Optimization and Computational Linear Algebra for Data Science
Homework 1: Vector spaces

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- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will note affect your grade).
- Late submissions will be graded with a penalty of 10% per day late. Weekend days do not count: from Friday to Monday count 1 day.
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact myself (leo.miolane@gmail.com) or to stop at the office hours.

Problem 1.1 (2 points)**.** *Let u, v be two vectors of $\mathbb{R}^2$. Show that either they are linearly dependent or that they span the whole of $\mathbb{R}^2$.*

Proof. If u, v are linearly dependent we can assume that there exists some scalar $\lambda \in \mathbb{R}^2$ such that:

$$u = \lambda v$$

Now that we know u can be written in terms of v we can show that:

$$\text{Span}(u, v) = \text{Span}(\lambda v, v) = \text{Span}(v) \neq \mathbb{R}^2$$

By contrast, if u, v are linearly independent (and thus neither can be the $\{0\}$) we know that:

$$\text{Span}(u) \neq \text{Span}(v)$$

Thus the intersection of Span(u,v) exists in two dimensions, specifically $\mathbb{R}^2$. □

Problem 1.2 (3 points)**.** *Are the following sets subspaces of $\mathbb{R}^3$? Justify your answer.*

(a) $E_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x - 2y + z = 0\}$.
Proof. *Yes, the line defined by E_1 passes through the origin at $E_1 = (0, 0, 0)$ and, as a linear span where $(x, y, z) \in \mathbb{R}^3$ it is a subspace of $\mathbb{R}^3$. More specifically, because all variable terms in E_1 are of the first power, we see that vector addition and scalar multiplication hold true over its linear span.* □

(b) $E_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x - 2y + z = 3\}$.
Proof. *No, $E_2 = (0, 0, 0)$ does not include the origin, thus it is not a subspace of $\mathbb{R}^3$.* □

(c) $E_3 = \{(x, y, z) \in \mathbb{R}^3 \,|\, 5x + y^2 + z = 0\}$.
Proof. *No, the curve E_3 can not be extended for any negative λ value of y (which when squared become positive), thus it is not a subspace of $\mathbb{R}^3$.* □

Problem 1.3 (3 points)**.** *Suppose that $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent. Let $x \in \mathbb{R}^n$ and assume that $x \notin \mathrm{Span}(v_1, \dots, v_k)$. Show that $(v_1, \dots, v_k, x)$ are linearly independent.*

Proof. Given that $x \notin \mathrm{Span}(v_1, \dots, v_k)$ we know, by the definition of span, that there exists no $\lambda \in \mathbb{R}^n$ such that any λv_i coincides with x. We can demonstrate this via contradiction by assuming there is a $\lambda_{k+1} x \neq 0$ such that the linear combination of $\lambda\{v_1, \dots, v_k, x\} = 0$

$$\lambda_1 v_1 + \cdots + \lambda_k v_k + \lambda_{k+1} x = 0$$
$$\frac{\lambda_1}{\lambda_{k+1}} v_1 + \cdots + \frac{\lambda_k}{\lambda_{k+1}} v_k + x = 0$$

This means that:

$$-\frac{\lambda_1}{\lambda_{k+1}} v_1 - \cdots - \frac{\lambda_k}{\lambda_{k+1}} v_k = x$$

But this means we have now written x as a linear combination of $(v_1, \dots, v_k)$ which contradicts our given than $x \notin \mathrm{Span}(v_1, \dots, v_k)$. Thus, because we cannot establish a λ which cancels out x using only $\lambda(v_1, \dots, v_k)$ and we know $(v_1, \dots, v_k)$ is already linearly independent, we can establish that $(v_1, \dots, v_k, x)$ must also be linearly independent and the only solution to

$$\lambda_1 v_1 + \cdots + \lambda_k v_k + \lambda_{k+1} x = 0$$

is

$$\lambda_1 = \cdots = \lambda_k = \lambda_{k+1} = 0$$

This confirms that the set $\{v_1, \dots, v_k, x\}$ is linearly independent. □

Problem 1.4 (2 points)**.** *Let S be a subspace of $\mathbb{R}^n$ and $v_1, \dots, v_k \in S$. We assume that $v_1, \dots, v_k$ are linearly independent. Show (using the result of Problem 1.3) that one can find vectors $v_{k+1}, \dots, v_{k+m}$ in S such that $(v_1, \dots, v_{k+m})$ is a basis of S.*

Proof. To prove set S is a basis we must demonstrate that it is linearly independent and that its Span is equal to the entire vector space (in this case $\mathbb{R}^n$). In the event that S is already the basis for $\mathbb{R}^n$, then there are no additional vectors that can be added to the set while maintaining linear independency. Otherwise, as seen by the result of Problem 1.3 we can add additional vectors and still maintain a linearly independent set of elements within $\mathbb{R}^n$. To find the basis you continue to add m new vectors until you arrive at at set where the $\mathrm{Span}(v_1, \dots, v_k, \dots, v_{k+m}) = \mathbb{R}^n$. Once this occurs you have obtained a basis for S. □

Problem 1.5 ($\star$)**.** *Let U and V be two subspaces of $\mathbb{R}^n$. Show that if*

$$\dim(U) + \dim(V) > n,$$

then there must exist a non-zero vector in their intersection, i.e. $U \cap V \neq \{0\}$.

Proof. Given that U and V are subspaces of $\mathbb{R}^n$, we know that their intersection must also be a subspace (by definition) between size V + W (smallest) and $\mathbb{R}^n$ (largest). Furthermore given $\dim(U) + \dim(V) > n$, we can use the definition of intersection to establish that at least one non-zero vector exists:

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

By using the maximum dimension for $\mathbb{R}^n$ we see that:

$$\dim(\mathbb{R}^n) = n \leq \dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

$$n \leq \dim(U) + \dim(V) - \dim(U \cap V)$$

Subtracting n from both sides ($\dim(U) + \dim(V) > n$) shows:

$$0 \leq c - \dim(U \cap V)$$

$$\dim(U \cap V) \leq c$$

Thus we've established that some constant c must exist which is greater than or equal to the dimension of $U \cap V$. To satisfy this condition, for any non-zero subspace, at least one non-zero vector must exist. $\square$