# Electrostatics

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## 1 Introduction

To begin a study of electrodynamics, we will review the principles of electrostatics. You should already be familiar with this topic from the perspective of Coulomb's law and electric fields, so we will take a more theoretical perspective. This will provide the foundations for examining all of Maxwell's equations from the same perspective.

## 2 Electric Potential

The electric potential $\phi$ is related to the electric field $\boldsymbol{E}$ just as potential energy is related to force:

$$-\boldsymbol{\nabla}\phi = \boldsymbol{E}.$$

Recall Coulomb's law for a point charge $q$ at the origin, which states that the electric field at a position $\boldsymbol{r}$ is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon r^2}\hat{\boldsymbol{r}}.$$

Therefore, the potential is given by

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r}.$$

This is one way of expressing Coulomb's law in terms of the potential. However, it is valid only for point charges. We can determine a more useful relationship by taking the Laplacian of the potential. At $r > 0$, this gives

$$\nabla^2\phi = \boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\boldsymbol{\nabla}\cdot\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle = 0.$$

However, at $r = 0$ this analysis does not hold. Instead, we can use the divergence theorem. We know from Gauss's law that for any surface enclosing the origin,

$$\oiint \boldsymbol{E}(\boldsymbol{r})\cdot d\boldsymbol{S} = \frac{q}{\epsilon_0}.$$

Additionally, the divergence theorem tells us that

$$\oiint \boldsymbol{E}(\boldsymbol{r})\cdot d\boldsymbol{S} = \iiint \boldsymbol{\nabla}\cdot\boldsymbol{E}\, dV.$$

This implies that

$$\iiint \boldsymbol{\nabla} \cdot \boldsymbol{E} \, dV = \frac{q}{\epsilon_0}.$$

Now, if this electric field is due to a point charge as we have assumed previously, this is a problem: the integral of a quantity which is 0 everywhere except a single point is not zero. This seeming contradiction can only be remedied using the $\delta$-function, which we will look at later. However, if the electric field is due to a body with finite charge density $\rho$, then we can use this integral equation to derive a differential equation. Let the surface of integration be a sphere of radius $a$ around the origin. As $a$ decreases, $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ becomes approximately constant inside the surface. In the limit, we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\boldsymbol{\nabla}^2 \phi = \frac{\rho}{\epsilon_0}.$$

This is how Maxwell's first equation is written in differential form. It is a special case of the *Poisson equation**, $\boldsymbol{\nabla}^2 \phi = f(\boldsymbol{r})$.

# 3 Uniqueness of Electric Potential

Given some charge distribution $\rho(\boldsymbol{r})$, we have shown that the electric potential satisfies the Poisson equation $\boldsymbol{\nabla}^2 \phi = -\frac{\rho}{\epsilon_0}$. We can show that, if we have a solution $\phi(\boldsymbol{r})$ to this equation, then this solution is unique. Consider a second solution $\psi(\boldsymbol{r})$. Then $\theta(\boldsymbol{r}) = \phi(\boldsymbol{r}) - \psi(\boldsymbol{r})$ satisfies

$$\boldsymbol{\nabla}^2 \theta = 0.$$

If $\phi \neq \psi$, then $\theta$ must reach a local maximum at some point. However, it is well known that *harmonic functions*, which are solutions to $\boldsymbol{\nabla}^2 f = 0$, can have no local maxima or minima (prove this to yourself). So, the Poisson equation can have only one solution. We can apply the same technique to a bounded region: if the potential is fixed on a boundary, then a given charge distribution within the boundary produces a unique potential.

This leads us to an interesting technique, the method of image charges. Consider a conducting plane $z = 0$ with a charge $q$ at $(0, 0, z)$. Since the sheet is conducting, the electric field at $z = 0$ must have only a $\hat{\boldsymbol{k}}$ component, so that it is perpendicular. Now, consider the potential due to a charge $q$ at $(0, 0, a)$ and a charge $-q$ at $(0, 0, -a)$,

$$\phi = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+a)^2}} \right).$$

You can verify that $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$ at $z = 0$, as shown in Figure 1. Additionally, this potential satisfies the Poisson equation for the original problem. So, the uniqueness theorem guarantees that it is the correct potential in $z > 0$. This tells us important information. First, we know that the force on the charge is $\frac{q^2}{16\pi\epsilon_0 a^2}$. Additionally, using Gauss's law with a short cylindrical surface around the plane, we can show that the induced charge density is given by

$$\sigma = -\epsilon_0 \left. \frac{\partial \phi}{\partial z} \right|_z = 0 = -\frac{qa}{2\pi \left(x^2 + y^2 + a^2\right)^{3/2}}.$$

Therefore, the total charge on the sheet is

$$-\frac{qa}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\left(x^2 + y^2 + a^2\right)^{3/2}} \, dx \, dy.$$

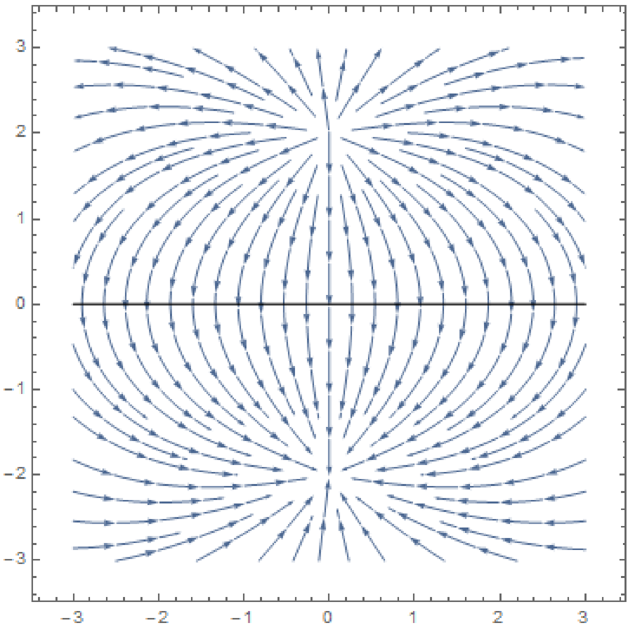


Figure 1

Rewriting this in polar coordinates, we have

$$-\frac{qa}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} \frac{r}{(r^2+a^2)^{3/2}}\, dr = (-qa)\left(\frac{1}{2a}\int_1^{\infty}\frac{1}{u^{3/2}}\, du\right) = -q.$$

So, as we might expect, the point charge $q$ induces a charge of $-q$ on the sheet.

**Example 1.** Find the potential inside a grounded sphere of radius $R$ due to a point charge $q$ placed at $(a, 0)$, with $a < R$.

**Solution:** We will solve this problem using the method of image charges. The image charge must be on the $x$-axis, due to symmetry, so let it be a charge of $q'$ at $(b, 0)$. Then the potential will be

$$\phi = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{\boldsymbol{r}^2+a^2-2\boldsymbol{r}\cdot\boldsymbol{r}_1}} + \frac{q}{\sqrt{\boldsymbol{r}^2+b^2-\boldsymbol{r}\cdot\boldsymbol{r}_2}}\right).$$

We need this potential to vanish when $|\boldsymbol{r}| = R$. That is, we need

$$\frac{q}{\sqrt{R^2+a^2-2aR\cos\theta}} = -\frac{q'}{\sqrt{R^2+b^2-2bR\cos\theta}}.$$

Squaring, this becomes

$$q^2(R^2+b^2-2bR\cos\theta) - q'^2(R^2+a^2-2aR\cos\theta) = 0.$$

Since $\theta$ can take any value, we must have the the coefficient of $\cos\theta$ be 0. So,

$$q^2 b = q'^2 a.$$

This leaves us with

$$q^2(R^2+b^2) - q'^2(R^2+a^2) = 0.$$

Substituting $b = \frac{q'^2 a}{q^2}$, we have

$$\left(a^2 \frac{q'^2}{q^2} - R^2\right) \left(q'^2 - q^2\right) = 0.$$

If we let $q^2 = q'^2$, then our equation above would imply $a = b$, which is a trivial case. So, we choose $q' = -\frac{R}{a} q$, making the first factor vanish. The equation above then gives $b = \frac{R^2}{a}$. This leaves us with the potential

$$\phi = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{\sqrt{\boldsymbol{r}^2 + a^2 - 2\boldsymbol{r} \cdot \boldsymbol{r}_1}} - \frac{R}{a} \frac{q}{\sqrt{\boldsymbol{r}^2 + \frac{R^4}{a^2} - \boldsymbol{r} \cdot \boldsymbol{r}_2}} \right).$$

The image charge technique is shown in Figure 2.

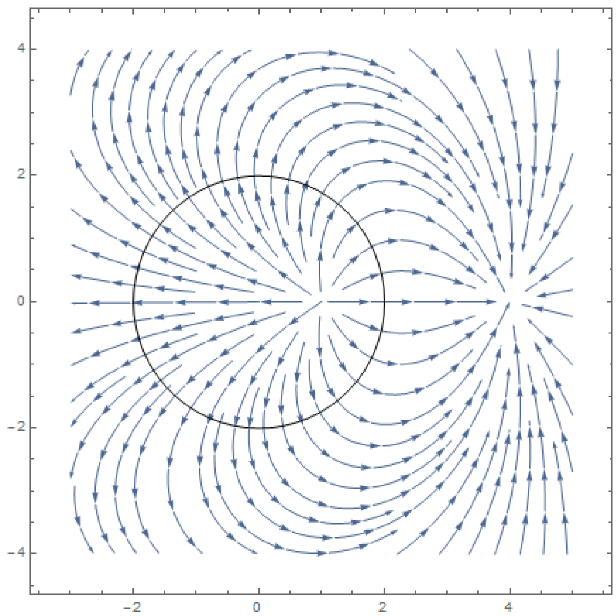


Figure 2

# 4 Capacitance

Consider a conductor which, when given a charge $Q$, has a potential $\phi$. If we instead give it a charge $aQ$, then a potential $a\phi$ would satisfy the corresponding Poisson equation, so it must be the correct potential. Therefore, the ratio of charge to potential for the conductor is a constant. We call it the self-capacitance of the conductor,

$$C \equiv \frac{Q}{\phi}.$$

For example, if the conductor is spherical with a radius $R$, then we know the potential is $\phi = \frac{Q}{4\pi\epsilon_0 R}$. Therefore, $C = 4\pi\epsilon_0 R$ for spheres.

Now, consider the case of two bodies, one with charge $Q$ and the other with charge $-Q$. Let the potentials be $\phi_1$ and $\phi_2$. If we replace the charges with $\pm aQ$, then the potentials become $a\phi_1$ and $a\phi_2$. So, the potential difference is multiplied by $a$. Therefore, the ratio of the charge to the potential difference is a constant, which we call the mutual capacitance:

$$C \equiv \frac{Q}{\phi_1 - \phi_2}.$$

For example, consider two conducting plates separated by a distance $d$, with charges $\pm Q$. Ignoring edge effects, the electric field between the plates is $E = \frac{Q}{\epsilon_0 A}$. Therefore, the potential difference is

$$\Delta\phi = Ed = \frac{Qd}{\epsilon_0 A}.$$

This means that the capacitance of the plates is $C = \frac{\epsilon_0 A}{d}$.

> **Example 2.** Determine the capacitance of two concentric spheres with radii $a < b$.
>
> **Solution:** Let the charge on the inner sphere be $Q$ and the charge on the outer sphere be $-Q$. The potential due to the outer sphere is constant inside the outer sphere. So, we only need to consider the potential difference due to the field from the inner sphere. The electric field will be given by $E = \frac{Q}{4\pi\epsilon_0 R^2}$, so the potential difference will be
>
> $$\delta\phi = \int_a^b \frac{Q}{4\pi\epsilon_0 R^2}\, dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$
>
> Therefore, the capacitance is $C = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}$.

We can now generalize to $N$ conductors. First, consider the case where every conductor but the first is grounded. Then, we have

$$\begin{aligned} Q_1 &= C_{11}\phi_1, \\ Q_2 &= C_{21}\phi_1, \\ &\vdots \\ Q_N &= C_{N1}\phi_1. \end{aligned}$$

We would obtain similar sets of equations if every conductor but the $i$th was grounded. In the general case, where all the $\phi_i$ can be nonzero, we can add these results to obtain

$$\begin{aligned} Q_1 &= C_{11}\phi_1 + C_{12}\phi_2 + \ldots + C_{1N}\phi_N, \\ Q_2 &= C_{21}\phi_1 + C_{22}\phi_2 + \ldots + C_{2N}\phi_N, \\ &\vdots \\ Q_N &= C_{N1}\phi_1 + C_{N2}\phi_2 + \ldots + C_{NN}\phi_N. \end{aligned}$$

As we can see, for a system of $N$ conductors, the capacitance becomes a matrix:

$$\boldsymbol{Q} = C\boldsymbol{\phi}.$$

As it turns out, this matrix is symmetric. This is a corollary of an important result, Green's reciprocity theorem.

# 5 Green's Reciprocity Theorem

Consider a charge distribution $\rho_1$ which gives rise to a potential $\phi_1$, and a separate charge distribution $\rho_2$ which gives rise to a potential $\phi_2$. Assume that both of these potentials are localized, so that they vanish at infinity. Then, consider the integral

$$\int \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \, dV.$$

Replacing $\boldsymbol{E}_1 = -\boldsymbol{\nabla}\phi_1$ and integrating by parts, we have

$$-\int (\boldsymbol{\nabla}\phi_1) \cdot \boldsymbol{E}_2 \, dV = -\oiint_\infty \phi_1 \boldsymbol{E}_2 \cdot d\boldsymbol{A} + \int \phi_1 (\boldsymbol{\nabla} \cdot \boldsymbol{E}_2) \, dV.$$

Since $\phi_1$ is 0 at infinity, the integrated part vanishes. Using Gauss's law, we finally have

$$\int \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \, dV = \frac{1}{\epsilon_0} \int \phi_1 \rho_2 \, dV.$$

The left hand side is symmetric with respect to interchange of indices, so the right hand side must be also. This gives us Green's reciprocity theorem:

$$\int \phi_1 \rho_2 \, dV = \int \phi_2 \rho_1 \, dV.$$

We can start by using this theorem to prove the symmetry of the capacitance matrix. Since the potential on each conductor is constant, the integrals become

$$\sum_{i=1}^{N} \phi_i \bar{Q}_i = \sum_{i=1}^{N} \bar{\phi}_i Q_i.$$

Since this holds for any set of potentials on either side, we can let $\phi_a$ and $\bar{\phi}_b$ be the only nonzero values. Then, we have

$$\phi_a C_{ab} \bar{\phi}_b = \bar{\phi}_b C_{ba} \phi_a,$$

which implies that $C_{ab} = C_{ba}$.

**Example 3.** Two infinite parallel conducting plates are separated by a distance $d$, and both are grounded. A point charge $q$ is placed at a distance $x$ from the left plate. Determine the charge induced on each plate.

**Solution:** For the first charge distribution, we take the system as described. For the second distribution, we remove the point charge and set the plates to have potentials 0 and $V_0$. Then the Green reciprocity theorem gives

$$0 = q\frac{x}{d}V_0 + q_r V_0.$$

This implies that $q_r = -q\frac{x}{d}$. We also must have $q_r + q_l = -q$, so $q_l = -q\left(1 - \frac{x}{d}\right)$.