Numerical Methods

home work - 4

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1) Lagrange polynomial to interpolate through points $(x,y) = \underset{x_1\ y_1}{(-1,6)}, \underset{x_2\ y_2}{(0,2)} \underset{x_3\ y_3}{(1,4)}$

$$P(x) = \sum_{i=0}^{n} y_i l_i(x)$$

$$P_{n-1} = y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x)$$

$$l_1(x) = \frac{x-x_2}{x_1-x_2} \times \frac{x-x_3}{x_1-x_3} = \frac{x-0}{-1-0} \times \frac{x-1}{-1-1} = \frac{x(x-1)}{2}$$

$$l_2(x) = \frac{x-x_1}{x_2-x_1} \times \frac{x-x_3}{x_2-x_3} = \frac{x+1}{0+1} \times \frac{x-1}{0-1} = \frac{(x+1)(x-1)}{-1} = -(x+1)(x-1)$$

$$l_3(x) = \frac{x-x_1}{x_3-x_1} \times \frac{x-x_2}{x_3-x_2} = \frac{x+1}{1+1} \times \frac{x-0}{1-0} = \frac{x(x+1)}{2}$$

$$P_2 = y_1 \frac{x(x-1)}{2} + y_2 (-1)(x+1)(x-1) + y_3 \frac{(x)(x+1)}{2}$$

$$= \frac{\overset{3}{\cancel{6}}(x^2-x)}{\cancel{2}} - 2(x^2-1) + \frac{\overset{2}{\cancel{4}}(x^2+x)}{\cancel{2}} = 3x^2-3x-2x^2+2+2x^2+2x$$

$$\boxed{P_2 = 3x^2 - x + 2}$$

2) a) Newton polynomial to interpolate through points $(x,y) = (-1,6)\ (0,2)\ (1,4)$

$$P_n(x) = a_1 + (x-x_1)a_2 + (x-x_1)(x-x_2)a_3$$

$$y_1 = a_1$$

$$y_2 = a_1 + (x_2-x_1)a_2$$

$$y_3 = a_1 + (x_3-x_1)a_2 + (x_3-x_1)(x_3-x_2)a_3$$

$$a_1 = 6$$

$$2 = 6+(0+1)a_2 \Rightarrow 2 = 6+a_2 \Rightarrow a_2 = -4$$

$$4 = 6+(1+1)(-4)+(1+1)(1-0)a_3$$

$$4 = 6-8+2a_3 \Rightarrow 2a_3 = 6 \Rightarrow a_3 = 3$$

$$P(x) = 6 + (x+1)(-4) + (x+1)(x-0)(3)$$

$$= 6 - 4x - 4 + 3x^2 + 3x$$

$$\boxed{P(x) = 3x^2 - x + 2}$$

3) a) natural piecewise cubic splines needed to interpolate through the points $(x,y) = (-1,6)(0,2)(1,4)$

General form of a cubic spline

(i)

$$S_i(x) = a_i x + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

$$S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3$$

$$S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3$$

$$S_0(-1) = 6 \quad a_0 + b_0(-1+1) + c_0(-1+1)^2 + d_0(-1+1)^3 = 6$$

$$\boxed{a_0 = 6}$$

(ii) $S_0(0) = 2 \Rightarrow a_0 + b_0(0+1) + c_0(0+1)^2 + d_0(0+1)^3 = 2$

$$a_0 + b_0 + c_0 + d_0 = 2$$

$$6 + b_0 + c_0 + d_0 = 2 \quad \boxed{b_0 + c_0 + d_0 = -4}$$

$S_1(0) = 2 \Rightarrow a_1 + b_1(0-0) + c_1(0-0)^2 + d_1(0-0)^3 = 2 \quad \boxed{a_1 = 2}$

$S_1(1) = 4 \Rightarrow a_1 + b_1(1-0) + c_1(1-0)^2 + d_1(1-0)^3 = 4$

$$a_1 + b_1 + c_1 + d_1 = 4$$

$$\Rightarrow 2 + b_1 + c_1 + d_1 = 4 \quad \boxed{b_1 + c_1 + d_1 = 2}$$

(iii) Smoothness Condition $\quad S_0'(x_1) = S_1'(x_1)$

$$S_0'(0) = b_0 + 2c_0(0+1) + 3d_0(0+1)^2 = b_0 + 2c_0 + 3d_0$$

$$S_1'(0) = b_1 + 2c_1(0-0) + 3d_0(0-0)^2 = b_1$$

$$\boxed{b_0 + 2c_0 + 3d_0 = b_1}$$

$$\left.\begin{array}{l} S_0''(0) = 2c_0 + 6d_0(0+1) = 2c_0 + 6d_0 \\ S_1''(0) = 2c_1 + 6d_1(0-0) = 2c_1 \end{array}\right\} 2c_0 + 6d_0 = 2c_1$$

iv) $S_0''(x_0) = S_1''(x_2) = 0$

$$S_0''(-1) = 2c_0 + 6d_0(-1+1) = 2c_0 \Rightarrow \boxed{c_0 = 0}$$

$\Rightarrow$

$a_0 = 6, \quad c_0 = 0$

$b_0 + d_0 = 4 \rightarrow (1)$

$a_1 = 2, \quad b_1 + c_1 + d_1 = 2 \rightarrow (2)$

$b_0 + 3d_0 - b_1 = 0 \rightarrow (3)$

$6d_0 - 2c_1 = 0 \rightarrow (4)$

$2c_1 + 6d_1 = 0 \rightarrow 5$

from 4 $\quad c_1 = 3d_0$

from 5 $\Rightarrow 2c_1 + 6d_1 = 0 \Rightarrow 2(3d_0) + 6d_1 = 0 \Rightarrow d_1 = -d_0$

from 1 & 3 $\quad b_0 + 3d_0 - b_1 = 0 \Rightarrow b_0 + d_0 + 2d_0 - b_1 = 0$

$\Rightarrow -4 + 2d_0 = b_1 \Rightarrow b_1 = 2d_0 - 4$

from (2) $\quad b_1 + c_1 + d_1 = 2 \Rightarrow 2d_0 - 4 + 3d_0 - d_0 = 2$

$\Rightarrow 4d_0 - 4 = 2 \Rightarrow d_0 = \frac{6}{4} \Rightarrow \frac{3}{2} \quad d_1 = -3/2$

(1) $b_0 + d_0 = -4, \quad b_0 = -4 - \frac{3}{2} = -\frac{11}{2}$

(4) $6d_0 - 2c_1 = 0 \Rightarrow c_1 = 3 \cdot \frac{3}{2} \Rightarrow \frac{9}{2}$

(2) $b_1 = 2 - c_1 - d_1 = 2 - \frac{9}{2} + \frac{3}{2} = -1$

Finally

$a_0 = 6$	$a_1 = 2$
$b_0 = -11/2$	$b_1 = -1$
$c_0 = 0$	$c_1 = 9/2$
$d_0 = \frac{3}{2}$	$d_1 = -3/2$

$$S_0(x) = 6 - \frac{11}{2}(x+1) + 0(x+1)^2 + \frac{3}{2}(x+1)^3$$

$$= 6 - \frac{11}{2}x - \frac{11}{2} + \frac{3}{2}x^3 + \frac{9}{2}x^2 + \frac{9}{2}x + \frac{3}{2}$$

$$S_0(x) = \frac{3}{2}x^3 + \frac{9}{2}x^2 - x + 2$$

$$S_1(x) = 2 - 1(x-0) + \frac{9}{2}(x-0)^2 - \frac{3}{2}(x-0)^3$$

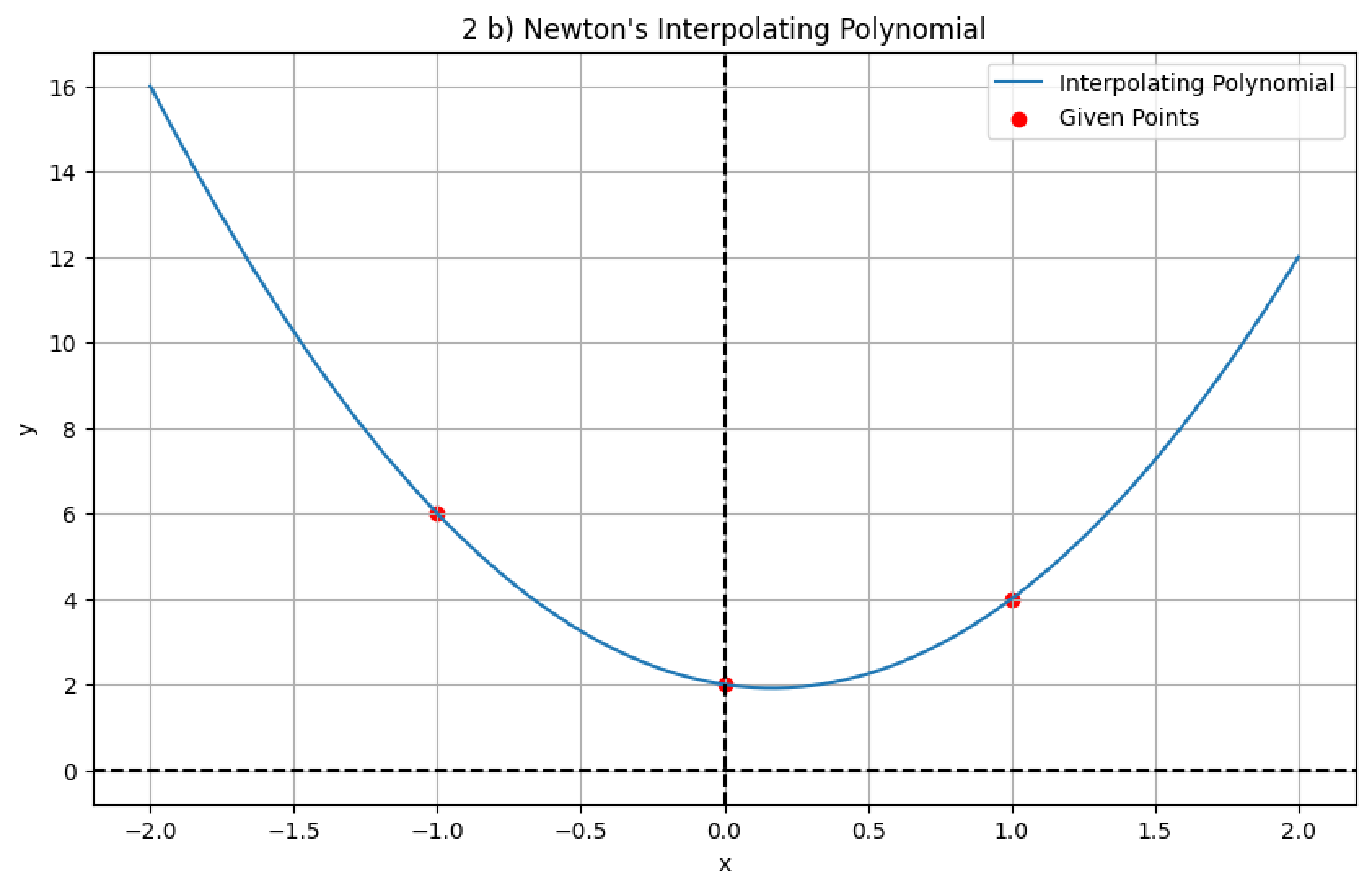
$$= 2 - x + \frac{9}{2}x^2 - \frac{3}{2}x^3$$

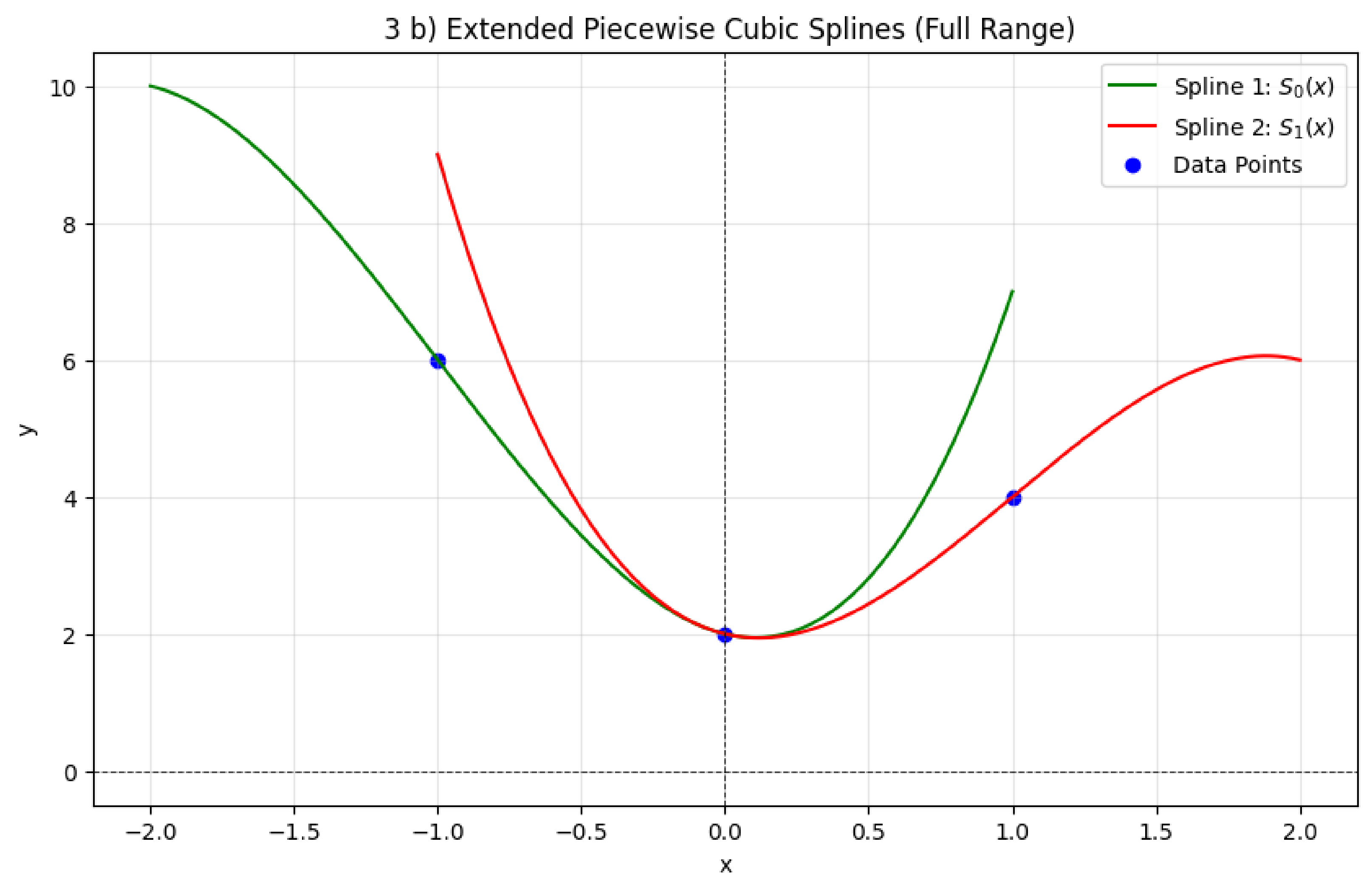
$$S_1(x) = -\frac{3}{2}x^3 + \frac{9}{2}x^2 - x + 2$$

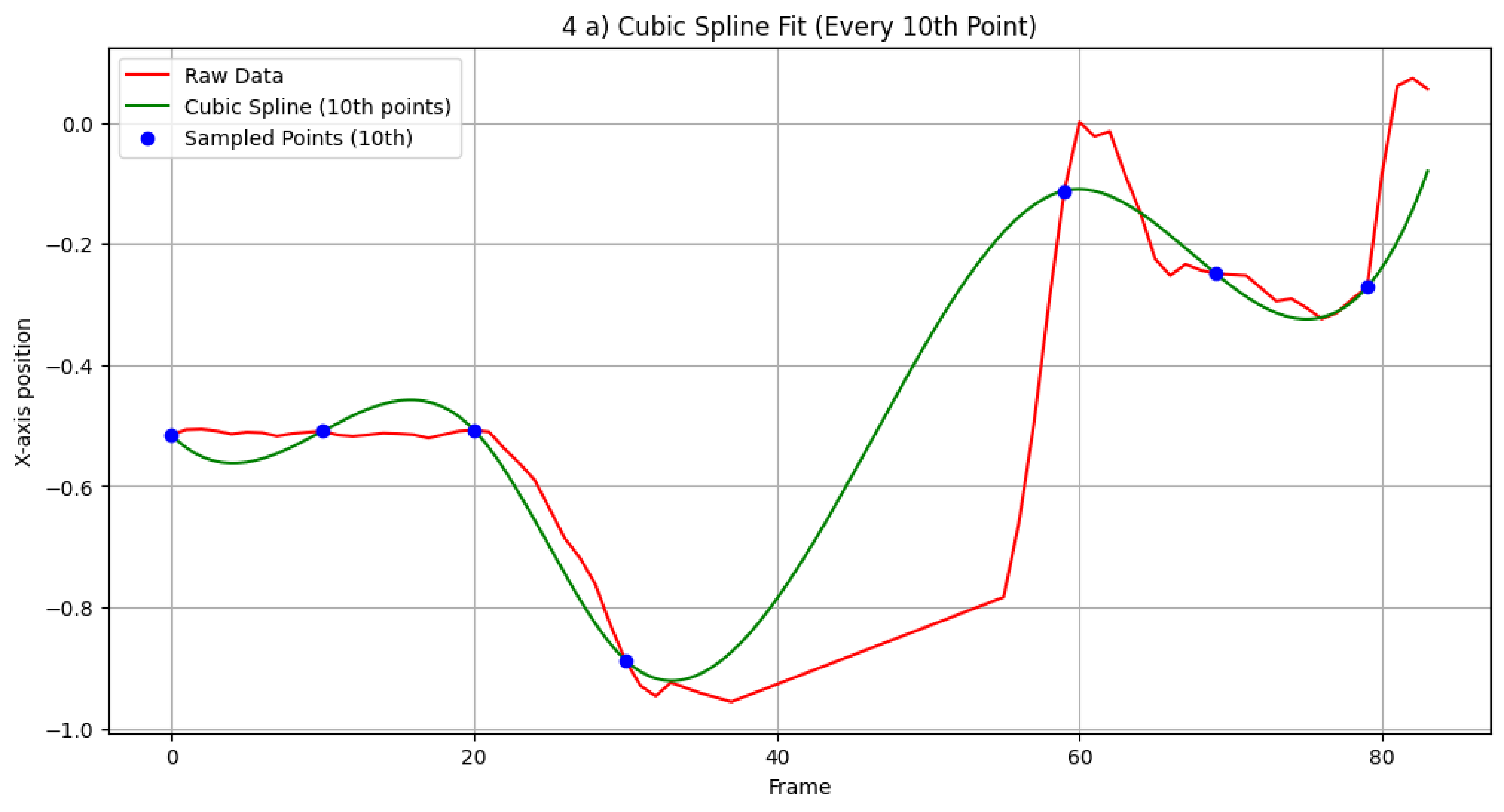
3.c) Newton interpolation is preferable due to its simplicity and direct fitting properties when the data is smooth and evenly distributed.

→ Piecewise cubic splines are preferred as they provide robust interpolation when the data is noisy and has irregular intervals.

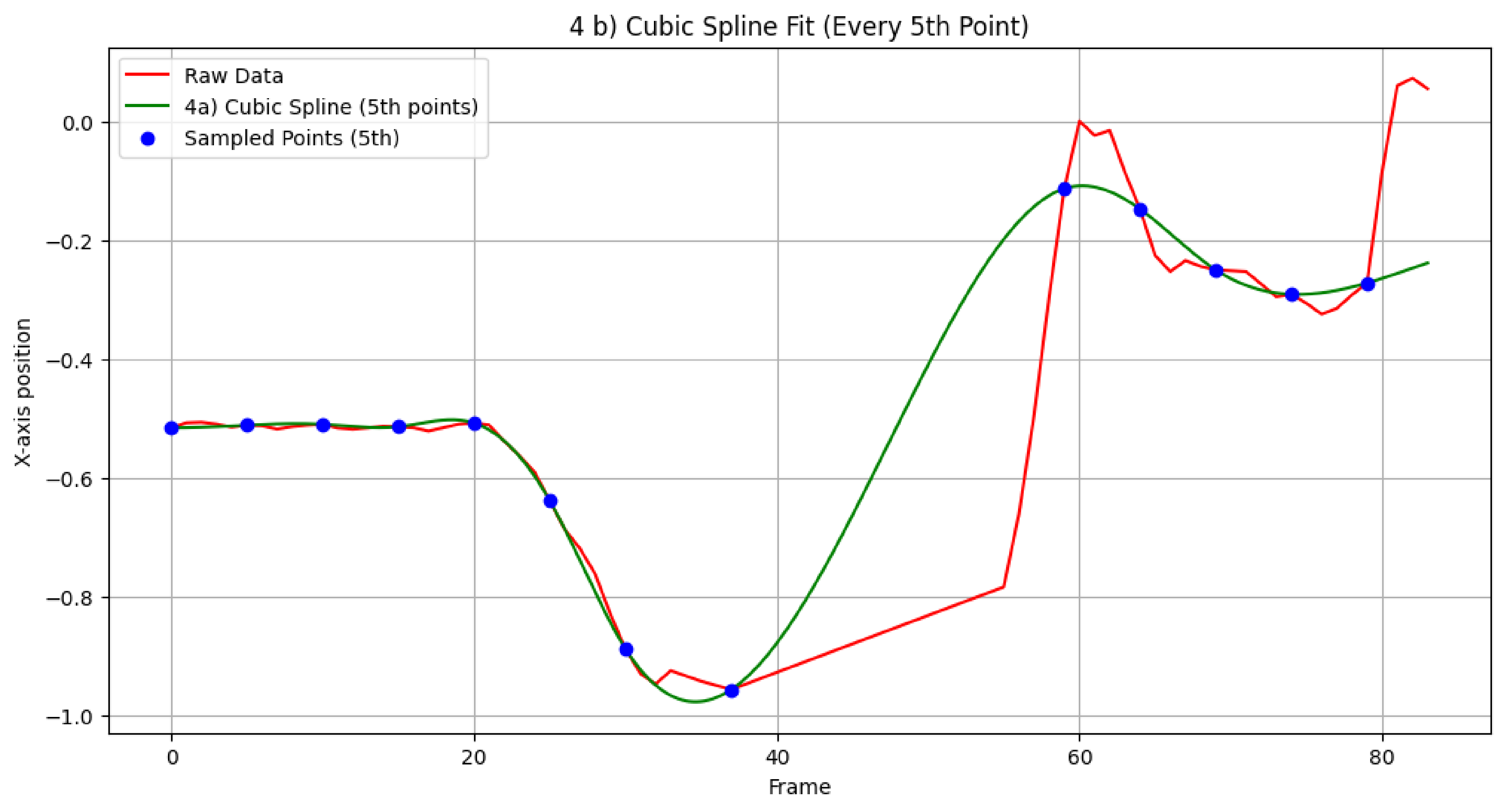
→ in our problem data is smooth and has no noise, missing values we would prefer newton polynomial interpolation.


2 b) Newton's Interpolating Polynomial
Interpolating Polynomial
Given Points
y
x
16
14
12
10
8
6
4
2
0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


3 b) Extended Piecewise Cubic Splines (Full Range)
Spline 1: $S_0(x)$
Spline 2: $S_1(x)$
Data Points
y
x
10
8
6
4
2
0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0



4 a) Cubic Spline Fit (Every 10th Point)
Raw Data
Cubic Spline (10th points)
Sampled Points (10th)
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0
20
40
60
80
Frame
X-axis position


4 b) Cubic Spline Fit (Every 5th Point)
Raw Data
4a) Cubic Spline (5th points)
Sampled Points (5th)
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
X-axis position
0
20
40
60
80
Frame