Optimization and Computational Linear Algebra for Data Science
Homework 3: Rank

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Problem 3.1 (2 points). *Let $A \in \mathbb{R}^{n\times n}$.*

(a) *Show that if $A = \alpha \mathrm{Id}_n$ for some $\alpha \in \mathbb{R}$, then for all $B \in \mathbb{R}^{n\times n}$ we have $AB = BA$.*

Proof. *Given:*

$$A, B \in \mathbb{R}^{nxn}$$

We see that:

$$AB = BA$$

Given $\alpha \in \mathbb{R}$ such that $A = \alpha \mathrm{Id}_n$, and that B is a square, full rank matrix:

$$B\mathrm{Id}_n = B = \mathrm{Id}_n B$$

Therefore:

$$B(\alpha \mathrm{Id}_n) = (\alpha \mathrm{Id}_n)B$$

$$\alpha(BA) = \alpha(AB)$$

□

(b) *Conversely, show that if for all $B \in \mathbb{R}^{n\times n}$ we have $AB = BA$, then there exists $\alpha \in \mathbb{R}$ such that $A = \alpha \mathrm{Id}_n$.*

Proof. *Given:*

$$A, B \in \mathbb{R}^{nxn}$$

We see that:

$$AB = BA$$

Given that B is a square, full rank matrix:

$$B\mathrm{Id}_n = B = \mathrm{Id}_n B$$

Therefore ***either*** *an $\alpha \mathrm{Id}_n$ exists such that $A = \alpha \mathrm{Id}_n$:*

$$B(\alpha \mathrm{Id}_n) = (\alpha \mathrm{Id}_n)B$$

$$\alpha(BA) = \alpha(AB)$$

Or *no $\alpha \mathrm{Id}_n$ exists such that $A = \alpha \mathrm{Id}_n$, intuitively:*

$$B(0) = 0$$

Thus, in any non-trivial case, an $\alpha \in \mathbb{R}$ must exist such that $A = \alpha \mathrm{Id}_n$.
To more rigorously (actually) prove this you'd probably want to use contradiction or induction...
□

Problem 3.2 (2 points)**.** *Let $M \in \mathbb{R}^{n \times m}$ and $r = \operatorname{rank}(M)$. Show that there exists $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{r \times m}$ such that $M = AB$.*

Proof.
Given that we know $r = \operatorname{rank}(M)$:
We know that we can construct a matrix A by isolating all linearly independent columns from M, such that $A \in \mathbb{R}^{n \times r}$. (A will be directly proportional to the basis of M.)
Then, if we then construct B, such that every column of B is a linear combination of A, we can construct any column $M_{n,k}$ where $k \in \mathbb{R}, 1 \le k \le m$. Thus there exists an $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{r \times m}$ such that $M = AB$. □

Problem 3.3 (3 points)**.** *Let $A \in \mathbb{R}^{n \times m}$.*

(**a**) *Let $M \in \mathbb{R}^{m \times m}$ be an invertible matrix. Show that*

$$\operatorname{rank}(AM) = \operatorname{rank}(A).$$

Proof. *Given that M is invertible we know that:*

$$(M^T)^{-1} = (M^{-1})^T \text{ and } \operatorname{rank}(M) = \operatorname{rank}(M^T)$$

Thus: $\operatorname{rank}(AM) = \operatorname{rank}((AM)^T) = \operatorname{rank}(A^T M^T) = \operatorname{rank}(A^T) = \operatorname{rank}(A)$ □

(**b**) *Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix. Show that*

$$\operatorname{rank}(MA) = \operatorname{rank}(A).$$

Proof. *Given that M is invertible we know that:*

$$MM^{-1} = \mathrm{Id}_n,\ \operatorname{rank}(M) = n \text{ and, } \operatorname{Ker}(M) = \{0\}$$

Let $x \in \operatorname{Ker}(MA), MAx = 0$, therefore: $Ax = 0$.

$$x \in \operatorname{Ker}(A) \text{ means that } \operatorname{Ker}(MA)) \subseteq \operatorname{Ker}(A)$$

$$\text{thus: } x \in \operatorname{Ker}(A) \Rightarrow Ax = 0 \Rightarrow MAx = 0 \Rightarrow x \in \operatorname{Ker}(MA)$$

$$\operatorname{Ker}(MA) = \operatorname{Ker}(A)$$

Via the rank-nullity theorem, we know that:

$$\operatorname{rank}(MA) + \operatorname{Ker}(MA) = n$$

$$\operatorname{rank}(A) + \operatorname{Ker}(A) = n$$

Since $\operatorname{Ker}(MA) = \operatorname{Ker}(A)$ *it follows that* $\operatorname{rank}(MA) = \operatorname{rank}(A)$ □

Problem 3.4 (3 points)**.** *Let $A \in \mathbb{R}^{n \times n}$ be an "upper triangular matrix", i.e. a matrix of the form*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n} \end{pmatrix}.$$

Show that A is invertible if and only if *its diagonal coefficients $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$ are all non-zero.*

Proof. *Given $A \in \mathbb{R}^{n \times n}$ is an invertible upper triangle matrix.*
Via contradiction, suppose that A has a diagonal coefficient $a_{k,k}$ equal to zero, such that $Ax = 0$:

$$a_{k,k} = 0 \textit{ where } k \in \mathbb{R}, 1 \leq k < n$$

$$A = \begin{pmatrix} a_{1,1} & & \cdots & & a_{1,n} \\ 0 & \ddots & & & \\ \vdots & & a_{k,k} & & \vdots \\ & & & \ddots & \\ 0 & & \cdots & 0 & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} = 0$$

We see the k^{th} row, by having a 0 on the main diagonal row, has a nontrivial (0) solution:

$$\underbrace{\overbrace{a_{k-1,k-1}x_{k-1}}^{k\neq 1} + a_{k-1,k}x_k}_{i} + \underbrace{a_{k-1,k+1}x_{k+t} + \cdots + a_{k-1,n}x_n}_{0} = 0$$

Since i implies a nontrivial solution, when $Ax = 0$ should only contain a trivial solution, we see via contradiction that if A is invertible if and only if its diagonal coefficients are all non-zero. □

Problem 3.5 ($\star$)**.** *The trace $\mathrm{Tr}(M)$ of a $k \times k$ matrix M is defined as the sum of its diagonal coefficients, i.e.*

$$\mathrm{Tr}(M) = \sum_{i=1}^{k} M_{i,i}.$$

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$. Show that

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

Proof.

$$\begin{aligned} \mathrm{Tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} B_{ji} \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} B_{ji} A_{ij} \\ &= \sum_{j=1}^{m} (BA)_{jj} \\ &= \mathrm{Tr}(BA) \end{aligned}$$

□