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THE DISTRIBUTION FUNCTIONS OF $\sigma(n)/n$ AND $n/\varphi(n)$

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ABSTRACT. Let $\sigma(n)$ be the sum of the positive divisors of n. We show that the natural density of the set of integers n satisfying $\sigma(n)/n \geq t$ is given by $\exp\left\{-e^{t\,e^{-\gamma}}\left(1+O\left(t^{-2}\right)\right)\right\}$, where γ denotes Euler's constant. The same result holds when $\sigma(n)/n$ is replaced by $n/\varphi(n)$, where φ is Euler's totient function.

1. Introduction

Let $\sigma(n)$ be the sum of the positive divisors of the natural number n. If $\sigma(n) \geq 2n$, then n is called abundant. More generally, we say that n is t-abundant if $\sigma(n) \geq t\,n$. Davenport [1] showed in 1933 that the sequence of t-abundant numbers has a natural density

$$A(t) := \lim_{N\to\infty} \frac{1}{N} \left|\{n \leq N : \sigma(n)/n \geq t\}\right|,$$

and that $A(t)$ is a continuous function of t. An estimate for the density of abundant numbers is $0.2474 < A(2) < 0.2480$, due to Deléglise [2].

A close relative of $A(t)$ is the distribution function

$$B(t) := \lim_{N\to\infty} \frac{1}{N} \left|\{n \leq N : n/\varphi(n) \geq t\}\right|,$$

where φ denotes Euler's totient function. The existence of this limit and its continuity with respect to t were established by Schoenberg [5] in 1928.

Erdös [3], and recently Tenenbaum and Toulmonde [6], studied the behavior of $B(t)$ near $t = 1$. Toulmonde [7, Thm. 1] showed that the behavior of $B(t)$ in the neighborhood of $t = 1$ determines the local behavior near any t in the image of $n/\varphi(n)$. The corresponding result for $A(t)$ holds as well [7, Sec. 10]. Using the fact that the values of $n/\varphi(n)$ and $\sigma(n)/n$ are dense in $[1, \infty)$, Toulmonde [7, Thm. 2, 3, 4] obtained estimates for the infinity and L^p norms of the modulus of continuity of $A(t)$ and $B(t)$.

In this note we are concerned with the size of $A(t)$ and $B(t)$ as t tends to infinity. In that direction, Erdös [3, Thm. 1] states that

$$\log(\log(1/B(t))) = t\,e^{-\gamma}(1 + o(1)) \qquad (t \to \infty),$$

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where γ is Euler's constant. With the help of a strong form of Mertens' theorem [4], it is not difficult to see that Erdös' argument [3, Eq.(2)] actually implies the stronger result

$$\log(1/B(t)) = e^{t\,e^{-\gamma}}(1+o(1)) \qquad (t \to \infty).$$

We will improve on this estimate by establishing a bound for the error term.

Theorem. *As t tends to infinity, we have*

$$A(t), B(t) = \exp\left\{-e^{t\,e^{-\gamma}}\left(1+O\left(t^{-2}\right)\right)\right\}$$

where $\gamma = 0.5772...$ is Euler's constant.

Since $\sigma(n)$ and $\varphi(n)$ are multiplicative functions, we have

$$\frac{\sigma(n)}{n} = \prod_{p^\nu||n} \frac{1+p+\ldots+p^\nu}{p^\nu} = \prod_{p^\nu||n} \frac{1-p^{-\nu-1}}{1-p^{-1}} < \prod_{p|n} \frac{1}{1-p^{-1}} = \frac{n}{\varphi(n)},$$

which shows that $A(t) \le B(t)$. To demonstrate the theorem, we will provide a lower bound for $A(t)$ and an upper bound for $B(t)$. Our strategy for obtaining the lower bound is to pick a relatively small t-abundant number n, and use the estimate $A(t) \ge 1/n$, which holds since all multiples of n are again t-abundant. The derivation of the upper bound for $B(t)$ is based on the observation

$$B(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\le N \\ n \ge t\,\varphi(n)}} 1 \le \lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \left(\frac{n}{t\,\varphi(n)}\right)^s \qquad (s \ge 0),$$

together with a product representation of the limit on the right hand side, and a careful choice of s depending on t.

We remark without proof that, for both $A(t)$ and $B(t)$, the correct order of magnitude of the error term implicit in the theorem is t^{-2}.

2. Lower bound for $A(t)$

Throughout we will use the notation

$$y = y(t) := e^{t\,e^{-\gamma}}$$

and

$$R(y) := \exp(\sqrt{\log y}).$$

Let

$$h_p = \left\lfloor \frac{\log y}{\log p} \right\rfloor,$$

and define the natural number n by

$$n = n(t) = \prod_{p\le y} p^{h_p}.$$

To obtain a lower bound for $A(t)$, we will estimate $\frac{\sigma(n)}{n}$ from below. We write

$$\frac{\sigma(n)}{n} = \prod_{p\le y} \frac{\sigma(p^{h_p})}{p^{h_p}} = \prod_{p\le y} \frac{1+p+p^2+\ldots+p^{h_p}}{p^{h_p}} = \prod_{p\le y} \frac{1-p^{-h_p-1}}{1-p^{-1}}. \tag{1}$$

Note that $p^{h_p+1} \geq y$, and, if $\sqrt{y} < p \leq y$, then $h_p = 1$. Thus

$$(2)\quad \prod_{p\leq y}\left(1-p^{-h_p-1}\right) > \prod_{p\leq\sqrt{y}}\left(1-y^{-1}\right)\prod_{\sqrt{y}<p\leq y}\left(1-p^{-2}\right) = 1+O\left(\frac{1}{\sqrt{y}\,\log y}\right).$$

A strong form of Mertens' theorem [4] implies

$$(3)\quad \prod_{p\leq y}\frac{1}{1-p^{-1}} = e^{\gamma}\log y + O(R^{-1}(y)) = t + O(R^{-1}(y)).$$

Combining (1), (2), and (3), we get

$$(4)\quad \frac{\sigma(n)}{n} > t - c\,R^{-1}(y),$$

for some constant c. If m is a multiple of n, then $\frac{\sigma(m)}{m} \geq \frac{\sigma(n)}{n}$, according to (1). Thus (4) shows that $A(t-c\,R^{-1}(y)) > 1/n(t)$, from which we conclude that $A(t) > 1/n(t+c\,R^{-1}(y))$. It remains to estimate $n(t)$. The prime number theorem in the form

$$(5)\quad \sum_{p\leq y}\log p = y\left(1+O\left(R^{-1}(y)\right)\right)$$

implies

$$\log n(t) \leq \sum_{p\leq\sqrt{y}}\log y + \sum_{\sqrt{y}<p\leq y}\log p = O(\sqrt{y}) + y\left(1+O\left(R^{-1}(y)\right)\right),$$

and therefore

$$\log(1/A(t)) \leq \log n(t+c\,R^{-1}(y)) \leq y\left(1+O\left(R^{-1}(y)\right)\right).$$

This completes the proof of the lower bound for $A(t)$.

3. Upper bound for $B(t)$

A classic result (see [5]) states that the limit in

$$W(s) := \lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}\left(\frac{n}{\varphi(n)}\right)^s$$

exists for every complex s. We will assume that s is real. Clearly, for $s \geq 0$,

$$(6)\quad B(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n\leq N\\ n\geq t\,\varphi(n)}} 1 \leq \lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}\left(\frac{n}{t\,\varphi(n)}\right)^s = \frac{W(s)}{t^s}.$$

It turns out (see [5]) that $W(s)$ has the product representation

$$(7)\quad W(s) = \prod_p\left(1+\frac{(1-p^{-1})^{-s}-1}{p}\right) = \prod_{p\leq y}\cdot\prod_{p>y} =: U\cdot V,$$

say. We choose s as

$$(8)\quad s = s(t) := y\log y.$$

Since $\log(1+x) \le x$, we have

$$\begin{aligned}\log V &= \sum_{p>y} \log\left(1 + \frac{(1-p^{-1})^{-s} - 1}{p}\right)\\ &\le \sum_{p>y} \frac{(1-p^{-1})^{-s} - 1}{p} = \sum_{p>y} \frac{1}{p}\left(e^{s\left(\frac{1}{p}+O\left(\frac{1}{p^2}\right)\right)} - 1\right).\end{aligned}$$

The contribution to the last sum from primes $p > s$ is

$$\ll \sum_{p>s} \frac{1}{p}\left(\frac{s}{p}\right) = O(1),$$

and from primes p with $y < p \le s$ it is

$$\le \sum_{y<p\le s} \frac{1}{p}\, e^{s\left(\frac{1}{p}+O\left(\frac{1}{p^2}\right)\right)} \ll \sum_{y<p\le s} \frac{1}{p}\, e^{s/p}.$$

From the prime number theorem and the definition of s in (8) we obtain

$$\begin{aligned}\sum_{y<p\le s} \frac{1}{p}\, e^{s/p} &\ll \int_y^s \frac{e^{s/x}}{x} \frac{dx}{\log x} \le \frac{1}{\log y} \int_y^s \frac{e^{s/x}}{x}\, dx = \frac{1}{\log y} \int_1^{s/y} \frac{e^u}{u}\, du\\ &\le \frac{1}{\log y} \int_1^{\frac{1}{2}\log y} e^u\, du + \frac{2}{\log^2 y} \int_{\frac{1}{2}\log y}^{\log y} e^u\, du \ll \frac{y}{\log^2 y}.\end{aligned}$$

Thus

$$\log V \ll \frac{y}{\log^2 y}. \tag{9}$$

To estimate U, we apply the bound $\log(1+x) \le \log x + 1/x$. We have

$$\begin{aligned}\log U &\le \sum_{p\le y} \log\left(1 + \frac{(1-p^{-1})^{-s}}{p}\right)\\ &\le -\sum_{p\le y} \log p - s \log \prod_{p\le y} (1-p^{-1}) + \sum_{p\le y} p\,(1-p^{-1})^s.\end{aligned} \tag{10}$$

The last sum can be estimated with the prime number theorem as follows:

$$\begin{aligned}\sum_{p\le y} p\,(1-p^{-1})^s &\le \sum_{p\le y} p\, e^{-s/p} \ll \int_2^y x\, e^{-s/x} \frac{dx}{\log x}\\ &\le ye^{-2s/y} \int_2^{y/2} dx + \frac{1}{\log(y/2)} \int_{y/2}^y \left(\frac{y^3}{x^3}\right) x\, e^{-s/x}\, dx\\ &\le 1 + \frac{y^3}{\log(y/2)} \frac{e^{-s/y}}{s} \ll \frac{y}{\log^2 y}.\end{aligned} \tag{11}$$

Now use (3), (5), (10), and (11), to conclude that

$$\log U \le -y + s \log t + O\left(\frac{y}{\log^2 y}\right). \tag{12}$$

Combining (6), (7), (9), and (12), we obtain

$$\log B(t) \le \log(W(s)\, t^{-s}) = \log U + \log V - s \log t \le -y + O\left(\frac{y}{\log^2 y}\right),$$

which completes the proof of the upper bound for $B(t)$.

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