C\mathbb{1}_A$. By DCT,

$$\lim ||\tau_{t_0} f - \tau_{t_n} f||_1 = \lim \int |f_n - f| \,\mathrm{d}x = 0.$$

So, $\tau_t(f)$ is continuous in t, and hence a continuous map from $\mathbb{R}$ to $L^1(\mathbb{R})$ over compactly supported continuous functions.

Now, suppose that f is an arbitrary $L^1(\mathbb{R})$ function. Compactly supported continuous functions are dense in $L^1(\mathbb{R})$, so there is some g such that $||f - g||_1 < \epsilon$. Observe that, by invariance of the Lebesgue measure under shifts, we have $||\tau_t(f) - \tau_t(g)||_1 < \epsilon$. Once again, let $t_n \to t_0$. Then,

$$\begin{aligned}
\lim ||\tau_{t_n}(f) - \tau_{t_0}(f)||_1 &\leqslant \lim ||\tau_{t_n}(f) - \tau_{t_n}(g)||_1 + ||\tau_{t_n}(g) - \tau_{t_0}(g)||_1 + ||\tau_{t_0}(g) - \tau_{t_0}(f)||_1 \\
&\leqslant 2\epsilon + \lim ||\tau_{t_n}(g) - \tau_{t_0}(g)||_1 \\
&= 2\epsilon.
\end{aligned}$$

Since this holds for all ϵ, then $\lim ||\tau_{t_n}(f) - \tau_{t_0}(f)||_1 = 0$. Hence, $\tau_{t_n}(f) \to \tau_{t_0}(f)$ in the L^1 norm for an arbitrary convergent sequence (t_n), so $\tau_t(f) : \mathbb{R} \longrightarrow L^1(\mathbb{R})$ is a continuous function, proving (i).

Take $f \in L^1(\mathbb{R})$, and suppose that g is a measurable function bounded by M. Then,

$$\begin{aligned}
|f * g(x) - f * g(y)| &= \left| \int f(x - z) g(z) \,\mathrm{d}z - \int f(y - z) g(z) \,\mathrm{d}z \right| \\
&\leqslant \int |f(x - z) - f(y - z)||g(z)| \,\mathrm{d}z \\
&\leqslant M \int |f(x - z) - f(y - z)| \,\mathrm{d}z.
\end{aligned}$$

We make the substitution $t = x - z$, obtaining

$$M \int |f(x - z) - f(y - z)| \,\mathrm{d}z = M \int |f(t) - f(y - x + t)| \,\mathrm{d}t = M||\tau_0(f) - \tau_{y-x}(f)||_1.$$

By continuity of the shift operator, there is some δ such that if $|t| < \delta$, then $||\tau_0(f) - \tau_t(f)||_1 < \epsilon/M$. Hence, for all x, y satisfying $|x - y| < \delta$, we have

$$|f * g(x) - f * g(y)| \leqslant M||\tau_0(f) - \tau_{y-x}(f)||_1 < \epsilon.$$

Therefore, $f * g$ is uniformly continuous, proving (ii). $\square$