Fourier Series – Chapter 2 Solutions

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1. Give another proof of the Schwarz inequality by considering the inequality

$$\int_a^b \int_a^b [f(x)g(y) - f(y)g(x)]^2 \, dx \, dy \geq 0.$$

Expanding the binomial in the integrand, the inequality becomes

$$\int_a^b f(x)^2 \, dx \int_a^b g(y)^2 \, dy - 2 \int_a^b f(x)g(x) \, dx \int_a^b f(y)g(y) \, dy + \int_a^b g(x)^2 \, dx \int_a^b f(y)^2 \, dy \geq 0.$$

Since all integrals are now over a single variable, we can relabel y as x and find

$$\int_a^b f(x)^2 \, dx \int_a^b g(x)^2 \, dx \geq \left(\int_a^b f(x)g(x) \, dx \right)^2.$$

This is the Schwarz inequality.

2. Prove the inequality

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2,$$

where a_i and b_i are arbitrary real numbers. This result, known as the *Cauchy inequality*, is the discrete analog of the Schwarz inequality.

For any t, we have $\sum_{i=1}^n (a_i + tb_i)^2 \geq 0$. Expanding the product, this means

$$\sum_{i=1}^n a_i^2 + 2t \sum_{i=1}^n a_i b_i + t^2 \sum_{i=1}^n b_i^2 \geq 0.$$

The left hand side is a quadratic function of t. If the inequality is satisfied for all t, the quadratic polynomial must have a negative or zero discriminant. Thus,

$$4 \left(\sum_{i=1}^n a_i b_i \right)^2 - 4 \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq 0,$$

which immediately gives the desired inequality.

3. Let the polynomial $P(x) = a_0 + a_1 x + \ldots + a_n x^n$ have coefficients satisfying the relation

$$\sum_{i=0}^n a_i^2 = 1.$$

Prove that

$$\int_0^1 |P(x)|\,dx \le \frac{\pi}{2}.$$

Show that this inequality continues to hold if $\pi/2$ is replaced by $\pi/\sqrt{6}$.

Using the triangle inequality, we have

$$\int_0^1 |P(x)|\,dx \le \int_0^1 (|a_0| + |a_1|x + \ldots + |a_n|x^n)\,dx = |a_0| + \frac{|a_1|}{2} + \ldots + \frac{|a_n|}{n+1}.$$

Using the Cauchy inequality, we then have

$$\left(\int_0^1 |P(x)|\,dx\right)^2 \le \sum_{i=0}^{n} a_i^2 \sum_{j=1}^{n+1} \frac{1}{j^2}.$$

We are given that $\sum_{i=0}^n a_i^2 = 1$, and $\sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6}$. Therefore,

$$\int_0^1 |P(x)|\,dx \le \frac{\pi}{\sqrt{6}} < \frac{\pi}{2}.$$

4. Show that if $n_1, n_2, \ldots, n_k$ are distinct nonzero integers, then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|\,dx \le \sqrt{k+1}.$$

By the Schwarz inequality with $g(x) = 1$,

$$\left(\int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|\,dx\right)^2 \le 2\pi \int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|^2\,dx.$$

Squaring the integrand, we obtain

$$\int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|^2\,dx = \int_0^{2\pi} \left(1 + e^{in_1x} + \ldots + e^{in_kx}\right)\left(1 + e^{-in_1x} + \ldots + e^{-in_kx}\right)\,dx.$$

Since all the terms are orthogonal functions, the cross terms vanish in the integral, and we are left with

$$\int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|^2\,dx = \int_0^{2\pi} (k+1)\,dx = 2\pi(k+1).$$

Therefore,

$$\left(\int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|\,dx\right)^2 \le 4\pi^2(k+1),$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} \left|1 + e^{in_1x} + \ldots + e^{in_kx}\right|\,dx \le \sqrt{k+1}.$$

5. Let f and g be square integrable functions. Prove that

$$\|f+g\| \le \|f\| + \|g\|.$$

This result is often called the *triangle inequality*, since it is the generalization of the familiar geometrical fact that the length of any side of a triangle is $\le$ the sum of the lengths of the other two sides.

We have

$$\begin{aligned}
\|f+g\| &= \left(\int (f+g)^2\,dx\right)^{1/2} = \left(\int f^2\,dx + \int g^2\,dx + 2\int fg\,dx\right)^{1/2} \\
&< \left(\int f^2\,dx + \int g^2\,dx + 2\left(\int f^2\,dx \int g^2\,dx\right)^{1/2}\right)^{1/2} \\
&= \left(\left(\left(\int f^2\,dx\right)^{1/2} + \left(\int g^2\,dx\right)^{1/2}\right)^2\right)^{1/2} \\
&= \left(\int f^2\,dx\right)^{1/2} + \left(\int g^2\,dx\right)^{1/2} \\
&= \|f\| + \|g\|\,.
\end{aligned}$$

6. Give an example of a sequence of functions which converges to 0 at each point of the interval $[0,1]$, but which does not converge in the mean.

   Let

$$f_n(x) = \begin{cases} 2^n & \text{for } 1-2^{1-n} < x < 1-2^{-n} \\ 0 \text{otherwise} & \end{cases}.$$

   Then for each x, the sequence $f_n(x)$ has the form $0,0,\ldots,0,2^n,0,\ldots$, so the functions converge to 0 at each point. However,

$$\left(\int_0^1 |f_n(x)|^2\,dx\right)^{1/2} = 2^{n/2}.$$

   Therefore the sequence does not converge in the mean.

7. A system of functions $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$ which is not necessarily orthogonal is said to be *complete* if every square integrable function can be approximated in the mean by a linear combination of the $\varphi_i(x)$, i.e., if given a square integrable function $g(x)$ and any $\epsilon > 0$, there exist numbers $a_0, a_1, \ldots, a_n$ such that

$$\int_a^b \left[g(x) - (a_0\varphi_0(x) + a_1\varphi_1(x) + \ldots + a_n\varphi_n(x))\right]^2\,dx < \epsilon.$$

   Show that if the system $\{\varphi_i(x)\}$ is complete, then any continuous function which is orthogonal to all the functions of the system must be zero.

   If $g(x)$ is orthogonal to all the $\varphi_i(x)$, then expanding the given inequality gives

$$\int_a^b g(x)^2\,dx + \int_a^b (a_0\varphi_0(x) + a_1\varphi_1(x) + \ldots + a_n\varphi_n(x))^2\,dx < \epsilon.$$

   If $g(x)$ is continuous and $g(x) \neq 0$, then let c be such that $g(c) \neq 0$. Then $g(c)^2 > 0$, and since g is continuous there exists δ such that for $|x-c| < \delta$, $g(x)^2 \geq g(c)^2/2$. Thus, $\int_a^b g(x)^2\,dx \geq g(c)^2\delta$. The second term on the left is positive, so picking $\epsilon = g(c)^2\delta/2$ gives a contradiction. Thus, $g(x)$ must be zero.

8. Let $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$ be a complete orthonormal system of functions. For which of the following systems is there no nonzero continuous function orthogonal to every function in the system:

   a) $\varphi_0+\varphi_1, \varphi_0+\varphi_2, \varphi_0+\varphi_3, \ldots$;

b) $\varphi_0+\varphi_1, \varphi_1+\varphi_2, \varphi_2+\varphi_3, \ldots$;

c) $\varphi_0+2\varphi_1, \varphi_1+2\varphi_2, \varphi_2+2\varphi_3, \ldots$?

In Part (c) we assume that the functions φ_n are continuous and uniformly bounded, i.e., $|\varphi_n(x)| \leq M$ for $a \leq x \leq b$.

Let $f(x)$ be some continuous function. Then, since the φ_k form a complete system, we have

$$f(x)=\sum_{k=0}^{\infty} a_k \varphi_k(x).$$

Then, taking the norm of both sides, we have

$$\sum_{k=0}^{\infty} |a_k|^2 = \|f\| < \infty.$$

Now, assume f is orthogonal to each of $\varphi_0+\varphi_1, \varphi_0+\varphi_2, \ldots$. This implies $a_0=-a_1$, $a_0=-a_2$, etc. If additionally $a_0 \neq 0$, then clearly $\sum |a_k|^2$ cannot be finite. Therefore, $a_k=0$ for all k, and $f(x)=0$.

In part (b), a similar argument implies that the coefficients of $f(x)$ satisfy $a_0=-a_1$, $a_1=-a_2$, etc. Again, if $a_0 \neq 0$ then the sum cannot be bounded, so $a_0=0$ and $f(x)=0$.

In part (c), we have $a_{k+1}=-\frac{1}{2}a_k$. Letting $a_0=1$, we have

$$f(x)=\varphi_0(x)-\frac{1}{2}\varphi_2(x)+\frac{1}{4}\varphi_2(x)-\ldots.$$

Since we assume the φ_k are uniformly bounded by M in this case, the kth term in the series is bounded in absolute value by $2^{-k}M$, so the sum converges pointwise. Additionally, we can show $f(x)$ is continuous. Let $\epsilon>0$, and for each k, let δ_k be such that $|\varphi_k(x)-\varphi_k(x_0)| \leq \epsilon/4$ whenever $|x-x_0| \leq \delta_k$. Then let $\delta=\min\{\delta_k \mid 0 \leq k \leq -\log_2(\epsilon/4)\}$. If $|x-x_0| \leq \delta$, then

$$\begin{aligned} |f(x)-f(x_0)| &\leq \left(1+\frac{1}{2}+\ldots+\frac{1}{2^{\lfloor -\log_2(\epsilon/4)\rfloor}}\right)\frac{\epsilon}{4}+\frac{1}{2^{\lfloor -\log_2(\epsilon/4)\rfloor}}\left(1+\frac{1}{2}+\ldots\right) \\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2} \leq \epsilon. \end{aligned}$$

Therefore, $f(x)$ is a continuous nonzero function orthogonal to each of the functions in the system of part (c).

9. A system of functions $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$ is said to be *linearly independent* if given any n, there is no set of numbers $a_0, a_1, \ldots, a_n$ which are not all zero such that the linear combination $a_0\varphi_0(x)+a_1\varphi_1(x)+\ldots+a_n\varphi_n(x)$ is identically zero. Show that

a) An orthogonal system of functions is linearly independent;

b) The functions $1, x, x^2, x^3, \ldots$ are linearly independent.

Let $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$ be orthogonal. Let $a_0, a_1, \ldots, a_n$ be such that

$$a_0\varphi_0(x)+a_1\varphi_1(x)+\ldots+a_n\varphi_n(x)=0.$$

Taking the inner product with φ_0, we find $a_0=0$. Similarly, for any k, taking the inner product with φ_k shows that $a_k=0$. Therefore all the coefficients are zero, so there is no set of coefficients not all zero which make the linear combination vanish.

Similarly, assume $a_0+a_1x+\ldots+a_nx^n=0$. Then the nth derivative must vanish, which means $a_n=0$. The $(n-1)$th derivative must also vanish, giving $na_n+a_{n-1}=0$, and since $a_n=0$ we must also have $a_{n-1}=0$. Proceeding in this way, we find that all the coefficients are zero. Therefore the functions are linearly independent.

10. Given a linearly independent system of functions $f_0, f_1, \dots, f_n, \dots$ defined on the interval $[a, b]$, we define a new system $g_0, g_1, \dots, g_n, \dots$ as follows:

$$\begin{aligned} g_0 &= f_0, \\ g_1 &= g_1 - \frac{(f_1, g_0)}{\|g_0\|^2} g_0, \\ g_2 &= f_2 - \frac{(f_1, g_1)}{\|g_1\|^2} g_1 - \frac{(f_2, g_0)}{\|g_0\|^2} g_0, \quad \text{etc.} \end{aligned}$$

This is the so-called *Gram-Schmidt orthogonalization process.* Interpret the process geometrically, and show that the new system $g_0, g_1, \dots, g_n, \dots$ is orthogonal and that $\|g_n\|^2 \neq 0$. Apply the process to the functions

$$1, x, x^2, x^3, \dots \quad (-1 \leq x \leq 1),$$

thereby generating the Legendre polynomials (except for numerical factors). Show that a nonzero function is orthogonal to all the f_i if and only if it is orthogonal to all the g_i, and show that the system $\{f_i\}$ is complete if and only if the system $\{g_i\}$ is complete.

Geometrically, g_k is formed by projecting f_k onto the orthogonal complement of the span of $\{g_0, \dots, g_{k-1}\}$. Therefore g_k is orthogonal to each of $g_0, \dots, g_{k-1}$ by construction, so the system is orthogonal. If $\|g_n\|^2 = 0$ then $g_n = 0$, and from the definition we see this implies that f_n can be written as a linear combination of $\{f_0, \dots, f_{n-1}\}$, a contradiction of the linear independence. Therefore $\|g_n\|^2 \neq 0$. The first few Legendre polynomials are, to within constant factors,

$$\begin{aligned} g_0 &= 1, \\ g_1 &= x - \frac{1}{2}\int_{-1}^{1} x \, dx = x, \\ g_2 &= x^2 - x\int_{-1}^{1} x^3 \, dx - \frac{1}{2}\int_{-1}^{1} x^2 \, dx = x^2 - \frac{1}{3}. \end{aligned}$$

The g_i are each linear combinations of the f_i, so if a function is orthogonal to all the f_i it is certainly orthogonal to all the g_i. If a function is orthogonal to all the g_i, then it is orthogonal to g_0, so it is orthogonal to f_0. Orthogonality to g_1 then implies orthogonality to f_1. Proceeding in this manner, we find it is orthogonal to all the f_i. This means the orthogonal complements of the two spaces of functions coincide, so the spaces coincide as well, and thus the f_i are complete if and only if the g_i are complete.

11. The *Legendre polynomials* are defined by the formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

Show that

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}.$$

Using the definition, we have

$$(P_n, P_m) = \int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{1}{2^{n+m} n! m!} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2 - 1)^n \frac{d^m}{dx^m}(x^2 - 1)^m \, dx.$$

Let $n < m$. Integrating once by parts, we find

$$2^{n+m} m! n! (P_n, P_m) = \left(\frac{d^n}{dx^n}(x^2 - 1)^n \frac{d^{m-1}}{dx^{m-1}}(x^2 - 1)^m \right)_{-1}^{1} - \int_{-1}^{1} \frac{d^{n+1}}{dx^{n+1}}(x^2 - 1)^n \frac{d^{m-1}}{dx^{m-1}}(x^2 - 1)^m \, dx.$$

Since $x = \pm 1$ are roots of $\frac{d^{m-k}}{dx^{m-k}}(x^2-1)^m$, the integrated part vanishes. We can integrate by parts n more times, and since $\frac{d^{2n+1}}{x^{2n+1}}(x^2-1)^n = 0$, we find $(P_n, P_m) = 0$.
If instead $n = m$, this argument fails. We instead arrive at

$$2^{2n}(n!)^2(P_n, P_n) = \int_{-1}^{1}(1-x^2)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n \, dx.$$

Only the highest order term in $(x^2-1)^n$ survives after $2n$ derivatives, so

$$(P_n, P_n) = 2^{-2n}\binom{2n}{n}\int_{-1}^{1}(1-x^2)^n \, dx.$$

Let $2u = 1 + x$. Then $(1-x^2)^n = 2^{2n}u^n(1-u)^n$, and $dx = 2\,du$. Therefore,

$$(P_n, P_n) = \frac{(2n)!}{(n!)^2}\int_0^1 u^n(1-u)^n(2\,du) = \frac{(2n)!}{(n!)^2}\frac{2\cdot(n!)^2}{(2n+1)!} = \frac{2}{2n+1}.$$

12. Expand the following functions in terms of Legendre polynomials.

a) $f(x) = \begin{cases} 0 & \text{for } -1 < x < 0; \\ 1 & \text{for } 0 < x < 1; \end{cases}$

b) $f(x) = |x|$.

The coefficient of $P_n(x)$ in the expansion of $f(x)$ is given by

$$a_n = \frac{(f, P_n)}{(P_n, P_n)} = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx.$$

The first few terms of the expansion for part (a) are thus

$$\begin{aligned} f(x) &= \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \dots \\ &= \frac{1}{2} + \frac{3}{4}x - \frac{7}{32}(5x^3 - 3x) + \dots . \end{aligned}$$

For part (b), we have

$$\begin{aligned} f(x) &= \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) + \dots \\ &= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) - \frac{3}{128}(35x^4 - 30x^2 + 3) + \dots . \end{aligned}$$