Electromagnetism

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1 Electrostatics

Two charges at rest experience a force very similar in form to gravity. The force exerted by a single charge can be represented as an electric field, which can be integrated to determine an electric potential. This potential can be used to analyze circuits containing resistor networks.

1.1 Coulomb Force

Two charges q_1 and q_2 at a distance r experience an attractive force if the charges are opposite in sign, and a repulsive force if they have the same sign:

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

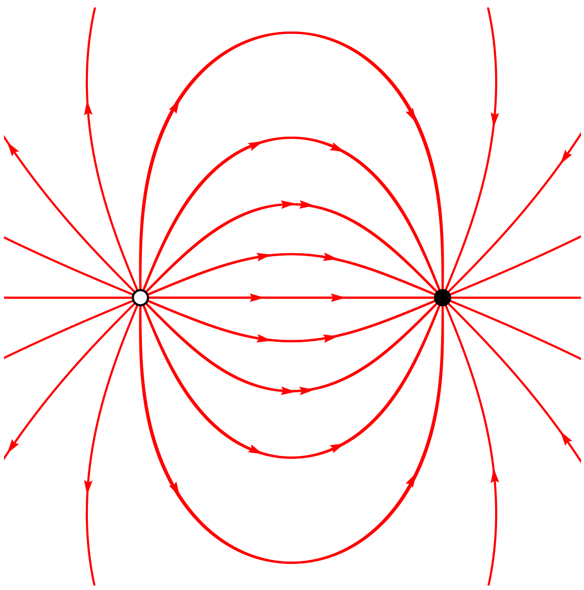
The quantity $\frac{1}{4\pi\epsilon_0}$ is often abbreviated k, so the force can be expressed as $\frac{kq_1q_2}{r^2}$. The value of ϵ_0 (called the permittivity of free space) is approximately $8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$, so k is approximately $8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$.

1.2 Electric Field

The electric field $\vec{E}$ is defined so that the force experienced by a particle with charge q at a point is given by $\vec{F} = q\vec{E}$. Thus, the electric field of a point charge q_1 is given by:

$$\vec{E} = \frac{\vec{F}}{q_2} = \frac{q_1}{4\pi\epsilon_0 r^2}\hat{r}$$

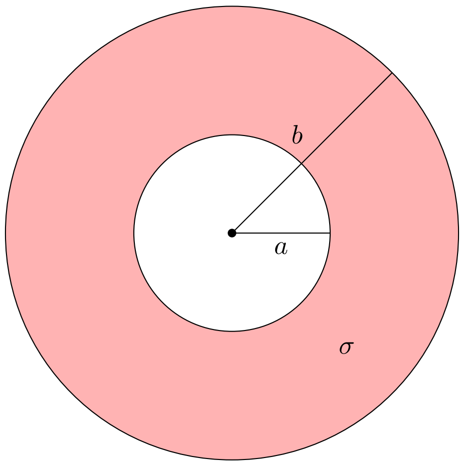
The electric field is useful in many ways, including visualizing the electric forces in a region. For example, consider two charges $+q$ and $-q$ separated by a distance a. This is called a dipole, and has a characteristic field.



These field lines are drawn along the electric field vectors. They point away from the positive (white) charge, and towards the negative (black) charge, as predicted by expression for the electric field.

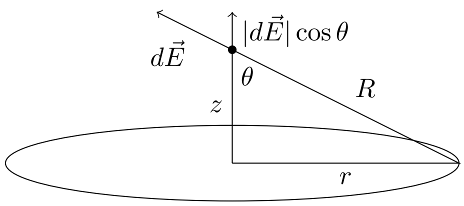
1.3 Example Electric Field

Electric fields for objects more complicated than point charges can be found in general by integrating the field for a point charge. A comprehensive example is an annulus of charge:



We will attempt to find the electric field on the axis of the annulus. First we will find the field due to a differential ring of charge, and then integrate that result to find the total field of the annulus.

Consider a ring with total charge q and radius r. The field at the axis of the ring will have only a component perpendicular to the plane of the ring, because of the symmetry in the other directions. Thus, we will integrate only that component of the differential field elements:



We can now make several substitutions to integrate this quantity around the ring:

$$\begin{aligned}
\int_0^{2\pi} |d\vec{E}|\cos\theta\, d\phi &= \int_0^{2\pi} |d\vec{E}|\frac{z}{R}\, d\phi \\
&= \int_0^{2\pi} |d\vec{E}|\frac{z}{\sqrt{z^2+r^2}}\, d\phi \\
&= \int_0^{2\pi} \frac{dq}{4\pi\epsilon_0(z^2+r^2)}\frac{z}{\sqrt{z^2+r^2}}\, d\phi \\
&= \frac{qz}{4\pi\epsilon_0(z^2+r^2)^{3/2}}
\end{aligned}$$

We now have our expression for the field due to a ring of radius r and charge q. We can use this to compute the total field due to the annulus. The area of a differential ring of radius r is $2\pi r\, dr$, so the charge on the ring is $2\pi r\sigma\, dr$. Thus, we can integrate:

$$
\begin{aligned}
\int_a^b \frac{2\pi r\sigma z}{4\pi\epsilon_0(z^2+r^2)^{3/2}}\,dr &= \frac{\sigma z}{2\epsilon_0}\int_a^b \frac{r}{(z^2+r^2)(3/2)}\,dr \\
&= \frac{\sigma z}{2\epsilon_0}\left(-\frac{1}{\sqrt{r^2+z^2}}\right)_a^b \\
&= \frac{\sigma z}{2\epsilon_0}\left(\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{\sqrt{b^2+z^2}}\right)
\end{aligned}
$$

The field due to other simple objects can be found by a similar method.

1.4 Gauss's Law

In addition to Coulomb's law, another relationship between charge and electric field is given by Gauss's law. Gauss's law states that the flux of the electric field through a closed surface is proportional to the enclosed charge.

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

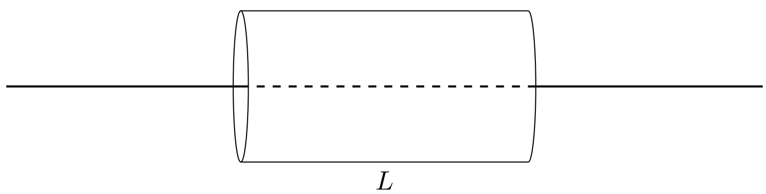
In general, this gives only the total surface integral of the field rather than the field at individual points, which is often less useful. However, in the simple case where $\vec{E}$ is parallel to $d\vec{A}$ and is constant on the surface, the equation can be greatly simplified:

$$\oiint \vec{E}\cdot d\vec{A} = |\vec{E}|\oiint d\vec{A} = |\vec{E}|A = \frac{Q_{\text{enc}}}{\epsilon_0}$$

For example, if we take the surface to be a sphere around a point charge q, so that $\vec{E}$ is parallel to $d\vec{A}$ on the sphere, we see that Gauss's law agrees with Coulomb's law:

$$
\begin{aligned}
|\vec{E}|(4\pi r^2) &= \frac{q}{\epsilon_0} \\
|\vec{E}| &= \frac{q}{4\pi\epsilon_0 r^2}
\end{aligned}
$$

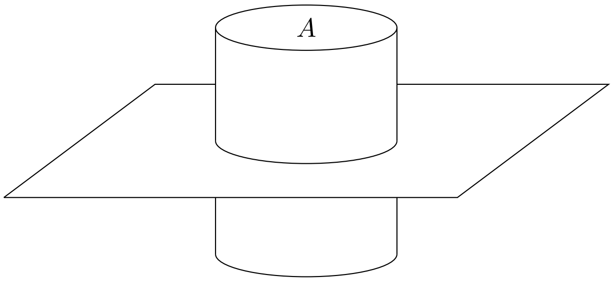
We can also use Gauss's law to find the field of an infinite line of charge. Let the linear density of charge be λ, and the distance from the line be r. Then we can construct a cylindrical surface with its axis on the wire:



Due to symmetry, the field must always be everywhere perpendicular to the wire, and thus $\vec{E}$ is perpendicular to $d\vec{A}$ on the ends of the cylinder. The surface integral is then only the integral over the round part of the cylinder. Again due to symmetry, the electric field is constant over this region, so we have:

$$
\begin{aligned}
|\vec{E}|(2\pi r L) &= \frac{\lambda L}{\epsilon_0} \\
|\vec{E}| &= \frac{\lambda}{2\pi\epsilon_0 r}
\end{aligned}
$$

We can also find the field produced by an infinite plane of charge. We will again use a cylinder as the Gaussian surface:



By a similar argument as before, the integral is non-zero only over the ends of the cylinders. Letting the surface density of the plane charge be σ, Gauss's law gives:

$$|\vec{E}|(2A) = \frac{\sigma A}{\epsilon_0}$$
$$|\vec{E}| = \frac{\sigma}{2\epsilon_0}$$

These three objects show a natural progression: a zero-dimensional charge has a field varying with the inverse square of distance, a one-dimensional charge has a field varying with the inverse of distance, and a two-dimensional charge does not vary at all with distance.

1.5 Voltage

We know that given a force, we can integrate it to arrive at a potential energy. For an electric field, we can integrate to arrive at a potential or voltage, which represents the potential energy per unit charge. Mathematically:

$$\Delta V = -\int_a^b \vec{E} \cdot d\vec{r}$$

This defines the potential difference between two points. To define an absolute potential, we must specify an arbitrary reference state, which we will take to be at ∞. Therefore:

$$V(x) = -\int_\infty^x \vec{E} \cdot d\vec{r}$$

For example, we can compute the potential due to a point charge:

$$\begin{aligned} V(x) &= -\int_\infty^x \vec{E} \cdot d\vec{r} \\ &= \int_\infty^x \frac{kq}{r^2}\, dx \\ &= \frac{kq}{x} \end{aligned}$$

Similar expressions can be computed for any electric field.

1.6 Capacitance

When a potential difference is applied to a conducting object, charge will accumulate in proportion to the potential applied. The proportionality constant is called the capacitance, and is measured in Farads:

$$C = \frac{Q}{V}$$

As an example, we can compute the capacitance of a parallel plate capacitor. Although a finite plane of charge will not have exactly the same field as an infinite plane, we can use the result for the infinite plane as an approximation. Consider two plates separated by a distance d with area A and charge Q on each (with opposite signs). Then we can find the capacitance by computing the potential difference between the plates. This is relatively simple:

$$\begin{aligned}\Delta V &= Ed \\ &= \frac{Qd}{A\epsilon_0}\end{aligned}$$

The capacitance is then:

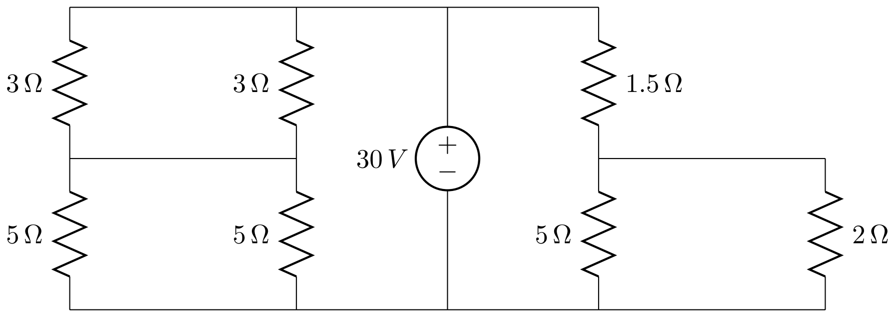
$$C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}$$

1.7 Circuits

We can now begin to analyze circuits. From Kirchoff's voltage law (the sum of the potential differences around a closed loop must be zero), we can derive formulas for the equivalent resistance or capacitance of resistor and capacitors in series and parallel:

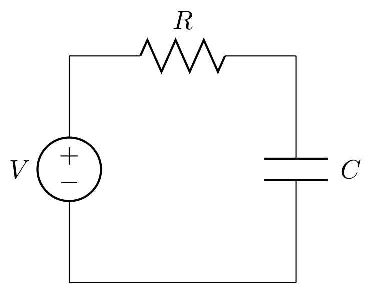
- Series Resistors: $R_{\text{eq}} = \sum R$
- Parallel Resistors: $\frac{1}{R_{\text{eq}}} = \sum \frac{1}{R}$
- Series Capacitors: $\frac{1}{C_{\text{eq}}} = \sum \frac{1}{C}$
- Parallel Capacitors: $C_{\text{eq}} = \sum C$

These rules are helpful in simplifying sections of circuits. To analyze a general resistor circuit on the whole, repeated application of Kirchoff's voltage law yields a system of equations which can be solved to give the current at any point in the circuit. For example, consider the following circuit:



The current through each of the 5 loops are the variables for a system of 5 equations. Solving these equations yields each current, which is a full description of the circuit.

A more interesting case is circuits containing both resistors and capacitors. We will look at a simple case of such circuits, and solve the differential equation for its behavior.



We can write Kirchoff's voltage law for the circuit:

$$V - V_R - V_C = 0$$

Substituting Ohm's law and the definition of capacitance gives:

$$V - IR - \frac{Q}{C} = 0$$

We know that $I = \frac{dQ}{dt}$, so we can substitute and solve the resulting differential equation:

$$\begin{aligned}
V - \frac{Q}{C} &= R\frac{dQ}{dt} \\
\frac{1}{R}\,dt &= \frac{1}{V - \frac{Q}{C}}\,dQ \\
\frac{t}{R} &= -C\ln\frac{V - \frac{Q}{C}}{V} \\
e^{-\frac{t}{RC}} &= \frac{V - \frac{Q}{C}}{V} \\
e^{-\frac{t}{RC}} &= 1 - \frac{Q}{CV} \\
Q &= CV\left(1 - e^{-\frac{t}{RC}}\right)
\end{aligned}$$

We see that the charge on the capacitor approaches a maximum value of CV exponentially. We can take the derivative to find that the current decays exponentially during this charging phase:

$$I = \frac{dQ}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}$$

We may now ask what occurs when a capacitor already charged is connected to a resistor (but no voltage source). We call this discharging the capacitor. By a similar process as above, we can arrive at the charge and current equations:

$$\begin{aligned}
Q &= Q_0 e^{-\frac{t}{RC}} \\
I &= \frac{Q_0}{RC}e^{-\frac{t}{RC}}
\end{aligned}$$

We see that in this case, both charge and current decay exponentially. Note that in this case $I = -\frac{dQ}{dt}$, because positive current is taken to be flowing away from the capacitor.

2 Magnetism

Having dealt thoroughly with charges at rest, we can now investigate the forces due to moving charges. A moving charge creates a magnetic field, and is influenced by other magnetic fields.

2.1 Lorentz Force

Since the magnetic field affects moving particles, the magnetic field (denoted $\vec{B}$) must combine with the velocity vector to yield a force vector. Naturally, this relation is given by the cross product. The Lorentz force is the sum of the electric and magnetic forces, and is the total electromagnetic force acting on a charged object:

$$F = F_E + F_B = q\vec{E} + q\vec{v} \times \vec{B}$$

As an example, we can compute the motion of a particle of charge q with a velocity perpendicular to a uniform magnetic field (and no electric field). The force is always perpendicular to the velocity of the particle, so it acts as a centripetal force, and the particle moves in a circle. We can easily compute the radius of the path:

$$\frac{mv^2}{r} = qvB$$
$$r = \frac{mv}{qB}$$

2.2 Biot-Savart Law

The expression above describes the magnetic force in terms of a field, but we need a way to compute the field itself. This is given by the Biot-Savart law, the magnetic analogy to Coulomb's law.

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2}$$

The value of μ_0 (called the permeability of free space) is exactly $4\pi \times 10^{-7}\,\mathrm{T\,m\,A^{-1}}$. The Biot-Savart law applied to a point charge with constant velocity gives a field wrapping around the velocity vector of the particle. The direction is given by the right-hand rule: if you point your right thumb in the direction of the particle's motion, your fingers curl in the direction of the $\vec{B}$ field.

The Biot-Savart law is more commonly used for currents, which contain many moving charges. The current at a point is the amount of charge that passes the point in a second. Therefore:

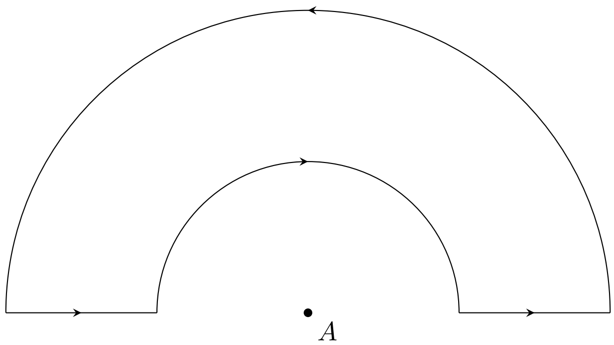
$$q\vec{v} = q\frac{d\vec{\ell}}{dt}$$
$$= \int I d\vec{\ell}$$

So we can rewrite the Biot-Savart law in terms of differential current elements:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$

2.3 Example Magnetic Field

The magnetic field due to any current-carrying wire can be found using the integral above. As an example, we will compute the magnetic field at the point labeled A below. The radius of the inner loop is r, and the radius of the outer loop is $2r$.



First, we note that the horizontal portions contribute nothing to the magnetic field at A because $d\vec{\ell}$ is parallel to $\hat{r}$. Therefore, we need only integrate along the semicircular portions. The direction $\hat{r}$ is always perpendicular to $d\vec{\ell}$, so the cross product is easily simplified. Taking into the page to be the positive $\hat{k}$ direction:

$$
\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi}\left(\int \frac{I\,d\vec{\ell}\times\hat{r}}{a^2} - \int \frac{I\,d\vec{\ell}\times\hat{r}}{4a^2}\right)\hat{k} \\
&= \frac{\mu_0}{4\pi}\left(\int_0^\pi \frac{Ia\,d\theta}{a^2} - \int_0^\pi \frac{2I\,d\theta}{4a^2}\right)\hat{k} \\
&= \frac{\mu_0}{4\pi}\left(\frac{Ia\pi}{a^2} - \frac{2I\pi}{4a^2}\right)\hat{k} \\
&= \frac{\mu_0 I}{8a}\hat{k}
\end{aligned}
$$

2.4 Ampere's Law

A magnetic analogy to Gauss's law is given by Ampere's law:

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I_{\text{enc}}$$

Like Gauss's law, Ampere's law only gives the actual magnetic field in conditions with a high degree of symmetry. One such case is an infinite wire. If we let the path be a circle with its center on the wire, then Ampere's law gives:

$$\oint \vec{B}\cdot d\vec{r} = |\vec{B}|(2\pi r) = \mu_0 I$$

And so the magnitude of the field is:

$$|\vec{B}| = \frac{\mu_0 I}{2\pi r}$$

The direction is given by the right-hand rule.

A solenoid is a dense coil of wire loops. Consider a solenoid with turn density n and current I. If we use a rectangle of length L that crosses the solenoid, so that one side lies inside the wire loops and one side lies outside, then the enclosed current is nLI. The only contribution to the integral comes from the side of the wire inside the solenoid (because there is vanishingly little field outside the solenoid, and the vertical sides are perpendicular to any field inside). Ampere's law then gives:

$$\oint \vec{B}\cdot d\vec{r} = |\vec{B}|L = \mu_0 nLI$$

And so the magnitude of the field inside the solenoid is:

$$|\vec{B}| = \mu_0 nI$$

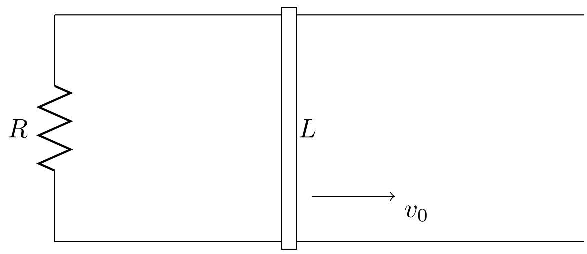
The direction is given by the right hand rule (if you curl the fingers on your right hand in the direction of current flow, your thumb points in the direction of the magnetic field).

2.5 Faraday's Law

A changing magnetic flux produces an electromotive force according to Faraday's law:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

The negative sign represents the fact that the induced EMF opposes the change in electromotive force; this is known as Lenz's law. Faraday's law can be illustrated with the following problem:



The circuit above contains a moving conductor of length L. There is a uniform magnetic field of magnitude B directed into the page. The conductor is given an initial velocity v_0 to the right. We will attempt to find the velocity of the bar as a function of time.

The magnetic flux through the circuit at any time is BA, where A, the area of the circuit, is given by Lx (where x is the position of the bar). The time derivative of the flux is then simply BLv. Faraday's law gives $\mathcal{E} = -BLv$, and so the magnitude of the current is $\frac{BLv}{R}$. This current must flow counter-clockwise in order to oppose the change in flux, so the current is moving up through the moving conductor. The force on the bar $I\vec{L}\times\vec{B}$ (which can be found by integrating the Lorentz force), and so a force of magnitude $ILB = \frac{B^2L^2v}{R}$ and directed to the left acts on the bar. Finally, we write Newton's second law and solve the differential equation:

$$F = -\frac{B^2L^2v}{R} = m\frac{dv}{dt}$$
$$\frac{dv}{dt} = -\frac{B^2L^2}{mR}v$$
$$v = e^{-\frac{B^2L^2}{mR}t}$$

2.6 Inductance

We have seen many cases where a current produces a magnetic field (and thus a magnetic flux), and we now know that a changing magnetic flux produces an EMF. We can connect these ideas by finding the EMF induced by a changing current which produces a changing magnetic flux. The EMF and the changing current can often be related by a constant inductance:

$$V = L\frac{dI}{dt}$$

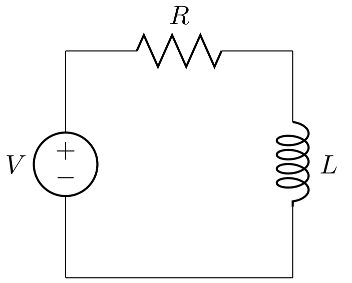
By far the most common and practical example of inductance is a solenoid. A solenoid with a loop area of A, a current I, a length ℓ, and N turns has a magnetic flux of $NA\left(\frac{\mu_0 NI}{\ell}\right) = \frac{\mu_0 N^2 AI}{\ell}$. Writing Faraday's law (and ignoring the negative sign, as is the convention for inductance) gives:

$$V = \frac{\mu_0 N^2 A}{\ell}\frac{dI}{dt}$$

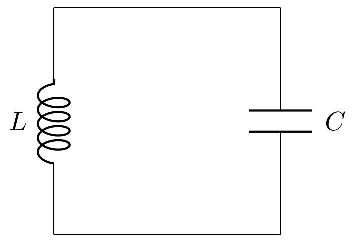
Thus, the inductance $L = \frac{\mu_0 N^2 A}{\ell}$.

2.7 Inductors in Circuits

The introduction of inductors as a third circuit elements leads to two more basic circuits, the RL circuit and the LC circuit. These are both pictured below.



RL Circuit



LC Circuit

The RL circuit has very similar dynamics as the RC circuit:

- RL Charging: $I = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$
- RL Discharging: $I = \frac{V}{R}e^{-\frac{R}{L}t}$

The LC circuit, however, has a different sort of differential equation:

$$\frac{Q}{C} + L\frac{dI}{dt} = 0$$
$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

This is the familiar equation for simple harmonic motion. We can go directly to its solution:

$$I = I_{\max}\cos(\frac{1}{\sqrt{LC}}t + \phi)$$

The value of $I_{\max}$ is determined by the initial conditions. The period of the oscillation is $2\pi\sqrt{LC}$.