# *Fourier Series* – Chapter 1 Solutions

Ross Dempsey

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1. Expand the following functions in Fourier series:

   a) $f(x) = e^{ax}$ $(-\pi < x < \pi)$, where $a \neq 0$ is a constant;

   b) $f(x) = \cos ax$ $(-\pi < x < \pi)$, where $a$ is not an integer;

   c) $f(x) = \sin ax$ $(-\pi < x < \pi)$, where $a$ is not an integer;

   d) $f(x) = \begin{cases} 0 & \text{for } -\pi < x \leq 0, \\ x & \text{for } 0 \leq x \leq \pi. \end{cases}$

   The coefficients in part (a) are

$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax}\, dx = \frac{1}{a\pi} \sinh a\pi, \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos kx\, dx = \frac{1}{\pi} \Re \int_{-\pi}^{\pi} e^{(a+ki)x}\, dx \\
&= (-1)^k \frac{2a}{\pi(a^2+k^2)} \sinh a\pi, \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin kx\, dx = \frac{1}{\pi} \Im \int_{-\pi}^{\pi} e^{(a+ki)x}\, dx \\
&= (-1)^{k+1} \frac{2k}{\pi(a^2+k^2)} \sinh a\pi.
\end{aligned}$$

   Therefore, on $-\pi < x < \pi$,

$$e^{ax} = \frac{1}{a\pi} \sinh a\pi + \frac{2}{\pi} \sinh a\pi \sum_{k=1}^{\infty} (-1)^k \frac{a \cos kx - k \sin kx}{a^2 + k^2}.$$

   The coefficients in part (b) are

$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax\, dx = \frac{1}{a\pi} \sin a\pi \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos ax \cos kx\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos(a+k)x + \cos(a-k)x\right)\, dx \\
&= \frac{1}{\pi(a+k)} \sin(a+k)\pi + \frac{1}{\pi(a-k)} \sin(a-k)\pi = (-1)^k \frac{2a}{\pi(a^2-k^2)} \sin a\pi, \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos ax \sin kx\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sin(a+k)x - \sin(a-k)x\right)\, dx \\
&= 0.
\end{aligned}$$

Therefore, on $-\pi < x < \pi$

$$\cos ax = \frac{1}{a\pi}\sin a\pi + \frac{2}{\pi}\sin a\pi \sum_{k=1}^{\infty}(-1)^k \frac{a}{a^2-k^2}\cos kx.$$

The coefficients in part (c) are

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin ax\,dx = 0\\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ax \cos kx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sin(a+k)x + \sin(a-k)x\right)\,dx\\
&= 0,\\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ax \sin kx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\cos(a-k)x - \cos(a+k)x\right)\,dx\\
&= \frac{1}{\pi(a-k)}\sin(a-k)\pi - \frac{1}{\pi(a+k)}\sin(a+k)\pi = (-1)^k\frac{2k}{\pi(a^2-k^2)}\sin a\pi.
\end{aligned}$$

Therefore, on $-\pi < x < \pi$,

$$\sin ax = \frac{2}{\pi}\sin a\pi \sum_{k=1}^{\infty}(-1)^k \frac{k}{a^2-k^2}\sin kx.$$

The coefficients in part (d) are

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{4},\\
a_k &= \frac{1}{\pi}\int_{0}^{\pi} x\cos kx\,dx = \frac{1}{k\pi}\left(x\sin kx + \frac{1}{k}\cos kx\right)_0^{\pi}\\
&= \frac{(-1)^k-1}{k^2\pi},\\
b_k &= \frac{1}{\pi}\int_{0}^{\pi} x\sin kx\,dx = \frac{1}{k\pi}\left(-x\cos kx + \frac{1}{k}\sin kx\right)_0^{\pi}\\
&= \frac{(-1)^{k+1}}{k}.
\end{aligned}$$

Therefore, on $-\pi < x < \pi$,

$$f(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k^2\pi}\cos kx + \frac{(-1)^{k+1}}{k}\sin kx\right).$$

2. Using the expansion of Prob. 1b, show that

$$\begin{aligned}
\frac{1}{\sin z} &= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right],\\
\cot z &= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right],
\end{aligned}$$

where $z$ is any number that is not a multiple of $\pi$.

In the expansion of Prob. 1b, let $x = 0$. Then we have

$$1 = \frac{1}{a\pi}\sin a\pi + \frac{2}{\pi}\sin a\pi \sum_{k=1}^{\infty}(-1)^k \frac{a}{a^2 - k^2}.$$

Letting $z = a\pi$, this gives

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{z/\pi}{(z/\pi)^2 - k^2} = \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^k \left[\frac{z}{z + k\pi} + \frac{1}{z - k\pi}\right].$$

Now let $x = \pi$. Then we have

$$\cos a\pi = \frac{1}{a\pi}\sin a\pi + \frac{2}{\pi}\sin a\pi \sum_{k=1}^{\infty} \frac{a}{a^2 - k^2}.$$

Again letting $z = a\pi$, this gives

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty}\left[\frac{1}{z + k\pi} + \frac{1}{z - k\pi}\right].$$

3. Using the expansion of Prob. 1a, expand the following functions in Fourier series:

   a) The hyperbolic cosine
   $$\cosh x = \frac{e^x + e^{-x}}{2} \quad (-\pi < x < \pi);$$

   b) The hyperbolic sine
   $$\sinh x = \frac{e^x - e^{-x}}{2} \quad (-\pi < x < \pi).$$

   These are the even and odd parts of $e^x$. Therefore,

$$\cosh x = \frac{1}{a\pi}\sinh a\pi + \frac{2}{\pi}\sinh a\pi \sum_{k=1}^{\infty}(-1)^k \frac{a\cos kx}{a^2 + k^2}, \sinh x \qquad = \frac{2}{\pi}\sinh a\pi \sum_{k=1}^{\infty}(-1)^{k+1} \frac{k\sin kx}{a^2 + k^2}.$$

4. Expand the following functions in Fourier cosine series:

   a) $f(x) = \sin ax$ $(0 < x < \pi)$, where $a$ is not an integer;

   b) $f(x) = \begin{cases} 1 & \text{for } 0 < x < h \\ 0 & \text{for } h \le x < \pi \end{cases};$

   c) $f(x) = \begin{cases} 1 - \frac{x}{2h} & \text{for } 0 < x < 2h \\ 0 & \text{for } 2h \le x < \pi \end{cases}.$

   The coefficients in part (a) are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} \sin ax\, dx = \frac{1}{a\pi}(1 - \cos a\pi), \\ a_k &= \frac{2}{\pi}\int_0^{\pi} \sin ax \cos kx\, dx = \frac{1}{\pi}\int_0^{\pi} \left(\sin(a+k)x + \sin(a-k)x\right)\, dx \\ &= \frac{1}{\pi(a+k)}(1 - \cos(a+k)\pi) + \frac{1}{\pi(a-k)}(1 - \cos(a-k)\pi) = \frac{2a}{\pi(a^2 - k^2)}(1 - (-1)^k \cos a\pi). \end{aligned}$$

Therefore, on $0 < x < \pi$,

$$\sin ax = \frac{1}{a\pi}(1 - \cos a\pi) + \sum_{k=1}^{\infty} \frac{2a(1 - (-1)^k \cos a\pi)}{\pi(a^2 - k^2)} \cos kx.$$

The coefficients in part (b) are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^h 1\,dx = \frac{h}{\pi}, \\ a_k &= \frac{2}{\pi}\int_0^h \cos kx\,dx = \frac{2}{k\pi}\sin kh. \end{aligned}$$

Therefore, on $0 < x < \pi$,

$$f(x) = \frac{h}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin kh}{k} \cos kx.$$

The coefficients in part (c) are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2h} \left(1 - \frac{x}{2h}\right)\,dx = \frac{h}{\pi}, \\ a_k &= \frac{2}{\pi}\int_0^{2h} \left(1 - \frac{x}{2h}\right)\cos kx\,dx, \\ &= \frac{2}{k\pi}\sin 2kh - \frac{1}{\pi kh}\left(x \sin kx + \frac{1}{k}\cos kx\right)_0^{2h} \\ &= \frac{1}{\pi k^2 h}(1 - \cos 2kh) = \frac{2}{\pi k^2 h}\sin^2 kh. \end{aligned}$$

Therefore, on $0 < x < \pi$,

$$f(x) = \frac{h}{\pi} + \frac{2}{\pi h}\sum_{k=1}^{\infty} \frac{\sin^2 kh}{k^2} \cos kx.$$

5. Expand the following functions in Fourier sine series:

a) $f(x) = \begin{cases} \sin \frac{\pi x}{l} & \text{for } 0 < x < \frac{l}{2} \\ 0 & \text{for } \frac{l}{2} \le x < l \end{cases};$

b) $f(x) = \begin{cases} \sin \frac{\pi x}{l} & \text{for } 0 < x < \frac{l}{2} \\ -\sin \frac{\pi x}{l} & \text{for } \frac{l}{2} \le x < l \end{cases}.$

The coefficients in part (a) are

$$\begin{aligned} b_k = \frac{2}{l}\int_0^{l/2} \sin\frac{\pi x}{l}\sin\frac{k\pi x}{l}\,dx &= \frac{1}{l}\int_0^{l/2}\left(\cos\frac{(1-k)\pi x}{l} - \cos\frac{(1+k)\pi x}{l}\right)dx \\ = \frac{1}{\pi(1-k)}\sin\frac{(1-k)\pi}{2} &- \frac{1}{2\pi(1+k)}\sin\frac{(1+k)\pi}{2} \\ = \frac{2k}{\pi(1-k^2)}\cos\frac{k\pi}{2}&. \end{aligned}$$

Note that this does not hold for $k = 1$; we can easily find $b_1 = \frac{1}{4}$. Otherwise, when $k$ is odd $b_k = 0$, and when $k$ is even $b_k = (-1)^{k/2}$. Therefore, on $0 < x < l$,

$$f(x) = \frac{1}{2}\sin\frac{\pi x}{l} + \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^j j}{1 - 4j^2}\sin\frac{2j\pi x}{l}.$$

The coefficients in part (b) are

$$
\begin{aligned}
b_k &= \frac{2}{l}\int_0^{l/2} \sin\frac{\pi x}{l}\sin\frac{k\pi x}{l}\,dx - \frac{1}{l}\int_{l/2}^{l} \sin\frac{\pi x}{l}\sin\frac{k\pi x}{l}\,dx \\
&= \frac{1}{l}\int_0^{l/2}\left(\cos\frac{(1-k)\pi x}{l} - \cos\frac{(1+k)\pi x}{l}\right)dx - \frac{1}{l}\int_{l/2}^{l}\left(\cos\frac{(1-k)\pi x}{l} - \cos\frac{(1+k)\pi x}{l}\right)dx \\
&= \frac{2}{\pi(1-k)}\sin\frac{(1-k)\pi}{2} - \frac{2}{\pi(1+k)}\sin\frac{(1+k)\pi}{2} \\
&= \frac{4k}{\pi(1-k^2)}\cos\frac{k\pi}{2}.
\end{aligned}
$$

Again this does not hold for $k=1$; in this case, $b_1 = 0$. Therefore, on $0 < x < l$,

$$
f(x) = \frac{8}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j j}{1-4j^2}\sin\frac{2j\pi x}{l}.
$$

6. Expand the periodic function

$$
f(x) = \left|\cos\frac{\pi x}{l}\right|, \quad l = \text{const}, l > 0
$$

in Fourier series.

This is an even function, so $b_k = 0$. The other coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{2l}\int_{-l}^{l}\left|\cos\frac{\pi x}{l}\right|dx = \frac{2}{\pi}, \\
a_k &= \frac{1}{l}\int_{-l}^{l}\left|\cos\frac{\pi x}{l}\right|\cos\frac{k\pi x}{l}\,dx \\
&= \frac{2}{l}\int_0^{l/2}\cos\frac{\pi x}{l}\cos\frac{k\pi x}{l}\,dx - \frac{2}{l}\int_{l/2}^{l}\cos\frac{\pi x}{l}\cos\frac{k\pi x}{l}\,dx \\
&= \frac{2}{\pi(1+k)}\sin\frac{\pi(1+k)}{2} + \frac{2}{\pi(1-k)}\sin\frac{\pi(1-k)}{2} \\
&= \frac{4}{\pi(1-k^2)}\cos\frac{\pi k}{2}.
\end{aligned}
$$

Therefore,

$$
\left|\cos\frac{\pi x}{l}\right| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j}{1-4j^2}\cos\frac{2j\pi x}{l}.
$$

7. Let $f(x)$ have period $2\pi$ and let $|f(x)-f(y)| \le c|x-y|^\alpha$ for some constants $c > 0$, $\alpha > 0$, and for all $x$ and $y$. Show that

$$
|a_n| \le \frac{c\pi^\alpha}{n^\alpha}, \quad |b_n| \le \frac{c\pi^\alpha}{n^\alpha},
$$

where $a_n$ and $b_n$ are the Fourier coefficients of $f(x)$.

The Fourier coefficients $a_n$ are given by

$$
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.
$$

Since $\cos(nx+\pi) = -\cos nx$ and $f(x)\cos nx$ has period $2\pi$,

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = -\int_{-\pi}^{\pi} f(x)\cos n\left(x - \frac{\pi}{n}\right)\,dx = -\int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right)\cos nx\,dx.$$

Therefore,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(x) - f\left(x+\frac{\pi}{n}\right)\right)\cos nx\,dx.$$

Using the Lipschitz condition, we have

$$|a_n| \le \frac{c}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi}{n}\right)^{\alpha}|\cos nx|\,dx \le \frac{c}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi}{n}\right)^{\alpha}\,dx = \frac{c\pi^{\alpha}}{n^{\alpha}}.$$

A very similar argument gives the same bound for $|b_n|$.

8. Expand the following functions in Fourier sine series:

   a) $f(x) = \cos x \quad (0 < x < \pi)$;

   b) $f(x) = x^3 \quad (0 \le x < \pi)$.

   The coefficients in part (a) are

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^{\pi}\cos x \sin kx\,dx = -\frac{1}{\pi}\left(\frac{1}{k+1}\cos(k+1)x + \frac{1}{k-1}\cos(k-1)x\right)_0^{\pi} \\ &= \frac{2k}{\pi(k^2-1)}\left((-1)^{k+1} - 1\right). \end{aligned}$$

   This does not hold for $k = 1$; we can easily show that $b_1 = 0$. Thus, on $0 < x < \pi$,

$$\cos x = \sum_{j=1}^{\infty}\frac{8j}{\pi(1-4j^2)}\sin 2jx.$$

   The coefficients in part (b) are

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^{\pi} x^3 \sin kx\,dx = \frac{2}{\pi}\left(-\frac{1}{k}x^3\cos kx + \frac{3}{k^2}x^2\sin kx + \frac{6}{k^3}x\cos kx - \frac{6}{k^4}\sin kx\right)_0^{\pi} \\ &= 2(-1)^k\left(\frac{6}{k^3} - \frac{\pi^2}{k}\right). \end{aligned}$$

   Therefore, on $0 \le x \le \pi$,

$$x^3 = 2\sum_{k=1}^{\infty}(-1)^k\left(\frac{6}{k^3} - \frac{\pi^2}{k}\right)\sin kx.$$

9. Let $f(x)$ be a function of period $2\pi$ defined for $-\pi < x < \pi$. Let $f(x)$ have the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx),$$

   and let

$$f_e(x) = \frac{f(x)+f(-x)}{2}, \quad f_o(x) = \frac{f(x)-f(-x)}{2}.$$

Show that $f_e(x)$ is an even function and $f_o(x)$ is an odd function, with Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx,\qquad \sum_{n=1}^{\infty}b_n\cos nx$$

respectively. Show that the function $f(x-\pi)$ has the Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(-1)^n(a_n\cos nx+b_n\sin nx).$$

Substituting $-x$ into the Fourier series gives

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx-b_n\sin nx),$$

from which the Fourier series for $f_e(x)$ and $f_o(x)$ follow immediately. Similarly, substituting $x-\pi$ and noting $\cos\theta-n\pi=(-1)^n\cos\theta$ and $\sin\theta-n\pi=(-1)^n\sin\theta$, we have

$$f(x-\pi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(-1)^n(a_n\cos nx+b_n\sin nx).$$

10. Sum the series

    a) $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$;
    b) $\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$;
    c) $\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$;
    d) $\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$

    by using Example 8 of Sec. 13 and the results of the preceding problem.

    From Example 8, we have for $-\pi<x<\pi$

$$x=-2\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}.$$

Therefore the sum in part (a) is $\frac{\pi-\tilde{x}}{2}$ and the sum in part (b) is $-\frac{\tilde{x}}{2}$, where $\tilde{x}=x-2\pi n$ and $n$ is chosen such that $\pi-\tilde{x}\in(-\pi,\pi)$ and $x\in(-\pi,\pi)$, respectively. Both sums vanish for $x=\pi n$.

Likewise, for $-\pi\le x\le\pi$ we have

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}.$$

Therefore the sum in part (c) is $(\frac{\tilde{x}-\pi}{2})^2-\frac{\pi^2}{12}$, and the sum in part (d) is $\frac{\tilde{x}^2}{4}-\frac{\pi^2}{12}$, where $\tilde{x}$ is chosen in the same way as before.

11. Find the sum of each of the following numerical series by evaluating at a suitable point a Fourier series given in the text or in the problems:

    a) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$;
    b) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2+a^2}$;
    c) $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\sin nh}{nh}$;

d) $\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)^2$.

From Problem 1(d), we have for $-\pi < x < \pi$

$$\begin{cases} 0 & \text{for } -\pi < x \leq 0, \\ x & \text{for } 0 \leq x \leq \pi. \end{cases} = \frac{\pi}{4} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2\pi}\cos kx + \frac{(-1)^{k+1}}{k}\sin kx\right).$$

Setting $x = 0$, this gives

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Therefore the sum in part (a) is $\frac{\pi^2}{8}$.

From Problem 1(a), we have for $-\pi < x < \pi$

$$e^{ax} = \frac{1}{a\pi}\sinh a\pi + \frac{2}{\pi}\sinh a\pi\sum_{k=1}^{\infty}(-1)^k\frac{a\cos kx - k\sin kx}{a^2+k^2}.$$

Setting $x = 0$, this gives

$$\frac{\pi}{2a\sinh a\pi} - \frac{1}{2a^2} = \sum_{k=1}^{\infty}(-1)^k\frac{1}{a^2+k^2}.$$

Thus, the sum in part (b) is $\frac{\pi}{2a\sinh a\pi} - \frac{1}{2a^2}$.

Substituting the result of Problem 10(a), we have for part (c)

$$\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin nh}{nh} = \frac{\pi}{2h}.$$

From Problem 4(c), we have for $-\pi < x < \pi$

$$\begin{cases} 1 - \frac{x}{2h} & \text{for } 0 < x < 2h \\ 0 & \text{for } 2h \leq x < \pi \end{cases} = \frac{h}{\pi} + \frac{2}{\pi h}\sum_{k=1}^{\infty}\frac{\sin^2 kh}{k^2}\cos kx.$$

Setting $x = 0$, this gives

$$1 = \frac{h}{\pi} + \frac{2h}{\pi}\sum_{k=1}^{\infty}\frac{\sin^2 kh}{k^2h^2}.$$

Therefore, the sum in part (d) is $\frac{\pi}{2h}$.

12. Show that the Fourier series for the function $f(x) = x$ on the interval $-\pi < x < \pi$ does not converge uniformly, but that the Fourier series for the function $f(x) = x^2$ does converge uniformly. Find the Fourier series for the function $f(x) = x^4$ by integrating the Fourier series for $f(x) = x^2$ between the limits 0 and $x$.

    Since $f(x) = x$ is odd, $a_0 = a_k = 0$, so the series is of the form $\sum_{n=1}^{\infty} b_n \sin nx$. Therefore, all partial sums $f_n(x)$ vanish at $x = \pi$. The partial sums of the Fourier series are continuous functions. Therefore, for each one, we can find $\delta_n$ such that, for $|x - \pi| \leq \delta_n$, $|f_n(x)| < \pi/2$. Therefore, at $x = \max(3\pi/4, \pi - \delta_n)$, $|f_n(x) - x| > \pi/4$, so the sequence does not converge uniformly to $x$.

    The Fourier series for $x^2$ is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}.$$

Using $M_n = \frac{1}{n^2}$, the Weierstrass $M$-test indicates that the sum converges uniformly. Integrating twice from 0 to $x$, we find

$$\begin{aligned} x^4 &= 12\left(\frac{\pi^2}{6}x^2 - 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx - 1}{n^4}\right) \\ &= \frac{2\pi^4}{5} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{8\pi^2}{n^2} - \frac{48}{n^4}\right)\cos nx. \end{aligned}$$