Fluid Mechanics

Ross Dempsey

January 2016

1 Introduction

Fluid mechanics is a vast topic. This is because there is more than one type of fluid that needs to be studied in order to garner an understanding of the various phenomena associated with non-solid objects. These questions include:

- Compressible or incompressible?
- Steady or unsteady?
- Viscous or inviscid?
- Rotational or irrotational?
- How many dimensions?

We will only cover a subset of the many possibilities here. However, this should prepare you for most elementary problems involving fluids.

2 Descriptions of Fluids

There are two ways of describing the behavior of a fluid. One is the *Lagrangian* description, which gives a full history of every particle in the fluid. In the Lagrangian particle, every position in space is assigned a function of time which gives the path of the particle originating at that point. So, we have a vector function

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t).$$

This description is not commonly used, since its detailed descriptions tend to be unwieldly.

Instead, the *Eulerian* description is employed. The Eulerian description of a fluid gives properties of a particular point in space, disregarding which particles of fluid end up where. So, for example, the velociy field of a fluid is given as a function of the spatial coordinates:

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t).$$

This leads us to a new kind of derivative. If we are simply interested in the change of the velocity field at a particular point in space, then we can use the ordinary derivative and obtain

$$\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{r}, t).$$

However, this derivative does not correspond to an acceleration of any particle, because the particles at $\boldsymbol{r}$ at time t will be replaced by new particles at time $t+\Delta t$. In order to determine the acceleration of actual particles, we need to consider both the change in the velocity field itself and the change in the velocity field as the particles move from $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{v}\Delta t$. For example, we have

$$v_x(t+\Delta t) - v_x(t) = \frac{\partial v_x}{\partial t}\Delta t + \frac{\partial v_x}{\partial x}(v_x \Delta t) + \frac{\partial v_x}{\partial y}(v_y \Delta t) + \frac{\partial v_x}{\partial z}(v_z \Delta t).$$

Dividing by Δt forms the *material derivative* of $\boldsymbol{v}$,

$$\frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}.$$

3 Pressure

One of the most important properties of a fluid is its pressure field. The pressure of a fluid is the force per unit area exerted at its boundary. For example, a stagnant cup of water has a pressure of 1 atm at its boundary with the air in order to balance the atmospheric pressure.

Underwater equipment always has to be designed to handle large pressures. This is because the weight of water is much greater than that of air, so descending even a few meters in water can lead to significant pressure increases. We can determine the change in pressure exactly simply by balancing forces. Consider a cubic section of water with side length a, with two of its faces aligned with the z axis. Then, the forces in each direction are

$$\begin{aligned}
F_x &= a^2(P(x_0) - P(x_0+a)), \\
F_y &= a^2(P(y_0) - P(y_0+a)), \\
F_z &= a^2(P(z_0) - P(z_0+a)) - \rho g a^3.
\end{aligned}$$

Clearly, in order to balance forces in the x and y directions, we need $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0$. In order to balance forces in the z direction, we need

$$\frac{P(z_0+a) - P(z_0)}{a} = -\rho g.$$

Taking the limit as $a \to 0$, we obtain

$$\frac{\partial P}{\partial z} = -\rho g.$$

Therefore, the pressure in a static fluid will increase linearly with depth. We can calculate the magnitude of this pressure for water. Water has a density of $1000\,\mathrm{kg/m^3}$, and we can let $g = 10\,\mathrm{m/s^2}$. Then, a descent of 10 m would lead to a change in pressure of $100\,\mathrm{kPa} \approx 1\,\mathrm{atm}$. Thus, the weight of the entire atmosphere is equivalent to the weight of only 10 meters of water.

Now that we have determined the pressure field, we can understand the forces on objects submerged in water. We can assume that the only pressure is vertical, and replace the pressure with $\boldsymbol{P} = P\hat{\boldsymbol{k}}$. If our object has a boundary with the water at the surface S, then the force on it will be given by

$$F = -\oiint \boldsymbol{P}\cdot d\boldsymbol{A},$$

where the negative sign accounts for the inward-facing force. We can use the divergence theorem to write this as

$$F = -\iiint \boldsymbol{\nabla}\cdot\boldsymbol{P}\,dV = \iiint \rho g\,dV.$$

Both ρ and g are constants, so we can pull them out of the integral and obtain a total force of $\rho g V$, the weight of the displaced fluid. This is known as *Archimedes' Principle*: an object submerged in fluid will feel a buoyant force equal to the weight of the displaced fluid.

4 Classification of Fluid Flows

Determining the velocity field of a fluid allows us to make two important classifications. These classifications can greatly simplify the analysis of a fluid.

4.1 Compressibility

Consider a one-dimensional fluid. At a given point x, if the velocity to the left of x and to the right of x differ, then there will be a net influx or outflux of fluid at x. This is indicative of a compressible fluid. However, we would not expect this type of behavior in water. In a thin tube of water, if one edge is moving with velocity v, then the entire column of water would move with the same velocity. This is because water is incompressible.

In more than one dimension, determinining compressibility requires looking at the influx and outflux of fluid in each direction. If we have a compressible fluid, then adding the net change in each direction should give 0. So,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$$

Or, more generally,

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0.$$

Thus, an incompressible fluid has a divergence-free velocity field.

4.2 Irrotational Flow

Most common fluid behavior exhibits rotational motion. For example, water going down a drain forms a familiar vortex pattern. However, sometimes we impose the condition that this rotational motion is absent. Mathematically, we require that the curl of the velocity field vanishes:

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 0.$$

We know that this holds whenever we write $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ for some potential function ϕ. If our fluid is also incompressible, then we require that

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla}^2\phi = 0.$$

This is known as *Laplace's equation*, and can be found in many branches of physics. There are well-known methods for solving it analytically as well as numerically. Additionally, complex analytic functions have real and imaginary parts which satisfy Laplace's equation, so two-dimensional problems involving incompressible and irrotational fluids can be solved using the methods of complex analysis.

5 Viscous Flow

Real fluids have an internal frictional force known as viscosity. This force acts between surfaces of fluid, and allows faster-moving surfaces to drag slower-moving ones along. The force is proportional to the area of contact and the difference in velocity. So, if a fluid is moving between two plates at $z = 0$ and $z = a$, and the top plate is moving in the y direction, then the force between layers would be given by

$$F = \eta S \frac{dv}{dz},$$

where S is the contact area. The constant η is the viscosity of the fluid.

We can directly apply this result to cases of laminar flow, in which a fluid moves in layers. For example, consider a pipe of radius R directed along the x axis, with a pressure p_1 at $x = 0$ and a pressure p_2 at $x = \ell$. Then, the force due to pressure on a cylindrical layer at a radius r is

$$F_p = 2\pi r\, dr(p_1 - p_2).$$

This force must be balanced with the forces due to viscosity. We know that the fluid will move fastest at the center since the walls of the pipe slow it down, so $\frac{dv}{dr}$ is negative. The layer just inside the layer at r should drag our layer forward, so

$$F_1 = -\eta(2\pi r\ell\, dr)\frac{dv}{dr}.$$

Likewise, the layer just outside our layer will drag it backwards, so

$$F_2 = \eta(2\pi(r + dr)\ell\, dr)\left(\frac{dv}{dr} + \frac{d^2v}{dr^2}\, dr\right).$$

Adding all thse, we obtain

$$r(p_1 - p_2) = \eta(r\ell)\frac{dv}{dr} - \eta((r + dr)\ell)\left(\frac{dv}{dr} + \frac{d^2v}{dr^2}\, dr\right) = -\eta\ell\left(r\frac{d^2v}{dr^2} + \frac{dv}{dr}\right).$$

We can rewrite this as

$$r(p_1 - p_2) = -\eta\ell\frac{d}{dr}\left(r\frac{dv}{dr}\right).$$

Integrating this gives

$$\frac{dv}{dr} = -\frac{(p_1 - p_2)r}{2\eta\ell}.$$

Integrating once more gives

$$v = -\frac{(p_1 - p_2)r^2}{4\eta\ell} + C.$$

Since the velocity must vanish at the wall, we have $C = \frac{(p_1 - p_2)R^2}{4\eta\ell}$. So, the velocity in the pipe is given by

$$v = \frac{(p_1 - p_2)(R^2 - r^2)}{4\eta\ell}.$$

We can integrate this to determine the total flow rate. This gives

$$\mathcal{F} = \int_0^R \frac{(p_1 - p_2)(R^2 - r^2)}{4\eta\ell} 2\pi r\, dr = \frac{\pi(p_1 - p_2)R^4}{8\eta\ell}.$$

This is known as *Poiseuille's law.*