Foundations of Mathematics – Chapter 3 Solutions

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1. In a given university, let $\mathfrak{C}$ denote the set of all sections (mathematics sections, history sections, etc.) in session at a given time, $\mathfrak{G}$ the set of all graduate sections, $\mathfrak{R}$ the set of all sections in graduate mathematics, C a section in graduate mathematics, and p a stuent in C. Set up the "$\in$" and "$\subset$" relations between $\mathfrak{C}$, $\mathfrak{G}$, $\mathfrak{R}$, C, and p. Explain the choice of symbols (cf. 3.1.1).

   From the definitions of the relationships, we have $p \in C \in \mathfrak{R} \subset \mathfrak{G} \subset \mathfrak{C}$.

2. Let $A = \{1, \varnothing, \{\varnothing\}\}$, $B = \{\varnothing\}$, $C = \{1, 2\}$. Show that $B \in A$, $B \subset A$, and that $B \cap C = \varnothing$; but that neither $A \cap B = \varnothing$ nor $A \subset B \cup C$ holds.

   We can clearly see that the set B is contained as an element of A, and separately that all the elements of B are contained in A. Likewise, since $1 \neq \varnothing$ and $2 \neq \varnothing$, we have $B \cap C = \varnothing$. Since $B \subset A$, $A \cap B = B \neq \varnothing$, and since $\{\varnothing\} \in A$ but $\{\varnothing\} \notin B \cup C$, we do not have $A \subset B \cup C$.

3. Use the $\{\,|\,\}$ symbol to define the following sets:

   (a) the set of all points in the coordinate pane that lie interior to the unit circle with center at the origin;
   (b) the set of points defined in (a) with those having negative abscissas deleted;
   (c) the set $(A \cap B) \cup C$ where A, B, and C are sets;
   (d) the set $A \cap (B \cup C)$;
   (e) the set $A - (B \cup C)$;
   (f) the set $(A - B) \cup C$.

   (a) $\{(x, y) \mid x^2 + y^2 < 1\}$
   (b) $\{(x, y) \mid x^2 + y^2 < 1 \wedge x \leq 0\}$
   (c) $\{x \mid (x \in A \wedge x \in B) \vee x \in C\}$
   (d) $\{x \mid x \in A \wedge (x \in B \vee x \in C)\}$
   (e) $\{x \mid x \in A \wedge x \notin B \wedge x \notin C\}$
   (f) $\{x \mid (x \in Ax \notin B) \vee x \in C\}$

4. Prove the following "distributive laws":

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

   Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$, which means that $x \in (A \cap B)$ or $x \in (A \cap C)$ (or both). Therefore, $x \in (A \cap B) \cup (A \cap C)$, so $[A \cap (B \cup C)] \subset [(A \cap B) \cup (A \cap C)]$. Similarly, let $y \in (A \cap B) \cup (A \cap C)$. Then $y \in (A \cap B)$ or $y \in (A \cap C)$. If $y \notin A$, then $y \notin (A \cap B)$ and $y \notin (A \cap C)$, a contradiction. Thus $y \in A$. We can likewise show that $y \in (B \cup C)$. Thus, $y \in A \cap (B \cup C)$, so $[(A \cap B) \cup (A \cap C)] \subset [A \cap (B \cup C)]$.

To prove the second relation, let $D = A \cup B$, $E = A$, and $F = C$. Then the first relation applied to D, E, and F gives

$$(A \cup B) \cap (A \cup C) = A \cup [(A \cup B) \cap C].$$

Since $(A \cup B) \cap C - B \cap C \subset A$, $A \cup [(A \cup B) \cap C] = A \cup (B \cap C)$. This proves the second relation.

5. Let the complement of a set A relative to a set S be denoted by $\mathscr{C}_S A$, or briefly, by $\mathscr{C} A$. Then prove:

$$\mathscr{C}(A \cap B) = \mathscr{C}A \cup \mathscr{C}B,$$
$$\mathscr{C}(A \cup B) = \mathscr{C}A \cap \mathscr{C}B.$$

If $x \in \mathscr{C}(A \cap B)$, then $x \notin (A \cap B)$, which implies $x \notin A \vee x \notin B$. Therefore, $x \in (\mathscr{C}A \cup \mathscr{C}B)$, so $\mathscr{C}(A \cap B) \subset (\mathscr{C}A \cup \mathscr{C}B)$. The reverse follows similarly, so the sets are equal.

By swapping X and $\mathscr{C}X$ in the first relation, we have $\mathscr{C}(\mathscr{C}A \cap \mathscr{C}B) = A \cup B$. Taking complements of both sides gives the second relation.

6. In the notation of Problem 5, notice that $\mathscr{C}(\mathscr{C}A) = A$. Use this to derive the second formula of Problem 5 from the first.

See Problem 5.

7. Show by mathematical induction (on the number, n, of set sumbols, counting repetitions) that, if M is an expression composed of set symbols $A_1, A_2, \ldots, A_n$ and the symbols $\mathscr{C}$, $\cap$, $\cup$, but containing no $\mathscr{C}(\,)$, then $\mathscr{C}M$ is the expression obtained by replacing, in M, each A_i, $\mathscr{C}A_i$, $\cap$, and $\cup$ by $\mathscr{C}A_i$, A_i, $\cup$, and $\cap$, respectively.

The statement clearly holds for $n = 1$. Let M be an expression of the specified form with n symbols; then we may write M as either $[A_1 \cdots A_{n-1}] \cup A_n$ or $[A_1 \cdots A_{n-1}] \cap A_n$ (ignoring the case where the last symbol is $\mathscr{C}A_n$, which follows in exactly the same manner). Then, by the induction hypothesis on the bracketed expression, $\mathscr{C}[A_1 \cdots A_{n-1}]$ can be obtained by making the specified replacements of symbols. By the results of problem 5, in both of the two cases, the final two symbols are also changed in the required manner. Therefore the statement holds for n, and thus for all natural numbers.

8. If $M = (A \cup \mathscr{C}B) \cap (A \cup C)$, find $\mathscr{C}M$.

By the result of Problem 7, $\mathscr{C}M = (\mathscr{C}A \cap B) \cup (\mathscr{C}A \cap \mathscr{C}C)$.

9. Some have insisted that the only type of set allowable in mathematics is one such that it is determinate of everything whether it is an element of the set or not. Criticize this criterion.

Whether a set is "determinate" in this sense is open to much interpretation. For example, let A be the set of positive integers n for which the Diophantine equation $a^n + b^n = c^n$ has nontrivial solutions. In some sense this is determinate of every positive integer whether it is an element; however, it gives no finite process for making such a determination, and absent Wiles' proof of Fermat's theorem, we would have no finite process.

10. The following problem is attributed to Tarski: Let N denote the set of all natural numbers, and for any two natural numbers m and n let "$m = n$" mean "m is identical to n." Consider any set S_1 having only one natural number as an element. Obviously, if $m, n \in S_1$, then $m = n$. Suppose it has been shown that, for any set S_n having exactly n natural numbers as elements, the relation $m, n \in S_n$ implies $m = n$. Consider any set S_{n+1} having exactly $n+1$ natural numbers as elements; let us denote its elements by $x_1, x_2, \ldots, x_n, x_{n+1}$. As the set $S'_{n+1} = S_{n+1} - x_{n+1}$ has exactly n natural numbersa s elements, the relation $m, n \in S'_{n+1}$ implies $m = n$; in particular, $x_1 = x_n$. Now consider the set $S''_{n+1} = S_{n+1} - x_n$. Since S''_{n+1} contains exactly n natural numbers, the relation $m, n \in S''_{n+1}$ implies $m = n$; in particular, $x_1 = x_{n+1}$. But, if $x_1 = x_n$ and $x_1 = x_{n+1}$, then $x_n = x_{n+1}$, and it follows that all elements of S_{n+1} are identical. Hence we have proved, in particular, that all natural numbers are identical! What is wrong with this "proof"?

The inductive step fails when $n+1=2$, since we cannot claim "in particular, $x_1 = x_{n+1}$" – x_1 is not a member of S_2''.

11. A famous fable recounts that the barber in a certain town shaved everyone who did not shave himself, and only those. To analyze this, use the symbolism of 3.2 for defining sets. [*Hint:* Let "$x$sy" denote that "x shaved y" and let "b" denote the barber. Then equate the set of townsmen who did not shave themselves with the set of townsmen that the barber shaved. What results when the "value" b is used for x?]

    The set of men who did not shave themselves is $S_1 = \{x \mid \neg(x\text{sx})\}$. The set of men who the barber shaved is $S_2 = \{x \mid b\text{sx}\}$. Let $S = S_1 = S_2$; then $S_1 \subset S_2$ and $S_2 \subset S_1$. We can prove two statements:

    - $b \notin S$: Assume for the sake of contradiction that $b \in S_1$. Since $S_1 \subset S_2$, $b \in S_2$, which implies that $b$sb. However, this implies that $b \notin S_1$, a contradiction. Thus $b \notin S$.
    - $b \in S$: Assume for the sake of contradiction that $b \notin S_2$. Then it must be the case that $\neg(b\text{sb})$, which implies that $b \in S_1$. Since $S_1 \subset S_2$, this means that $b \in S_2$, a contradiction. Thus, $b \in S$.

    Since these two statements are clearly contradictions, the situation in the problem is paradoxical.

12. Analyze the Russell contradiction in the same manner as the barber fable.

    By replacing the operator s with $\ni$, and replacing b with A, the set of all sets containing themselves, then the analysis in Problem 11 applies directly to the Russell contradiction.

13. The following is due to Grelling (1908): Divide all adjectives into two classes, calling those that describe themselves "autological" and those that do not "heterological." Thus "English" and "short" are autological, and "French" and "long" are heterological. What is the result if we ask whether the adjective "heterological" is either autological or heterological?

    If "heterlogical" were autological, then it would describe itself, meaning that it is heterological, a contradiction. Likewise, if "heterological" were heterological, then it would describe itself and be autological, a contradiction.

14. Prove that N is not ordinary finite.

    We will prove by induction that there is no k for which N can be placed in (1-1)-correspondence with N_k. The statement holds for $k=1$ because the only correspondence would map all natural numbers to 1, which is not one-to-one. Now, assume for the sake of contradiction that there exists a (1-1)-correspondence between N and N_{k+1}, which we denote $\phi : N_{k+1} \to N$. This means that there is a (1-1)-correspondence ψ between N_k and $N - \phi(k+1)$. But there is also a (1-1)-correspondence ϕ between $N - \phi(k+1)$ and N. Thus, $\phi\psi$ is a (1-1) correspondence between N_k and N, a contradiction of the inductive hypothesis. Thus, there is no (1-1)-correspondence between N_{k+1} and N, and thus for any value of k, there is no (1-1)-correspondence between N_k and N. Thus N is not ordinary finite.

15. Prove by mathematical induction that if S is ordinary infinite, then for every natural number n, S has a subset consisting of exactly n elements. Is the Choice Axiom necessary for the proof?

    Since S is ordinary infinite, it cannot be placed in (1-1)-correspondence with N_0. Thus S contains at least one element x_1. The existence of the subset $\{x_1\}$ proves the statement for $n=1$. Now, assume the statement holds for $n=k$, and let $\{x_1, \ldots, x_k\}$. If S has no elements beyond these, then it can clearly be put in (1-1)-correspondence with N_k, a contradiction. Thus there is at least one other element, which we call x_{k+1}. Then $\{x_1, \ldots, x_{k+1}\}$ is a subset of S, and the statement holds for $n = k+1$. This proves that S has a subset of n elements for every natural number n. The Choice Axiom is necessary for this argument, since it relies on selecting arbitrary elements of S.

16. Show that if a set S has a proper subset S_1 such that there exists a (1-1)-correspondence between S and S_1, then S has subsets S_n, $n = 1, 2, 3, \ldots$, such that for each natural number n, S_{n+1} is a proper subset of S_n and there exists a (1-1)-correspondence between S_n and S_{n+1}.

Let $\phi : S \to S_1$ be a (1-1)-correspondence between the sets. Since S_1 is a proper subset of S, there exists $x \in S$ such that $x \notin S_1$. Thus there is no $y \in S$ such that $\phi(y) = x$. This implies that there is no y such that $\phi(\phi(y)) = \phi(x)$. Thus, the set $S_2 = \phi(S_1)$ is a proper subset of S_1 (since it cannot contain $\phi(x)$). Following the same argument, the sets $S_n = \phi^n(S)$ meet the given criteria.

17. Using the set N of natural numbers for S, and the set $N - \{1\}$ for S_1, show that the sequence $\{S_n \,|\, n \in N\}$ of the preceding problem is the "longest" that can be generally proved to exist (under the hypothesis of Problem 16), in that $\bigcap_{n=1}^{\infty} S_n$ may be empty.

    Using the (1-1)-correspondence $\phi(n) = n+1$, we see that $S_n = \{n+1, n+2, \ldots\}$. Thus, for any natural number n, $n \notin S_n$. Hence no natural number is an element of $\bigcap_{n=1}^{\infty} S_n$, so the set is empty.

18. Prove Theorem 5.2.2 and its corollary (5.2.3).

    Assume that $S_1 = S \cup \{e\}$ has a (1-1)-correspondence ϕ with a proper subset A of itself. By definition, there must be some $x \in S_1$ such that $x \notin A$. We can compose ϕ with a map swapping e and x, so that we have a (1-1)-correspondence between S_1 and $S_2 \subset S$. The same (1-1)-correspondence acting on S maps to a proper subset of S_2, and therefore a proper subset of S. Thus, if S_1 has a (1-1)-correspondence with a proper subset of itself, so does S. This proves Corollary 5.2.3, and the contrapositive Theorem 5.2.2.

19. Young [Y, 63] gives the following argument for 5.2.3: S has a proper subset S' such that there exists a (1-1)-correspondence T between the elements of S and the elements of S'. Let x' be the element of S' paired with x in the correspondence T. The set $S'_1 = S' - \{x'\}$ is then a proper subset of S_1 $(= S - \{x\})$ such that there exists a (1-1)-correspondence between the elements of S_1 an the elements of S'_1. The set S'_1 is, therefore, Dedekind infinite. What is missing from this argument?

    It is not guaranteed that S'_1 is a proper subset of S_1. In particular, it may be the case that $x \neq x'$ and $x \in S'$, so that $x \in S'_1$ but $x \notin S_1$.

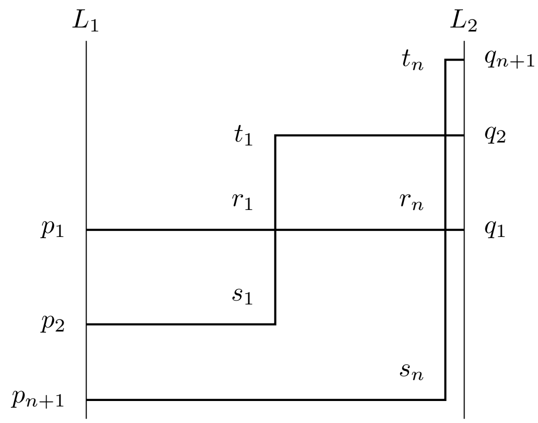
20. In the same connection, Young states "an infinite class cannot be exhausted by removing its elements one at a time." But suppose that between 12:00 NOON and 12:30 P.M., 1 is deleted from N; between 12:30 and 12:45, 2 is deleted from N; during the next $\frac{1}{8}$ hour 3 is deleted; during the next $\frac{1}{16}$ hour 4 is deleted; and so on. What is left of N at 1:00 P.M.? Discuss the statement of Young.

    In this construction, all of N is deleted by 1:00 P.M., because for every natural number x there exists an interval ending before 1:00 P.M. in which it is to be deleted. This is possible only because an infinite number of deletions occurred in finite time. Young's statement can be made more precise by stating that an infinite class cannot be exhausted by any finite number of removals.

21. Prove the converse of 5.3.2. [*Hint*: Let S' be a proper subset of S and let $\{(x, f(x))\}$ be a (1-1)-correspondence such that $x \in S$, $f(x) \in S'$. Let $x_1 \in S - S'$, and for each natural number $n > 1$ let $x_n = f(x_{n-1})$. Then $S_1 = x_n$ is the desired subset of S.]

    Let S be Dedekind infinite. Then there exists a (1-1)-correspondence $f : S \to S'$ between S and a proper subset S' of S. Since S' is a proper subset of S, there exists $x_1 \in S - S'$. We know that $f(x_1) \neq x_1$, since $f(x_1) \in S'$. Likewise, since f is (1-1), $f(f(x_1)) \neq f(x_1)$, and in general, $f^m(x_1) = f^n(x_1)$ implies $m = n$. Therefore, $\{x_n\}$ is a subset of S which obviously has a (1-1)-correspondence with the natural numbers.

22. In an imaginary world W, a man starts from his home at 12:00 NOON to walk to a post office. He walks at a uniform rate, reaching the post office at 1:00 P.M.. He leaves home with 1 cent, intending to buy a stamp. However, between 12:00 and 12:30 a friend offers him a brighter cooin than the one he has, and, intrigued by it, he trades his original coin for it. Between 12:30 and 12:45 a similar event occurs, so that at 12:45 he has a still different coin. During the next $\frac{1}{8}$ hour a similar exchange occurs; during the next $\frac{1}{16}$ hour another occurs; and so on. Assuming that W allows such unlimited possibilities, does the man have a coin at 1:00 P.M. to buy the stamp he set out to buy? [In case the pennies are



all different, a logical conclusion is possible; but, if the same two pennis are involved throughout, no conclusion is possible.]

If the pennies are all different, then after a penny is traded away it can never return to the man. Then, let S be the set of all pennies; let $S_1 = S - \{P_1\}$, where P_1 is the first penny traded; and so on. Then, $\bigcap S_i = \varnothing$, so the man does not end up with a penny.

23. If the set S is Problem 16 is the set of all pennies involved in the anecdote of Problem 22, and S_n is the set of all pennies involved from the nth interval on, show that the resulting collection of sets S_n satisfies the conditions of Problem 17 and is a case where $\bigcap_{n=1}^{\infty} S_n = \varnothing$ if the pennies are all different.

    This was shown in the solution to Problem 22.

24. Prove that, if A is a non-empty subset of N (N the set of all natural numbers), then A has an element a which precedes ($<$) every other element of A in the natural order (4.1, footnote) of N.

    Surely, the statement holds when A has one element. Now let A have $k+1$ elements, and assume the statement holds for all cardinalities less than or equal to k. Then take some element b which is not the minimal element, so there exists $\{c_1, \ldots, c_j\} \subset A$ such that $c_i < b$ for all $1 \leq i \leq j$. Since $j \leq k$, there must be an element c_i less than all the others. But c_i is also less than b, and since b is less than the remaining elements, c_i precedes every other element in A. Hence, by the principle of strong induction proved in Problem 26, the statement holds.

25. Suppose we wish to define some concept $D(n)$ for every natural number n. We first define $D(1)$ and $D(2)$. then we give a general definition of $D(n+2)$ in terms of $D(n)$ and $D(n+1)$.

    This is a direct application of the principle of strong induction, proved in Problem 26, on the statement "$D(n)$ is defined."

26. Suppose that $T(n)$ is a theorem about the natural number n such that we can prove $T(1)$, and that if we had proved $T(1), T(2), \cdots, T(n)$, then a proof for $T(n+1)$ can be given. Show that $T(n)$ is thus proved for every natural number n.

    Let $T^*(n)$ be the theorem that $T(k)$ holds for $1 \leq k \leq n$. Then, $T^*(1)$ holds, and if $T^*(n)$ holds then $T^*(n+1)$ holds. Then $T^*(n)$ holds for all n, which in particular implies that $T(n)$ holds for all n.

27. Consider the accompanying figure. It consists of:

    (1) Two parallel line segments L_1 and L_2 of unit length one unit distance apart. On L_1, $p_1, p_2, \cdots, p_n, \cdots$, are points such that p_n is at a distance $1/(n+1)$ from the base of L_1; and, on L_2, $q_1, q_2, \cdots, q_n, \cdots$ are points such that q_n is at a distance $1/(n+1)$ from the top of L_2.

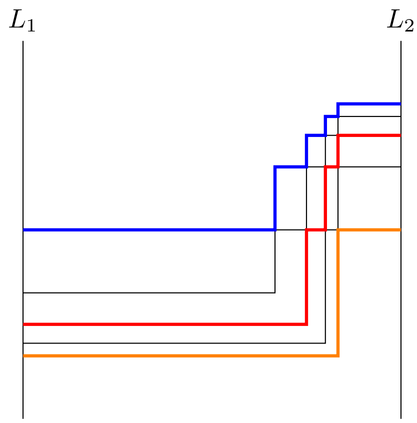


Figure 1:

(2) The line segment p_1q_1, on which $r_1, r_2, \cdots, r_n, \cdots$ are points such that r_n is at a distance $1/(n+1)$ from q_1.

(3) A broken line $p_2s_1r_1t_1q_2$ as shown.

(4) In general, a broken line $p_{n+1}s_nr_nt_nq_{n+1}$ as shown. Let a *path* from L_1 to L_2 consist of *any* broken line – $p_2s_1r_1r_6t_6q_7$ for instance – made up of segments of the given broken lines, but with only its endpoints, p_2 and q_7, in the instance cited, on L_1 and L_2 respectively. Let two paths be called *disjoint* if they have no point in common.

Show that, for every natural number n, there exist n disjoint paths from L_1 to L_2. Show also that there does not exist an *infinite* number of disjoint paths from L_1 to L_2 (even though there are infinitely many different but intersecting paths). What general "moral" can be drawn from this example in regards to existance theorems provable for every natural number n?

In order to draw n paths, we draw the first $2n-1$ broken lines and then traverse them starting at p_{2i-1} by turning upwards whenever possible, as shown by the colored paths in Figure 1.