Symmetric Matrix

$$\begin{bmatrix} 2 & 4 & 5 \\ 4 & 7 & 8 \\ 5 & 8 & 0 \end{bmatrix}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

$$\underset{A}{\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}} \times \underset{A^T}{\underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}} = \underset{AA^T}{\underset{2\times 2}{\begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}}}$$

$A^{m\times n}$

$AA^T \; m\times m \qquad A^TA \; n\times n$

$\Downarrow \qquad\qquad \Downarrow$

$S^L \qquad\qquad S^R$

$$\underset{A^T}{\underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}} \times \underset{A}{\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}} = \underset{A^T\times A}{\underset{3\times 3}{\begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}}}$$

S^L, S^R are symmetric matrices.

$$\underset{2\times 2}{\begin{bmatrix} S_L \end{bmatrix}} \qquad \underset{3\times 3}{\begin{bmatrix} S_R \end{bmatrix}}$$

– These eigen vectors are related to original matrix A.

$\downarrow \qquad\qquad \downarrow$

Eigen Vectors $\quad \vec{u_1} \; \vec{u_2} \qquad \vec{v_1} \; \vec{v_2} \; \vec{v_3}$

(left singular vectors) (Right singular vectors)

Eigen values $\quad \lambda_2 \; \lambda_1 \qquad \lambda_3 \; \lambda_2 \; \lambda_1 \qquad \boxed{\lambda_i \geq 0}$

$\lambda_1 = \lambda_1$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3$

$\lambda_2 = \lambda_2$

$\downarrow$

left over eigen value will be 0.

Singular Values $\quad \sqrt{\lambda_1} \quad \sqrt{\lambda_2}$

$\downarrow \qquad \downarrow$

$\sigma_1 \qquad \sigma_2$

$$A = U \Sigma V^T$$

$$\underset{\substack{4\times 2\\ m\ n}}{\begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \end{bmatrix}} = \overset{AA^T}{\underset{\substack{4\times 4\\ m\ m}}{\begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \\ | & | & | & | \end{bmatrix}}} \underset{4\times 2}{\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \\ & \\ & \end{bmatrix}} \overset{A^TA}{\underset{\substack{2\times 2\\ n\ n}}{\begin{bmatrix} - & \vec{v}_1^{\,T} & - \\ - & \vec{v}_2^{\,T} & - \end{bmatrix}}}$$

$\downarrow$ Singular Values

$$\underset{A}{\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}} = \underset{U}{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & -\frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}}$$

$V^T \rightarrow$ orthagonal matrix. Applies rotation such that right singular vectors are on standard bases.

$A = U \Sigma V^T$

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$$A^TA - \lambda I = \begin{bmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{bmatrix}$$

$$\det(A^TA - \lambda I) = ((25-\lambda)(25-\lambda)) - (-15 \times -15)$$

$$= (625 - 25\lambda - 25\lambda + \lambda^2) - (225)$$

$$= \lambda^2 - 50\lambda + 400$$

$$\Rightarrow \lambda^2 - 40\lambda - 10\lambda + 400 = 0$$

$$\lambda(\lambda - 40) - 10(\lambda - 40) = 0$$

$$(\lambda - 40)(\lambda - 10)$$

eigen values = 10, 40

$$A^TA - 10I = \begin{bmatrix} 25-10 & -15 \\ -15 & 25-10 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

Row 1 + Row 2

$$\begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix}$$

Row 1 $\times \frac{1}{15}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$v_1 - v_2 = 0$

$v_1 = v_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $v_2 = 1$

$$\sqrt{1^2 + 1^2} = \sqrt{2} \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^TA - 40I = \begin{bmatrix} 25-40 & -15 \\ -15 & 25-40 \end{bmatrix} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

Row 1 + (-1) Row 2

$$\begin{bmatrix} -15 & -15 \\ 0 & 6 \end{bmatrix}$$

$-\frac{1}{15}$ Row 1

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|V_1| + |V_2| = 0$$

$$V_1 = -V_2$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

$$A = U\Sigma V^T$$

$$U = \underbrace{AV\Sigma^T}$$

$$\begin{bmatrix} -\frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ -\frac{8}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

$$U = \begin{bmatrix} -2\sqrt{2} & 2\sqrt{2} \\ -4\sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$