Fourier Series – Chapter 3 Solutions

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1. Derive formula (3.1) from the formula

$$1 + e^{ix} + \ldots + e^{inx} = \frac{e^{i(n+1)x} - 1}{e^{ix} - 1}.$$

Taking the real part of each side, we have

$$1 + \cos x + \ldots + \cos nx = \frac{1}{2}\left(\frac{e^{i(n+1)x} - 1}{e^{ix} - 1} + \frac{e^{-i(n+1)x} - 1}{e^{-ix} - 1}\right) = \frac{1}{2}\frac{e^{i(n+1)x} - e^{inx} + e^{ix} - 1}{e^{ix} - 1}.$$

Factoring $e^{ix/2}$ out of the numerator and denominator of the right hand side,

$$\frac{1}{2} + \cos x + \ldots + \cos nx = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin(x/2)}.$$

2. For each of the following functions, find the right hand derivative at zero, $f'_+(0)$, if it exists, and find $\lim\limits_{\substack{x\to 0\\ x>0}} f'(x)$ if this limit exists:

   (a) $f(x) = x \sin \frac{1}{x}$

   (b) $f(x) = x^2 \sin \frac{1}{x}$

   (c) $f(x) = x^3 \sin \frac{1}{x}$

In part (a), we have

$$f'_+(0) = \lim_{h\to 0} \frac{h \sin \frac{1}{h}}{h},$$

which does not exist. The derivative is $f'(x) = \sin \frac{1}{x} + \frac{1}{x} \cos \frac{1}{x}$, which does not have a right-sided limit at 0.

In part (b), we have

$$f'_+(0) = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = 0.$$

The derivative is $f'(x) = 2x \sin \frac{1}{x} + \cos \frac{1}{x}$, which does not have a right-sided limit at 0.

In part (c), we have

$$f'_+(0) = \lim_{h\to 0} \frac{h^3 \sin \frac{1}{h}}{h} = 0.$$

The derivative is $f'(x) = 3x^2 \sin \frac{1}{x} + x \cos \frac{1}{x}$, so $\lim\limits_{\substack{x\to 0\\ x>0}} f'(x) = 0$.

3. Show that the theorem of Sec. 6 can be generalized in the following way: If $f(x)$ is an absolutely integrable function of period 2π, if $f(x)$ is continuous at the point x_0, and if there are numbers $c > 0$, $\alpha > 0$ such that

$$|f(x) - f(x_0)| \le c|x - x_0|^\alpha$$

for all x in some neighborhood of x_0, then the Fourier series of $f(x)$ converges to the value $f(x_0)$ at the point x_0.

We repeat here the argument of Sec. 6 with the slight generalization. The nth partial sum of the Fourier series for $f(x)$ is given by the integral representation

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin(n+1/2)u}{2\sin(u/2)}\,du.$$

Therefore, we aim to show that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}[f(x+u) - f(x)]\frac{\sin(n+1/2)u}{2\sin(u/2)}\,du = 0.$$

Consider the function

$$g(x) = \frac{f(x+u) - f(x)}{u}\frac{u}{2\sin(u/2)}.$$

We will show that $g(u)$ is absolutely integrable. Let $(x_0 - \delta, x_0 + \delta)$ be a neighborhood of x_0 for which the Lipschitz condition holds. Then, for $|u| < \delta$, we have

$$\left|\frac{f(x+u) - f(x)}{u}\right| \le c|u|^{\alpha-1}.$$

Since $\alpha > 0$, this function is absolutely integrable. For $|x - x_0| > \delta$, the function is trivially absolutely integrable, so it is absolutely integrable on $[-\pi, \pi]$. The function $\frac{u}{2\sin(u/2)}$ is bounded on $[-\pi, \pi]$. Therefore, $g(u)$ is absolutely integrable. Then,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} g(u)\sin(n+1/2)u\,du = 0,$$

and the result follows.

4. Let $f(x)$ and $g(x)$ be absolutely integrable functions with period 2π, whose Fourier series are

$$f(x) \sim \sum_{n=1}^{\infty} a_n e^{inx}, \qquad g(x) \sim \sum_{n=1}^{\infty} b_n e^{inx},$$

and let

$$h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)g(t)\,dt.$$

Show that

$$\frac{1}{2\pi}\int_0^{2\pi}|h(x)|\,dx \le \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|\,dx\right)\left(\frac{1}{2\pi}\int_0^{2\pi}|g(x)|\,dx\right)$$

and that $c_n = a_n b_n$. In particular, if $f(x)$ and $g(x)$ are square integrable, show that

$$\sum_{n=1}^{\infty}|c_n| < \infty,$$

i.e. that the Fourier series for $h(x)$ converges absolutely.

We have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|h(x)|\,dx &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\left|\int_0^{2\pi} f(x-t)g(t)\,dt\right|\,dx \\ &\le \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|f(x-t)||g(t)|\,dt\,dx \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}|g(t)|\int_0^{2\pi}|f(x-t)|\,dx\,dt \\ &= \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|\,dx\right)\left(\frac{1}{2\pi}\int_0^{2\pi}|g(x)|\,dx\right).\end{aligned}$$

The Fourier coefficients of $h(x)$ are

$$\begin{aligned}c_n &= \frac{1}{2\pi}\int_0^{2\pi} h(x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} f(x-t)g(t)\,dt\right)e^{-inx}\,dx \\ &= \left(\frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-int}\right)\left(\frac{1}{2\pi}\int_0^{2\pi} f(u)e^{-inu}\,du\right) \\ &= a_n b_n.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}|c_n| = \sum_{n=1}^{\infty}|a_n||b_n| \le \left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{1/2}.$$

Thus, if $f(x)$ and $g(x)$ are square integrable, the sum on the left is bounded.

5. Show that the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

can be written in the form

$$\frac{1}{2}\rho_0 + \sum_{n=1}^{\infty}\rho_n\cos(nx+\theta_n),$$

where $\rho_n = \sqrt{a_n^2+b_n^2}$. Express θ_n in terms of a_n and b_n.
Using the complex representation, we have

$$a_n\cos nx + b_n\sin nx = \frac{(a_n - ib_n)e^{inx} + (a_n + ib_n)e^{-inx}}{2}.$$

Let

$$\theta_n = \arg\sqrt{\frac{a_n - ib_n}{a_n + ib_n}} = -\tan^{-1}\frac{b_n}{a_n}.$$

Then we have

$$a_n\cos nx + b_n\sin nx = \sqrt{a_n^2+b_n^2}\,\frac{\sqrt{\frac{a_n-ib_n}{a_n+ib_n}}e^{inx} + \sqrt{\frac{a_n+ib_n}{a_n-ib_n}}e^{-inx}}{2} = \rho_n\cos(nx+\theta_n).$$

6. Suppose that $\rho_n \geq 0$ and that

$$\sum_{n=1}^{\infty} \rho_n |\cos(nx + \theta_n)| \leq M$$

for $a \leq x \leq b$. Show that

$$\sum_{n=1}^{\infty} \rho_n < \infty.$$

Integrating the given bound, we have

$$\sum_{n=1}^{\infty} \rho_n \int_a^b |\cos(nx + \theta_n)| \leq M(b-a).$$

Using $|\cos x| \geq \cos^2 x$, and $\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin 2x$, this gives

$$\sum_{n=1}^{\infty} \rho_n \left(\frac{b-a}{2} + \frac{\sin(2nb + \theta_n) - \sin(2na + \theta_n)}{4n} \right) \leq M(b-a).$$

The second term in parentheses is bounded in absolute value by $\frac{1}{2n}$. Therefore,

$$\sum_{n=1}^{\infty} \rho_n \left(1 - \frac{1}{n(b-a)} \right) \leq 2M.$$

Finally, letting $N = \left\lceil \frac{2}{b-a} \right\rceil$, we have

$$\sum_{n=N}^{\infty} \rho_n \leq 4M,$$

which establishes the convergence of $\sum_{n=1}^{\infty} \rho_n$.

7. According to equation (13.7) of Ch. 1, the function

$$f(x) = \frac{1}{2}(\pi - x) \quad (0 < x < 2\pi)$$

has the Fourier expansion

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k}.$$

Let $s_n(x)$ be the nth partial sum of the series, i.e.,

$$s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k},$$

and let

$$D_n(x) = \frac{\sin(n + 1/2)x}{2\sin(x/2)}.$$

Show that

a) $\frac{x}{2} + s_n(x) = \int_0^x D_n(t)\,dt$;

b) $\int_0^x D_n(t)\,dt = \int_0^x \frac{\sin nt}{t}\,dt + \omega_n(x) = \int_0^{nx} \frac{\sin t}{t}\,dt + \omega_n(x)$, where $\omega_n(x) \to 0$ as $n \to \infty$;

c) $\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

We showed in Problem 1 that

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kt = D_n(t).$$

Integrating this from 0 to x gives the result in part (a).

The integral in part (b) can be approximated using the trapezoidal rule with n intervals as

$$\int_0^x \frac{\sin nt}{t}\,dt \approx \frac{x}{n}\left(\frac{n}{2} + \frac{\sin nx}{2x} + \sum_{k=1}^{n-1} \frac{\sin kx}{kx/n}\right) = \frac{x}{2} + s_n(x) - \frac{\sin nx}{2n}.$$

The error in the approximation is bounded by $\frac{x^3}{12n^2} \max \frac{d^2}{dx^2}\left(\frac{\sin nx}{x}\right)$. Comparing this with the result of part (a), we find

$$\int_0^x D_n(t)\,dt = \int_0^x \frac{\sin nt}{t}\,dt + \frac{\sin nx}{2n} + \frac{cx^3}{12n^2}.$$

The end terms on the right certainly decay to zero as $n \to \infty$, which establishes the result of part (b).

Part (c) follows by combining parts a and b and setting $x = \pi$ while letting $n \to \infty$.

8. Using the notation of the previous problem, show that

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{n}\right) = \int_0^\pi \frac{\sin t}{t}\,dt > \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

Using the results of the previous problem, we have

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(\int_0^{\pi/n} D_n(t)\,dt - \frac{\pi}{n}\right) = \int_0^\pi \frac{\sin t}{t}\,dt.$$

By writing

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\pi \frac{\sin t}{t}\,dt + \left(\int_\pi^{2\pi} \frac{\sin t}{t}\,dt + \int_{2\pi}^{3\pi} \frac{\sin t}{t}\,dt\right) + \ldots$$

and noting that all terms in parentheses are negative, the inequality readily follows. The final equality is a result of the previous problem.