Foundations of Mathematics – Chapter 4 Solutions

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1. If $S = \varnothing$, why does 3.2.1.3 hold?

   Theorem 3.2.1.3 states that if S is a set and $\mathfrak{S}$ is the collection of all subsets of S, then S and $\mathfrak{S}$ do not have the same cardinal number. If $S = \varnothing$, then $\overline{\overline{S}} = 0$ while $\overline{\overline{\mathfrak{S}}} = 1$ (since $\mathfrak{S} = \{\varnothing\}$.

2. Show that the set of all finite subsets of N is countable.

   We can order the finite subsets of N by their largest element. There are 2^{n-1} subsets with largest element n. This ordering begins $\{1\}, \{2\}, \{1, 2\}, \ldots$.

3. Show that in the coordinate plane the set $K = \{(x, y) \,|\, (x \in F)\&(y \in F)\}$ is denumerable. Then use Theorem 3.1.6 to show that the set of all circles with raddi elements of F and centers in K is denumerable. Work the analogous problem for three-dimensional space.

   Let $f(x) : F \to N$ be a denumeration of the rationals. Define the index of $(x, y) \in K$ by $f(x) + f(y)$. There are a finite number of pairs (x, y) with a given index, so ordering the pairs by their indices shows that they are denumerable. By Theorem 3.1.6, since the set of circles is a denumerable collection of circles at given centers, and each of these individual sets is denumerable, the entire set of circles is denumerable. The extension to three-dimensional space is trivial; the extension to n-dimensional space follows by induction.

4. Use the results of Problem 3 to show that, if physical space is assumed to be euclidean, then the set of all physical objects (assuming some kind of a division of the material universe into things called "objects") is countable.

   Every object can be bounded by a sphere in three dimensions with rational coordinates and radius, and no two distinct objects need be bounded by the same sphere. Thus, the collection of objects can be put in (1-1)-correspondence with a subset of the set of circles, which we have shown to be denumerable, so it must be countable.

5. That 2.5.3 is stated as a corollary of Theorem 2.5.2 implies that the set I must be Dedekind infinite. Justify this by showing that I is Dedekind infinite; base your proof on III 5.3.2, thus avoiding use of the Choice Axiom.

   The numbers $r_{\omega+k}$ derived in the chapter form a subset of I in (1-1)-correspondence with N. Thus, by 5.3.2, I is Dedekind infinite.

6. To carry out the proof implied for Corollary 2.5.3, it is necessary to know that the set of transcendental numbers is Dedekind infinite. Prove this.

   Applying the diagonal procedure to the algebraic numbers produces a (1-1)-correspondence between a subset of the transcendentals and N, so the transcendentals are Dedekind infinite.

7. Show that if B is a Dedekind infinite set, and A a set such that $A \supset B$, then A is Dedekind infinite.

   Let $f(a) = a$ if $a \in A - B$ and $f(b) = g(b)$, where $g : B \to B'$ is the map that must exist for the Dedekind infinite set B. Then $f : A \to A'$ where $A' \subset A$, so A is Dedekind infinite.

8. Show that F may also be defined as the set of all elements of R whose decimal expansions ultimately begin to repeat (such as $\frac{1}{5} = 0.19999\cdots$, $\frac{1}{7} = 0.142857142857142857\cdots$).

   If a number has a repeating decimal expansion, then a simple geometric sum shows that it is rational. Conversely, given $x \in F$, let $x = \frac{a}{b}$ by the usual definition. Let $b = c \times 2^m \times 5^n$, with c being an integer and m and n being the largest possible integers. Let $10 \equiv k \pmod{c}$; then $10^\ell \equiv 1 \pmod{c}$ for some ℓ. Thus, $b \mid 10^{\max(m,n)}\left(10^\ell - 1\right)$. Hence, we can write

   $$\frac{a}{b} = \frac{A}{10^{\max(m,n)}\left(10^\ell - 1\right)} = \frac{X}{10^{\max(m,n)}} + \frac{Y}{10^\ell - 1},$$

   where the split into a sum is possible because $\gcd(10^{\max(m,n)}, 10^\ell - 1) = 1$. Clearly, the first fraction is a terminating decimal. Since $\frac{Y}{10^\ell - 1} = \frac{Y}{10^\ell} + \frac{Y}{10^{2\ell}} + \cdots$, it is a repeating decimal. Thus the sum eventually repeats, so x is a repeating decimal.

9. Show that if in the proof of 2.4 we made each block a single digit, there would not result a (1-1)-correspondence of the type desired.

   Every terminating rational number has two decimal expansions; for example, $0.5 = 0.499\cdots$. Thus, we have the obvious problem: $\left(\frac{1}{2}, 0\right)$ could be mapped to either $0.500\cdots$ or $0.40909\cdots$, depending on the representation chosen for $\frac{1}{2}$. We can alleviate this by arbitrarily picking the $0.499\cdots$ representation, but then no element of $[0,1]^2$ is mapped to, for example, $0.510101\cdots$ (since we would need $x = 0.500\cdots$, the disallowed representation).

10. Modify the proof of 2.4 to show that the subset $E^3 = \{(x,y,z) \,|\, (0 < x \leq 1)\&(0 < y \leq 1)\&(0 < z \leq 1)\}$ of coordinate 3-dimensional space has the cardinal number c. How about space of four dimensions, in which each point has *four* coordinates (x,y,z,w); five dimensions; etc.?

    In n dimensions, we simply interleave blocks in cycles of n. For example, for E^3, we map $(x,y,z) \to 0.a_1b_1c_1a_2b_2c_2\cdots$. This holds for any finite number of dimensions.

11. In a Hilbert space, each point is represented by an infinite sequence of coordinates $(x_1, x_2, \cdots, x_n, \cdots)$, where x_n is a real number. Show that the set $E^\omega = \{(x_1, x_2, \cdots, x_n, \cdots) \,|\, 0 < x_n \leq 1 \quad \text{for all } n\}$ has the cardinal number c.

    Let the ith block of x_k be $x_k^(i)$. Let the index of a block be $i + k$. Then ordering the blocks by index gives a bijection $E^\omega \to E$, in which a point in E^ω is taken to $0.x_1^(1)x_1^(2)x_2^(1)\cdots$.

12. Show that every infinite subset of N has the cardinal number $\aleph_0$ (cf. Problem 24 of Chapter III).

    According to Problem 24, every infinite subset of N has an element which precedes all the others. Let the infinite subset be A and let this element be $\min(A) = a$; then let $f(1) = a$. Similarly, let $f(2) = \min(A - \{a\})$, and define f for all values by induction. Then $f : N \to A$ is a bijection, so $\overline{\overline{A}} = \overline{\overline{N}} = \aleph_0$.

13. Show that the set of positive real numbers ahs the cardinal number c. As a corollary, the set of non-negative real numbers has the cardinal number c.

    The map $f(x) = e^x$ is a bijection from the real numbers to the positive numbers. The bijection $g(x) = \begin{cases} x - 1 & \text{if } x \in N \\ x & \text{if } x \notin N \end{cases}$ maps the positive numbers to the non-negative numbers. Thus, all these sets have the same cardinality c.

14. Show that the set of all real numbers between 0 and 1 has the cardinal number c. As a corollary, the subset $\overline{R}^1 = \{x \,|\, 0 \leq x \leq 1\}$ of R has the cardinal number c.

The map $f(x) = \frac{1}{\pi}\tan^{-1} x + \frac{1}{2}$ is a bijection from the real numbers to $(0,1)$, so this set has cardinality c. We also have $g(x) = \begin{cases} \frac{a-1}{b-1} & \text{if } x = \frac{a}{b} \\ x & \text{if } x \notin F \end{cases}$ which maps $(0,1) \to [0,1)$, and an analogous function which maps $[0,1) \to [0,1]$. Thus, $[0,1]$ also has cardinality c.

15. In the set $\overline{R}^1 = \{x \,|\, (x \in R)\&(0 \leq x \leq 1)\}$, let $x \equiv y$, where $x, y \in \overline{R}^1$, mean that $x - y \in F$. Show that $\equiv$ is an equivalence relation in $\overline{R}^1$. How many elements are there in each class of the class decomposition of $\overline{R}^1$ corresponding to $\equiv$? What would you guess is the cardinal number of the set of all classes in this class decomposition?

    The relation $\equiv$ satisfies reflexivity because 0 is rational, symmetry because z is rational if and only if $-z$ is rational, and transitivity because rationals are closed under addition. Each class of $\equiv$ has $\aleph_0$ elements. The set of classes cannot have cardinality $\aleph_0$, because this would imply countability of the reals, and it cannot be greater than c, because this would contradict $\overline{\overline{R}} = c$. Thus, a reasonable guess for the cardinal number is c.

16. Contrast (especially as regards effectiveness) the decomposition of $\overline{R}^1$ defined in Problem 15 with the following (due to Sierpinski): We define a function $f(x), x \in \overline{R}^1$ as follows: First express x as a "non-finite decimal" in the ternary scale, $x = (0.a_1 a_2 \cdots a_n \cdots)_3$, such that for no n are all $a_n, a_{n+1}, a_{n+2}, \cdots$ zeros unless $x = 0$. Then, if $x = 0$, or infinitely many of the digits a_n are 2's, let $f(x) = 0$. Otherwise, let n be the smallest (Problem 24 of Chapter III) natural number such that all digits $a_n, a_{n+1}, a_{n+2}, \cdots$ are *all* 0 or 1; then let $f(x) = (0.a_n a_{n+1} a_{n+2} \cdots)_2$. Finally, for every t such that $0 \leq t \leq 1$, let $R_t = \{x \,|\, f(x) = t\}$.

    Many answers are possible. It is easy to see that $f(x)$ maps surjectively onto $\overline{R}^1$, so the cardinality of classes is definitely c.

17. Show that if a set S has a proper subset S_1, such that there exists a (1-1)-correspondence between the elements of S and the elements of S_1, then S_1 is Dedekind infinite.

    Let $f : S \to S_1$ be the (1-1)-correspondence. If $f(S_1) \subset S_1$, we are done. If $f(S_1) = S_1$, then let $\alpha \in S$ and $\alpha \notin S_1$, and let $\beta, \gamma \in S_1$. For any element x such that $f(x) = \beta$, redefine $f(x) = \gamma$. Then let $f(\alpha) = \beta$. The edited f now surely satisfies $f(S_1) \subset S_1$, because $\beta \in S_1$ and $\beta \notin f(S_1)$.

18. Make the extension of the map in 1.2.1 to define a (1-1)-correspondence between N and F.

    We have a bijection $g : N \to F^+$, and a bijection $N \to Z$, where Z^* is the set of integers. This gives us a bijection from N to F, namely

$$h\left(n + \frac{a}{b}\right) = \frac{a}{b} + \begin{cases} 0 & \text{if } n = 0 \\ f(n) & \text{if} n > 0 \end{cases}.$$

19. Denote the ith prime by p_{i-1}; thus $p_0 = 2$, $p_2 = 3$, etc. Show that if we express each rational number (in its lowest terms) in the form $p_0^{a_0}.p_1^{a_1}.\cdots.p_n^{a_n}$ where p_n is the largest prime for which $a_n \neq 0$, then we can effectively define a (1-1)-correspondence between F and the set P of all polynomials in x with integral coefficients.

    The bijection is given by $f(p_0^{a_0}.p_1^{a_1}.\cdots.p_n^{a_n}) = a_n x^n + a_{n_1} x^{n-1} + \cdots + a_1 x + a_0$.

20. Convert the mapping defined in Problem 19 to a (1-1)-correspondence between N and the set P.

    We have a bijection $h : N \to F$ and a bijection $f : F \to P$, so $f(h(n))$ is a (1-1)-correspondence between N and P.

21. Show that the set of all rational powers of rational numbers is countable. Note that we can conclude that not all irrational numbers are rational powers of rationals (such as $\sqrt{2}$, for instance).

    Let $f : N \to F$ be a denumeration of the rationals. Then let the index of a rational power of a rational α^β be $f^{-1}(\alpha)+f^{-1}(\beta)$. Ordering the rational powers of rationals by their indices gives a denumeration.

22. Show that the class of all numbers x of the form a^b, where a is a rational number and b a rational power of a rational, is countable.

    Using the same argument as in the previous problem, except using the denumeration of the rational powers of rationals to define the part of the index corresponding to b, we obtain a denumeration of these numbers.

23. It is known that all numbers of the form a^b, where a is an algebraic number different from 0 or 1, and b is an irrational algebraic number, are transcendental. Show that not all transcendental numbers are of this form. (Give a constructive definition of one such.)

    Since algebraic numbers are countable, pairs (x, y) of algebraic numbers are countable. The set of numbers described obviously has the cardinality of a subset of this set, so it is countable as well. Since the transcendental numbers are uncountable, they cannot all be of this form. To find a transcendental not of this form, we can list all algebraic numbers and all a^b as defined (by interleaving), and then use the diagonal method.

24. Let f map N into F by the formula $f(n) = n$. Let g map F into N by the formula $g(p/q) = 2^p.3^q$ if $p \leq 0$ and $g(p/q) = 2^{|p|}.3^q.5$ if $p < 0$, where it is assumed that $q > 0$ and p and q are integers prime to one another. If h is the resulting (1-1)-mapping of N onto F worked out in the proof of the Bernstein equivalence theorem (4.2.6), find $h(24)$, $h(360)$, and $h^{-1}(33/49)$.

    In this case, the subset A_1' is the set of natural numbers that can be written as either $2^a \cdot 3^b$ or $2^a \cdot 3^b \cdot 5$. The subset A_1 is the set of natural numbers that can be written as either $2^a \cdot 3$ or $2^a \cdot 3 \cdot 5$. Thus, $360 \in A_1' - A_1$, so $h(360) = g^{-1}(360) = -\frac{3}{2}$. The forms of numbers in the next few subsets are:

    $$\begin{aligned} A_2' &: \quad 2^\alpha \cdot 3, \alpha \in A_1', \\ A_2 &: \quad 2^\alpha \cdot 3, \alpha \in A_1, \\ A_3' &: \quad 2^\beta \cdot 3, \beta \in A_2', \\ A_3 &: \quad 2^\beta \cdot 3, \beta \in A_2. \end{aligned}$$

    Clearly 3 is an element of all of these sets, and therefore so is 24. Thus, $h(24) = g^{-1}(24) = 3$. Finally, since $2^3 3 \cdot 3^4 9 \in A_1' - A_1$, we have $h(2^3 3 \cdot 3^4 9) = g^{-1}(2^3 3 \cdot 3^4 9) = 33/49$.

25. Apply the Bernstein equivalence theorem to prove: *If* $A \subset B$, *then* $\overline{\overline{A}} \leq \overline{\overline{B}}$. Conversely, show that this theorem has the Bernstein equivalence theorem as a consequence.

    Assume to the contrary that $\overline{\overline{A}} > \overline{\overline{B}}$. Then there is a bijection between B and some subset of A. But then, by the Bernstein equivalence theorem applied to this map and the identity map on A into B, we have $\overline{\overline{A}} = \overline{\overline{B}}$, a contradiction. Thus, $\overline{\overline{A}} \leq \overline{\overline{B}}$.

    Now, to prove the Bernstein theorem, take A and B with $f : A \to B' \subset B$ and $g : B \to A' \subset A$. We have $\overline{\overline{A}} = \overline{\overline{B'}}$ and, by the theorem stated in the problem, $\overline{\overline{B'}} \leq \overline{\overline{B}}$. So $\overline{\overline{A}} \leq \overline{\overline{B}}$, and likewise $\overline{\overline{B}} \leq \overline{\overline{A}}$. Therefore $\overline{\overline{A}} = \overline{\overline{B}}$, and thus a (1-1)-correspondence between the sets exists.

26. Prove that, if $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ are cardinal numbers such that $\alpha_n < \alpha_{n+1}$ for all n, then the cardinal number α of the set $\bigcup_{n=1}^\infty A_n$, where $\overline{\overline{A_n}} = \alpha_n$, satisfies the relation $\alpha_n < \alpha$ for all n.

    We have $A_n \subset \bigcup_{n=1}^\infty A_n$, so by the solution of the previous problem, $\alpha_n \leq \alpha$. If $\alpha_n = \alpha$, then there exists a (1-1)-correspondence between A_n and the union, and in particular a (1-1)-correspondence between $B \subset A_n$ and A_{n+1}. But this implies $\overline{\overline{B}} = \alpha_{n+1}$, and thus $\alpha_{n+1} \leq \alpha_n$, a contradiction. Thus $\alpha_n < \alpha$.

27. Show that, if E' is the set of all single-valued functions defined over $\overline{R}^1$ with values restricted to 0 and 1, then the cardinal number of E' is f.

    Clearly, an element of E' can be corresponded to a subset of $\overline{R}^1$. We know that $\overline{R}^1$ has cardinality c, and $2^c = f$, so $\overline{\overline{E'}} = f$.

28. Show that, if we assume there exists a "set of all sets" U and that α is its cardinal number, we can apply Theorem 4.2.3.1 to show that our assumption leads to a contradiction.

    If U is indeed the set of all sets, then U must contain as a subset all subsets of U. But this set has cardinality 2^α, and if it is contained in U then $2^\alpha \leq \alpha$, a contradiction of Theorem 4.2.3.1.

29. Suppose we assume that there exists a set, U, whose elements are all those sets having exactly one element (i.e., U is the set of all "singletons"). Then if S is a subset of U, the set $\{S\}$ having the single element S must be an element of U. Show that we can apply Theorem 4.2.3.1 to arrive at a contradiction.

    Let the cardinality of U be α. Then the set of subsets of U has cardinality 2^α. But U contains an element corresponding to each of these subsets by the construction given; thus $2^\alpha \leq \alpha$, a contradiction of Theorem 4.2.3.1.

30. Is the set of all sets each of which has exactly two elements self-contradictory?

    It is self-contradictory. We could use essentially the same construction as above, except using pairs of subsets; i.e. for $S_1 \neq S_2$ where $S_1 \subset U$ and $S_2 \subset U$, U must contain $\{S_1, S_2\}$. There exists an injective mapping of the set of subsets into the set of pairs of subsets (namely $S \to \{S, \varnothing\}$), so the cardinality of the set of pairs of subsets is at least 2^α. Again, we have $2^\alpha \leq \alpha$, a contradiction.

31. Compare the material in 3.1, especially 3.1.8.1, with the following ("Richard paradox"): If a specific English dictionary, D, is used to form sentences, some of them may designate natural numbers; e.g., "Let N be the number of moons of the earth" would designate the number 1. However, consider the sentence "Let N be the smallest natural number not definable in twenty words or less from the dictionary D."

    It would seem that this definition is inherently paradoxical. However, in analogy with the discussion in 3.1.8.1, we must be careful about the word "definable." The class of numbers definable in twenty words or less is not itself well-defined; there is no decision procedure for determining whether a sentence defines a number at all, or whether it defines a particular number.

32. Let r be any real number. In the sequence (F) of 1.2 (any other such ordering of the rational numbers, for instance, that of 1.2.1, would do as well, however), let f_1 be the first rational number in the sequence such that $f_1 < r$; let f_2 be the first rational number in the sequence such that $f_1 < f_2 < r$; and, in general, having defined f_n, let f_{n+1} be the first rational number in (F) such that $f_n < f_{n+1} < r$. Show that the limit of the sequence $f_1, f_2, \cdots, f_n, \cdots$ in the real numbers is r.

    The sequence $\{f_i\}$ is monotonic and bounded, so it must have a limit. Assume that the limit L is less than r. This would imply that there are only finitely many values for which $L < f_i < r$. But let $L < f_m < f_n < r$; then we can define infinitely many rationals in the interval by taking $g_k = f_m + \frac{f_n - f_m}{k}$ for $k > 1$. Thus, the limit must be $L = r$.

33. Is *the set of all cardinal numbers* a self-contradictory notion?

    Let C be such a set, and let C' be a set containing a representative set for each cardinal in C. Then by the result of Problem 26, $\cup_{S \in C'} S$ has a cardinal number exceeding all those in C, a contradiction. Thus, there is no set of all cardinal numbers.