Calculus

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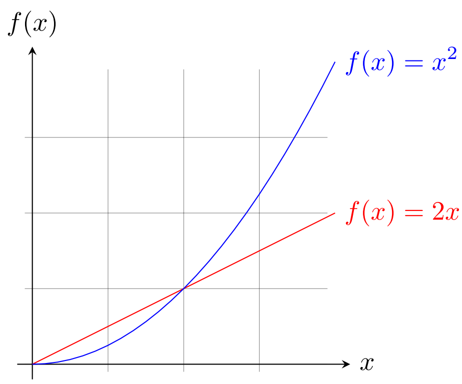
1 Introduction

Calculus is the study of infinitesimal change and how it relates to the the standard arithmetic of functions. There are two main operations studied in calculus, and one of the most important results that will be presented is their hidden complementarity. These operations are the *derivative* and the *integral*. The derivative will be examined first, followed by the integral, and then the theorem that relates them.

2 The Differential Calculus

The two functions shown to the right may appear disconnected: one is a quadratic function, and the other is linear. However, one may note that the slope, or the rate of change, of x^2 increases with x. Likewise, the value of $2x$ increases with x. We may conjecture that the slope of x^2 is a function of x itself, and moreover that we may be able to find an expression for this function. However, we only know how to find the slope of a line, which must be determined by two points; although the tangent line to a curve is intuitively well-defined, we have only one point on it. Thus, we must view the slope of the tangent line as the limit of the slopes of secant lines in the neighborhood of x. Formally,



$$\begin{aligned}
\text{slope} &= \lim_{h\to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h = 2x
\end{aligned}$$

Therefore, we find that the two functions plotted above are in fact related: the slope of $f(x) = x^2$ at any point is equal to $2x$. This idea of a slope-function is the first major concept in calculus: the derivative.

> **Definition:** *Derivative* – Given a function $f(x)$, the derivative of f gives the slope of the curve $y = f(x)$ at x according to the limit $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

We denote the derivative of $y = f(x)$ as either $f'(x)$ or $\frac{dy}{dx}$. The definition given above is a very inconvenient way of computing the derivative. It can be used to derive more straightforward rules for differentiating a function. For example, the process for differentiating x^2 above can be applied to a general monomial ax^n to yield the derivative anx^{n-1}. The most commonly used rules and their names are given below.

- Sum Rule: $\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x)$
- Power Rule: $\frac{d}{dx}\left[ax^n\right] = anx^{n-1}$
- Product Rule: $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x)+f(x)g'(x)$
- Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
- Chain Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$

In addition, some derivatives cannot be readily obtained using the above rules and should be memorized.

$$\frac{d}{dx}\left[\sin x\right] = \cos x \qquad \frac{d}{dx}\left[\cos x\right] = -\sin x \qquad \frac{d}{dx}\left[\tan x\right] = \sec^2 x \qquad \frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$$

In addition, the derivative of a function of the form $\left(f(x)\right)^{g(x)}$ cannot be found with the rules above. Through a technique known as logarithmic differentiation, the following rule can be obtained.

$$\frac{d}{dx}\left[\left(f(x)\right)^{g(x)}\right] = \left(f(x)\right)^{g(x)}\left(g'(x)\ln f(x)+\frac{g(x)f'(x)}{f(x)}\right)$$

The Mean Value Theorem, stated below, is a useful result about the slope of a secant between two points on a function and the value of the derivative on the interval between those two points.

Mean Value Theorem. Given a function f differentiable over the interval $[a,b]$, there exists $c \,\epsilon\, [a,b]$ such that $f(c) = \frac{f(b)-f(a)}{b-a}$.

3 The Integral Calculus

Having discovered how to find the infinitesimal rate of change at every point of a function, we now must attempt a different sort of infinitesimal calculation: finding the area under a function. The animation to the right illustrates how this computation must invoke the infinitesimal. Because the tools of elementary geometry only enable us to find the areas of simple regions such as rectangles, we must determine the area of a curved region by summing the areas of increasingly thin rectangles. To make this approximation exact, we must take the limit of the sum as the number of rectangles approaches infinity. This notion is formalized by a Riemann sum:

$$\text{area} = \lim_{N\to\infty}\sum_{i=0}^{N}\frac{b-a}{N}f\left(a+i\frac{b-a}{N}\right)$$

This expression is heavily inconvenient; in calculus, this operation of taking an infinite sum over infinitesimal parts is so important that we assign to it a new notation and name.

Definition: *Definite Integral* – Given a function f continuous over $[a, b]$, the definite integral of f over the interval $[a, b]$ is denoted as $\int_a^b f(x)\,dx$, and is defined as $\lim_{N\to\infty} \sum_{i=0}^{N} \frac{b-a}{N} f\left(a + i\frac{b-a}{N}\right)$.

Like the definition of the derivative, this definition suggests only a very inefficient method of finding definite integrals. However, an extremely important result connecting the derivative and the definite integral allows for a simpler method of integration.

The Fundamental Theorem of Calculus. Given a continuous function f defined on $[a, b]$, and let $F(x) = \int_a^x f(x)\,dx$. Then F is differentiable on (a, b) and $F'(x) = f(x)$ over this open interval. Furthermore, $\int_a^b f(x)\,dx = F(b) - F(a)$.

In other words, differentiation is the inverse operation of integration, and the anti-derivative can be used to compute definite integrals. However, writing the anti-derivative of a function in terms of a definite integral is cumbersome and requires the introduction of an arbitrary constant a. Instead, we define a new type of integration that refers to an anti-derivative instead of an area.

Definition: *Indefinite Integral* – Given a continuous function f defined on $[a, b]$, the indefinite integral $\int f(x)\,dx = \int_a^x f(x)\,dx$ for some arbitrary constant a.

In this definition, the constant remains, but is not included in the notation. Instead, $\int f(x)\,dx$ refers to *an* anti-derivative of $f(x)$, of which there are infinitely many all differing by a constant. For example, consider $f(x) = 2x$. Any function $F(x) = x^2 + C$ satisfy $F'(x) = f(x)$, because the constant C disappears in the process of differentiation. While the fundamental theorem of calculus does make integration simpler, the question remains of how to find an anti-derivative of a function. Unfortunately, the process is not as straight-forward as for differentiation; in fact, not every elementary function possesses an integral that can be defined in terms of elementary functions. Nonetheless, every rule of differentiation has an analog in integral form (although the quotient rule is redundant, because it is simply the product rule for a reciprocal of a function). These rules are presented here:

- Sum Rule: $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- Power Rule: $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C$
- Chain Rule: $\int f'(g(x))g'(x)dx = f(g(x)) + C$
  Note: this rule is useful when both a function and its derivative appear within a function to be integrated. For example, $\int 2x(x^2+3)^3\,dx = \int u^3\,du$ where $u = x^2 + 3$. The integral in terms of u can be evaluated by the power rule, and then $x^2 + 3$ can be substituted back into the expression.

- Product Rule: $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$
  Note: this rule is useful when one factor of the integrand can be differentiated away to yield a simpler integral. For example, $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$.

Learning to apply these rules in an efficient path to the solution is a skill that may take considerable practice.

4 Taylor Approximation

Considering the rules above, polynomials are by far the simplest functions to differentiate and integrate. Therefore, it is often very important to be able to approximate functions by polynomials. This can be done by using the *Taylor polynomial* of a function.

Definition: *Taylor Polynomial* – Analytic functions (which include most elementary functions) can be approximated in the neighborhood of a point $\big(x_0, f(x_0)\big)$ by their Nth order Taylor polynomial: $f(x) \approx \sum_{i=0}^{N} \frac{f^i(x-x_0)}{i!} x^i$. In this expression, $f^i(x-x_0)$ denotes the ith derivative of f evaluated at $x - x_0$.

A Taylor polynomial of infinite order, which is exactly equal to its associated analytic function, is called a Taylor series. A Taylor polynomial where $x_0 = 0$ is called a Maclaurin polynomial (or likewise a Maclaurin series). Some common examples of Maclaurin series follow.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots$$
$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots$$
$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots$$

One may notice that the sum of the sine and cosine expansions closely resemble the exponential expansion: the two series contain the same terms, differing only in sign. While the exponential series has all positive terms, the series for $\sin x + \cos x$ has a sign pattern of positive, positive, negative, negative. This is equivalent to the sign pattern for the powers of i: i, -1, $-i$, 1, i, ... The powers of i can easily be inserted into the exponential expansion by substituting ix for x:

$$e^{ix} = 1 + ix - \frac{1}{2}x^2 - \frac{i}{6}x^3 + \frac{1}{24}x^4 + \dots$$

This series resembles the two trigonometric series much more closely. The expansion of $\cos x$ can be seen immediately, and the expansion of $\sin x$ multiplied by i accounts for the remaining terms. We thus have the following important result:

Euler's Formula. For any real number x, $e^{ix} = \cos x + i \sin x$.

Furthermore, letting $x = \pi$ yields one of the most famous equations in mathematics:

$$e^{i\pi} + 1 = 0$$

5 Exercises

1. Prove the power rule for positive integer powers using the definition of the derivative. *Hint:* use the binomial theorem.

2. Find the derivative of $\sin\left(\frac{x}{k}\right) \ln(kx)$.

3. Find an integral of $\frac{1}{x^2+9}$.

4. Find the first 4 terms of the Taylor series of $\frac{1}{1+x}$.