Applications of Integration

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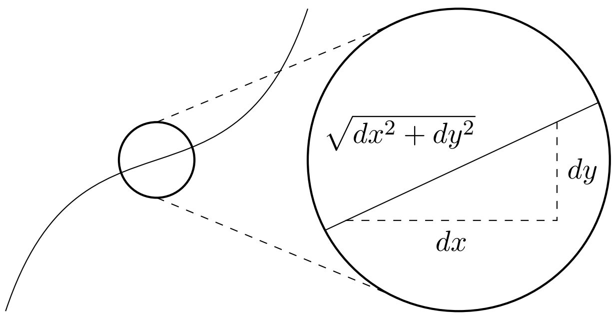
Definite integration, which was motivated as a method for finding the areas under plane curves, is in fact a far more powerful construct. The integral should be understood as a tool for taking infinite sums of differential quantities. Although this seems very abstract, it is commonly useful for concrete purposes. For example, the volume of a complicated region may be computed as an infinite sum of more simply expressed differential volumes. Likewise, a complicated area can be expressed as a sum of differential areas, and the length of a complicated curve can be expressed as a sum of differential lengths. These simple geometric applications are examined in more detail here; however, it is far more important to develop an intuition for how to effectively utilize the integral than to gain skill in solving these particular problems.

1 Arc Length

Given a plane curve, we wish to determine its length. The only length we can compute in the general case, without the use of calculus, is the length of the hypotenuse of a right triangle given its legs. We can therefore try to write the length as a sum of many hypotenuses, like so:



We see that the differential arc length is $\sqrt{dx^2 + dy^2}$. However, we must integrate with respect to a single free differential, not two differentials both inside a radical. We could do this in one of two ways: factoring out either a dx, for finding the arc length of a function $y = f(x)$, or a dt, for finding the arc length of a parametric curve $x = x(t)$, $y = y(t)$. The integrals that result are the following:

$$\int_C \sqrt{dx^2 + dy^2} = \int_{x=a}^{b} \sqrt{\frac{dx^2 + dy^2}{dx^2}}\, dx = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\int_C \sqrt{dx^2 + dy^2} = \int_{t=t_1}^{t_2} \sqrt{\frac{dx^2 + dy^2}{dt^2}}\, dt = \int_{t=t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For any curve more complicated than a line, integrals of the first type are always at least moderately difficult, and are often computed numerically. The second integral, in terms of dt, is sometimes tractable by hand, such as in the following example:

Example. Verify with calculus that the circumference of an circle with radius r is $2\pi r$.

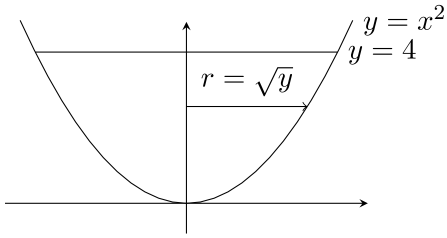
We start by parameterizing the cicle in terms of a variable t. We use the common cicle parameterization $x = r\cos t$ and $y = r\sin t$, for $0 < t < 2\pi$. We see easily that $\frac{dx}{dt} = -r\sin t$ and $\frac{dy}{dt} = r\cos t$. The integration is then simple:

$$\int_{t=0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t=0}^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = \int_{t=0}^{2\pi} r\, dt = 2\pi r$$

2 Volumes of Revolution

When a region in the plane is revolved about an axis, it forms a region in space composed of many radially symmetric differential pieces. To find its volume, we must integrate these differential volumes. In many cases, this is simply a matter of adding the volumes of infinitesimally short cylinders, or "disks". The following example illustrates this method.

Example. Find the volume of the region between $y = x^2$ and $y = 4$ revolved about the y axis.

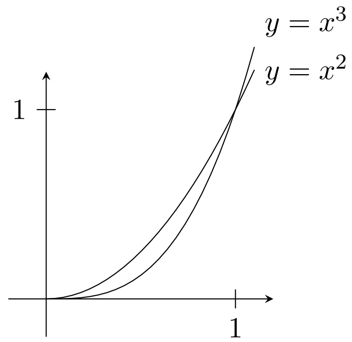


The figure above shows the radius of the disk formed at a given y. The volume of such a disk is the area of its base, πr^2, times its height, dy. Therefore, we carry out the following integration:

$$\int_{y=0}^{4} \pi r^2\, dy = \pi \int_{y=0}^{4} y\, dy = 8\pi$$

A similar method is to use differential slices shaped like "washers," or more specifically, a cylinder with infinitesimal height and a base formed by the difference of two concentric circles. This is best illustrated by another example.

Example. Find the volume of the region between $y = x^2$, $y = x^3$, and $x = 0$ revolved about the x axis.

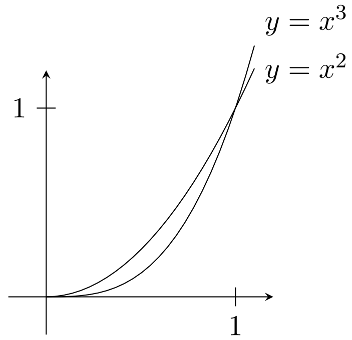


At a given x, we can envision a "washer" formed with inner radius $r_1 = x^3$ and outer radius $r_2 = x^2$. The volume of this washer is the area of its base, $\pi r_2^2 - \pi r_1^2$, multiplied by its height, dx. We thus have the following integral:

$$\int_{x=0}^{1} \pi r_2^2 - \pi r_1^2 \, dx = \pi \int_{x=0}^{1} x^4 - x^6 \, dx = \frac{2\pi}{35}$$

There is a separate way to express a three-dimensional region in terms of washer-like shapes. Instead of letting their height be the differential unit, we can let their thickness $r_2 - r_1$ be differential. Such a region is called a "shell" instead of a washer for clarity. A final example shows a problem where these shells are of use.

Example. Find the volume of the region between $y = x^2$, $y = x^3$, and $x = 0$ revolved about the y axis.



This is the same region as in the previous example, but revolved about the y axis instead of the x axis. However, because we are using shells, the integration will still be over dx, the thickness of the shell. The height will be $x^2 - x^3$. The volume is $2\pi r h dx$. This is found by multiplying the circumference of the cylinder, $2\pi r$ by the height h to obtain the surface area, and finally multiplying by the thickness dx. The radius r in this case is simply x, and the integral is simple to compute:

$$\int_{x=0}^{1} 2\pi r h \, dx = 2\pi \int_{x=0}^{1} x(x^2 - x^3) \, dx = \frac{\pi}{20}$$

These three methods are sufficient to handle any problem you may encounter involving volumes of revolution.