# Relativistic Electrodynamics

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## 1 Introduction

Last week, we explored Coulomb's law in its potential form, $\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\epsilon_0}$. This equation allows us to determine the electric potential given a static charge distribution. We will see this week that the existence of an electric field in a Lorentz-invariant world implies the existence of a magnetic field. We will then look at all of Maxwell's equations, in both the familiar vector form and the more suggestive tensor form.

## 2 Electricity implies Magnetism

Consider an infinite wire of radius $r$ on the $x$-axis with current $I$ flowing to the right. From previous studies, you may recall that the magnetic field has a magnitude of

$$|\boldsymbol{B}| = \frac{\mu_0 I}{2\pi R}$$

and is directed in circles around the wire, with the direction determined by the right hand rule. So, a charge $q$ at a distance $d$ from the wire moving to the right will experience a force

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = \frac{\mu_0 q v I}{2\pi d}\hat{\boldsymbol{\jmath}}.$$

We have previously accepted the Lorentz force law without much scrutiny. Using relativity, we can show that the charge must experience this force, even if we do not assume the magnetic field. First, however, we need a relativistic way of looking at charges. Our single assumption, aside from the postulates of relativity, is that charge is Lorentz invariant. If this was not the case, we would certainly notice: a slight fluctuation in the charge distribution of an object due to the thermal motion of its atoms would lead to gargantuan and seemingly random forces. So, with this assumption, we can determine the transformation law of charge density $\rho$. If we have a block of length $L_0$, width $W_0$, height $H_0$, and uniformly distributed charge $Q$, then an observer flying by at speed $v$ must also observe a charge $Q$. However, he observes the length to be

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}.$$

Thus, the charge density must be

$$\rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

so that $\rho_0 L_0 W_0 H_0 = \rho L W H$. As a simple corollary, we can determine the transformation for current density. Since $\boldsymbol{j} = \rho \boldsymbol{v}$, we have

$$\boldsymbol{j} = \rho \boldsymbol{v} = \frac{\rho_0 \boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Now, note the formal similarity between expressions for charge density and current density and the expressions for mass-energy and momentum:

$$\rho' = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad\qquad m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boldsymbol{j'} = \frac{\rho_0 \boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad\qquad \boldsymbol{p} = \frac{m_0 \boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

We already know that the quantities $\begin{pmatrix} mc \\ \boldsymbol{p} \end{pmatrix}$ transform as a four-vector. Therefore, the quantities $\begin{pmatrix} \rho c \\ \boldsymbol{j} \end{pmatrix}$ transform in the same way. This is known as the four-current, which we will denote by $\boldsymbol{J}$.

We can use this result to look at our wire in the frame of the moving charge. The four-current in this frame is

$$\boldsymbol{J} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \frac{I}{r^2} \end{pmatrix} = \begin{pmatrix} -\beta\gamma \frac{I}{r^2} \\ \gamma \frac{I}{r^2} \end{pmatrix}.$$

Remarkably, we see that there is a negative charge density inside the wire in the frame of the moving charge. The linear charge density is simply $\lambda = -\beta\gamma \frac{I}{c}$, so according to Gauss's law, the electric field at distance $d$ is

$$\boldsymbol{E}(d) = -\frac{vI}{2\pi\epsilon_0 d c^2 \sqrt{1 - \frac{v^2}{c^2}}} \hat{\boldsymbol{r}}.$$

Therefore, the force on the charge in its own frame is

$$\boldsymbol{F} = \frac{1}{\epsilon_0 c^2} \frac{qvI}{2\pi d \sqrt{1 - \frac{v^2}{c^2}}} \hat{\boldsymbol{j}}.$$

Since this is the force in the frame of the charge, we have

$$\frac{dp_{y,0}}{d\tau} = \frac{1}{\epsilon_0 c^2} \frac{qvI}{2\pi d \sqrt{1 - \frac{v^2}{c^2}}},$$

where $\tau$ is the proper time and $p_{y,0}$ is the proper transverse momentum. You can show that $p_y = p_{y,0}$ since the transformation of $v_y$ balances the change in $\gamma$, so that the force in the lab frame is simply

$$\frac{dp_y}{dt} = \frac{dp_{y,0}}{d\tau} \frac{d\tau}{dt} = \frac{1}{\epsilon_0 c^2} \frac{qvI}{2\pi d}.$$

This is in perfect agreement with our result before, except that we have a constant factor $\frac{1}{\epsilon_0 c^2}$ in place of $\mu_0$. This is one way of showing a fact we will prove more rigorously later: the speed of light can be written in terms of magnetic and electric susceptibilities as

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

# 3 Four-Potential

Previously, we have seen how a charge density $\rho$ leads to a scalar potential $\phi$. Since $\rho$ is the temporal component of the four-current, we might expect that there is also a potential related to the current density $\boldsymbol{j}$. We know that, when all else fails, we can solve Poisson's equation for a given charge distribution by integrating the differential potential, obtaining

$$\phi(\boldsymbol{r}) = \frac{1}{\epsilon_0} \int \frac{\rho(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \, d^3\boldsymbol{r'}.$$

You can verify that this is a solution to $\boldsymbol{\nabla}^2 \phi = -\frac{\rho}{\epsilon_0}$. We might attempt to perform an analogous integral on each of the components of $\boldsymbol{j}$, in order to obtain a vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \, d^3\boldsymbol{r'}.$$

Ignoring the constant factor $\frac{\mu_0}{4\pi}$ for now[1], we see that each component of $\boldsymbol{j}$ appears in an integral of the same form as the $\rho$ integral. You can show that $\phi/c$ and $\boldsymbol{A}$ together form the components of a four-vector. We call this the four-potential,

$$A_\alpha = \begin{pmatrix} \phi/c \\ \boldsymbol{A} \end{pmatrix}.$$

Note that we are using index notation here: the equation above indicates that $A_0 = \phi/c$, $A_1 = \boldsymbol{A} \cdot \hat{\boldsymbol{\imath}}$, and so on.

The importance of $\boldsymbol{A}$ is not immediately apparent. We can cast it into a more familiar form by taking the curl. Remember that $\boldsymbol{r'}$ is a dummy variable in the integral, so all the derivatives are with respect to the $\boldsymbol{r}$ coordinates and we can move the curl inside the integral:

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \times \left( \frac{\boldsymbol{j}(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \right) dV.$$

Using the identity $\boldsymbol{\nabla} \times (f\boldsymbol{v}) = (\boldsymbol{\nabla} f) \times \boldsymbol{v} + f(\boldsymbol{\nabla} \times \boldsymbol{v})$ and noting that $\boldsymbol{j}(\boldsymbol{r'})$ is a constant with respect to the $\boldsymbol{r}$ coordinates, this becomes

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \left( \frac{1}{|\boldsymbol{r} - \boldsymbol{r'}|} \right) \boldsymbol{j}(\boldsymbol{r'}) \, dV = \frac{\mu_0}{4\pi} \int \frac{(\boldsymbol{r'} - \boldsymbol{r}) \times \boldsymbol{j}}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, dV.$$

This is the Biot-Savart law. You may recognize it more readily by replacing $\boldsymbol{j}\, dV$ with $I \, d\boldsymbol{\ell}$. So, we have

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B}.$$

# 4 The Faraday Tensor

We will find that the electromagnetic field is nicely summarized by an object known as the Faraday tensor. Before we develop this object, we will need a brief aside on tensors.

[1] This factor is essentially a unit conversion factor. In Gaussian units, $\epsilon_0$ and $\mu_0$ do not appear at all.

## 4.1 Tensors

Tensors are characterized by their rank and dimension. A tensor with rank $r$ and dimension $n$ is written as $X_{\alpha_1\alpha_2\cdots\alpha_r}$, where $0 \leq \alpha_i < n$ for $1 \leq i \leq r$. You are familiar with tensors of rank 1 as vectors and tensors of rank 2 as matrices.

Not all collections of $n^r$ values are tensors. For an object to be a tensor, its components must obey a specific transformation law. Whether an index is raised or lowered in the tensor determines the form of the transformation law[2], but for now we will only deal with *covariant* tensors. If we have a covariant tensor $X_{\alpha_1\alpha_2\cdots\alpha_r}$, and we change from our original coordinates $x^i$ $(1 \leq i \leq n)$ to coordinates

$$\begin{aligned}
\bar{x}^1 &= \bar{x}^1(x_1, x_2, \ldots, x_n), \\
\bar{x}^2 &= \bar{x}^2(x_1, x_2, \ldots, x_n), \\
&\vdots \\
\bar{x}^n &= \bar{x}^n(x_1, x_2, \ldots, x_n),
\end{aligned}$$

then the tensor components must transform according to

$$\bar{X}_{\beta_1\beta_2\cdots\beta_n} = \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{n}\cdots\sum_{\alpha_n=1}^{n} \frac{\partial x^{\alpha_1}}{\partial \bar{x}^{\beta_1}}\frac{\partial x^{\alpha_2}}{\partial \bar{x}^{\beta_2}}\cdots\frac{\partial x^{\alpha_n}}{\partial \bar{x}^{\beta_n}} X_{\alpha_1\alpha_2\cdots\alpha_r}.$$

This expression should be sufficient motivation for Einstein summation convention, in which repeated indices are summed over. In this notation, the transformation law becomes

$$\bar{X}_{\beta_1\beta_2\cdots\beta_n} = \frac{\partial x^{\alpha_1}}{\partial \bar{x}^{\beta_1}}\frac{\partial x^{\alpha_2}}{\partial \bar{x}^{\beta_2}}\cdots\frac{\partial x^{\alpha_n}}{\partial \bar{x}^{\beta_n}} X_{\alpha_1\alpha_2\cdots\alpha_r}.$$

**Example 1.** Given that $P_\alpha$ is a covariant vector, show that

$$Q_{yz} = \frac{\partial P_y}{\partial x^z} - \frac{\partial P_z}{\partial x^y}$$

is a covariant tensor of rank 2.

**Solution:** In order to show that these components form a tensor, we must directly verify that the necessary transformation law holds. If we transform to a coordinate system $\bar{x}^i = \bar{x}^i(x_1, x_2, \ldots, x_n)$, then the components become

$$\bar{Q}_{yz} = \frac{\partial \bar{P}_y}{\partial \bar{x}^z} - \frac{\partial \bar{P}_z}{\partial \bar{x}^y}.$$

But, we are given that $P_\alpha$ is a contravariant vector. So, we have

$$\bar{P}_s = \frac{\partial x^t}{\partial \bar{x}^s} P_t.$$

Making this substitution above, we obtain

$$\bar{Q}_{yz} = \frac{\partial}{\partial \bar{x}^z}\left(\frac{\partial x^t}{\partial \bar{x}^y} P_t\right) - \frac{\partial}{\partial \bar{x}^y}\left(\frac{\partial x^u}{\partial \bar{x}^z} P_u\right) = \left(\frac{\partial^2 x^t}{\partial \bar{x}^z \partial \bar{x}^y} P_t - \frac{\partial^2 x^u}{\partial \bar{x}^z \partial \bar{x}^y} P_u\right) + \frac{\partial x^t}{\partial \bar{x}^y}\frac{\partial P_t}{\partial \bar{x}^z} - \frac{\partial x^u}{\partial \bar{x}^z}\frac{\partial P_u}{\partial \bar{x}^y}.$$

Since $t$ and $u$ are dummy indices for summation, the two terms in the parentheses are equal and their difference vanishes. We can apply the chain rule to the remaining to terms to obtain

$$\bar{Q}_{yz} = \frac{\partial x^t}{\partial \bar{x}^y}\frac{\partial P_t}{\partial x^u}\frac{\partial x^u}{\partial \bar{x}^z} - \frac{\partial x^u}{\partial \bar{x}^z}\frac{\partial P_u}{\partial x^t}\frac{\partial x^t}{\partial \bar{x}^y} = \frac{\partial x^t}{\partial \bar{x}^y}\frac{\partial x^u}{\partial \bar{x}^z} Q_{yz}.$$

This is the required transformation law, so $Q_{yz}$ is a tensor.

[2] More detailed notes on tensor algebra and possibly tensor analysis will be posted shortly.

## 4.2 Components of the Faraday Tensor

We have proved (in the case of static current density) that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. In terms of the four-potential, this is

$$\begin{aligned} B_1 &= \frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}, \\ B_2 &= \frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}, \\ B_3 &= \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}. \end{aligned}$$

Additionally, recall from electrostatics that for a static electric potential, $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$. In terms of the four-potential, this becomes

$$\begin{aligned} cE_1 &= -\frac{\partial A_0}{\partial x^1}, \\ cE_2 &= -\frac{\partial A_0}{\partial x^2}, \\ cE_3 &= -\frac{\partial A_0}{\partial x^3}. \end{aligned}$$

Since we have assumed that nothing is time dependent, we are free to add time derivatives in order to case this into a form similar to the expressions for the $B_i$:

$$\begin{aligned} cE_1 &= \frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}, \\ cE_2 &= \frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2}, \\ cE_3 &= \frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3}. \end{aligned}$$

Following this pattern, consider the covariant tensor

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}.$$

In matrix form, this is

$$F = \begin{pmatrix} 0 & \frac{1}{c}\left(-\frac{\partial\phi}{\partial x^1} - \frac{\partial A_1}{\partial t}\right) & \frac{1}{c}\left(-\frac{\partial\phi}{\partial x^2} - \frac{\partial A_2}{\partial t}\right) & \frac{1}{c}\left(-\frac{\partial\phi}{\partial x^3} - \frac{\partial A_3}{\partial t}\right) \\ -\frac{1}{c}\left(-\frac{\partial\phi}{\partial x^1} - \frac{\partial A_1}{\partial t}\right) & 0 & -(\boldsymbol{\nabla}\times\boldsymbol{A})_3 & (\boldsymbol{\nabla}\times\boldsymbol{A})_2 \\ -\frac{1}{c}\left(-\frac{\partial\phi}{\partial x^2} - \frac{\partial A_2}{\partial t}\right) & (\boldsymbol{\nabla}\times\boldsymbol{A})_3 & 0 & -(\boldsymbol{\nabla}\times\boldsymbol{A})_1 \\ -\frac{1}{c}\left(-\frac{\partial\phi}{\partial x^3} - \frac{\partial A_3}{\partial t}\right) & -(\boldsymbol{\nabla}\times\boldsymbol{A})_2 & (\boldsymbol{\nabla}\times\boldsymbol{A})_1 & 0 \end{pmatrix}.$$

This is the Faraday tensor. It is an antisymmetric tensor, so it has six independent components. However, a species that cannot reach velocities near the speed of light might see things differently. When $v/c \ll 1$, a Lorentz transformation is approximately a Euclidean rotation. So, the components of $F_{\mu\nu}$ with $\mu \geq 1$ and $\nu \geq 1$ will transform together, and the components with $\mu\nu = 0$ will transform together. Thus, this stunted species might model the electromagnetic field with two vectors, the electric and magnetic fields, instead of a single tensor:

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \\ \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A}. \end{aligned}$$

In these terms, the Faraday tensor becomes

$$F = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix}.$$

## 4.3 Maxwell's Equations

We now have all the machinery required to give a simple derivation of the Maxwell equations. We will start with Gauss's law,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

We can write this in terms of the Faraday tensor and the four-current as

$$\frac{\partial F_{0a}}{\partial x_a} = -\frac{j_0}{c^2 \epsilon_0} = -\mu_0 j_0.$$

Note that the minus sign comes from our choice of metric signature $(+1, -1, -1, -1)$. The quantity on the right is the first component of a four-vector, and you can verify that the divergence of a four-tensor gives a four-vector as well when the only transformations have constant components (like the Lorentz transformations). Thus, we must have the general relationship

$$\frac{\partial F_{ia}}{\partial x_a} = -\mu_0 j_i.$$

For $1 \le i \le 3$, this gives the equations

$$\begin{aligned} -\frac{1}{c^2}\frac{\partial E_1}{\partial t} + \frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} &= -\mu_0 j_1, \\ -\frac{1}{c^2}\frac{\partial E_2}{\partial t} - \frac{\partial B_3}{\partial x^1} + \frac{\partial B_1}{\partial x^3} &= -\mu_0 j_2, \\ -\frac{1}{c^2}\frac{\partial E_3}{\partial t} + \frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2} &= -\mu_0 j_3. \end{aligned}$$

Combining these into a vector equation, we obtain[3]

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \left(\mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right).$$

This is Ampere's law, with the displacement current term added.

For the remaining two Maxwell laws, we can simply take the curl and divergence (respectively) of our expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of $\phi$ and $\boldsymbol{B}$. This gives

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0. \end{aligned}$$

The first equation is Faraday's law, and the second equation is Gauss's law for magnetism.

[3] Another change in sign has occurred here, because lowering the index of the four-current changes it to $(\rho c, -\boldsymbol{j})$. Apologies for the rampant minus signs, but they are almost unavoidable in Minkowski space.

## 4.4 The Lorentz Force

We began by showing that a magnetic force can be viewed as an electric force in a particle's rest frame. We can now make this argument more generally to prove the Lorentz force law $F = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}$. Consider a particle of charge $q$ moving with velocity $v$ in the $x$ direction in a lab with Faraday tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix}.$$

In order to calculate the force on the particle, we need to determine the Faraday tensor in the rest frame of the particle. We use the Lorentz transformation[4]

$$L^{\alpha}_{\beta} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to obtain

$$\begin{aligned} F'_{\alpha\beta} = L^{\alpha}_{\mu} F_{\mu\nu} L^{\nu}_{\beta} &= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E_1/c & \gamma(E_2 - vB_3)/c & \gamma(E_3 + vB_2)/c \\ -E_1/c & 0 & -\gamma B_1 - \frac{\beta\gamma}{c}E_2 & \gamma B_2 - \frac{\beta\gamma}{c}E_3 \\ -\gamma(E_2 - vB_3)/c & \gamma B_1 + \frac{\beta\gamma}{c}E_2 & 0 & -B_1 \\ -\gamma(E_3 + vB_2)/c & -\gamma B_2 + \frac{\beta\gamma}{c}E_3 & B_1 & 0 \end{pmatrix}. \end{aligned}$$

Now, the force is simply given by

$$\boldsymbol{F}' = q\boldsymbol{E'} = qE_1\hat{\boldsymbol{\imath}} + q\gamma(E_2\hat{\boldsymbol{\jmath}} + E_3\hat{\boldsymbol{k}}) + qv\gamma(-B_3\hat{\boldsymbol{\jmath}} + B_2\hat{\boldsymbol{k}}) = q\boldsymbol{E}_{\parallel} + q\gamma\boldsymbol{E}_{\perp} + q(v\hat{\boldsymbol{\imath}} \times \boldsymbol{B})\gamma.$$

This force is equal to $\frac{d\boldsymbol{p'}}{d\tau}$. Remembering that $\tau = \gamma t$, $p_{\parallel} = \gamma p'_{\parallel}$, and $p_{\perp} = p'_{\perp}$, we obtain

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E}_{\parallel} + q\boldsymbol{E}_{\perp} + q(v\hat{\boldsymbol{\imath}} \times \boldsymbol{B}) = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}.$$

This is the familiar Lorentz force law.

[4]The change in sign on the off-diagonal terms is due to index business.