Formalizing the Edmonds-Karp Algorithm

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Maximum Flow Problem

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  - Value: flow transported from source to sink (=Outflow of source)

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- Problem: given a network, find a flow with maximum value

Min-Cut/Max-Flow Theorem

- Consider network with flow
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  - Intuition: flow that can be moved between nodes
    - By either increasing or decreasing flow on network edge
  - For network edge $u \xrightarrow{c,f} v$, residual graph has edges

    $u \xrightarrow{c-f} v$ and $v \xrightarrow{f} u$

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    $u \xrightarrow{c-f} v$ and $v \xrightarrow{f} u$
- Min-Cut/Max-Flow Theorem
  - Value of minimal cut = value of maximum flow
  - Corollary: maximum flow iff no source–sink path in residual graph

Ford-Fulkerson Method

Path from source to sink in residual graph is called *augmenting path*.
Flow is maximal iff there is no augmenting path.

- Greedy Algorithm to compute maximum flow

  set flow to zero
  while *exists augmenting path*
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- Partial correctness: obvious
- Termination: only for integer/rational capacities
- Edmonds/Karp: choose shortest augmenting path
  - $O(VE)$ iterations for real-valued capacities
  - Using BFS to find path: $O(VE^2)$ algorithm

Our Contributions

Verified in Isabelle/HOL

- Min-Cut/Max-Flow Theorem
  - Human-readable proof
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  - Human-readable presentation of algorithms
  - Proved correctness and complexity

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  - Human-readable proof
  - Closely following Cormen et al.
- Ford-Fulkerson and Edmonds Karp algorithms
  - Human-readable presentation of algorithms
  - Proved correctness and complexity
- Efficient Implementation
  - Using stepwise refinement down to Imperative/HOL
  - Isabelle's code generator exports to SML
  - Benchmark: comparable to Java (from Sedgewick et al.)

Human-Readable Proofs

- Used Isar proof language

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  Proof fragment from Cormen at al.:

$$
\begin{aligned}
(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) && \text{(definition of } \uparrow\text{)} \\
&\leq f(u,v) + f'(u,v) && \text{(because flows are nonnegative)} \\
&\leq f(u,v) + c_f(u,v) && \text{(capacity constraint)} \\
&= f(u,v) + c(u,v) - f(u,v) && \text{(definition of } c_f\text{)} \\
&= c(u,v).
\end{aligned}
$$

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Our Isar version:

have (f↑f')(u,v) = f(u,v) + f'(u,v) - f'(v,u)
  by (auto simp: augment_def)
also have . . . $\leq$ f(u,v) + f'(u,v) **using** f'.capacity_const **by** auto
also have . . . $\leq$ f(u,v) + cf(u,v) **using** f'.capacity_const **by** auto
also have . . . = f(u,v) + c(u,v) - f(u,v)
  by (auto simp: residualGraph_def)
also have . . . = c(u,v) **by** auto
finally show (f↑f')(u, v) $\leq$ c(u, v) **.**

And Automatic Proofs

- Cormen et al. also give more complicated proofs

First part of proof that $|f \uparrow f'| = |f| + |f'|$:

$$
\begin{aligned}
|f \uparrow f'| & \\
&= \sum_{v \in V_1} (f(s,v) + f'(s,v) - f'(v,s)) - \sum_{v \in V_2} (f(v,s) + f'(v,s) - f'(s,v)) \\
&= \sum_{v \in V_1} f(s,v) + \sum_{v \in V_1} f'(s,v) - \sum_{v \in V_1} f'(v,s) \\
&\qquad - \sum_{v \in V_2} f(v,s) - \sum_{v \in V_2} f'(v,s) + \sum_{v \in V_2} f'(s,v) \\
&= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) \\
&\qquad + \sum_{v \in V_1} f'(s,v) + \sum_{v \in V_2} f'(s,v) - \sum_{v \in V_1} f'(v,s) - \sum_{v \in V_2} f'(v,s) \\
&= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) + \sum_{v \in V_1 \cup V_2} f'(s,v) - \sum_{v \in V_1 \cup V_2} f'(v,s) . \qquad (26.6)
\end{aligned}
$$

And Automatic Proofs

- Cormen et al. also give more complicated proofs
- We sometimes chose to use more automatic proofs

lemma augment_flow_value: Flow.val c s (f↑f') = val + Flow.val cf s f'
proof -
  interpret f'': Flow c s t f↑f' **using** augment_flow_presv[OF assms] **.**

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  - Using some simplifier setup

```
note setsum_simp_setup[simp] =
  sum_outgoing_alt[OF capacity_const] s_node
  sum_incoming_alt[OF capacity_const]
  cf.sum_outgoing_alt[OF f'.capacity_const]
  cf.sum_incoming_alt[OF f'.capacity_const]
  sum_outgoing_alt[OF f''.capacity_const]
  sum_incoming_alt[OF f''.capacity_const]
  setsum_subtractf setsum.distrib
```

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  - And auxiliary statements

```
have aux1: f'(u,v) = 0 if (u,v)∉E (v,u)∉E for u v
proof -
  from that cfE_ss_invE have (u,v)∉cf.E by auto
  thus f'(u,v) = 0 by auto
qed
```

And Automatic Proofs

- Cormen et al. also give more complicated proofs
- We sometimes chose to use more automatic proofs
  - Using some simplifier setup
  - And auxiliary statements
  - We reduce the displayed proof's complexity

have f''.val = ($\sum$ u$\in$V. augment f' (s, u) - augment f' (u, s))
  unfolding f''.val_def **by** simp
also have . . . = ($\sum$ u$\in$V. f (s, u) - f (u, s) + (f' (s, u) - f' (u, s)))
  — Note that this is the crucial step of the proof, which Cormen et al. leave as an exercise.
  by (rule setsum.cong) (auto simp: augment_def no_parallel_edge aux1)
also have . . . = val + Flow.val cf s f'
  unfolding val_def f'.val_def **by** simp
finally show f''.val = val + f'.val **.**

qed

Main Result

- Finally, we arrive at

context NFlow **begin**
...
theorem ford_fulkerson:
  isMaxFlow f $\longleftrightarrow$ ($\nexists$ p. isAugmentingPath p)

Ford-Fulkerson Method

- We use the Isabelle Refinement Framework

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  - Based on nondeterminism monad + refinement calculus
  - Provides proof tools + Isabelle Collection Framework

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```
definition ford_fulkerson_method ≡ do {
 let f = (λ(u,v). 0);

 (f,brk) ← while (λ(f,brk). ¬brk)
  (λ(f,brk). do {
   p ← selectp p. is_augmenting_path f p;
   case p of
    None ⇒ return (f,True)
   | Some p ⇒ return (augment c f p, False)
  })
  (f,False);
 return f
}
```

Correctness Proof

- First, we add some assertions and invariant annotations

```
definition fofu ≡ do {
 let f₀ = (λ_. 0);

 (f,_) ← while^fofu_invar
  (λ(f,brk). ¬brk)
  (λ(f,_). do {
   p ← find_augmenting_spec f;
   case p of
    None ⇒ return (f,True)
   | Some p ⇒ do {
     assert (p≠[]);
     assert (NFlow.isAugmentingPath c s t f p);
     let f = NFlow.augment_with_path c f p;
     assert (NFlow c s t f);
     return (f, False)
    }
  })
  (f₀,False);
 assert (NFlow c s t f);
 return f
}
```

Correctness Proof

- First, we add some assertions and invariant annotations
- Then, we use the VCG to prove partial correctness

theorem fofu_partial_correct: fofu $\leq$ (**spec** f. isMaxFlow f)

```
unfolding fofu_def find_augmenting_spec_def
 apply (refine_vcg)
 apply (vc_solve simp:
  zero_flow
  NFlow.augment_pres_nflow
  NFlow.augmenting_path_not_empty
  NFlow.noAugPath_iff_maxFlow[symmetric])
 done
```

Correctness Proof

- First, we add some assertions and invariant annotations
- Then, we use the VCG to prove partial correctness
- This also yields correctness of the unannotated version

theorem (**in** Network) ford_fulkerson_method $\leq$ (**spec** f. isMaxFlow f)

Edmonds-Karp Algorithm

- Specify shortest augmenting path

definition find_shortest_augmenting_spec f $\equiv$
 assert (NFlow c s t f) $\gg$
 (**selectp** p. Graph.isShortestPath (residualGraph c f) s p t)

Edmonds-Karp Algorithm

- Specify shortest augmenting path
- This is a refinement of augmenting path

lemma find_shortest_augmenting_refine:
find_shortest_augmenting_spec $\leq$ find_augmenting_spec

Edmonds-Karp Algorithm

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- Replace in algorithm

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Edmonds-Karp Algorithm

- Specify shortest augmenting path
- This is a refinement of augmenting path
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definition edka_partial ≡ **do** {

...
p ← find_shortest_augmenting_spec f;
...

Edmonds-Karp Algorithm

- Specify shortest augmenting path
- This is a refinement of augmenting path
- Replace in algorithm
- New algorithm refines original one

```
lemma edka_partial_refine[refine]: edka_partial ≤ ⇓Id fofu

unfolding find_shortest_augmenting_spec_def find_augmenting_spec_def
  apply (refine_vcg)
  apply (auto
    simp: NFlow.shortest_is_augmenting
    dest: NFlow.augmenting_path_imp_shortest)
  done
```

Total Correctness and Complexity

- Next, we define a total correct version

```
definition edka_partial ≡ do {
...
(f,_) ← while^fofu_invar

...
```

Total Correctness and Complexity

- Next, we define a total correct version

definition edka $\equiv$ **do** {
...
(f,_) $\leftarrow$ $\textbf{while}_T{}^{\textit{fofu\_invar}}$

...

Total Correctness and Complexity

- Next, we define a total correct version
- And show refinement

theorem edka_refine[refine]: edka $\leq$ $\Downarrow$Id edka_partial

Total Correctness and Complexity

- Next, we define a total correct version
- And show refinement
- We also show $O(VE)$ bound on loop iterations
  - Instrumenting the loop with a counter

Towards Efficient Implementation

Several refinement steps lead to final implementation:

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    - Using pre-computed map of adjacent nodes in network

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5. Imperative Data Structures
   - Tabulate capacity matrix and adjacency map to array
   - Maintain residual graph in array

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   - Tabulate capacity matrix and adjacency map to array
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6. Export to SML code

Assembling Overall Correctness Proof

- Correctness statement
  - As Hoare Triple using Separation Logic

```
context Network_Impl begin
 theorem edka_imp_correct:
  assumes Graph.V c ⊆ {0..<N}
  assumes is_adj_map am
  shows
   <emp>
    edka_imp c s t N am
   <λfi. ∃A f. is_rflow N f fi * ↑(isMaxFlow f)>t
```

Assembling Overall Correctness Proof

- Correctness statement
  - As Hoare Triple using Separation Logic
- Proof by transitivity

proof -
    interpret Edka_Impl **by** unfold_locales fact

    note edka5_refine[OF ABS_PS]
    also note edka4_refine
    also note edka3_refine
    also note edka2_refine
    also note edka_refine
    also note edka_partial_refine
    also note fofu_partial_correct
    finally have edka5 am $\leq$ SPEC isMaxFlow **.**
    from hn_refine_ref[OF this edka_imp_refine]
    show ?thesis
      by (simp add: hn_refine_def)
qed

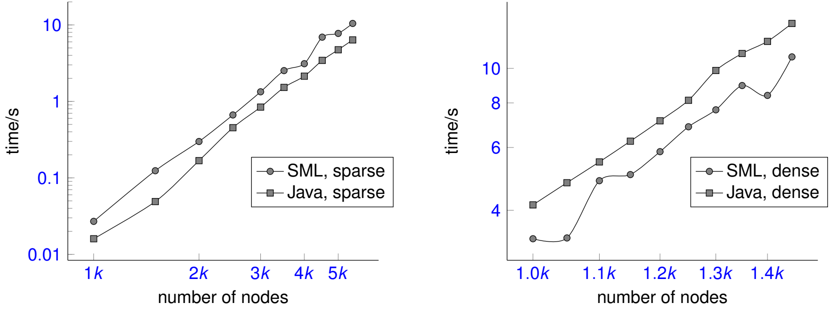
Assembling Overall Correctness Proof

- Correctness statement
  - As Hoare Triple using Separation Logic
- Proof by transitivity
- Also integrated with check for valid network
  - Input: list of edges, source node, sink node

```
theorem
 fixes el defines c ≡ ln_α el
 shows
  <emp>
   edmonds_karp el s t
  <
   λNone ⇒ ↑(¬ln_invar el ∨ ¬Network c s t)
   | Some (_,_,N,cf) ⇒
     ↑(ln_invar el ∧ Network c s t ∧ Graph.V c ⊆ {0..<N})
    * (∃A f. is_rflow c s t N f cf * ↑(Network.isMaxFlow c s t f))
  >t
```

Benchmarking

- Against Java version of Sedgewick et al., on random networks
- Two data sets: Sparse ($D = 0.02$) and dense ($D = 0.25$) graphs
  - Sparse: Java is (slightly) faster
  - Dense: we are (slightly) faster
  - Supposed reason: different 2-dimensional array implementations



Conclusion

- Proof of Min-Cut/Max-Flow Theorem
  - Human readable proofs following textbook presentation
  - Showing off Isar proof language
- Verified Edmonds-Karp algorithm
  - From abstract pseudo-code like version ...
  - ... down to imperative implementation
  - Showing off Isabelle Refinement Framework
- Our implementation is pretty efficient

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Questions?