Ziig Institute
Mechanical Engineering and Applied Mathematics

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Mechanics for first years

by

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1 Vectors and Scalars

1.1 Some basic definitions

Before we get into the fun stuff we must define basic concepts that are important to perform calculations.

1. A scalar is a **physical quantity** that only has a **magnitude**, examples can include electric charge, time, mass, density etc. The mathematical operation of addition can be used to add or subtract scalar quantities.Units of these quantities include kilograms(kg) or pounds (lbs), here is an example:

   (a)
   $$18kg - 9kg$$
   $$= 9kg$$

   (b) Here is another one but including electrical charge
   $$27nC + (-15nC)$$
   $$= 12nC$$

2. Vectors are physical quantities possess both a **magnitude** and **direction** examples include acceleration, displacement, velocity etc. Vectors can be graphical represented by an arrow pointing in any direction, the arrow head represents the sense (direction), the length of the arrow represents the magnitude. In these notes a vector quantity will be represented in two ways with an alphabetic character with a boldface type shown as such **X** and with a bar on top of the character shown as such $\vec{X}$. The magnitude can be represented as such | **X** | or X.

2 Vector mathematics

We use the parallelogram law to add two vectors **K** and **P** together giving us a resultant vector **X** as the diagonal of a parallelogram formed by **K** and **P** as the adjacent sides with the two green dotted lines parallel to the respective sides.The figure below illustrates:

The resultant of a system of vectors is a vector that can replace the system of vectors with a single vector or the net effect of the systems of vectors is the same as the resultant vector.

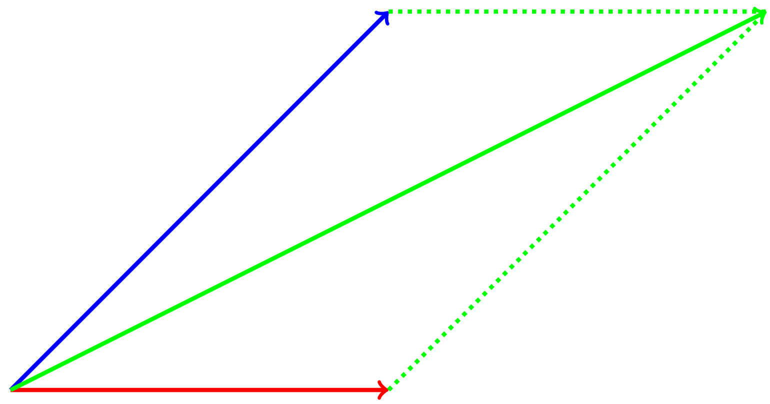

Figure 1: Vector Addition:Parallelogram law

There is a special case of the parallelogram law where the vectors **K** and **P** are perpendicular to each other. In this case **K** and **P** are called components of vector **X**.The figure below illustrates:

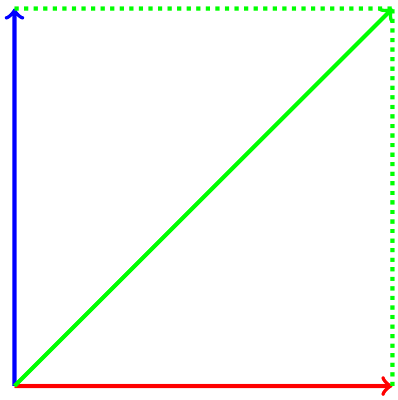

Figure 2: Vector Addition:Rectangular law

The magnitudes of | **K** | and | **P** | are given by the following formulas:

1.
$$| \mathbf{K} | = \mathbf{X} sin(\theta)$$

   and

2.
$$| \mathbf{P} | = \mathbf{X} cos(\theta)$$

The triangle law is basically the parallelogram law drawn by using the head to tail method. We place the tail end of either vector on the head of the other vector then resultant will be drawn from the tail of the first vector to the head of the second vector.

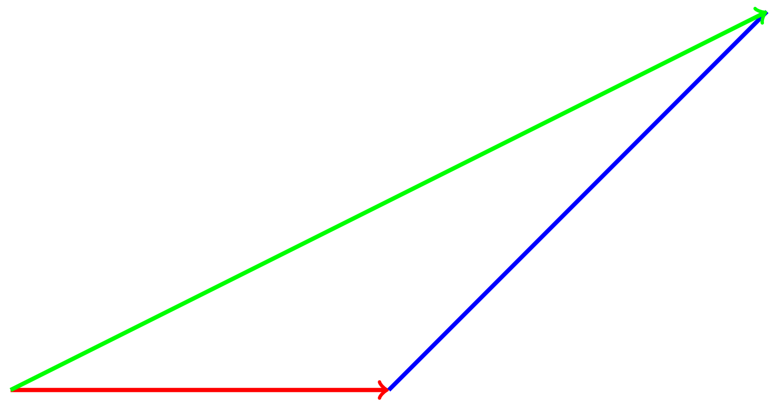

Figure 3: Vector Addition:Triangle law

When vector **-K** is added to **K** results in a **zero vector**, **-K** is the negative of vector **K**.

> The negative of vector is a vector acting on the same line of action and has the same magnitude but is in the opposite direction.

$$\mathbf{K}\text{-}\mathbf{K} = \mathbf{0}$$

2.1 Vector Laws

Commutative law-A law that states that adding **K** to **P** is the same as adding **P** to **K** (1) .

Associative law-which states that adding **K** and **P** then adding **x** is the same as adding **P** and **x** then adding **K**(2)

As shown below:

1.
$$\mathbf{P} + \mathbf{K} = \mathbf{K} + \mathbf{P}$$

2.
$$(\mathbf{P} + \mathbf{K}) + \mathbf{x} = (\mathbf{P} + \mathbf{x}) + \mathbf{K}$$

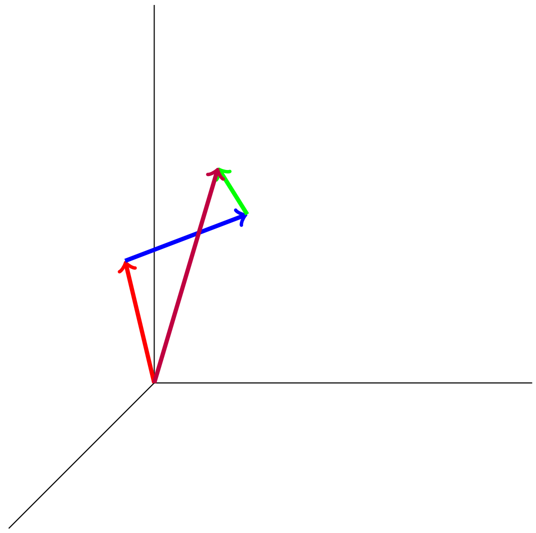

Figure 4: Vector Addition:Composition of a system of vectors

2.2 Polygon addition

This is an extension of the triangle law and making use of the associative law of vectors to finding the resultant of a system of vectors. Each vector is drawn from the head of the previous vector then the resultant vector will be drawn from the tail of the initial vector. Consider the following vectors **P**, and with the respective coordinates (0,2,1),(2,1,1) and (0,1,1). We'll find the resultant of this system of vectors illustrated above in figure 4:

2.3 Scalar and Vector products

> N.B- Some of these concepts feel alien but relax. The fields of mathematics and engineering overlap all the time, especially applied mathematics. Mathematics models are a way to communicate real world elements by simplifying the complexity problems. .

2.4 Scalar multiplication

Let's revisit the concepts of **scalars** but this time we'll focus on the multiplication of a scalar and a vector.Multiplying a **K** with some scalar **a** have the following results depending on the sign and magnitude of the scalar:

1. If scalar **a** is +ve (positive) and has a magnitude greater than 1 then a new vector**K** larger in magnitude but in the same direction will form.
2. If scalar **a** is -ve (negative) and has a magnitude greater than 1 then a new vector **K** larger in magnitude but in the opposite direction will form.
3. If scalar **a** is +ve (positive) and has a magnitude less than 1 then a new vector **K** smaller in magnitude but in the same direction will form.
4. If scalar **a** is -ve (negative) and has a magnitude greater than -1 then a new vector **K** smaller in magnitude but in the opposite direction will form.

2.4.1 Dot product

Scalar product is the multiplication of two vectors' magnitude and the cosine of the angle between them. Later on will introduce unit vectors , magnitude of a vector and how they can simplify scalar product.

$$\mathbf{P} \cdot \mathbf{Q} = |\mathbf{P}||\mathbf{Q}| \, cos(\theta)$$

Unit vectors are vectors with a magnitude of 1 and are in the same direction as the axes (x,y,z). We use these triad of orthogonal vectors to indicate the magnitude of each component of a vector in each direction of the axes

Unit vectors can be written as directional cosines which is more helpful in three dimensional problems. For simple two dimensional problems that involve dotted two vectors the define of a scalar product is much useful.

Here is a three dimensional representation of a vector where x, y, z are the magnitudes and i, j, k are the unit vectors:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

2.4.2 Dot product laws

$$A \cdot B = B \cdot A$$

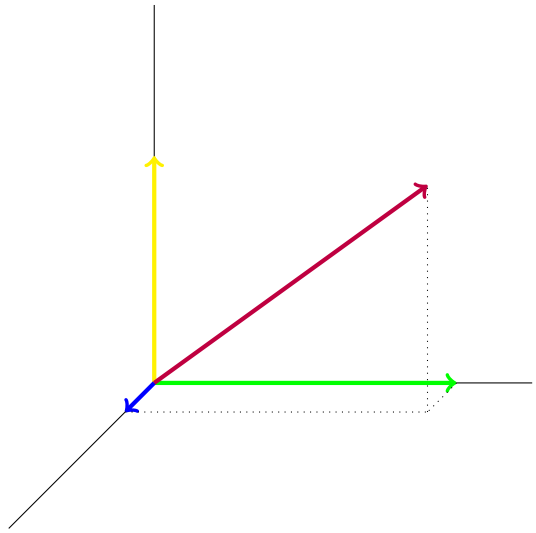

Figure 5: Three dimensional vector representation

$$A \cdot (B + C) = A \cdot B + A \cdot C$$
$$(A + B) \cdot (C + D) = A \cdot (C + D) + B \cdot (C + D) = A \cdot C + A \cdot D + B \cdot C + B \cdot D$$
$$n(A \cdot B) = (nA) \cdot B = A \cdot (nB)$$
$$\hat{i} \cdot \hat{j} = \hat{k} \cdot \hat{i} = \hat{j} \cdot \hat{k} = (1)(1)cos(90^\circ) = 0$$
$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = (1)(1)cos(0^\circ)$$

$\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ then:

$$A \cdot B = x_1x_2 + y_1y_2 + z_1z_2$$
$$B \cdot B = x_2^2 + y_2^2 + z_2^2$$

$\mid K \mid = \sqrt{x_2^2 + y_2^2 + z_2^2}$ then:

$$x_2 = K \cdot \hat{i} \quad y_2 = K \cdot \hat{j} \quad z_2 = K \cdot \hat{k}$$

The position vector can be used to give a unit vector (normalization of a vector),subsequently even other vectors can be used to give the unit vector but the magnitude of the vector in question will always equal one and the normalization of the vector kis by the following equation where $k = x\hat{i} + y\hat{j} + z\hat{k}$:

$$\frac{k}{\mid k \mid} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

2.5 Cross Product

The cross product of two vectors **a** and **b** is denoted as $a \times b$, resulting in a vector **U** with a magnitude equal to the product of the magnitudes of **a** and **b** multiplied by the sine of the angle between them. Vector **U** is perpendicular to the plane that is defined by vectors **a** and **b**.

Using the right hand rule your thumb would face in the direction of the the vector **U** and the curl of your fingers would show the direction moving from a to b in the plane which contains both vectors.

For the cross product the law of commutativity doesn't apply as seen in the following equation:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

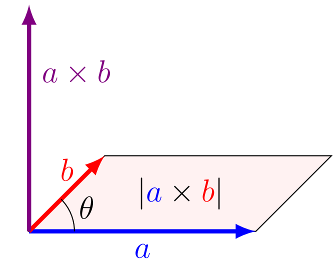

Figure 6: Three dimensional vector representation

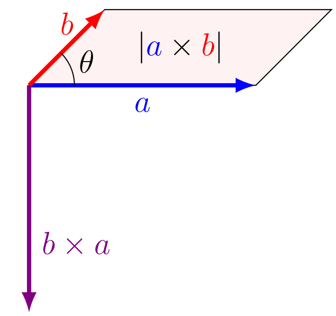

Figure 7: Three dimensional vector representation

2.5.1 Matrix representation of Cross product

Given vectors $\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then the cross product is given by the following:

$$\vec{A} \times \vec{B} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \end{pmatrix} \hat{i} - \begin{pmatrix} x_1 & z_1 \\ x_2 & z_2 \end{pmatrix} \hat{j} + \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \hat{k}$$

$$= (y_1 z_2 - y_2 z_1)\hat{i} - (x_1 z_2 - z_1 x_2)\hat{j} + (x_1 y_2 - y_1 - x_2 y_2)\hat{k}$$

2.5.2 Cross product laws

$$A \times (B + C) = A \times B + A \times C$$

$$(A + B) \times (C + D) = A \times (C + D) + B \times (C + D)$$
$$= A \times C + A \times D + B \times C + B \times +D$$
$$m \times (B + C) = (mB) \times C = B \times (mC)$$
$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$
$$\hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}$$