Referee's report

on

Ideals of nowhere dense sets in some topologies on positive integers

by

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This paper is a quite thorough investigation of the ideal of nowhere dense subsets of $\mathbb{N}$ under three well known topologies (all of which were introduced in the *American Mathematical Monthly*). It is carefully written; I found no mathematical errors. There appears to be enough interest in this subject to warrant publication. However, I have a large number of comments.

To begin with, writing $\{an+b\} = \{an+b : n \in \mathbb{N}_0\}$ is absurd. (I am restraining myself from using stronger language.) Would the authors write "for $n, m \in \mathbb{N}$, by $n+m$ we denote the least common multiple of n and m"? I hope not, because $n+m$ already means something to everybody. Likewise, $\{an+b\}$ is absolutely standard as a set with one member. I suggest the authors follow their reference [14] and write $\langle b, a\rangle = \{b + an : n \in \mathbb{N}_0\}$, as I will do throughout the rest of this report.

The next remark is that the topology with the base $\mathcal{B}_k$ is not Kirch's topology (which has a subbase consisting of $\{\langle b, p\rangle : p \in \mathbb{P}$ and $b < p\}$). For example, the set $\langle 6, 15\rangle \in \mathcal{B}_k$ and there is no member U of Kirch's topology with $6 \in U \subseteq \langle 6, 15\rangle$. I realize that the authors are following their reference [24] where the same mistake is made, and I haven't tried to figure out who first made that mistake.

It is easy enough to handle the problem. In fact, they can continue to use the name "Kirch's topology" if they do like they do with "Furstenberg's topology". That is, they can explain that the topology genertated by $\mathcal{B}_K$ has the essential properties of Kirch's topology even though it is strictly larger. Once one establishes (which is pretty easy) that if $\langle a, P\rangle$ is as produced by [14, Theorem 4], then $a < P$ and $(a, P) = 1$, one has that Kirch's topology is contained in the topology generated by $\mathcal{B}_K$. Further, the proof of [14, Theorem 5] only uses that the topology has a base consisting of sets of the form $\langle a, P\rangle$ where P is square free and $(a, P) = 1$.

In the first full paragraph of page 2, "the Tychonoff-Urysohn's metrization theorem" should either be "the Tychonoff-Urysohn metrization theorem" or "Tychonoff-Urysohn's metrization theorem", preferably the former.

In the small section titled Three ideals there is a reference to “decent” topologies. This is the first time in my long life that I have ever heard the discrete topology referred to as “indecent”.

Since it has already been established that the three topologies are all Hausdorff, in view of Proposition 1.2, whose proof is short and simple, it is silly to include Proposition 1.1.

Proposition 1.2 should say that “Any base for any ...”. A given topology usually has many different bases. Likewise, the proof should start “Let $\mathcal{B}$ be a base...”.

Section 2 consists primarily of a determination of the relations among the ideals $\mathcal{I}_F$, $\mathcal{I}_G$, and $\mathcal{I}_K$. As such, Theorem 2.9 should be moved up at least before Example 2.4. Then Example 2.4 could be stated as “is in $\mathcal{I}_K$”. If they wish, the authors can add “hence in $\mathcal{I}_G$”. But in any event, the proof should be given for $\mathcal{I}_K$ rather than for $\mathcal{I}_G$.

The proof of Example 2.1 can be significantly simplified and simultaneously made more complete, eliminating “proof ... goes similarly”, as follows.
Proof. We show that A is closed with respect to $\mathcal{T}_K$, and thus is closed with respect to each of the three topologies. To see this, suppose that x is an accumulation point of A with respect to $\mathcal{T}_K$. Pick $p \in \mathbb{P}$ with $p > x$. Then $\langle x, p\rangle$ is a neighborhood of x which hits A in only finitely many points.

Next note that the interior of A is empty with respect to each topology because A contains no three term arithmetic progression. □

In the proof of Example 2.3, the displayed equations are overly detailed. (I hesitate to complain, since it is far more common for authors to omit necessary details. But this is computation that any high school student can do.) That is,

$$an_0 + b = a(a^2 + ab + 2a + 2b + 1) + b = (a + 1)^2(a + b)$$

is quite adequate.

At the conclusion of the proof of Example 2.3, “every its superset” should be “every superset of $\mathbb{P}$”. And rather than appeal to Example 2.2, which was presented without proof, for the fact that $\mathbb{P}$ is dense in $\mathcal{T}_G$, it should be pointed out that this fact is an immediate consequence of Dirichlet’s Theorem.

Assuming that Theorem 2.9 was moved forward as requested, Example 2.5 and Corollary 2.6 should refer to $\mathcal{I}_K \setminus \mathcal{I}_F$ rather than $\mathcal{I}_G \cap \mathcal{I}_K \setminus \mathcal{I}_F$.

There doesn't seem to be any motive for including Proposition 2.7.

Theorem 2.10 can be strengthened to read $(\mathcal{I}_G \cap \mathcal{I}_F) \setminus \mathcal{I}_K \neq \emptyset$.

Proof. Proceed verbatim down through "as it contains x_{k_0}", except replace "We will construct a set $X \in \mathcal{I}_G \setminus \mathcal{I}_K$." by "We will construct a set $X \in (\mathcal{I}_G \cap \mathcal{I}_F) \setminus \mathcal{I}_K$." Then proceed as follows.

Take any $\langle b, a\rangle \in \mathcal{B}_F$. We will show that there exists nonempty $V \subseteq \langle b, a\rangle$ with $X \cap V = \emptyset$ such that $V \in \mathcal{T}_F$ and, if $\langle b, a\rangle \in \mathcal{B}_G$, then $V \in \mathcal{T}_G$.

Assume first that $2 \not| a$. Let $V = (\langle b, a\rangle \cap \langle 3, 4\rangle) \setminus \{x_1\}$. Then $V \neq \emptyset$ because $(a, 4) = 1$ so $V \in \mathcal{T}_F$ and, if $\langle b, a\rangle \in \mathcal{B}_G$, then $V \in \mathcal{T}_G$. If $k > 1$, then $x_k \in \langle 1, 4\rangle$ so $V \cap X = \emptyset$.

Now assume that $2|a$ and pick the largest $m \in \mathbb{N}$ such that $2^m|a$. If b is even, then $\langle b, a\rangle \cap X = \emptyset$ and we may let $V = \langle b, a\rangle$. Thus we may assume that b is odd. If $b \not\equiv 1(\text{mod } 2^m)$, let $V = \langle b, a\rangle \setminus \{x_1, x_2, \ldots, x_{m-1}\}$. Then $\langle b, a\rangle \cap \langle 1, 2^m\rangle = \emptyset$ so $V \cap X = \emptyset$.

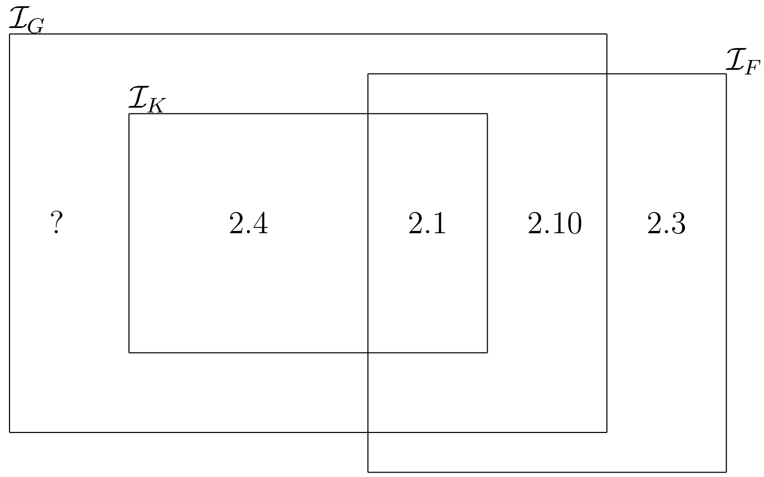
Thus we may assume that $b \equiv 1(\text{mod } 2^m)$. Let

$$V = (\langle b, a\rangle \cap \langle 1 + 2^m, 2^{m+1}\rangle) \setminus \{x_1, x_2, \ldots, x_m\}.$$

Then $V \neq \emptyset$ because $\left(\frac{a}{2^m}, \frac{2^{m+1}}{2^m}\right) = 1$ and $\langle b, a\rangle$ and $\langle 1 + 2^m, 2^{m+1}\rangle$ are contained in $\langle 1, 2^m\rangle$. If $k > m$, then $x_k \in \langle 1, 2^{m+1}\rangle$ so $V \cap X = \emptyset$. Also $V \in \mathcal{T}_F$ and, if $\langle b, a\rangle \in \mathcal{B}_G$, then $V \in \mathcal{T}_G$. $\square$

Notice that in the above proof there was no need to consider $m = 1$ and $m = 2$ separately. That sort of thing can be justified if it clarifies a difficult idea. There is no difficult idea here to clarify.

At this stage, one has established the relationships that hold among the three ideals as indicated in the following diagram. I would suggest spending a little time trying to see whether $\mathcal{I}_G \setminus (\mathcal{I}_K \cup \mathcal{I}_F) \neq \emptyset$, and if unsuccessful, ask the question.



The first sentence of Section 3 has been used several times before. It should either be omitted, or moved to before its first use.

The second sentence of the proof of Theorem 3.6 says "Without loss of generality, we can assume that $A \notin S^0(\mathcal{F})$." That is not "without loss of generality". The phrase "without loss of generality" applies when there are two or more cases and the proofs for each case are identical. Here one should write something like "The case that $A \in S^0(\mathcal{F})$ is trivial, so we assume that $A \notin S^0(\mathcal{F})$.

In the sentence after Corollary 3.8, I think it would be helpful to define an $F_{\sigma\delta}$ ideal.

In Definition 4.1, σ-ideal is not defined. I am guessing it means closed under countable unions. Similarly, in Definition 4.2(iii) *selector* is not defined. I found a definition in [17], but it depended on the undefined $\mathcal{I}^+$. As a consequence *weakly selective* remains undefined. It is mentioned in Corollary 4.6. Also Corollary 4.6 depends on a result in an unpublished paper. I hope that the authors have at least personally verified the proof of [17, Proposition 4.3].

In [8] NWD($\mathbb{Q}$) $= \{A \subseteq \mathbb{Q} \cap [0,1] : A$ is nowhere dense$\}$ and NULL($\mathbb{Q}$) $= \{A \subseteq \mathbb{Q} \cap [0,1] : cl_{\mathbb{R}}(A)$ is of Lebesgue measure $0\}$. (The authors of [8] did not mention whether "nowhere dense" referred to $\mathbb{Q}$ or $\mathbb{R}$ which is more-or-less OK since they are equivalent for subsets of $\mathbb{Q}$.) The current authors in their definition of NWD($\mathbb{Q}$) do not specify whether they are taking the closure of A in $\mathbb{Q}$ or in $\mathbb{R}$. It has to be in $\mathbb{R}$ since every subset of $\mathbb{Q}$ is meager. It is true that their definition is equivalent, by the Baire Category Theorem, but

I don't see the point of introducing a different definition.

In the definition of NWD and NULL on page 12, I presume the authors wanted to have b as a bijection between $\mathbb{N}$ and $\mathbb{Q} \cap [0, 1]$. Otherwise it is not obvious to me that NWD is isomorphic to $\mathcal{I}_F$. And in the definition of NWD it is silly to take the closure of $b[A]$ since any set is nowhere dense if and only if its closure is.

On page 15 in item (ii) of the proof of Theorem 5.4, it is true, and trivial, that $\langle b, a\rangle = \langle b, 2a\rangle \cup \langle b + a, 2a\rangle$, but it doesn't "follow from the splitting property of $\mathcal{B}_F$". It follows from *how* $\langle b, a\rangle$ splits.