Lagrangian Mechanics

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1 Introduction

To start this year, we will look at the beginnings of a new way of formulating Newtonian mechanics. Lagrangian mechanics is one topic studied in the broader realm of analytical mechanics, sometimes called variational mechanics or theoretical mechanics.

Before we begin, we should answer an important question: why reformulate mechanics? In some sense, there is no need to. Newton's laws are sufficient to predict the future state of any system of masses given its initial conditions. This is the chief problem of mechanics, and it was solved long before Lagrange or any of the other characters behind analytical mechanics. The reason we use Lagrangian mechanics, or any of the other tools of variational mechanics, is because they are better. This is not a matter of opinion. Newton's laws for N particles require $3N$ second-order and (usually) coupled differential equations. The Lagrangian principle is formulated in terms of just two scalar quantities: the kinetic and the potential energies. This may seem impossible, but it can be accomplished by using these quantities in a variational principle rather than in a simple equation.

2 Calculus of Variations

One of the most useful applications of calculus in physics is determining extrema of functions. More precisely, calculus allows us to find *stationary values* of functions, where their derivatives are zero. It is easy to extend this technique to functions of multiple variables; we simply require that all the partial derivatives vanish.

The calculus of variations deals with a similar but much more difficult problem: finding the stationary values of an integral by varying the path. There is no way to extend the usual techniques to this case: a path cannot be described by a finite collection of variables. Instead, we must start from scratch. So, consider a function $F(x, y, y')$. We specify two endpoints, (a, α) and (b, β), and seek to find the minimum value of the integral

$$I = \int_a^b F(x, y, y')\, dx.$$

We are free to choose the function $y = f(x)$, so long as it is differentiable and satisfies the boundary conditions $f(a) = y_1$ and $f(b) = y_2$. We will start with a non-rigorous derivation of the central result, due to Euler. The definite integral can be approximated by a right Riemann sum,

$$I \approx \sum_{i=0}^{n} F\left(x_i, y_{i+1}, \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)(x_{i+1} - x_i),$$

where $x_0 = a$, $x_{n+1} = b$, $y_0 = \alpha$, and $y_{n+1} = \beta$. Now we can resort to our old techniques for finding stationary values, since the path is described by the n quantities $y_1, y_2, \ldots, y_n$. So, we can simply set the derivatives with respect to each one to be zero. The derivative with respect to y_{i+1} is

$$\frac{\partial I}{\partial y_{i+1}} = \left(\frac{\partial F}{\partial y}\right)_{x=x_i} (x_{i+1} - x_i) + \left(\frac{\partial F}{\partial y'}\right)_{x=x_i} - \left(\frac{\partial F}{\partial y'}\right)_{x=x_{i+1}}.$$

These derivatives must vanish for $i = 0, 1, \ldots, n-1$. If we divide through by $\Delta x = x_{i+1} - x_i$, then we obtain

$$\frac{\partial F}{\partial y} - \frac{1}{\Delta x}\Delta\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Now, we allow Δx to become arbitrarily small, and this becomes the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

This is called the *Euler-Lagrange equation*. Solving it gives the function $y = f(x)$ which makes the integral stationary.

As a trivial example, we can use the Euler-Lagrange equation to determine the path between two points which minimizes the distance. The distance, or arc length, is found by integrating

$$\int_a^b \sqrt{1 + (y')^2}\, dx,$$

so we let $F(x, y, y') = \sqrt{1 + (y')^2}$. The Euler-Lagrange equation gives

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}} = 0.$$

So, y'' must be zero. This is the condition for a straight line. The two endpoints determine a unique curve with constant y', the line between them.

Example 2.1. Determine the path that minimizes the integral

$$\int_a^b \frac{yy'}{x}\, dx.$$

Solution: We can immediately determine the Euler-Lagrange equation,

$$\frac{y'}{x} = \frac{y'}{x} - \frac{y}{x^2}.$$

Conveniently, y' cancels, and we have

$$y = 0.$$

You can verify by making small perturbations to this function that it is a saddle point in function space.

3 d'Alembert's Principle[1]

We can use the idea of an inertial force to rewrite Newton's second law in a form which gives it new meaning. We know that

$$\boldsymbol{F} = m\boldsymbol{a},$$

so we can write

$$\boldsymbol{F} - m\boldsymbol{a} = 0.$$

If we label the inertial force $-m\boldsymbol{a}$ as $\boldsymbol{I}$, then this becomes

$$\boldsymbol{F} + \boldsymbol{I} = 0.$$

This shows that the sum of all forces, real and inertial, is zero. This is a condition for equilibrium; in the language of acclerated frames, it is equivalent to stating that a particle is in equilibrium in a frame moving with it. This is d'Alembert's principle; it allows us to look at dynamical problems in static terms.

We usually write this principle in terms of *virtual work*. Equilibrium occurs when no infinitesimal displacement of a system will result in first order quantities of work. We write this as

$$\boldsymbol{F}_{\text{tot}} \cdot \delta\boldsymbol{r} = 0,$$

where $\delta\boldsymbol{r}$ is an arbitrary *virtual displacement*. In the case of d'Alembert's principle, this becomes

$$(\boldsymbol{F} - m\boldsymbol{a}) \cdot \delta\boldsymbol{r} = 0.$$

In a more general case, where the system contains many particles, we simply say that the total virtual work vanishes:

$$\sum_i (\boldsymbol{F}_i - m_i\boldsymbol{a}_i) \cdot \delta\boldsymbol{r}_i = 0,$$

where all the $\delta\boldsymbol{r}_i$ are arbitrary virtual displacements.

We can illustrate the utility of this principle by using it to prove the conservation of energy. We will start by assuming that the forces are conservative, so that

$$\sum_i (\boldsymbol{F}_i \cdot \delta\boldsymbol{r}_i) = -\delta V.$$

Then d'Alembert's principle becomes

$$\delta V + \sum_i m_i\boldsymbol{a}_i \cdot \delta\boldsymbol{r}_i = 0.$$

Since this holds for arbitrary displacements, we can substitute $\delta\boldsymbol{r}_i = d\boldsymbol{r}_i = \dot{\boldsymbol{r}}_i\, dt$, the actual displacements. Then $\delta V = dV$, and we have

$$dV + \sum_i m_i\boldsymbol{a}_i \cdot \boldsymbol{v}_i\, dt = 0.$$

The second term on the left side can be rewritten as $d\left(\frac{1}{2}\sum_i m_i\boldsymbol{v}_i^2\right)$. If we call this quantity T, then we have

$$d(T + V) = 0,$$

so that the total energy $T + V$ is constant.

[1] Excerpted with minor modifications from *Mechanics: An Introduction with Advanced Topics*.

4 The Lagrangian Equations

So far, we have developed the calculus of variations for minimizing integrals and d'Alembert's principle as a formulation of mechanics. We will now manipulate d'Alembert's principle so that we can use the calculus of variations with it. We start by integrating the virtual work with respect to time, forming the variation of a quantity called *action*:

$$\delta A = \int_{t_1}^{t_2} (\boldsymbol{F} - m\boldsymbol{a}) \cdot \delta \boldsymbol{r}\, dt = 0.$$

The last equality follows from d'Alembert's principle. We can evaluate the two terms of the integral separately. The first is trivial:

$$\int_{t_1}^{t_2} \boldsymbol{F} \cdot \delta \boldsymbol{r}\, dt = -\int_{t_1}^{t_2} \delta V\, dt = -\delta \int_{t_1}^{t_2} V\, dt.$$

Note that we have assumed that the forces $\boldsymbol{F}$ are conservative and have a potential energy V. The second term can be evaluated with an integration by parts, giving

$$-m \int_{t_1}^{t_2} \frac{d\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r}\, dt = -m\boldsymbol{v} \cdot \delta \boldsymbol{r}\Big|_{t_1}^{t_2} + m \int_{t_1}^{t_2} \boldsymbol{v} \cdot \delta \frac{d\boldsymbol{r}}{dt}\, dt.$$

We now impose an important condition: the virtual displacement must vanish at the boundaries. So, we are concerned with paths varied between fixed boundaries. This causes the integrated part to vanish. We are left with

$$m \int_{t_1}^{t_2} \boldsymbol{v} \cdot \delta \frac{d\boldsymbol{r}}{dt}\, dt = \delta \int_{t_1}^{t_2} \frac{1}{2} m v^2\, dt = \delta \int_{t_1}^{t_2} T\, dt,$$

where $T = \frac{1}{2} m v^2$ is the kinetic energy.

So, in total, we have

$$\delta A = \delta \int_{t_1}^{t_2} (T - V)\, dt = 0.$$

The quantity $T - V$ clearly holds some significance. We call it the *Lagrangian* $\mathcal{L}$, so that we have

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0.$$

This is clearly a problem for the calculus of variations. We can start by considering a particle in one dimension, with a potential $V(x)$. Then the Lagrangian is $\frac{1}{2} m \dot{x}^2 - V(x)$, so the Euler-Lagrange equation gives

$$-\frac{dV}{dx} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}.$$

This is, of course, Newton's second law for the particle.

In general, though, we would like to consider many dimensions. In Cartesian coordinates, it is clear that we can consider the variation in each direction separately. So, we have three Euler-Lagrange equations corresponding to the three components of Newton's second law:

$$\begin{aligned}
\frac{d\mathcal{L}}{dx} &= \frac{d}{dt}\frac{d\mathcal{L}}{dv_x}, \\
\frac{d\mathcal{L}}{dy} &= \frac{d}{dt}\frac{d\mathcal{L}}{dv_y}, \\
\frac{d\mathcal{L}}{dz} &= \frac{d}{dt}\frac{d\mathcal{L}}{dv_z}.
\end{aligned}$$

So far, though, this gives no improvement on the standard Newtonian formulation. The main advantage comes when we move to a new coordinate system. If we have

$$\begin{aligned} x &= f(q_1, q_2, q_3), \\ y &= g(q_1, q_2, q_3), \\ z &= h(q_1, q_2, q_3), \end{aligned}$$

then the Lagrangian becomes

$$\mathcal{L}\left(x, y, z, \sum_{i=1}^{3} \frac{dx}{dq_i}\dot{q}_i, \sum_{i=1}^{3} \frac{dy}{dq_i}\dot{q}_i, \sum_{i=1}^{3} \frac{dz}{dq_i}\dot{q}_i, t\right).$$

So, we have

$$\frac{d\mathcal{L}}{dq_1} = \frac{d\mathcal{L}}{dx}\frac{dx}{dq_1} + \frac{d\mathcal{L}}{dy}\frac{dy}{dq_1} + \frac{d\mathcal{L}}{dz}\frac{dz}{dq_1}$$

and

$$\frac{d\mathcal{L}}{d\dot{q}_1} = \frac{d\mathcal{L}}{d\dot{x}}\frac{dx}{dq_1} + \frac{d\mathcal{L}}{d\dot{y}}\frac{dy}{dq_1} + \frac{d\mathcal{L}}{d\dot{z}}\frac{dz}{dq_1}.$$

Since $\frac{dx}{dq_1}$ and similar differentials do not vary with time, it is clear that the Euler-Lagrange equations in x, y, and z imply that

$$\frac{d\mathcal{L}}{dq_1} = \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{q}_1}.$$

The same holds for q_2 and q_3. In general, the Euler-Lagrange equations are invariant under well-behaved coordinate transformations.

Example 4.1. Determine the equations of motion for a general double pendulum, as shown in Figure 1.

Solution: Since the Lagrangian equations are invariant under coordinate transformations, we can choose from the start to work with θ_1 and θ_2, since these are the natural parameters of the state of the system. You can verify that the potential energy is simply

$$V = -(m_1 + m_2)g\ell_1 \cos\theta_1 - m_2 g\ell_2 \cos\theta_2.$$

The kinetic energy is slightly more difficult, since the motion of m_2 is influenced by the motion of m_1. We can write the positions of the masses as Cartesian vectors:

$$\begin{aligned} \boldsymbol{r}_1 &= \langle \ell_1 \sin\theta_1, \ell_1 \cos\theta_1 \rangle, \\ \boldsymbol{r}_2 &= \langle \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2, \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2 \rangle. \end{aligned}$$

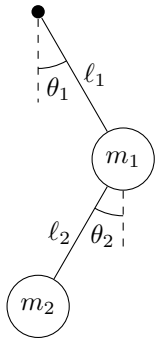

Figure 1

Then the squared velocities are clearly

$$\begin{aligned} v_1^2 &= \ell_1^2\dot{\theta}_1^2, \\ v_2^2 &= \ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\cos(\theta_1 - \theta_2). \end{aligned}$$

So, we have

$$T = \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right).$$

Now, we can form the Euler-Lagrange equations. With respect to θ_1, we have

$$-(m_1 + m_2)g\ell_1 \sin\theta_1 = (m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2\ell_1\ell_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2).$$

With respect to θ_2, we have

$$-m_2 g\ell_2 \sin\theta_2 = m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_2^2\sin(\theta_1-\theta_2).$$

These are the correct equations of motion. Note that we did not need to use Newton's second law anywhere. In particular, the tensions did not come into play whatsoever, since they do not contribute to the potential. Determining these equations by balancing forces would be very tedious.

4.1 Cyclic Coordinates

When working in Cartesian coordinates, the momentum is

$$p_x = mv_x = \frac{d\mathcal{L}}{dv_x},$$

with similar relations holding in the other directions. When working with generalized coordinates, we can extend this definition to form the generalized momentum:

$$p_i = \frac{d\mathcal{L}}{d\dot{q}_i}.$$

In terms of the momentum, the Euler-Lagrange equation becomes $\dot{p}_i = \frac{d\mathcal{L}}{dq_i}$. This is simply a relabeling of terms. However, this equation takes an important form when the Lagrangian $\mathcal{L}$ does not depend on q_i. In this case, we have

$$\dot{p}_i = 0,$$

so that p_i is a constant of the motion. In this case, q_i is called a *cyclic coordinate*.

Example 4.2. Consider a body of mass m in gravitational orbit around a star of mass M, which is fixed at the origin. Determine the Lagrangian in polar coordinates and show that θ is a cyclic coordinate.

Solution: The kinetic energy is given by $\frac{1}{2}m(r\dot{\theta})^2$, and the potential energy is $-\frac{GMm}{r}$. So, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(r\dot{\theta})^2 - \frac{GMm}{r}.$$

Since θ does not appear in this expression, it must be a cyclic coordinate. This means that p_θ is conserved. So,

$$p_\theta = \frac{d\mathcal{L}}{d\dot{\theta}} = mr^2\dot{\theta} = \text{const.}$$

This is familiar as the angular momentum. Note that the Lagrangian formulation gives a clear way to determine whether angular momentum is conserved. Rather than verifying that all torques vanish, we simply show that the potential is independent of θ.

Example 4.3. Consider a point mass constrained to move on a sphere under the influence of gravity. It is launched at the equator with a horizontal velocity v_0. Determine $\dot{\phi}$ as a function of θ.

Solution: The kinetic energy is given by $\frac{1}{2}mr^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$, and the potential energy is $mgr\cos\theta$. So, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mr^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgr\cos\theta.$$

Since ϕ does not appear in this expression, we can conclude that it is a cyclic coordinate and that

$$p_\phi = \frac{d\mathcal{L}}{d\dot{\phi}} = mr^2\dot{\phi}\sin^2\theta = \text{const.}$$

Additionally, we have the initial conditions $\theta = \frac{\pi}{2}$ and $\dot{\phi} = \frac{v_0}{r}$. So,

$$mr^2\dot{\phi}\sin^2\theta = mv_0 r,$$

and we have $\dot{\phi} = \frac{v_0}{r}\csc^2\theta$.