Chapter 1 Solutions

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Problem #1: *Consider a standard deck of 52 playing cards dealt into four hands of 13 cards each. If a given suit distribution within a hand represents a macrostate while a specific set of cards within a suit represents a microstate, find (a) the number of possible macrostates for each hand, (b) the number of microstates allowed for each macrostate, and (c) the most probable macrostate.*

The number of macrostates is the number of ways to divide the 13 cards among four suits. This is equivalent to sorting 13 identical cards and 3 identical dividers, which gives

$$N = \frac{16!}{13!3!} = \binom{16}{3} = 560.$$

For a macrostate (a_1, a_2, a_3, a_4), with $\sum a_i = 13$, there are

$$n(a_1, a_2, a_3, a_4) = \binom{13}{a_1}\binom{13}{a_2}\binom{13}{a_3}\binom{13}{a_4}$$

possible microstates.

The most probable macrostate is given by

$$\underset{(a_1,a_2,a_3,a_4)}{\arg\max} \; n(a_1, a_2, a_3, a_4).$$

We can see by exhaustive search that this gives $(3,3,3,4)$, or permutations thereof. We can also show analytically that a nearly uniform distribution is the most probable macrostate. The Lagrange function for this optimization problem is

$$\mathcal{L} = n(a_1, a_2, a_3, a_4) - \mu\left(\sum a_i - 13\right).$$

This gives the equations

$$\frac{\partial n}{\partial a_i} - \mu = 0, \qquad \sum a_i = 13.$$

In the Stirling approximation, we have

$$\frac{\partial n}{\partial a_i} = n \log \frac{a_i}{13 - a_i}.$$

Thus, in order to have $\frac{\partial n}{\partial a_i} - \mu = 0$ for all i, we must have $a_i = \text{const}$.

Problem #2: *Consider a space with three cells of size $2h^3$ and nine particles. Find the number of macrostates, the total number of microstates, and the most probable macrostate, assuming the particles are (a) "Maxwellons," (b) fermions, and (c) bosons.*

We have three cells and nine particles, so the number of macrostates for any type of particle is

$$N = \binom{9+3-1}{3-1} = 66.$$

For "Maxwellons," the total number of microstates is the number of ways to assign each of the nine particles to one of the three cells, or $3^9 = 19683$. The most probable macrostate is the one with three particles in each cell; it is straightforward to show that a uniform distribution of particles is most probable in the general case.

For fermions, the total number of microstates is the number of ways to place the nine indistinguishable particles into the $2 \times 2 \times 3 = 12$ half-compartments, or $\binom{12}{9} = 220$. The most probable macrostate is again the one with three particles in each cell, since it has $4^3 = 64$ microstates.

For bosons, the total number of microstates is the number of ways to place the nine indistinguishable particles into the 6 compartments, or $6^9 = 10{,}077{,}696$. The most probable macrostate has three particles in each cell, since it has

$$W = \left(\binom{3+4-1}{4-1}\right)^3 = 8000$$

microstates.

Problem #3: *Given that*

$$dN = B\left(\exp\frac{w}{kT} + \phi\right)^{-1} dp_x\, dp_y\, dp_z$$

and that $w = (p_x^2 + p_y^2 + p_z^2)/2m$, find an expression for B in terms of N for the case where $\phi = 0, \pm 1$.

Since dN depends only on p^2, we can convert the integral to spherical coordinates, obtaining

$$N = \int dN = B \int_0^\infty \left(\exp\frac{p^2}{2mkT} + \phi\right)^{-1} (4\pi p^2)\, dp.$$

If $\phi = 0$, then this becomes

$$N = 4\pi B \int_0^\infty p^2 \exp\left(-\frac{p^2}{2mkT}\right) dp.$$

Substituting $u = \frac{p^2}{2mkT}$, we have

$$N = 2\pi(2mkT)^{3/2} B \int_0^\infty u^{1/2} e^{-u}\, du = (2\pi mkT)^{3/2} B.$$

Thus we have $B = \frac{N}{(2\pi mkT)^{3/2}}$.

If $\phi = -1$, then we have

$$N = 2\pi(2mkT)^{3/2} B \int_0^\infty \frac{u^{1/2} e^{-u}}{1 - e^{-u}}\, du.$$

A geometric series expansion gives

$$N = 2\pi(2mkT)^{3/2} B \sum_{j=1}^{\infty} \int_0^\infty u^{1/2} e^{-ju}\, du.$$

The integral is $\frac{\Gamma(3/2)}{j^{3/2}} = \sqrt{\pi} 2 j^{3/2}$, so we have

$$N = (2\pi mkT)^{3/2} B \sum_{j=1}^{\infty} \frac{1}{j^{3/2}}.$$

Therefore we have $B = \frac{N}{(2\pi mkT)^{3/2}\zeta(3/2)}$.

If $\phi = +1$, then we have

$$N = 2\pi(2mkT)^{3/2} B \int_0^\infty \frac{u^{1/2} e^{-u}}{1 + e^{-u}}\, du.$$

A geometric series expansion gives

$$N = 2\pi(2mkT)^{3/2} B \sum_{j=1}^{\infty} (-1)^j \int_0^\infty u^{1/2} e^{-ju}\, du.$$

The integral is $\frac{\Gamma(3/2)}{j^{3/2}} = \frac{\sqrt{\pi}}{2 j^{3/2}}$, so we have

$$N = (2\pi mkT)^{3/2} B \sum_{j=1}^{\infty} \frac{(-1)^j}{j^{3/2}}.$$

Therefore we have $B = \frac{\sqrt{2}}{\sqrt{2}-1} \frac{N}{(2\pi mkT)^{3/2}\zeta(3/2)}$.

Problem #4: *Derive the equation of state for a Fermi gas from first principles.*

We assume a non-relativistic degenerate gas, so $n(p)\,dp = \frac{8\pi p^2}{h^3}\,dp$ and $\boldsymbol{v} = \boldsymbol{p}/m$. A given particle delivers momentum at a rate $2\boldsymbol{v}\cdot\boldsymbol{p}$, but this is distributed over the 6 walls of a cubical box, so we must integrate $\frac{1}{3}\boldsymbol{v}\cdot\boldsymbol{p} = \frac{p^2}{3m}$. Therefore, we have

$$P = \frac{1}{3m}\int_0^{p_0} p^2 n(p)\,dp = \frac{8\pi}{15mh^3}p_0^5.$$

We can also write the density in terms of the maximum momentum as

$$\rho = m\int_0^{p_0} n(p)\,dp = \frac{8\pi m p_0^3}{3h^3}.$$

We then eliminate p_0 and obtain

$$P = \frac{h^2}{20m^2}\left(\frac{3}{\pi m}\right)^{2/3}\rho^{5/3}.$$

Problem #5: *Given that $f(x)$ is an analytic function in the interval $0 \le x \le \infty$, show that $f(x)$ can be represented in terms of the moments of the function $M_i[f(x)]$, where*

$$M_i[f(x)] \equiv \int_0^\infty x^i f(x)\,dx.$$

Let $\operatorname{supp} f(x) = [0,\infty)$. We can formally represent $f(x)$ as an inverse Laplace transform of its moment generating function. The moment-generating function is

$$M_f(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^\infty e^{tx} f(x)\,dx = \sum_{n=0}^{\infty}\frac{t^n}{n!}M_n[f(x)].$$

Then the inverse Lapalce transform gives

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{tx} M_f(t)\,dt,$$

where the notation reprsents a line integral in the complex plane along a vertical path and a is greater than the maximum real part of the singularities of M_f.

Problem #6: *If the pressure tensor $\boldsymbol{P}$ has the form specified by equation (1.2.25), show that it can be rewritten as it appears in equation (1.2.24) (i.e., as the tensor operated on by the divergence operator in the third term on the left-hand side).*

The problem asks us to show the tensor equality

$$\frac{\int \boldsymbol{v}\boldsymbol{v} f(v)\,dv}{\int f(v)\,dv} - \boldsymbol{u}\boldsymbol{u} = \frac{\int(\boldsymbol{v}-\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u})f(v)\,d\boldsymbol{v}}{\int f(v)\,d\boldsymbol{v}},$$

where $\boldsymbol{u}$ is the mean flow velocity

$$\frac{\int \boldsymbol{v} f(v)\,d\boldsymbol{v}}{\int f(v)\,d\boldsymbol{v}}.$$

We can show this by expanding the right hand side, obtaining

$$\frac{\int \boldsymbol{vv} f(v)\,dv}{\int f(v)\,dv} - 2\boldsymbol{u}\frac{\int \boldsymbol{v} f(v)\,dv}{\int f(v)\,dv} + \boldsymbol{uu} = \frac{\int \boldsymbol{vv} f(v)\,dv}{\int f(v)\,dv} - \boldsymbol{uu}.$$

Problem #7: *Show that the virial theorem holds in the form given by equation (1.2.35) even if the forces of interaction include velocity-dependent terms (i.e., Lorentz forces or viscious drag forces).*

We will start by taking the first velocity moment of the Boltzmann equation with the assumption of a velocity-dependent force. We have

$$\int \boldsymbol{v}\frac{\partial f}{\partial t}\,d\boldsymbol{v} + \int \boldsymbol{v}(\boldsymbol{v}\cdot\nabla f)\,d\boldsymbol{v} + \int \boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\nabla_v f)\,d\boldsymbol{v} = \int \boldsymbol{v} S\,d\boldsymbol{v}.$$

The first term is $\frac{\partial(n\boldsymbol{u})}{\partial t}$. Integrating by parts in the second term, and noting that $\nabla\cdot\boldsymbol{v} = 0$, we have

$$\int \boldsymbol{v}(\boldsymbol{v}\cdot\nabla f)\,d\boldsymbol{v} = \int \boldsymbol{v}\nabla\cdot(\boldsymbol{v} f)\,d\boldsymbol{v}.$$

The third term differs in the case of a velocity-dependent force; we cannot move $\dot{\boldsymbol{v}}$ outside the integral. However, we can still write the integrand as the dot product of a vector and a tensor:

$$\int \boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\nabla_v f)\,d\boldsymbol{v} = \int \dot{\boldsymbol{v}}\cdot((\nabla_v f)\boldsymbol{v})\,d\boldsymbol{v}.$$

We then have $(\nabla_v f)\boldsymbol{v} = \nabla_v(f\boldsymbol{v}) - \mathbf{1}f$, so the integral becomes

$$\int \dot{\boldsymbol{v}}\cdot\nabla_v(f\boldsymbol{v})\,d\boldsymbol{v} - \int \dot{\boldsymbol{v}} f\,d\boldsymbol{v}.$$

We can then perform a similar manipulation on the remaining integral, using $\dot{\boldsymbol{v}}\cdot\nabla_v(f\boldsymbol{v}) = \nabla_v(f\boldsymbol{v}\cdot\dot{\boldsymbol{v}}) - f\boldsymbol{v}(\nabla_v\cdot\dot{\boldsymbol{v}})$. Noting that the distribution function vanishes at high velocities, the final result is

$$n\boldsymbol{g} = -\left(\int f\boldsymbol{v}(\nabla_v\cdot\dot{\boldsymbol{v}})\,d\boldsymbol{v} + \int \dot{\boldsymbol{v}} f\,d\boldsymbol{v}\right).$$

Note that we have $\boldsymbol{g} = \frac{\nabla\Phi}{m}$ when the force is not velocity-dependent. The expression for hydrodynamic flow thus becomes

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\boldsymbol{g} - \frac{\nabla P}{\rho}.$$

The first position moment then gives

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2(T+U) - \int_V \rho\boldsymbol{r}\cdot\boldsymbol{g}\,dV.$$

Thus the virial equation takes the same form, with the virial of Clausius replaced by

$$\Omega' = -\int_V \rho\boldsymbol{r}\cdot\boldsymbol{g}\,dV.$$

Problem #8: *Show that the second velocity moment of the Boltzmann transport equation leads to an equation describing the conservation of energy.*

Taking the second velocity moment of the Boltzmann equation gives the tensor relation

$$\int \boldsymbol{v}\boldsymbol{v}\frac{\partial f}{\partial t}\,d\boldsymbol{v} + \int \boldsymbol{v}\boldsymbol{v}(\boldsymbol{v}\cdot\nabla f)\,d\boldsymbol{v} + \int \boldsymbol{v}\boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\nabla_v f)\,d\boldsymbol{v} = \int \boldsymbol{v}\boldsymbol{v}S\,d\boldsymbol{v}.$$

We can write this in terms of the mean velocity tensor

$$\boldsymbol{U} = \frac{\int \boldsymbol{v}\boldsymbol{v}f\,d\boldsymbol{v}}{\int f\,d\boldsymbol{v}}$$

as

$$\frac{\partial(n\boldsymbol{U})}{\partial t} + \int \boldsymbol{v}\boldsymbol{v}\nabla\cdot(f\boldsymbol{v})\,d\boldsymbol{v} - 2\dot{\boldsymbol{v}}\boldsymbol{u} = \int \boldsymbol{v}\boldsymbol{v}S\,d\boldsymbol{v}.$$

Multiplying by $\frac{m}{2}$ gives

$$\frac{1}{2}\frac{\partial(\rho\boldsymbol{U})}{\partial t} + \int \frac{m\boldsymbol{v}\boldsymbol{v}}{2}\nabla\cdot(f\boldsymbol{v})\,d\boldsymbol{v} + \nabla\phi\boldsymbol{u} = \int \frac{m\boldsymbol{v}\boldsymbol{v}}{2}S\,d\boldsymbol{v}.$$

Taking the trace gives

$$\frac{\partial\epsilon}{\partial t} + \int \frac{mv^2}{2}\nabla\cdot(f\boldsymbol{v})\,d\boldsymbol{v} + \nabla\phi\cdot\boldsymbol{u} = \int \frac{mv^2}{2}S\,d\boldsymbol{v},$$

where ϵ is the energy density. In the second term we can move $\frac{mv^2}{2}$ inside the divergence since velocity and position coordinates are independent, and obtain

$$\frac{\partial\epsilon}{\partial t} + \nabla\cdot(\epsilon\boldsymbol{u}) = \boldsymbol{F}\cdot\boldsymbol{u} + \Im_E,$$

where $\Im_E$ is the creation rate of energy. This equation clearly represents the conservation of energy.