Texas A&M University

Research Project 1

Some Conjectures on Fibonacci Numbers

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1 Introduction

The *Fibonacci sequence*, $\{F_n\}$, is defined as follows:

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 3.$$

In this paper we analyze two conjectures: one about which Fibonacci numbers are perfect squares, and one about factorials (put reference and more info here).

2 Theoretical Analysis

It was a long-standing conjecture, until it was proved in 1964 [2], that the only Fibonacci numbers that are perfect squares are $F_1 = 1$, $F_2 = 1$, and $F_12 = 144$. M. Wunderlich[1] described an ingenious "exclusion" method to calculate which Fibonacci numbers, out of the first million, can possibly be perfect squares. The method does not prove that any Fibonacci numbers are perfect squares, rather it rules out those that are not.

Consider a prime p. We first calculate the period of the Fibonacci numbers $\pmod p$, denoted here as the *Fibonacci period* of p. For example, the Fibonacci period of $p = 7$ is 16, as the Fibonacci sequence $\pmod 7$ repeats after 16 steps. Specifically, the first 16 Fibonacci numbers $\pmod 7$ are:

$$1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0,$$

after which it repeats $1, 1, 2, 3, 5, \ldots$ again. Because the numbers $\pmod 7$ that are quadratic non-residues are 3, 5, and 6, replacing each quadratic residue by a 1 and each quadratic non-residue by a 0 gives the binary string

$$111001010001101.$$

Take those residues of Fibonacci numbers $\pmod p$ that are quadratic non-residues $\pmod p$. If any Fibonacci number is congruent to a quadratic non-residue $\pmod 7$, then it can never be a perfect square of any integer. Repeat, until the number of possible square Fibonacci numbers is down to a reasonable number to analyze.

The computation in [1] was somewhat limited for two reasons. First, for each prime p, the quadratic residues and non-residues mod p had to be reiterated all the way to the computation limit (in the paper, the computation limit is 10^6). This means that a loop of size 10^6 had to be performed each time a new prime was considered. Second, because 10^6 is not perfectly divisible by each of the Fibonacci periods considered, the computation cannot easily be extended to all positive integers.

Suppose the primes considered are $p_1, p_2, \ldots, p_n$, and let the Fibonacci periods of each prime be $d_1, d_2, \ldots, d_n$. If a certain positive integer k is such that F_k is a quadratic residue modulo each of the p_i, then we know that it is a quadratic residue modulo $(p_1, \ldots, p_n)$. Similarly, if F_k is a quadratic non-residue modulo *one* of the p_i, it is a quadratic non-residue modulo $(p_1, \ldots, p_n)$. Thus, we can say that everything that does not work modulo one of the p_i

We also created a data structure to check Lucas number and factorial conjecture... (insert more detail) We also did identities for Lucas numbers...

3 Computational Approach

[1] found that the only Fibonacci numbers up to $F_{1000000}$ are the three described above using a computational approach. The researchers here took a similar approach, but used optimization techniques to decrease running time and to give more conclusive results. The programming language used was predominantly C++.

Functions to check which numbers are quadratic residues mod a certain prime, to calculate GCDs and LCMs, and to calculate the power of a number modulo another were implemented. The basic algorithm from [1] was also implemented as a preliminary version.

4 Results

5 Conclusion

6 Individual Contributions

1. Sarah:
2. Daniel: I wrote a significant portion of the code for
3. Taylor:

7 References