Foundations of Mathematics – Chapter 1 Solutions

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1. Obtain other meanings for "point" and "line" which make the axioms of Γ true statements. Also find a collection of *nine* objects (instead of four, as in Section 2.5) "satisfying" these axioms.

   Let a *point* be a vertex of a regular tetrahedron, and let a *line* be an edge of the tetrahedron containing each of its endpoints. This is isomorphic to the set of four points a, b, c, and d given Section 2.5, so it satisfies the axioms of Γ.

   We can also satisfy the axioms with nine points. Let a point be an ordered pair of integers (a, b), with $1 \leq a, b \leq 3$. Let a line be one of the following three types of collections:

   (a) {(a,1), (a,2), (a,3)} with $1 \leq a \leq 3$ (3 lines)
   (b) {(1,a), (2,a), (3,a)} with $1 \leq a \leq 3$ (3 lines)
   (c) {(1,a), (2,b), (3,c)} with $a \neq b \neq c$ (6 lines)

   Axioms 1, 2, and 4 are satisfied by definition. To prove Axiom 3, consider two distinct points (a, b) and (c, d). If $a = c$, then we have a unique line of type (a). If $b = d$, we have a unique line of type (b). If $a \neq c$ and $b \neq d$, then we must have a line of type (c), which implies that the third point on the line is (e, f) where $e = 6 - a - c$ and $f = 6 - b - d$. Thus, in each case, we have a unique line containing the two points.

   To prove Axiom 5, we note that lines of type (a) can only be parallel to lines of type (a), and so on. If we have a line of type (a) or (b), then the statement follows readily. Given a line ℓ of type (a) and a point p not on ℓ, there is a unique line of type (a) through p, and this line is parallel to ℓ. The same statement holds replacing (a) with (b). The same argument does not follow directly for lines of type (c), since through every point there are two lines of type (c). In this case, let ℓ be the collection $\{(1, a), (2, b), (3, c)\}$. Without loss of generality, let p be the point $(1, d)$ with $a \neq d$. We must have either $b = d$ or $c = d$; let $b = d$ without loss of generality. Then, to form a line of type (c) through p parallel to ℓ, we must include $(3, a)$: including $(3, b)$ would not give a line of type (c), and including $(3, c)$ is disallowed by parallelism. By Axiom 3, we have a unique parallel line.

2. Why is not the corollary to Theorem 2 a direct consequence of Axiom 3 of the system Γ?

   Axiom 3 is a statement about points, guaranteeing that there can be only one line through two points. However, it does not exclude the possibility of lines containing only one point, or no points at all. Theorem 2 disallows these cases, which then allows us to apply Axiom 3 to two distinct points on a line to prove the corollary.

3. Do the lines M and M_1 defined in the proof of Theorem 4 in Section 3.8 necessarily have a point in common?

   M and M_1 can be parallel. We show this by using the example given in Section 2.5. Following the proof, let $L = ab$ and $L_1 = cd$. Then $K = ac$, $K_1 = bd$, $M = ad$, and $M_1 = bc$.

4. Is Axiom 5 *necessary* for the validity of the corollary to Theorem 1 in the system Γ? [*Hint:* Consider a collection containing three points p, q, and r, and let each pair of these be a line; also let the empty collection be a line.]

The points and lines given in the hints satisfy Axioms 1-4, but not Axiom 5. We also see that the corollary to Theorem 1 does not hold, since the empty collection is a line. Thus, Axiom 5 is necessary for the validity of this corollary.

5. May a term appear in one axiom system as an "undefined technical term" and in another axiom system as a "universal logical term"?

   Certainly we could write an axiom system which repurposes some of the universal logical terms as technical terms, but this would be artificial. It would not occur in practice because 1) we would still need to use the term in its logical sense, so the meanings would be confounded; and 2) grammatically, universal logical terms are adjectives whereas undefined technical terms are usually nouns.

6. What fundamental differences distinguish the Greek conception of axioms and the modern conception?

   The Greeks had two groups of fundamental statements: axioms, which were pure logical statements, and postulates, which were statements about the subject matter at hand (geometry, in the case of the Greeks). The postulates were considered to derive from physical necessities, i.e., the properties of actual space. Today, we make no distinction between axioms and postulates; the laws of logic are assumed, and axioms define a mathematical system. Moreover, we distinguish between mathematics itself and applications of mathematics; the axioms are not meant to directly imply anything about the physical world.

7. Would it be feasible to use the axiomatic method in order to describe ethical, political, or other social systems?

   Generally, no. Such systems do not have the precision or the logical simplicity that lends mathematics to the axiomatic method. While it may be interesting to attempt to formulate axioms for such a system, there would probably be a plethora of undefined terms, axioms that oversimplify the real systems, and provable theorems that conflict with results obtained by more human methods.

8. Would we necessarily become involved in contradictions in an axiom system if, in formulating an axiom as a statment which "seems to hold" for some concept, the statement is actually a false statement about the concept?

   A contradiction would not necessarily result. For example, in formulating axioms for groups, we might mistakenly think that the group operation must be commutative. While this is not the case, adding such an axiom would give a consistent axiom system for abelian groups.

9. In what respect to definitions such as those given in Definitions 2.3 and 2.4 differ from those given in Sections 4.1 and 4.2? [*Hint:* Note that 2.3 and 2.4 are especially designed for Γ.]

   Definitions 2.3 and 2.4 are statements made in a particular axiom system, defining new terms for that system. The definitions in Section 4 concern the study of axiom systems themselves, and allow us to make statements about axiom systems in general.

   In Problems 10-14, theorems are stated which are to be proved as theorems of the axiom system Γ. It is assumed that the number of points is finite (although in Problems 10 and 11 this is an unnecessary assumption if familiarity with the notion of cardinal number is assumed; see Chapter IV).

10. The number of points on a line is constant; that is, any two lines have the same number of points.

    Let ℓ be a line with r points and P be a point not on the line. Then every line through P contains either none of the points of ℓ or exactly one. By Axiom 5, there is one line through P of the first type; by Axiom 3, there are r lines through P of the second type. Now consider a line ℓ' distinct from ℓ. Again, all the lines through P contain either no points of ℓ' or exactly one point of ℓ'. Again there is exactly one line of the first type, which leaves r lines, each of which intersect a unique point of ℓ'. Thus, ℓ' has r points as well.

11. The number of lines containing a given point is constant.

    Let P and Q be points, and let there be m lines passing through P. By Theorem 3 and Axiom 2, there must exist a line ℓ not containing P or Q. Exactly one of the m lines is parallel to ℓ by Axiom 5, and the remaining $m-1$ lines each intersect unique points of ℓ by Axiom 3. The lines through Q follow similar conditions, so we can determine that there is one for each of the $m-1$ points of ℓ and another parallel to ℓ. Thus, there are m lines through Q.

12. If m is the number of lines containing a given point, and r is the number of points on a line, then $m = r + 1$.

    This is established as a lemma in the solutions to Problems 10 and 11.

13. If n is the total number of points, then $n = (r-1)m + 1$; and hence $n = r^2$.

    Consider a point P, with m lines through it. Each of these m lines contain r points, or $r-1$ points distinct from P. By Axiom 3, all of these $(r-1)m$ points are distinct. By Axiom 3, every point lies on one of these lines. Thus, there are a total of $n = (r-1)m + 1$ points when P is included. By the result of Problem 12, $n = r^2$.

14. The number of lines is $n + r$ and hence, in view of Problem 13, mr.

    By Axiom 3, each of the $\binom{n}{2}$ pairs of points defines a line. Since each line contains $\binom{r}{2}$ pairs, there must be $\binom{n}{2}/\binom{r}{2} = \frac{n(n-1)}{r(r-1)}$ lines. Substituting $n = r^2$, this gives $r(r+1) = r^2 + r = n + r$ lines.

15. In the light of the theorems stated in Problem 10-14, show that a collection of nine things considered as "points" yields several different interpretations of Γ according to how we selectthe possible combinations of points to form lines.

    Unlike in the case of four points, where there were $\binom{4}{2} = 6$ possible lines and $4 + 2 = 6$ actual lines, in the case of nine points we have $\binom{9}{3} = 84$ possible lines and only $9 + 3 = 12$ actual lines.