Optimization and Computational Linear Algebra for Data Science
Homework 2: Linear transformations & matrices

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Problem 2.1 (2 points)**.** *Which of the following are linear transformations? Justify.*

(a) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x^2+y^2, x-y) \end{array} \right.$

Proof. *No, given* $(x_1, y_1) = (0, -2)$ *and* $(x_2, y_2) = (-2, 1)$ *we see via contradiction that*

$$T = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \neq T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$T = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \neq T \begin{pmatrix} 0 \\ -2 \end{pmatrix} + T \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

□

(b) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+y+1, x-y) \end{array} \right.$

Proof.

$$No,\ T(0,0) \to (1,0).$$

Furthermore, for T(x,y) to pass through the origin there must be some (x,y) such that (x+y+1) = 0 and (x-y) = 0. Solving the linear equations we find no such (x,y) exist, thus T(x,y) never passes through the origin. □

(c) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times m} & \to & \mathbb{R}^{m\times n} \\ A & \mapsto & A^{\mathsf{T}} \end{array} \right.$ *where* A^{T} *is transpose of* A*, i.e. the* $m \times n$ *matrix defined by*

$$\left(A^{\mathsf{T}}\right)_{i,j} = A_{j,i} \qquad \text{for all} \quad (i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}.$$

Proof. *Yes. Given* $A,\ B \in \mathbb{R}^{m \times n}$*:*

$$((A+B)^T)_{i,j} = A_{j,i} + B_{j,i} = A^T + B^T$$

$$(\lambda A^T)_{i,j} = \lambda A_{j,i} = \lambda (A^T)_{i,j}$$

As both the additive and scalar multiplication properties for the transpose of A (from A) hold true, it is a valid linear transformation. □

(d) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R} \\ A & \mapsto & \mathrm{Tr}(A) \end{array} \right.$ *where* $\mathrm{Tr}(A)$ *is the trace of the matrix* A*, defined by*

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{i,i}\,.$$

Proof. *Yes. Given* A*,* $B \in \mathbb{R}^{n\times n}$*:*

$$Tr(A+B) = \sum_{i=1}^{n}(A_{i,i} + B_{i,i}) = \sum_{i=1}^{n} A_{i,i} + \sum_{i=1}^{n} B_{i,i} = Tr(A) + Tr(B)$$

$$Tr(\lambda A) = \sum_{i=1}^{n} \lambda A_{i,i} = \lambda \sum_{i=1}^{n} A_{i,i} = \lambda Tr(A)$$

As both the additive and scalar multiplication properties for the Trace of A (from A) hold true, it is a valid linear transformation. □

Problem 2.2 (3 points)**.** *Let* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear transformation such that*

$$f(1,2) = (1,2,3) \qquad \text{and} \qquad f(2,2) = (1,0,1).$$

(a) *Compute the matrix (canonically) associated to* f*.*
Proof. f *is a 3 x 2 matrix whose columns are described by:*

$$f = \begin{pmatrix} \vdots & \vdots \\ f_1 & f_2 \\ \vdots & \vdots \end{pmatrix} \in \mathbb{R}^{3x2}$$

$$f(1,2) = (1,2,3) \Rightarrow f_1 + 2f_2 = (1,2,3)$$
$$f(2,2) = (1,0,1) \Rightarrow 2f_1 + 2f_2 = (1,0,1)$$

Now we have two equations with two variables, solving the system of equations gives:

$$f_1 = (0,-2,-2)$$

$$f_2 = (\frac{1}{2}, 2, \frac{5}{2})$$

$$f = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 2 \\ -2 & \frac{5}{2} \end{pmatrix}$$

□

(b) *Compute the set* $\{x \in \mathbb{R}^2 \mid f(x) = (1,4,5)\}$*.*
Proof.

$$f = \begin{pmatrix} 0f_1 & \frac{1}{2}f_2 \\ -2f_1 & 2f_2 \\ -2f_1 & \frac{5}{2}f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$

Solving the system of equations gives $f_1 = 0$ *and* $f_2 = 2$ *thus x = (0,2)* □

(**c**) *Compute the set* $\{x \in \mathbb{R}^2 \mid f(x) = (2,4,5)\}$.
Proof.

$$f = \begin{pmatrix} 0f_1 & \frac{1}{2}f_2 \\ -2f_1 & 2f_2 \\ -2f_1 & \frac{5}{2}f_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

The system of equations does not resolve, suppose $f_2 = 4$*, thus resolving row 1. Then* f_1 *for row 2 = 2 and row 3 =* $\frac{5}{2}$*. As* $2 \neq \frac{5}{2}$ *there is no solution* x *such that f(x) = (2,4,5).* □

Problem 2.3 (2 points)**.** *Let* $B \in \mathbb{R}^{4\times3}$ *be a matrix with arbitrary entries:*

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,3} \end{pmatrix}.$$

Find two matrices A *and* C *such that*

$$ABC = \begin{pmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} + B_{3,2} & B_{2,1} + B_{3,1} & B_{2,3} + B_{3,3} & B_{2,2} + B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{pmatrix}$$

holds for any B *defined above.*

Proof. To complete the transformation described above the following actions must take place:

(**a**) column 2 goes to column 1

(**b**) column 1 goes to column 2

(**c**) column 3 goes to column 3

(**d**) column 2 goes to column 4

(**e**) row 3 is added to row 2

(**f**) row 3 is removed

We can complete steps a-d by rearranging columns of the identity matrix as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To complete step e, add row 3 to row 2, we add row 3 into row 2 for our modified identify matrix, matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, to complete step f, to transform the dimensions of the matrix and remove row 3, we can use the identity matrix without the row 3, as matrix, "C":

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

Problem 2.4 (3 points)**.**

(**a**) *Let A be a $n \times m$ matrix. Show that the image* $\mathrm{Im}(A)$ *and the kernel* $\mathrm{Ker}(A)$ *of A are subspaces of respectively $\mathbb{R}^n$ and $\mathbb{R}^m$.*

(**b**) *Let*

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Proof.

$$\mathrm{Ker}(A) = x \in \mathbb{R}^m | Ax = 0$$

(a) Let $u, v \in \mathrm{Ker}(A) \Rightarrow Au = 0$ *and* $Av = 0$

$$Au + Av = 0 \Rightarrow A(u + v) = 0 \Rightarrow u + v \in \mathrm{Ker}(A)$$

(b) Let $u \in \mathrm{Ker}(A) \Rightarrow Au = 0$*, for any scalar* $\alpha \in \mathbb{R}$*,*

$$\alpha Au = \alpha.0 = 0$$

$$A(\alpha u) = 0 \Rightarrow \alpha u \in \mathrm{Ker}(A)$$

(c) $A.0 = 0 \Rightarrow \in \mathrm{Ker}(A)$

$$\mathrm{Im}(A) = Ax \in \mathbb{R}^n | x \in \mathbb{R}^m$$

(a) Let $u, v \in \mathrm{Im}(A) \Rightarrow$*, there exists* $x, y \in \mathbb{R}^m$ *such that* $Ax = u$ *and* $Ay = v$*.*

$$\Rightarrow Ax + Ay = u + v \Rightarrow A(x + y) = u + v \Rightarrow u + v \in \mathrm{Im}(A)$$

(b) Let $u \in \mathrm{Im}(A) \Rightarrow$*, there exists* $x \in \mathbb{R}^m$ *such that* $Ax = u$*, for any scalar* $\alpha \in \mathbb{R}$*,*

$$\alpha Ax = \alpha.u$$

$$\Rightarrow A(\alpha x) = \alpha u \Rightarrow \alpha u \in \mathrm{Im}(A)$$

(c) $A.0 = 0 \Rightarrow 0 \in \mathrm{Im}(A)$

□

Compute a basis of $\mathrm{Ker}(A)$ *and show that* $\mathrm{Im}(A) = \mathbb{R}^3$*.*

Proof.

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Solving the system of linear equations

$$c1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + c3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c4 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0$$

Yields the following basis for $\mathrm{Ker}(A)$*:*

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Additionally, because we see that columns 1 and 3 are identical, we know know that the $\mathrm{Im}(A)$ *or unique column space, can be reduced from* $\mathbb{R}^4$ *to* $\mathbb{R}^3$. $\square$

Problem 2.5 ($\star$)**.** *Let* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{k\times n}$*. Prove that there exists a matrix* $C \in \mathbb{R}^{m\times k}$ *such that* $A = CB$ *if and only if* $\mathrm{Ker}(B)$ *is a subspace of* $\mathrm{Ker}(A)$.