Foundations of Mathematics – Chapter 2 Solutions

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1. Show that if Σ_1 and Σ_2 are axiom systems such that Σ_1 is consistent and Σ_2 has a model in Σ_1, then Σ_2 is consistent.

   If Σ_2 has a model in Σ_1, then all provable Σ_2-statements are also provable Σ_1-statements. There are no contradictions among Σ_1-statements, so Σ_2 must therefore be consistent.

2. If an axiom system that is known to be consistent has a model in a system Σ, need Σ be consistent? (Illustrate by means of an example.)

   A simple example suffices: Γ is consistent and has a model in $\Gamma + \text{A} + \sim \text{A}$, but $\Gamma + \text{A} + \sim \text{A}$ is inconsistent.

3. Show that an empty collection is a model of the axiom system O of Section 7. Does the existence of this model prove O consistent?

   Satisfiability only implies consistency when 2.2.2 (Contradictory I-Σ-statements cannot both hold true of $\mathfrak{R}(I)$) is valid. Since the empty collection only vacuously satisfies the axioms of O, this is not necessarily the case. For example, if from O it were possible to prove "If $x < y$, then $x = y$" (a clear contradiction of Axiom 2), both statements would still hold true of the empty collection, and we would not detect the inconsistency.

4. Show that axiom (1) of the system E of Section 8 is independent in E.

   We show independence by finding a model for $(E - E1) + \sim E1$. This is given by the collection $\{a\}$, with $a \not\approx a$.

5. Prove on the basis of Γ_6 that "line" may be defined in terms of "point." (Recall that since the axioms of Γ are included among those of Γ_6, the theorems proved in Chatper I from Γ are valid for Γ_6.

   Theorem 3 of Chapter I guarantees that there are at least four points, so there must be exactly four points. The result of Problem 13 implies that each of the lines contains two points. Axiom 3 states that every pair of points defines a line. Thus, every pair of points defines a line and every line is a pair of points, so we may define a line as a pair of points.

6. Show that the system O' is satisfiable.

   Let the two points be x and y, with $x < y$. This obviously satisfies all four axioms.

7. Is the system O' consistent by Definition 1.1?

   Since O' is satisfiable, and all of the axioms are more than vacuously satisfied, it is consistent.

8. Show that Axiom 3 is independent in the system O'.

   If we replace Axiom 3 by its negation, the same model as above satisfies all the axioms. Thus, Axiom 3 is independent.

9. Show that the axiom system obtained by deleting Axiom 3 from O' is not categorical. (Note, incidentally, that although Axiom 3 is only vacuously satisfied in every model of O', it does give information about the model.)

   In addition to the model above, we could use the model in which $x < y$ and $y < x$. This is not isomorphic to our first model, so the new system is not categorical.

10. Show that if in axiom system O' Axiom 3 is replaced by the axiom "Not both $x > y$ and $y < x$ hold," then the resulting axiom system is categorical.

    In the previous problem we enumerated the two models satisfying Axioms 1, 2, and 4. The new axiom given disallows one of these possibilities. Thus, only one model remains up to isomorphism, so the system is categorical.

11. Show that if the axiom system O of Section 7 is augmented by the axiom "There exist three and only three points," then the resulting system O_3 is categorical.

    The axioms of O imply that the three points can be uniquely ordered. The two models $x < y < z$ and $a < b < c$ are obviously isomorphic by mapping $x \to a$, $y \to b$, and $z \to c$. Thus, O_3 is categorical.

12. Let O_3' denote the system O_3 of Problem 11 augmented by some axiom of the form "If a, b, c, d are four distinct points, then ..." Is this axiom independent in O_3'?

    Since the condition can never be met in a model of O_3, neither the new axiom nor its negation lead to any inconsistencies. Therefore, this axiom is independent in O_3'.

13. If $\mathfrak{G}$ is a collection whose elements are themselves collections (for example, $\mathfrak{G}$ might be a collection of libraries, each library bein gitself a collection of books), and A, B are elements of $\mathfrak{G}$, let $A \leftrightarrow B$ mean that there exists a (1-1)-correspondence between the elements of A and the elements of B (cf. 4.4.2). Show that $\leftrightarrow$ is an equivalence relation in the collection $\mathfrak{G}$.

    Reflexivity follows trivially; the identity map is a (1-1)-correspondence between a collection and itself. Symmetry follows from bijectivity; if $A \leftrightarrow B$, then there exists a bijection $\phi : A \to B$, which implies the existence of $\phi^{-1} : B \to A$, which implies $B \leftrightarrow A$. For transitivity, we recognize that if $\phi : A \to B$ and $\psi : B \to C$ are bijections, then $\psi \circ \phi : A \to C$ is a bijection.

14. Show that if two simply ordered sets A and B have the same number, n, of elements, then they are isomorphic relative to the simple order axioms.

    We can order the elements of A as $a_1 < a_2 < \cdots < a_n$, and the elements of B as $b_1 < b_2 < \cdots < b_n$. Then the map defined by $\phi(a_i) = b_i$ forms an isomorphism relative to O.

15. Let Σ be an axiom system and suppose that, if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are any two models of Σ, there exists a (1-1)-correspondence between the elements of $\mathfrak{R}_1$ and the elements of $\mathfrak{R}_2$ which preserves those Σ-statements that form the axioms of Σ. Is Σ categorical? [*Hint:* Consider axioms system O' with Axiom 3 deleted.]

    No: the two models defined in the solution to Problem 9 are not isomorphic relative to Σ, but the (1-1)-correspondence $x \to x, y \to y$ does preserve the statements of the axioms.

16. If C is any collection and $\leq$ a binary relation between certain pairs of elements of C, then C is called *parially ordered with respect to* $\leq$ if the following axioms hold:

    (1) For every element of C, $x \leq x$.

    (2) If $x \leq y$ and $y \leq x$, then x and y denote the same element of C.

    (3) If $x \leq y$ and $y \leq z$, then $x \leq z$.

[Note that (1) and (3) are, respectively, the reflexiveness and transitivity of $\leq$. The property of the binary relation stated in (2) is called *anti-symmetry.*]

Denote this axiom system by P. Prove P consistent and non-categorical; establish independence of each of its axioms.

The numbers 1, 2, and 3 under the normal meaning of $\leq$ form a model, so P is satisfiable and therefore consistent. The numbers 1, 2, 3, and 4 form another model, which is clearly not isomorphic to the first, so P is not categorical.

To show the independence of Axiom 1, consider the model of 1, 2, and 3 except where $1 \not\leq 1$. This is consistent with the negation of Axiom 1, and with Axioms 2 and 3.

To show the independence of Axiom 2, consider the model of 1, 2, and 3 except where $a \leq b$ for all a and b. This is consistent with the negation of Axiom 2, and with Axioms 1 and 3.

To show the independence of Axiom 3, let the relations be the reflexive ones in addition to $1 \leq 2$, $2 \leq 3$, and $3 \leq 1$. This is consistent with the negation of Axiom 3, and with Axioms 1 and 2.

17. (a) If M is a collection of objects of varous ages, and the relation $\leq$ is interpreted to mean "is at least as old as," is M partially ordered with respect to this relation? (b) Show that every collection of sets is partially ordered with respect to the relation "is contained in" (compare III 3.1.2.)

    (a) Axiom 2 is not satisfied, because two distinct objects could have exactly the same age.

    (b) Clearly the relation is reflexive, anti-symmetric, and transitive, so every collection of sets is partially ordered under this relation.

18. Let F be an axiom system with undefined terms *figure* and *rectangular*, the former being an undefined object and the latter an undefined property of figures. The axioms of F are:

    (a) There exist at most twenty figures.

    (b) If x is a figure, then x is rectangular.

    (c) If x is a figure, then x is not rectangular.

    Show that F is satisfiable and categorical, and that its axiom (c) is independent. How would the system be affected if in (a) "at most" were replaced by "at least"?

    If a figure exists, then axioms (b) and (c) lead to a contradiction. Thus, only the empty set satisfies F, so F is satisfiable and clearly categorical.

    If "at most" were replaced with "at least," then there would necessarily be contradictions, so F would be inconsistent and not satisfiable.

19. Let Σ be an axiom system and T a theorem in the proof of which an axiom A of Σ is used. Show that if we wish to demonstrate that T cannot be proved without using axiom A, it suffices to give a model of $\Sigma - A$ in which T fails to hold.

    If T can be proved in Σ without relying on axiom A, then it can be proved from the axiom system $\Sigma -$ A. Thus, T will hold in every model of $\Sigma -$ A. The statement given is the contrapositive: if there exists a model of $\Sigma -$ A in which T fails to hold, then T cannot be proved in Σ without relying on axiom A.

20. Is the axiom system Γ'_6 of 4.4.5 categorical? Is it a monotransformable system? [A determinant D of order 3 yields a model of Γ'_6 if the triples used in evaluating D, as well as rows and columns, are called "lines."]

    We can show that any choice of lines is isomorphic to the ones given in the solution to Problem 1 of Chapter I. Let r_1 be an arbitrary line; there are three points on r_1 and six points not on it. Let P be one of the points not on r_1. Axiom 5 implies that there exists a line r_2 through P parallel to r_1. There are three points not on r_1 or r_2, and Axiom 5 implies that they form a line r_3. These sets of points will form the rows of our lattice. We may order the first and second rows arbitrarily; call the first point of r_1 (1,1), the third point of r_2 (2,3), etc. Then there exists a line c_1 through $(1, 1)$ and $(2, 1)$ which must contain a third point; let this point be the first of r_3. We similarly order the remaining points of r_3. We have now shown that there are three lines of type (a) and three lines of type (b), in the terminology used in Problem 1; it remains to show that the 6 remaining lines are all of type (c). Let ℓ be a line through $(1, a)$ and $(2, b)$. The third point must be on r_3 and on c_{6-a-b}, because otherwise a contradiction of Axiom 5 would result. Thus, the line is type (c).

It is clear from this proof that Γ_6' is not monotransformable. For example, swapping r_1 and r_2 would still yield a (1-1)-correspondence that forms an isomorphism.

21. The definition of independence in 3.1.1 is stated in terms of satisfiability. Suppose, however, that the word "satisfiable" is replaced by "consistent" in 3.1.1. Then show that, as a result, Definition 4.3.1 becomes equivalent to: "An axiom system Σ is complete if the addition of a Σ-statement, A, not implied by Σ, results in contradiction."

    When "satisfiable" is replaced by "consistent," the criterion for completeness of Σ becomes "there exists no Σ-statement A for which $\Sigma + \text{A}$ and $\Sigma + \sim\text{A}$ are both consistent." We can prove that each criterion implies the other. If the addition of any Σ-statement A which is not implied by Σ results in a contradiction, then it must be the case that for every Σ-statement A, Σ implies either A or $\sim$A. Thus, clearly one of $\Sigma + \text{A}$ and $\Sigma + \sim\text{A}$ is inconsistent. In the other direction, if there exists no Σ-statement A for which $\Sigma + \text{A}$ and $\Sigma + \sim\text{A}$ are both consistent, then again, it must be the case that for every Σ-statement A, Σ implies either A or $\sim$A. Therefore, if A is not implied by Σ, $\sim$ A is, so $\Sigma + \text{A}$ is inconsistent.

22. Show (under assumptions 2.2.1 and 2.2.2) that completeness of an axiom system Σ in the new sense given in Problem 21 in quotations implies that Σ is complete in the sense of Definition 4.3.1 (where "independent" is understood in terms of satisfiability as given in 3.1.1).

    It was proved in the chapter that, under assumptions 2.2.1 and 2.2.2, a satisfiable axiom system is consistent. Therefore, if there is no A for which $\Sigma + \text{A}$ and $\Sigma + \sim\text{A}$ are both consistent, then there must be no A for which the two systems are both satisfiable. This shows that the criterion in Problem 21 implies the criterion in Definition 4.3.1.

23. If Σ is an axiom system, define a Σ-question to be a question of the form "Does A hold?" where A is a Σ-statement. Then define Σ to be complete if it implies an answer to every Σ-question. Show that this definition is equivalent to that given in Problem 21 (in quotations).

    If Σ implies an answer to every Σ-question, then for every Σ-statement A, Σ implies either A or $\sim$A. This implies the criterion of Problem 21. In the other direction, if the addition of any Σ-statement A which is not implied by Σ results in a contradiction, then it must be the case that for every Σ-statement A, Σ implies either A or $\sim$A. This gives an answer to every Σ-question.

24. If Σ_1 and Σ_2 are axiom systems having the same set, T, of undefined terms, define $\Sigma_1 \approx \Sigma_2$ to mean that the collections $\mu(\Sigma_1)$ and $\mu(\Sigma_2)$ are the same. Is this relation $\approx$ an equivalence relation in the collection of all axiom systems based on T (i.e., having T as their sets of undefined terms)?

    The relation $\approx$ is an equivalence relation. It satisfies all three axioms trivially, because it is based upon the notion of sameness of collections, which is an equivalence relation.

25. If Σ_1 and Σ_2 are axiom systems based on T (as in Problem 24), define $\Sigma_1 \leq \Sigma_2$ to mean that $\mu(\Sigma_1)$ contains $\mu(\Sigma_2)$ as a subcollection. Is the collection of all axiom systems based on T partially ordered with respect to this relation $\leq$? (See Problem 16).

    Since $\leq$ is based on a subset relation, we can trivially show reflexivity and transitivity. To prove antisymmetry, we recognize that $\mu(\Sigma_1) \subseteq \mu(\Sigma_2)$ and $\mu(\Sigma_2) \subseteq \mu(\Sigma_1)$ implies $\mu(\Sigma_1) = \mu(\Sigma_2)$. Thus, $\Sigma_1 \approx \Sigma_2$, in the sense of Problem 24, so $\leq$ partially orders axiom systems over T.

26. Let Σ_1 and Σ_2 be axiom systems as in Problem 25. Define $\Sigma_1 \Rightarrow \Sigma_2$ to mean that the axioms of Σ_2 are provable (as theorems) in the axiom system Σ_1 (we may read "$\Sigma_1 \Rightarrow \Sigma_2$" as "$\Sigma_1$ implies Σ_2"). Show that $\Sigma_1 \Rightarrow \Sigma_2$ implies that $\Sigma_2 \leq \Sigma_1$ (the latter being defined as in Problem 25).

    If the axioms of Σ_2 are provable from Σ_1, then all models of Σ_1 also satisfy Σ_2. Thus, $\mu(\Sigma_1)$ is contained in $\mu(\Sigma_2)$, so $\Sigma_2 \leq \Sigma_1$.

27. With Σ_1, Σ_2, etc., as in Problem 26, define $\Sigma_1 \Leftrightarrow \Sigma_2$ to mean that both $\Sigma_1 \Rightarrow \Sigma_2$ and $\Sigma_2 \Rightarrow \Sigma_1$. Show that $\Leftrightarrow$ is an equivalence relation in the collection of all axiom systems based on T.

    Since $\leq$ is a partial order, $\Rightarrow$ is as well. The second axiom of partially ordered sets implies that $\Leftrightarrow$ is an equivalence relation.

28. Let P' denote the axiom system obtained when axiom (1) of the system P (Problem 16) is replaced by the axiom "If x and y are elements of C, then $x \leq y$ or $y \leq x$." If, in P', "$\leq$" is replaced by "$<$ or identical with," what relation is there between the resulting axiom system and the axiom system for simple order (Section 7)?

    The axioms of P' are:

    (1) If x and y are elements of C, then either $x < y$, $y < x$, or x is the same as y.

    (2) If either $x < y$ and $y < x$ or x is the same as y, then x and y denote the same element of C.

    (3) If either $x < y$ or x is the same as y, and either $y < z$ and y is the same as z, then either $x < z$ or x is the same as z.

    It is easy to show that each of these axioms is equivalent to the corresponding axiom of O. Thus, $P' \Leftrightarrow O$.

29. Problem 28 suggests generalizations if the "$\Rightarrow$" and "$\Leftrightarrow$" of Problems 26 and 27, such as the following: If Σ_1 and Σ_2 are axiom systems based respectively on collections T_1 and T_2 of undefined technical terms, let $\Sigma_1 \Rightarrow \Sigma_2$ mean that the elements of T_2 may be so defined in terms of the elements of T_1 as to make the axioms of Σ_2 provable theorems in the axiom system Σ_1. Discuss.

    In this interpretation, the result of Problem 28 tells us that $O \Rightarrow P$. This is a sensible generalization; using familiar models for ordered sets and partially ordered sets, we would think of the ordered sets as a subclass of the partially ordered sets. We can formalize this intuition by allowing the redefining of $\leq$ in terms of $<$, as is done in this definition.

30. Let us alter axiom system P of Problem 16 as in Problem 28 to obtain system P'. Show that if we define $x < y$ to mean "x is not identical with y and $x \leq y$," then $P' \Rightarrow O$ in the sense of Problem 29. (Does $O \Rightarrow P'$ hold?)

    To show that $P' \to O$, we must prove the axioms of O (with $<$ redefined) using the axioms of P'. When written in terms of $\leq$, the axioms of P' are

    (1) If x and y are elements of C, then either $x \leq y$ or $y \leq x$.

    (2) If $x \leq y$ and $y \leq x$, then x and y denote the same element of C.

    (3) If $x \leq y$ and $y \leq z$, then $x \leq z$.

    In terms of $\leq$, axiom (1) of O says "If x and y are distinct points of C, then either $x \leq y$ or $y \leq x$, and additionally, x and y are distinct." This is obviously a consequence of axiom (1) of P'. Axiom (2) of O', "If $x < y$, then x and y are distinct," follows trivially from the redefinition. Axiom (3), written in terms of $\leq$, says "If $x \leq y$ and $y \leq z$, and x is distinct from y, and y is distinct from z, then $x \leq z$ and x is distinct from z." The first implication is an obvious consequence of axiom (3) of P'. To prove in addition that x and z are distinct, assume that they are the same element. Then $x \leq y$ and $y \leq x$, and from axiom (2) of P', we have that x is the same as y, a contradiction.

Problems 31-35 will be based on the following axiom system:

Let S be a collection, of which certain subcollections are designated as *m-classes*. By definition, two such m-classes are called *conjugate* if they have no element in common. The following axioms are to hold (we denote elements of S by small letters x, y, z, $\cdots$):

(a) If x and y are distinct, there is one and only one m-class containing x and y.
(b) For every m-class, there is one and only one conjugate m-class.
(c) There exists at least one m-class.
(d) No m-class is empty.
(e) Each m-class contains only a finite number of elements of S.

31. Show that every m-class has at least two elements.

First, we show that S contains at least three elements. By axioms (3) and (4), there exists at least one m-class with at least one element. By axiom (2), this m-class has a conjugate, which must also be non-empty. Thus, S contains at least two elements. By axiom (1), there is an m-class containing both of these elements. By axiom (2), this m-class has a conjugate, which must have a third element of S.

Let α denote an m-class with exactly one element, x. By our lemma, there exist at least two other elements of S, y and z. By axiom (1), there is an m-class xy and an m-class xz. By axiom (2), each of these m-classes have conjugates. Both of those conjugates are conjugate to α, in contradiction with axiom (2). Thus, there must be no such m-class α. This in addition with axiom (4) implies that every m-class has at least two elements.

32. Show that S contains at least four elements.

We have already shown that S contains at least two elements. By axiom (1), there exists an m-class containing these two elements. By Problem 31, the conjugate to this m-class contains at least two elements. Thus, S contains at least four elements.

33. Show that S contains at least six m-classes.

Let two elements of S be w, x, y, and z. By axiom (1), there is an m-class α containing w and x. By axiom (2), there exists an m-class β conjugate to α. Let α_1 and β_1 be elements of α and β, respectively. Then by axiom (1) there exists an m-class γ containing α_1 and β_1, which can not be either α or β by the definition of conjugate. The conjugate of γ, which we call δ, also cannot be identical to either α or β. Since α and β are conjugate, δ must share at least one element with each; let these be α_2 and β_2, respectively. By axiom (1), α_1 and β_2 both belong to an m-class ϵ, and this is not identical to any of the m-classes we have defined previously. Likewise, α_2 and β_1 define a distinct m-class ζ. Thus, we have at least six m-classes.

34. Show that no m-class has more than two elements.

Let α be an m-class of at least three elements. By axiom (2) it has a conjugate, β, which by Problem 31 has at least two elements. Consider the two m-classes containing β_1 and α_1, and β_1 and α_2, respectively. These must be distinct, because otherwise there would be two m-classes containing α_1 and α_2. For a similar reason, they must both be disjoint with the m-class containing β_2 and α_3. This is a contradiction with axiom (2), so there must not be any m-class with at least three elements.

35. Prove that the given axiom system is categorical.

Because of the results of Problems 31 and 34, along with axiom (1), the m-classes are all the pairs of distinct elements of S. If S has more than four elements, then every pair will have more than two conjugates, a contradiction of axiom (2). Thus, S has exactly four elements, and all pairs are m-classes. Clearly, given any two models, any (1-1)-correspondence constitutes an isomorphism, so the system is categorical.

36. Show that the axiom system on which Problems 31-35 are based is equivalent (as in Problem 27) to the system Γ_6 of Section 4.3, if elements of S are called *points*, and *line* is substituted for m-class.

We have already shown that Γ_6 is categorical, admitting only the model with four points in which every pair of points is a line. If we call elements of S points and m-classes lines, then this is the same as the sole model admitted by the five axioms given. Thus, calling the new axiom system Σ, we have both $\Gamma_6 \leq \Sigma$ and $\Sigma \leq \Gamma_6$. By Problem 26, this is equivalent to $\Sigma \Leftrightarrow \Gamma_6$.