

Construction of coarse approximations for problems with highly oscillating coefficients

Simon Ruget

Joint work with Claude Le Bris and Frédéric Legoll

GAMM Seminar on Microstructures

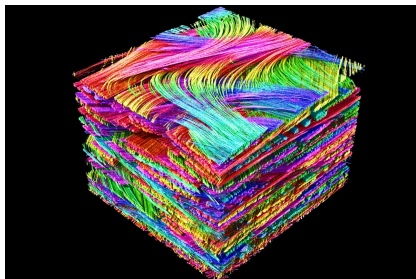
Young Researcher's Meeting

Bochum, January 25 2024

An Inverse Multiscale Problem

Our study focuses on **multiscale** systems (e.g. composite materials).

Determining the fine-scale structure from measurements is an **ill-posed inverse problem** (see [Lions78]¹) but identifying **effective parameters** is possible !



¹J.-L. Lions, *Some aspects of modelling problems in distributed parameter systems*, 1978.

Multiscale Context : ill-posed inverse problem

Consider the prototypical linear equation with characteristic **length scale** ε

$$\mathcal{L}_\varepsilon u_\varepsilon = (-\Delta + \varepsilon^{-1} V(\varepsilon^{-1} \cdot)) u_\varepsilon = f \text{ in } \Omega, \quad u_\varepsilon = 0 \text{ on } \partial\Omega,$$

with **periodic** coefficient V such that $\langle V \rangle = 0$, and RHS $f \in L^2(\Omega)$.

Homogenization theory² assesses the existence of a limit equation when $\varepsilon \rightarrow 0$:

$$\mathcal{L}_\star u_\star = (-\Delta + V_\star) u_\star = f \text{ in } \Omega, \quad u_\star = 0 \text{ on } \partial\Omega,$$

and we have $u_\varepsilon \rightarrow u_\star$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ when $\varepsilon \rightarrow 0$.

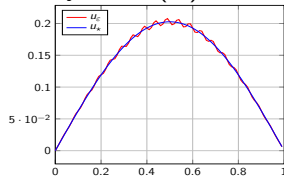
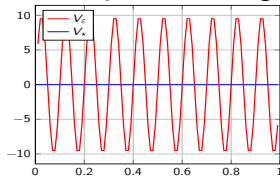


Figure: Two similar solutions associated to two very distinct coefficients.

²A. Bensoussan, J.-L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, 1978

Our Approach

Consider the problem (1) involving a periodic coefficient V :

$$\mathcal{L}_\varepsilon u_\varepsilon = (-\Delta + \varepsilon^{-1} V(\varepsilon^{-1} \cdot)) u_\varepsilon = f \text{ in } \Omega, \quad u_\varepsilon = 0 \text{ on } \partial\Omega. \quad (1)$$

From the knowledge of solutions u_ε for various RHS f , our aim is to propose a **numerical methodology** to build an **effective operator** $\overline{\mathcal{L}}$ approaching \mathcal{L}_ε with satisfying L^2 error on the solutions.

Our strategy

- is inspired by homogenization theory,
- does not rely at its core on classical hypothesis for homogenization (such as periodicity), which may be too restrictive in practical situations,
- is valid outside the regime of homogenization (i.e. $\varepsilon \rightarrow 0$),
- can be adapted to a wide range of other equations (see [LeBris18]³).

³C. Le Bris, F. Legoll, and S. Lemaire, *On the best constant matrix approximating an oscillatory matrix-valued coefficient in divergence-form operators*, 2018.

Recovering an effective coefficient

Let $\overline{V} \in \mathbb{R}$ be a *constant* coefficient, and $\overline{u} = u(\overline{V}, f)$ be the solution to (2) with RHS $f \in L^2(\Omega)$.

$$\overline{\mathcal{L}}\overline{u} = (-\Delta + \overline{V})\overline{u} = f \text{ in } \Omega, \quad \overline{u} = 0 \text{ on } \partial\Omega. \quad (2)$$

The quality of the approximation of \mathcal{L}_ε by $\overline{\mathcal{L}}$ can be quantified through the functional

$$\sup_{\|f\|_{L^2(\Omega)}=1} \|u_\varepsilon(f) - u(\overline{V}, f)\|_{L^2(\Omega)}^2$$

Hence we can **minimize** the **worst case scenario** with the optimization problem

$$I_\varepsilon = \inf_{\overline{V} \in \mathbb{R}} \sup_{\|f\|_{L^2(\Omega)}=1} \|u_\varepsilon(f) - u(\overline{V}, f)\|_{L^2(\Omega)}^2 \quad (3)$$

The choice of an $L^2(\Omega)$ norm is reminiscent of the fact that $\|u_\varepsilon - u_\star\|_{L^2(\Omega)}$ tends to 0 with ε .

(In practice, we slightly modify (3) to recover a quadratic problem.)

Consistency with homogenization in the regime of separated scales

The problem

$$I_\varepsilon = \inf_{\overline{V} \in \mathbb{R}} \sup_{\|f\|_{L^2(\Omega)}=1} \|u_\varepsilon(f) - u(\overline{V}, f)\|_{L^2(\Omega)}^2 \quad (3)$$

is **consistent with homogenization theory** in the sense that :

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0. \quad (4)$$

Furthermore, for $\varepsilon > 0$ fixed (sufficiently small), there **exists a unique minimizer** $\overline{V}_\varepsilon^0 \in \mathbb{R}$. The following convergence holds :

$$\boxed{\lim_{\varepsilon \rightarrow 0} \overline{V}_\varepsilon^0 = V_\star,} \quad (5)$$

where V_\star is the homogenized coefficient.

Numerical results

To solve

$$I_\varepsilon = \inf_{\bar{V} \in \mathbb{R}} \sup_{\|f\|_{L^2(\Omega)}=1} \|u_\varepsilon(f) - u(\bar{V}, f)\|_{L^2(\Omega)}^2,$$

we use an Uzawa algorithm (i.e. $\min_{\bar{V}}$ intertwined with \max_f) in 2D ($\Omega = [0, 1]^2$) using the coefficient

$$V(x, y) = c (\sin(2\pi x) + \sin(2\pi y)).$$

We approximate the supremum by a maximization over the first eigenmodes of $(-\Delta)$ -operator. In practice, a *single* mode is often sufficient in order to find the *single* coefficient \bar{V} .

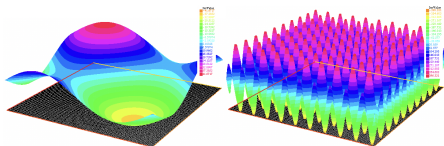


Figure: Coefficient V and Oscillating Coefficient $V_\varepsilon = \varepsilon^{-1}V(\varepsilon^{-1}\cdot)$.

Relationship with homogenized potential

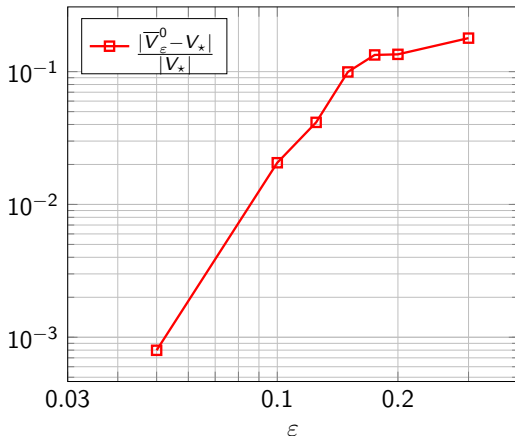


Figure: Error between the homogenized coefficient V_* and the effective coefficient $\overline{V}_\varepsilon^0$ as a function of ε .

Beyond the regime of separated scales

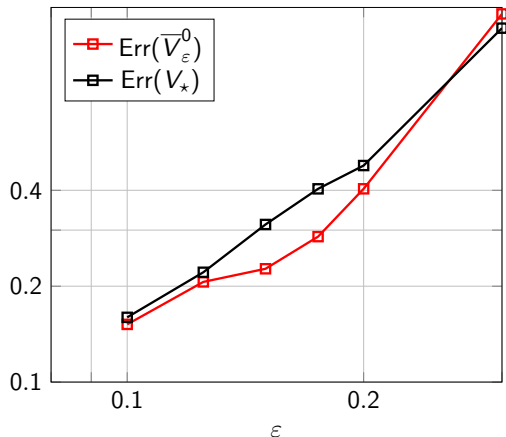








Figure: Error $\text{Err}(\mathbf{V}) = \sup_{f \in \text{Span}_{1 \leq p \leq 10}(f_p)} \left(\frac{\|u_\varepsilon(f) - u(\mathbf{V}, f)\|_{L^2(\Omega)}}{\|u_\varepsilon(f)\|_{L^2(\Omega)}} \right)$ as a function of ε for $\mathbf{V} = V_\star$ and $\mathbf{V} = \bar{V}_\varepsilon^0$.

In progress :

- Exploring **other type of data** : **boundary measurements**, **macroscopic data** (e.g. energy $\int_{\Omega} |\nabla u_{\varepsilon}|^2 + V_{\varepsilon} u_{\varepsilon}^2$) instead of microscopic ones, ...
- **Robustness analysis** : what happens if the data is blurred/perturbed/deteriorated/... ?
- Exploring **pertubative setting**.
- Performing **classification** rather than identification of the effective coefficient.

References

-  H. Ammari, *An introduction to mathematics of emerging biomedical imaging*, Springer, Vol. 62., 2008.
-  H. Ammari, J. Garnier, L. Giovangigli, W. Jing, J. Seo, *Spectroscopic imaging of a dilute cell suspension*, Journal de Mathématiques Pures et Appliquées 105, 2016.
-  A. Caiazzo, R. Maier, and D. Peterseim *Reconstruction of quasi-local numerical effective models from low-resolution measurements*, Journal of Scientific Computing, 2020.
-  E. Cherkaev, *Inverse homogenization for evaluation of effective properties of a mixture*, Inverse Problems, 2001.
-  B. Engquist, and C. Frederick, *Numerical methods for multiscale inverse problems*, ArXiv preprint, 2014.
-  J. Garnier, L. Giovangigli, Q. Goepfert, and P. Millien, *Scattered wavefield in the stochastic* ArXiv preprint, 2023.

References



C. Le Bris, F. Legoll, and K. Li, *Coarse approximation of an elliptic problem with highly oscillating coefficients*, Comptes Rendus Mathématique, 2013.



C. Le Bris, F. Legoll, and S. Lemaire, *On the best constant matrix approximating an oscillatory matrix-valued coefficient in divergence-form operators*, ESAIM: Control, Optimisation and Calculus of Variations, 2018.



J.-L. Lions, *Some aspects of modelling problems in distributed parameter systems*, Springer, 2005.



J. Nolen, G.A. Pavliotis, and A.M. Stuart, *Multiscale modelling and inverse problems*, Numerical analysis of multiscale problems, Springer, 2012.



G. Bal, and G. Uhlmann, *Reconstruction of Coefficients in Scalar Second Order Elliptic Equations from Knowledge of Their Solutions*, Communications on Pure and Applied Mathematics, 2013.

An Inverse Multiscale Problem

How to tackle such problems ?

General approach :

- **Direct Inversion** (see [Uhlmann13]),
- **Perturbation** (see [Ammari08]),

Approach based on homogenization (see [Nolen12]):

- **Features Determination** (see [Engquist14]),
- **Regularization at order 0**, identifying effective quantity (see [Ammari16, Caiazzo20]),
- **Regularization at order 1**, beyond effective quantity : H^1 reconstruction (see [Garnier23, LeBris18]).
- **Inverse Homogenization** (see [Cherkaev01])