NM - Homework-4

Chandana Budaati (1002087303)

1 Bisection Method:

Sol: Given $f(x) = x^3 - 10x^2 + 5$, initial interval - $[0.6, 0.8]$ (a_1, b_1), $\epsilon = 2\times10^{-4}$

Iteration ①:- $f(a_1) = f(0.6) = 1.616$ $f(b_1) = f(0.8) = -0.888$

$f(a_1)\cdot f(b_1) = (+ve)(-ve) = (-ve)$.

∴ Root lies between 0.6 & 0.8.

$c_1 = \frac{a_1+b_1}{2} = \frac{0.6+0.8}{2} = 0.7$ $\boxed{\frac{b_1-a_1}{2} = 0.05 > \epsilon}$

$f(c_1) = f(0.7) = 0.443 > 0$.

$f(a_1)\cdot f(c_1) = (+ve)(+ve) = (+ve)$; $f(c)>0$ - Left Interval

New Interval $= [a_2, b_2] = [c_1, b_1] = \boxed{[0.7, 0.8]}$

Iteration ②:- $f(0.7) = 0.443$ $f(0.8) = -0.888$.

$f(a_2)\cdot f(b_2) = (+ve)(-ve) = (-ve)$

Root lies between 0.7 & 0.8.

$c_2 = \frac{0.7+0.8}{2} = 0.75$ $\boxed{\frac{b_2-a_2}{2} = 0.025 > \epsilon}$

$f(c_2) = f(0.75) = -0.203 < 0$ $f(a_2)\cdot f(c_2) = (+)(-) = -ve$ $f(c)<0$ -- Right Interval

New Interval $= [a_3, b_3] = [a_2, c_2] = \boxed{[0.7, 0.75]}$

Iteration ③:- $f(0.7) = 0.443$ $f(0.75) = -0.203$

$f(a_3)\cdot f(b_3) = -ve$

∴ Root lies between 0.7 and 0.75 $f(c)>0$, left

$c_3 = \frac{0.7+0.75}{2} = 0.725$, $\boxed{\frac{b-a}{2} = 0.0125 > \epsilon}$

$f(c_3) = 0.1248 > 0$, $f(a_3)\cdot f(c_3) = (+)(+) = (+)$

New Interval $= [a_4, b_4] = [c_3, b_3] = \boxed{[0.725, 0.75]}$

Iteration ④:- $f(0.725) = 0.1248$ $f(0.75) = -0.203$.

$f(a_4)\cdot f(b_4) = (+ve)(-ve) = (-ve)$ $f(c)<0$ - Right

∴ Root lies between 0.725 & 0.75.

$c_4 = \frac{a_4+b_4}{2} = 0.7375$.

$f(c_4) = f(0.7375) = -0.037932 < 0$. $\boxed{\frac{b-a}{2} = 0.00625 > \epsilon}$

New Interval $= [a_5, b_5] = \boxed{[0.725, 0.7375]}$

Iteration ⑤ :- $f(0.725) = 0.1248$ $f(0.7375) = -0.0379$ &

$f(a_5)\cdot f(b_5) = (+)(-) = (-)$

∴ Root lies between 0.725 and 0.7375

② Chandana Budaati

$f(c)>0$ - Left

$c_5 = \frac{a_5+b_5}{2} = 0.73125$ | $f(c_5) = 0.04375 > 0$

$\boxed{\frac{b-a}{2} = 0.00312 > \epsilon}$

New Interval $= [a_6, b_6] = [c_5, b_5] = [0.73125, 0.7375]$

Iteration ⑥ :- $f(a_6) = 0.04375$ $f(b_6) = -0.0379$

$f(a_6) \cdot f(b_6) = (+)(-) = (-ve)$.

∴ Root lies between 0.73125 and 0.7375.

$f(c)>0$ left

$c_6 = \frac{a_6+b_6}{2} = 0.734375 \cong 0.7343$ | $f(c_6) = 0.00298 > 0$

$\boxed{\frac{b-a}{2} = 0.00156 > \epsilon}$

New Interval $= [a_7, b_7] = [c_6, b_6] = [0.7343, 0.7375]$

Iteration ⑦ : $f(a_7) = 0.00298$ $f(b_7) = -0.0879$

$f(a_7) \cdot f(b_7) = (+)(-) = (-ve)$

∴ Root lies between 0.7343875 and 0.7375.

$f(c) < 0$ - Right

$c_7 = \frac{a_7+b_7}{2} = 0.7359375$.

$\boxed{\frac{b-a}{2} = 0.0008 > \epsilon}$

$f(c_7) = f(0.7359375) = -0.01745 < 0$

New Interval $= [a_8, b_8] = [0.734375, 0.735975]$
(a_7, c_7)

Iteration ⑧ :- $f(a_8) = f(0.734375) = 0.0029869$ | $f(b_8) = f(0.735975) = -0.01745$

$f(a_8) \cdot f(b_8) = (+)(-) = (-ve)$

∴ Root lies between 0.734375 and 0.735975.

$f(c) < 0$ - Right

$c_8 = \frac{a_8+b_8}{2} = 0.735175$

$\boxed{\frac{b-a}{2} = 0.0004 > \epsilon}$

$f(c_8) = f(0.735175) = -0.0074737 < 0$

∴ New Interval $= [a_9, b_9] = [a_8, c_8] = [0.734375, 0.735175]$.

Iteration ⑨ :- $f(a_9) = f(0.734375) = 0.00298$ $f(b_9) = f(0.735175) = -0.0074737$

$f(a_9) \cdot f(b_9) = (+)(-ve) = (-ve)$

∴ Root lies between 0.734375 and 0.735175.

$c_9 = \frac{a_9+b_9}{2} = 0.734775$

$f(c) < 0$ - Right

$f(c_9) = f(0.734775) = -0.002242 < 0$

$\boxed{\frac{b-a}{2} = 0.0002 \ngtr \epsilon}$

∴ New Interval $= [a_{10}, b_{10}] = [a_9, c_9] = [0.734375, 0.734775]$

Iteration ⑨ :- has Interval - $[0.734375, 0.734775]$ with $\frac{b-a}{2} = \epsilon$

$\therefore$ Root of the Equation $= \frac{a_{10}+b_{10}}{2} = \frac{0.734375 + 0.734775}{2}$

$$\boxed{Root = 0.734575}$$

ⓑ To get required No. of iterations, we have formula.

$$i \geq \log_2\left(\frac{b_0 - a_0}{\epsilon}\right)$$

$b_0 = 0.8$
$a_0 = 0.6$
$\epsilon = 0.0002$

$$\geq \log_2(1000)$$

$$\geq 9.96578$$

$$\boxed{i \geq 10}$$

$\therefore$ No. of iterations are, $i \geq 10$.

② Find root of following function using Newton Method.
$f(x) = x^3 - 10x^2 + 5 = 0$.
Consider Starting guess as $x_0 = 0.7$ and error tolerance, $e = 2 \times 10^{-4}$.

Sol:- Given, $f(x) = x^3 - 10x^2 + 5 = 0$, $x_0 = 0.7$, $\epsilon = 2 \times 10^{-4} = 0.0002$.

Newton Method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$f'(x) = \frac{d}{dx} f(x) = 3x^2 - 20x.$$

Iteration ①:- $n = 0$, $x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$
$x_0 = 0.7$.

$$x_1 = 0.7 - [-0.035355]$$

$$x_1 = 0.735355.$$

Iteration ②:- $n = 1$, $x_1 = 0.735355$:
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Chandana Budaati

(3)

$E_1 = |x_0 - x_1|$
$= |0.7 - 0.735355|$
$= 0.035355 > E$

$x_2 = 0.735355 - [7.51156 \times 10^{-4}]$

$= 0.735355 - 0.000751156$

$x_2 = 0.734604$

Iteration (3). $\boxed{x_2 = 0.734604}$

$x_{2+1} = x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$E_2 = |x_1 - x_2|$
$= |0.735355 - 0.734604|$
$= 0.000751156 > E$

$= 0.734604 - [4.9221 \times 10^{-7}]$

$= 0.734604 - 0.00000049$

$E_3 = |x_2 - x_3|$
$= 0.00000065 < E$

$x_3 = 0.73460\cancel{4}335$

$\boxed{x_3 = 0.73460}$

Here $E_3 < E$. Hence we stop the iteration and the required root will be $\boxed{0.7346}$

(3) Find the root of following function using Secant Method. Initial guesses are $x_0 = 1$, $x_1 = 2$ and error tolerance $f(x) = x^2 - 2$. $e = 2 \times 10^{-5}$.

Sol:- Given, $f(x) = x^2 - 2$, $x_0 = 1$, $x_1 = 2$, $E = 0.00002$.

Secant Method formula, $x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$

$f(x_0) = -1$
$f(x_1) = 2$

Iteration (1) - $n = 1$, $x_0 = 1$, $x_1 = 2$

$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$

$E_1 = |x_0 - x_1|$
$= |2 - 1|$
$E_1 = 1 > E$

$= 2 - \frac{(2-1)}{(2) - (-1)} (2)$

$= 2 - \frac{2}{3}$

$= \frac{4}{3}$

$\boxed{x_2 = 1.3333}$

$E_2 = |x_2 - x_1|$
$= |1.333 - 2|$
$= 0.666666666\cancel{67} > E$

$$f(x_2) = f(1.33333) = -0.2222222$$

$$x_3 = x_2 - \frac{(x_2 - x_1)\, f(x_2)}{f(x_2) - f(x_1)}$$

$$= 1.33333 - \left[\frac{(1.3333 - 2)(-0.2222)}{-0.2222 - 2}\right]$$

$$= 1.333 - [-0.06667\cancel{0}0067]$$

$$= \cancel{1.22666}$$

$$\boxed{x_3 = 1.39997}$$

$$E_3 = |x_3 - x_2| = |1.39997 - 1.333| = 0.0666401 > \epsilon$$

$$\boxed{f(x_3) = -0.04008}$$

$$x_4 = x_3 - \frac{(x_3 - x_2)\, f(x_3)}{f(x_3) - f(x_2)}$$

$$= 1.39997 - \left[\frac{(1.39997 - 1.3333)(-0.04008)}{\cancel{1.39997}(-0.04008) - (-0.2222)}\right]$$

$$= 1.39997 - \left[\cancel{-0.51226664}\; -0.0146724\right]$$

$$E_4 = |x_4 - x_3| = 0.01494 > \epsilon$$

$$\boxed{x_4 = 1.414642} \quad \boxed{f(x_4) = 0.001206}$$

$$x_5 = x_4 - \frac{(x_4 - x_3)\, f(x_4)}{f(x_4) - f(x_3)}$$

$$= 1.41464 - \left[\frac{(1.41464 - 1.39997)(0.001206)}{(0.001206) - (-0.04008)}\right]$$

$$= 1.41464 - [0.0004285]$$

$$E_5 = |x_5 - x_4| = 0.00043. > \epsilon$$

$$\boxed{x_5 = 1.4142115} \quad \boxed{f(x_5) = -0.00000583}$$

$$x_6 = x_5 - \frac{(x_5 - x_4)\, f(x_5)}{f(x_5) - f(x_4)}$$

$$E_6 = |x_6 - x_5| = 0.000002 = \epsilon$$

$$= 1.4142115 - \left[\frac{(1.4142115 - 1.41464)(-0.00000583)}{(-0.00000583) - (0.001206)}\right]$$

$$\boxed{x_6 = 1.414213561} \quad \boxed{f(x_6) = -0.000000000388}$$

Chandana Budaati (1002087323)

Here $E_6 \le E$ so we can stop iteration.

Required Root $= 1.414211$.

(4) Given $f_1(x,y) = 6x^3 + xy - 3y^3 - 4 = 0$ $(x_0, y_0) = (2,2)$

Sol:- $f_2(x,y) = x^2 - 18xy^2 + 16y^3 + 1 = 0$

Formula: $X_{K+1} = X_K - DF^{-1} F_K$

$$\begin{pmatrix} x_{K+1} \\ y_{K+1} \end{pmatrix} = \begin{pmatrix} x_K \\ y_K \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x_K} & \frac{\partial f_1}{\partial y_K} \\ \frac{\partial f_2}{\partial x_K} & \frac{\partial f_2}{\partial y_K} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_K, y_K) \\ f_2(x_K, y_K) \end{pmatrix}$$

For X_1,

Iteration (1):-

$$X_1 \Rightarrow \begin{pmatrix} x_{0+1} \\ y_{0+1} \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial y_0} \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial y_0} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{pmatrix}$$

$\frac{\partial f_1}{\partial x} = 18x^2 + y$ $\frac{\partial f_1}{\partial y} = x - 9y^2$ $f_1(x_0, y_0) = f_1(2,2) = 24$

$\frac{\partial f_2}{\partial x} = 2x - 18y^2$ $\frac{\partial f_2}{\partial y} = -36xy + 48y^2$ $f_2(x_0, y_0) = f_2(2,2) = -11$

Using Calculator i.e substituting (2,2)

$$DF = \begin{pmatrix} 74 & -34 \\ -68 & 48 \end{pmatrix} \cdot DF^{-1} = \begin{pmatrix} 6/155 & 17/620 \\ 17/310 & 37/620 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.038709 & 0.02 \\ 0.0548 & 0.0596 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 0.038709 & 0.02704 \\ 0.0548 & 0.0596 \end{pmatrix} \begin{pmatrix} 24 \\ -11 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 0.6274193 \\ 0.6596774 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1.3725 \\ 1.3403 \end{pmatrix}$$

Iteration (2):-

$$X_2 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_1, y_1) \\ f_2(x_1, y_1) \end{pmatrix}$$

$\frac{\partial f_1}{\partial x_1} = 35.2479125$ $\frac{\partial f_1}{\partial y_1} = -14.7923786$ $f_1(x_1, y_1) = 6.1291333$

$\frac{\partial f_2}{\partial x_1} = -29.590274$ $\frac{\partial f_2}{\partial y_1} = 20.0031733$ $f_2(x_1, y_1) = -2.9728804.$

$$DF = \begin{pmatrix} 35.2479125 & -14.7951368 \\ -29.590274 & 20.0031733 \end{pmatrix}$$

$$DF^{-1} = \begin{pmatrix} 0.07484034 & 0.05535487 \\ 0.110709746 & 0.1318773677 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1.3725 \\ 1.3403 \end{pmatrix} - \begin{pmatrix} 0.07484034 & 0.05535487 \\ 0.110709746 & 0.1318773677 \end{pmatrix} \begin{pmatrix} 0.129133 \\ -2.9728804 \end{pmatrix}$$

$$= \begin{pmatrix} 1.3725 \\ 1.3403 \end{pmatrix} - \begin{pmatrix} 0.27414298 \\ 0.286499116 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1.078357 \\ 1.0538008 \end{pmatrix}$$

Iteration ③. $X_3 = X_2 - DF_{x_2}^{-1} F_2$

$$X_3 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} f_1(x_2, y_2) \\ f_2(x_2, y_2) \end{pmatrix}$$

$$DF = \begin{pmatrix} 21.9851696 & -8.916108135 \\ -17.843216 & 13.548187 \end{pmatrix}$$

$f_1(x_2, y_2) = 1.14947768$

$f_2(x_2, y_2) = -0.6684817$

$$DF^{-1} = \begin{pmatrix} 0.09768 & 0.064313 \\ 0.1286483 & 0.1585114 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1.078357 \\ 1.0538008 \end{pmatrix} - \begin{pmatrix} 0.09768 & 0.064313 \\ 0.1286483 & 0.1585114 \end{pmatrix} \begin{pmatrix} 1.14947768 \\ -0.6684817 \end{pmatrix}$$

$$= \begin{pmatrix} 1.078357 \\ 1.0538008 \end{pmatrix} - \begin{pmatrix} 0.0692889 \\ 0.041916379 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1.0090681 \\ 1.01188442 \end{pmatrix}$$

(5) Iteration (4):- $X_4 = X_3 - DF_3^{-1} F_K$ Chandana Budaati (1002087323)

$$X_4 \Rightarrow \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial y_3} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_3, y_3) \\ f_2(x_3, y_3) \end{pmatrix}$$

$\frac{\partial f_1}{\partial x_3} = 19.3398162$ $\quad \frac{\partial f_1}{\partial y_3} = -8.20612262$ $\quad f_1(x_3, y_3) = 0.07753474$

$\frac{\partial f_2}{\partial x_3} = -16.412245$ $\quad \frac{\partial f_2}{\partial y_3} = 11.812282$ $\quad f_2(x_3, y_3) = -0.002033032$

$$X_4 = \begin{pmatrix} 1.0090681 \\ 1.01188442 \end{pmatrix} - \begin{pmatrix} 19.3398162 & -8.20612262 \\ -16.412245 & 11.812282 \end{pmatrix}^{-1} \begin{pmatrix} 0.07753474 \\ -0.002033032 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 1.0090681 \\ 1.01188442 \end{pmatrix} - \begin{pmatrix} 0.125975546 & 0.08751661 \\ 0.175033201 & 0.20625514 \end{pmatrix} \begin{pmatrix} 0.077534774 \\ -0.002033032 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0090681 \\ 1.01188442 \end{pmatrix} - \begin{pmatrix} 0.0099145096 \\ 0.0131508365 \end{pmatrix}$$

$$\boxed{X_4 = \begin{pmatrix} 0.9991230 \\ 0.9987325 \end{pmatrix}}$$

(∵) Given a hint, once $(x, y) <= (1.005, 1.002)$, we can stop iteration. So we stop iteration.

Here, $X_4 \leq (1.005, 1.002)$.

(5) Golden Section Search (GSS)

Sol Given, $f(x) = x^6 - 11x^3 + 17x^2 - 7x + 1$, initial interval ~~(a₀,b₀)~~ = $[0, 1]$

Error tolerance, $\epsilon \leq 0.02$.

Iteration 1:- $x_1 = a_0 + \rho(b_0 - a_0)$ $\quad x_2 = a_0 + (1-\rho)(b_0 - a_0)$

$\rho = 0.382$ $\quad = 0 + (0.382)(1-0)$ $\quad = 0 + (0.618)(1-0)$

$(1-\rho) = 0.618$ $\quad x_1 = 0.382$ $\quad x_2 = 0.618$

$F(x_1) = F(0.382)$ $\quad f(x_2) = f(0.618)$

$= 0.1966426305$ $\quad = 0.626098352$

$f(x_1) < f(x_2)$ <- Right

New Interval, $[a_1, b_1] = [a_0, x_2]$ $\quad |b_0 - a_0| = 0.236 > \epsilon$

$= [0, 0.618]$

Iteration ②:- $x_1 = a_1 + \rho(b_1 - a_1)$
$= 0 + (0.382)(0.618 - 0)$
$x_1 = 0.236076$

$x_2 = a_1 + (1-\rho)(b_1 - a_1)$
$= 0 + (0.618)(0.618 - 0)$
$x_2 = 0.381924$

$F(x_1) = F(0.236076)$
$= 0.1503564812$

$F(x_2) = F(0.381924)$
$= 0.1965498379$

$F(x_1) < F(x_2)$

New Interval $= [a_2, b_2] = [a_1, x_2] = [0, 0.381924]$.

Iteration ③:- $x_1 = a_2 + \rho(b_2 - a_2)$
$= 0 + (0.382)(0.381924)$
$x_1 = 0.14589497$
$F(x_1) = \cancel{1.21458608}$
$F(x_1) = 0.3064360042$

$x_2 = a_2 + (1-\rho)(b_2 - a_2)$
$= 0 + (0.618)(0.381924)$
$x_2 = 0.23602903$
$F(x_2) = 0.1503944613$

$\boxed{|b_3 - a_3| = 0.23602903 > \epsilon}$

$F(x_1) > F(x_2)$, New Interval $= [a_3, b_3] = [x_1, b_2]$
$= [\cancel{0.3064360042}\ 0.14589497, 0.381924]$

~~Iteration ④:-~~ $\cancel{x_1 = a_3 + \rho(b_3 - a_3)}$
$= \cancel{0.306436004 + (0.382)(0.075487996)}$
$\cancel{0.14}$
$= \cancel{0.33527242}$
$=$
$\cancel{F(x_1) = 0.1508836753}$

$\cancel{x_2 = a_3 + (1-\rho)(b_3 - a_3)}$
$= \cancel{(0.3064360042) + (0.618)(0.075487996)}$
$= \cancel{0.353087586}$
$\cancel{F(x_2) = 0.1655119735}$

$\cancel{F(x_1) < F(x_2)}$. ~~New Interval~~ $\cancel{= [a_4, b_4] = [a_3,}$

Iteration ④:- $x_1 = a_3 + \rho(b_3 - a_3)$
$= 0.14589497 + (0.382)(0.23602903)$
$= 0.2360580595$
$F(x_1) = 0.1503709831$

$x_2 = a_3 + (1-\rho)(b_3 - a_3)$
$= 0.14589497 + (0.618)(0.23602903)$
$= 0.2917609105$
$F(x_2) = 0.1322099954$

$F(x_1) > F(x_2)$.

New Interval, $[a_4, b_4] = [0.2360580595, 0.381924]$

$\boxed{> \text{— left} \quad |b_4 - a_4| = 0.145865941 > \epsilon}$

Iteration ⑤:- $x_1 = a_4 + \rho(b_4 - a_4)$
$= 0.2917788488$
$F(x_1) = 0.1449085923$

$x_2 = a_4 + (1-\rho)(b_4 - a_4)$
$= 0.3262032107$
$F(x_2) = 0.1322121709$

$\boxed{|b_5 - a_5| = 0.090145 > \epsilon}$

<- right

$F(x_1) < F(x_2)$, New Interval $= [a_5, b_5] = [0.2360580595, 0.326203211]$

Chandana Budaath

Iteration ⑥:- $x_1 = a_5 + P(b_5 - a_5)$ $x_2 = a_5 + (1-P)(b_5 - a_5)$

$= 0.2704935074$ $= 0.2917676305$

$F(x_1) = 0.1330692731$ $F(x_2) = 0.132210784$

$|b-a|$ $\Rightarrow 0.0557$

>- left

$F(x_1) > F(x_2)$. New Interval $= [a_6, b_6] = [x_1, b_5]$

$= [0.2704935074,\ 0.3262032117]$

Iteration ⑦:- $x_1 = a_6 + P(b_6 - a_6)$ $x_2 = a_6 + (1-P)(b_6 - a_6)$

~~= 0.3049228933~~ $= 0.2704935074 + (0.618)(0.0557097043)$

$x_1 = 0.2917746143$ $x_2 = 0.3049221042$

$F(x_1) = 0.1322116471$ $F(x_2) = 0.1351065578$

<- right

$|b-a|$ $= 0.0344 > \epsilon$

$F(x_1) < F(x_2)$. New Interval $= [a_7, b_7] = [a_6, x_2]$

$= [0.2704935074,\ 0.3049221042]$

Iteration ⑧:- $x_1 = a_7 + P(b_7 - a_7)$ $x_2 = a_7 + (1-P)(b_7 - a_7)$

$= 0.2836452314$ $= 0.29177038.$

$|b-a| = 0.0212 > \epsilon$

<- right $F(x_1) = 0.1317063986$ $F(x_2) = 0.1322111237$

$F(x_1) < F(x_2)$. New Interval $= [a_8, b_8] = [a_7, x_2]$

$= [0.2704935074,\ 0.29177038]$

Iteration ⑨:- $x_1 = a_8 + P(b_8 - a_8)$ $x_2 = a_8 + (1-P)(b_8 - a_8)$

$= 0.2786212727$ $= 0.2836426147$

$|b-a| = 0.013149 < \epsilon$

>- left $F(x_1) = 0.1319032576$ $F(x_2) = 0.131981$

$F(x_1) < F(x_2)$. New Interval $= [a_9, b_9] = [a_8, x_2]$

~~= [0.2786212727, 0.29177038]~~

$= [0.2704935074,\ 0.283643]$

Here, $|b_9 - a_9| = |0.28364338 - 0.27049327| = 0.0131491 < \epsilon$

so we stop iterations. Hence,

$$x^* = \frac{a_9 + b_9}{2} = ~~0.28519583~~.$$

$$\therefore \boxed{x^* = 0.2770681}$$

(6) Given data points: $\overset{x_0\ y_0}{(0,1)}$ $\overset{x_1\ y_1}{(3,4)}$ $\overset{x_2\ y_2}{(5,7)}$

Sol: (i) Lagrange Polynomial.

Formula: $P_{n-1}(x) = \sum_{i=1}^{n} y_i\, l_i(x)$.

$$l_i(x) = \frac{x-x_1}{x_i-x_1} \cdot \frac{x-x_2}{x_i-x_2}, \dots \frac{x-x_n}{x_i-x_n}$$

Here, $n=3$, $P_{3-1}(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$.

$$l_0(x) = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2}$$

$$= \left(\frac{x-3}{-3}\right)\left(\frac{x-5}{-5}\right)$$

$$l_0(x) = \frac{x^2-8x+15}{15}, \quad y_0 \cdot l_0(x) = \frac{1(x^2-8x+15)}{15}$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2}$$

$$= \left(\frac{x-0}{3-0}\right)\left(\frac{x-5}{3-5}\right)$$

$$l_1(x) = \frac{x^2-5x}{-6}, \quad y_1 \cdot l_1(x) = \frac{4(x^2-5x)}{-6} = \frac{-2x^2+10x}{3}$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1}$$

$$= \left(\frac{x}{5}\right)\left(\frac{x-3}{5-3}\right)$$

$$l_2(x) = \frac{x^2-3x}{10}, \quad y_2 \cdot l_2(x) = \frac{7(x^2-3x)}{10} = \frac{7x^2-21x}{10}$$

$$\therefore P_2(x) = \frac{x^2-8x+15}{15} + \frac{(-2x^2+10x)}{3} + \frac{(7x^2-21x)}{10}$$

$$= \frac{1}{30}(2x^2-16x+30 - 20x^2 + 100x + 21x^2 - 63x)$$

$$= \frac{3x^2+21x+30}{30}$$

$$\boxed{P_2(x) = \frac{x^2+7x+10}{10}}$$

100
63
37
16
21

⑥ Given data points: $\overset{x_0\ y_0}{(0,1)}\ \overset{x_1\ y_1}{(3,4)}\ \overset{x_2\ y_2}{(5,7)}$

(a) Lagrange Polynomial.

Formula: $P_{n-1}(x) = \sum_{i=1}^{n} y_i\, l_i(x)$.

$$l_i(x) = \frac{x-x_1}{x_i-x_1}\cdot\frac{x-x_2}{x_i-x_2},\ \cdots\ \frac{x-x_n}{x_i-x_n}$$

Here, $n = 3$, $P_{3-1}(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$.

$$l_0(x) = \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2}$$

$$= \left(\frac{x-3}{-3}\right)\left(\frac{x-5}{-5}\right)$$

$$l_0(x) = \frac{x^2-8x+15}{15}, \quad y_0\cdot l_0(x) = \frac{1\,(x^2-8x+15)}{15}$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2}$$

$$= \left(\frac{x-0}{3-0}\right)\left(\frac{x-5}{3-5}\right)$$

$$l_1(x) = \frac{x^2-5x}{-6}, \quad y_1\cdot l_1(x) = \frac{4\cdot(x^2-5x)}{-6} = \frac{-2x^2+10x}{3}$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1}$$

$$= \left(\frac{x}{5}\right)\left(\frac{x-3}{5-3}\right)$$

$$l_2(x) = \frac{x^2-3x}{10}, \quad y_2\cdot l_2(x) = \frac{7(x^2-3x)}{10} = \frac{7x^2-21x}{10}$$

$$\therefore P_2(x) = \frac{x^2-8x+15}{15} + \frac{(-2x^2+10x)}{3} + \frac{(7x^2-21x)}{10}$$

$$= \frac{1}{30}(2x^2-16x+30-20x^2+100x+21x^2-63x)$$

$$= \frac{3x^2+21x+30}{30}$$

$$\boxed{P_2(x) = \frac{x^2+7x+10}{10}}$$

100
62
37
16
21

Chandana Budaati

6 (ii) Newton Polynomial.

Sol: Given data points - $\overset{x_1, y_1}{(0,1)}$, $\overset{x_2\ y_2}{(3,4)}$ & $\overset{x_3\ y_3}{(5,7)}$

$$P_{n-1}(x) = a_1 + (x-x_1)a_2 + (x-x_1)(x-x_2)a_3 + \dots$$
$$\dots (x-x_1)(x-x_2)\dots(x-x_{n-1})a_n$$

$$\therefore \quad P_2(x) = a_1 + (x-x_1)a_2 + (x-x_1)(x-x_2)a_3.$$

$y_1(x) = a_1$

$y_1 \Rightarrow 1 = a_1$

$y_2(x) \Rightarrow a_1 + (x_2-x_1)a_2$

$y_2 \Rightarrow (1) + (3-0)a_2 = 4$

$\Rightarrow 1 + 3a_2 = 4$

$\Rightarrow \boxed{a_2 = 1}$

$y_3(x) = a_1 + (x_3-x_1)a_2 + (x_3-x_1)(x_3-x_2)a_3$

$\Rightarrow \cancel{1+4(x)+x(x-3)}$

$\Rightarrow a_1 + (5-0)a_2 + (5-0)(5-3)a_3 = 7$

$\Rightarrow 1 + 5(1) + (5)(2)(a_3) = 7$

$\Rightarrow 1+5+10a_3 = 7$

$\Rightarrow \boxed{a_3 = 1/10}$

$\cancel{y(x) = k+}$

$$P_2(x) = (1) + (x-0)(1) + (x-0)(x-3)\left(\frac{1}{10}\right)$$

$$= 1 + x + \frac{x^2-3x}{10}$$

$$= (x^2 - 3x + 10x + 10)/10$$

$$P_2(x) = (x^2+7x+10)/10$$

$$\therefore \ y(x) = P_2(x) = \frac{x^2+7x+10}{10}$$

we know that, $Ax = b$,

$\therefore$ Linear system of equations.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 5 & 10 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

b $\qquad$ A $\qquad$ X.

Lower Triangular Matrix.

6

(iii) Piecewise Cubic Polynomial.

Given data points, $(x_0, y_0) = (0,1)$, $(x_1, y_1) = (3,4)$, $(x_2, y_2) = (5,7)$

Conditions, $S_0(x_0) = f(x_0) = y_0$.

$S_1(x_1) = f(x_1) = y_1$

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3.$$

$$y_0 \Rightarrow S_0(x_0) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3 \rightarrow (1)$$

$$y_1 \Rightarrow S_1(x_1) = a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3 \rightarrow (2)$$

substitute, $(x_0, y_0) = (0,1)$

$$(1) \Rightarrow S_0(0) = a_0 + b_0(0-0) + c_0(0-0)^2 + d_0(0-0)^3$$

$$\Rightarrow a_0 = S_0(0) \quad [\because S_0(x_0) = y_0.]$$

$$\Rightarrow y_0 = a_0$$

$$\Rightarrow \boxed{a_0 = 1}$$

Substitute (x_1, y_1) in (1). $\overset{x_1\ y_1}{(3, 4)}$.

$$(1) \Rightarrow S_0(3) = a_0 + b_0(3-0) + c_0(3-0)^2 + d_0(3-0)^3$$

$$\Rightarrow S_0(3) = 1 + 3b_0 + 9c_0 + 27d_0 \quad [\because a_0 = 1 \text{ \& } S_0(3) = y_1]$$

$$\Rightarrow 4 - 1 = 3b_0 + 9c_0 + 27d_0.$$

8

Chandana Budaati

$\therefore \quad b_0 + 3c_0 + 9d_0 = 1 \rightarrow (3)$

Substitute $(x_1, y_1) = (3, 4)$ in (2)

$(2) \Rightarrow S_1(x) = a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3.$

$S_1(3) = a_1 + b_1(0) + c_1(0)^2 + d_1(0)^3$

$S_1(3) = a_1 \quad [\because \; S_1(3) = y_1 = 4]$

$\therefore \quad S_1(3) = y_1 = a_1 = 4.$

~~from (2)~~.

Substitute, $(x_2, y_2) = (5, 7)$ in (2),

$S_1(5) = a_1 + b_1(5-3) + c_1(5-3)^2 + d_1(x-3)^3$

$7 = 4 + 2b_1 + 4c_1 + 8d_1 \quad [\because \; a_1 = 4 \; \& \; S_1(5) = y_1 = 7]$

$3 = 2b_1 + 4c_1 + 8d_1 \rightarrow (4)$

we find derivatives of eq (1),

$S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3.$

$S_0'(x) = b_0 + 2c_0(x) + 3d_0(x)^2$

$S_0''(x) = 2c_0 + 6d_0x$

Derivatives of eqn (2),

$S_1(x) = a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3$

$S_1'(x) = b_1 + 2c_1(x-3) + 3d_1(x-3)^2$

$S_1''(x) = 2c_1(1) + 6d_1(x-3)$

$(x-3)^2$
$\Rightarrow x^2 - 6x + 9$
$\Rightarrow 2x - 6.$

$x-3$

Let, $S_0'(3) = S_1'(3)$

$S_0'(3) = b_0 + 6c_0 + 27d_0$

$S_1'(3) = b_1 + 0 + 0 = b_1$

$\therefore \quad b_0 + 6c_0 + 27d_0 = b_1 \rightarrow (5)$

$S_0''(3) = S_1''(3)$

$S_0''(3) = 2c_0 + 18d_0$

$S_1''(3) = 2c_1 + 0 = 2c_1$

$2c_0 + 18d_0 = 2c_1$

$\Rightarrow c_0 + 9d_0 = c_1 \rightarrow (6)$

Find Boundary Condition:-

$S_0''(x) = S''(x_n) = 0$

$S_0''(0) = S_1''(5)$

$S_0''(0) = 2c_0 + 6d_0(0) \Rightarrow 2c_0.$

$S_1''(5) = 2c_1 + 6d_1(5-3).$

$\Rightarrow 2c_1 + 12d_1$

$S_1''(0) = 0 \Rightarrow 2c_0 = 0 \Rightarrow c_0 = 0$

$S_1''(5) = 0 \Rightarrow 2c_1 + 12d_1 = 0.$

$\Rightarrow c_1 + 6d_1 = 0 \rightarrow (7)$

We have a couple of equations numbered.

$b_0 + 9d_0 = 1$

we found, $a_0 = 1$, $a_1 = 4$, $c_0 = 0$.

$2b_1 + 4c_1 + 8d_1 = 3.$

$b_0 + 27d_0 - b_1 = 0$

$9d_0 - c_1 = 0$

$c_1 + 6d_1 = 0$

Let's write the matrix form, $AX = b$,

$$\begin{array}{ccccc} b_0 & d_0 & b_1 & c_1 & d_1 \end{array}$$

$$\begin{bmatrix} 1 & 9 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 8 \\ 1 & 27 & -1 & 0 & 0 \\ 0 & 9 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} b_0 \\ d_0 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

using Jacobi / Guass elimination

Rearrange equations & solve, we get,

$b_0 = 17/20$, $d_0 = 1/60$, $b_1 = 13/10$, $c_1 = 3/20$, $d_1 = -1/40$.

Now, $S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3$

$= 1 + \left(\frac{17}{20}\right)x + 0 + \left(\frac{1}{60}\right)x^3$

$\therefore S_0(x) = 1 + \frac{17}{20}x + \frac{1}{60}x^3$

(a)

$S_1(x) = a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3.$

$S_1(x) = 4 + \frac{13}{10}(x-3) + \frac{3}{20}(x-3)^2 + \left(\frac{-1}{40}\right)(x-3)^3$ $\quad (a^3 - b^3 - 3a^2b + 3ab^2)$

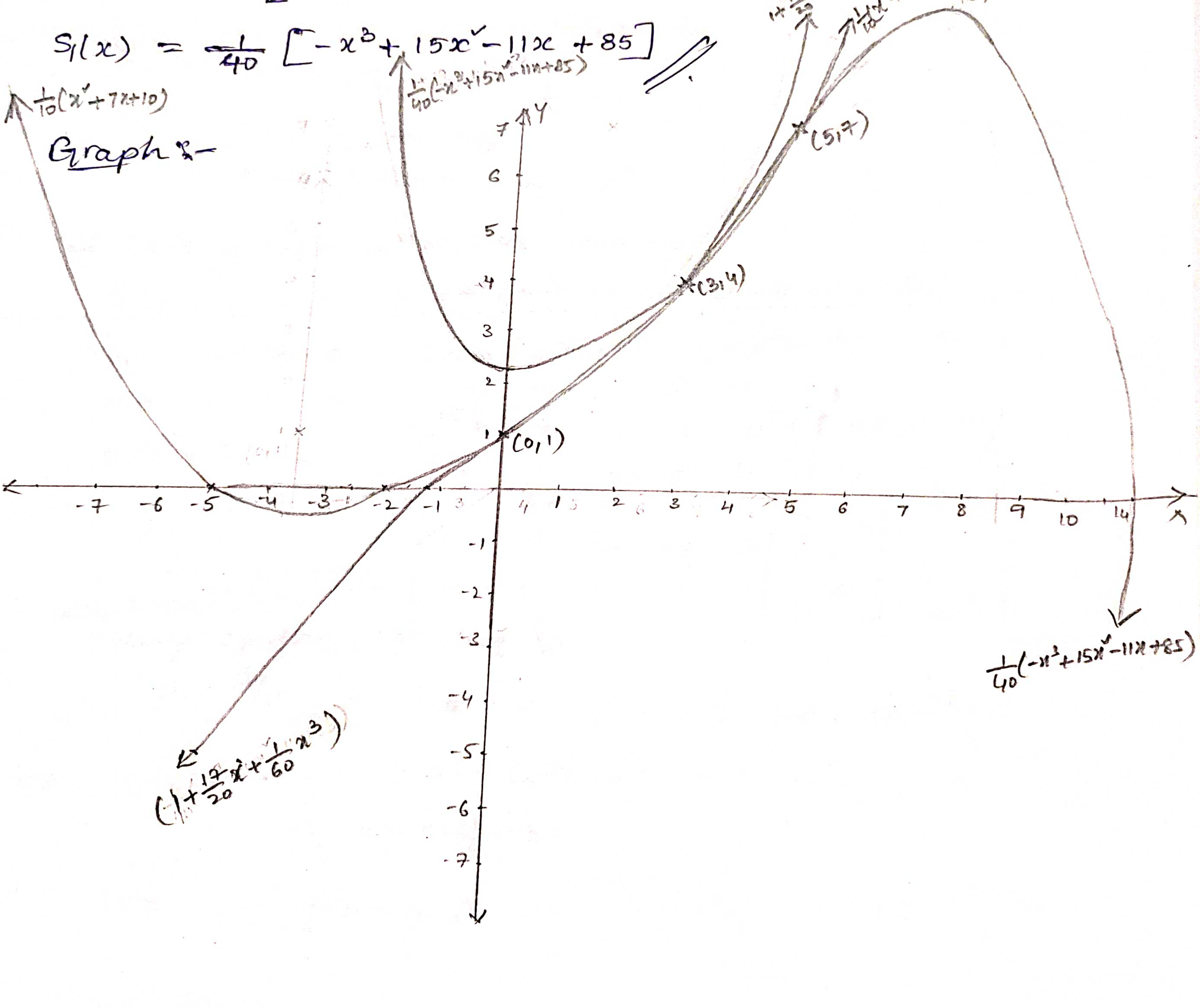
$= 4 + \left(\frac{13}{10}x - \frac{39}{10}\right) + \frac{3}{20}(x^2 - 6x + 9) - \frac{1}{40}(x^3 - 27 - 9x^2 + 27x)$

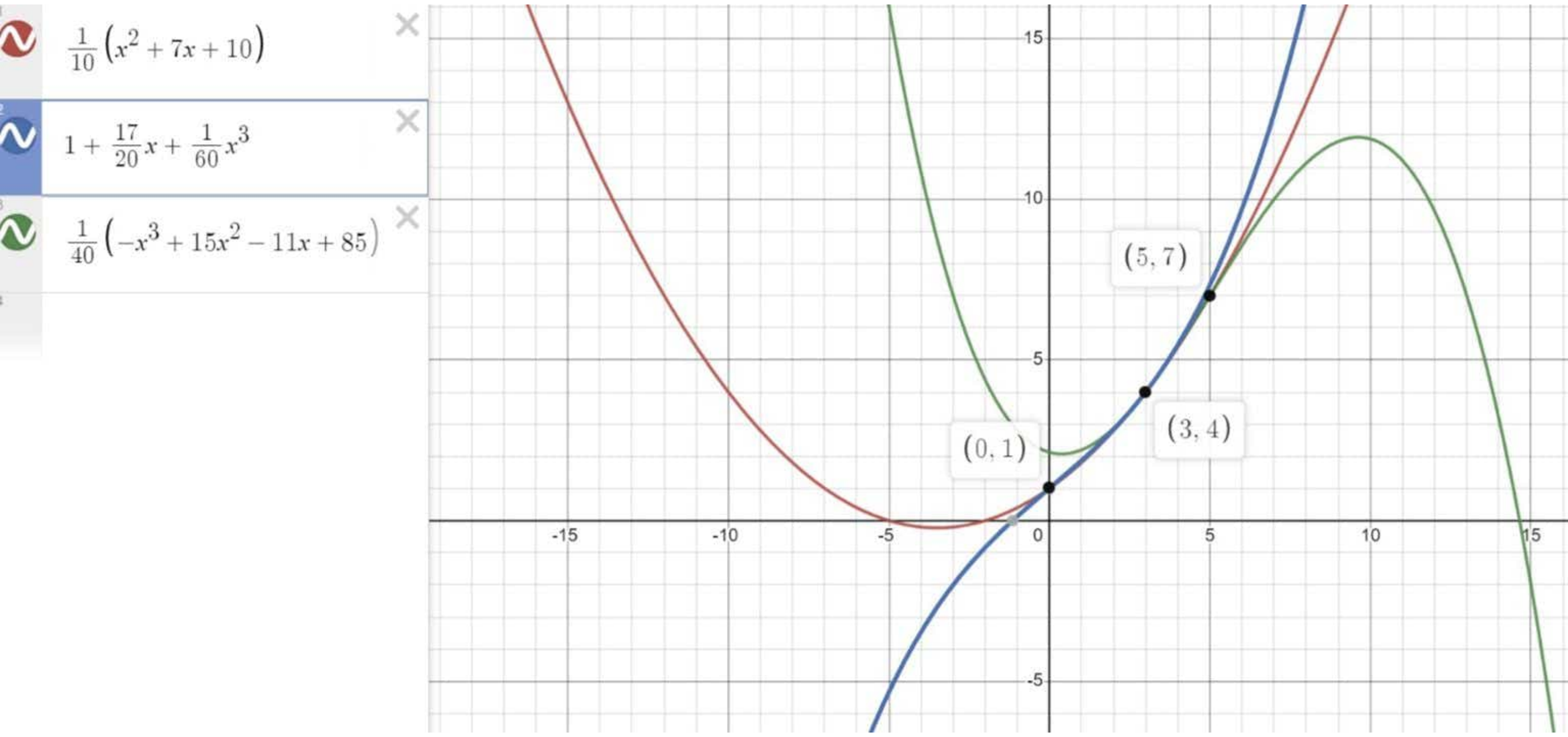
$= \frac{1}{10} + \frac{13}{10}x + \frac{3}{20}x^2 - \frac{9}{10}x + \frac{27}{20} - \frac{x^3}{40} + \frac{27}{40} + \frac{9}{40}x^2 - \frac{27}{40}x$

$= \frac{1}{40}\left[4 + 52x + 6x^2 - 36x + 54 - x^3 + 27 + 9x^2 - 27x\right]$

$S_1(x) = \frac{1}{40}\left[-x^3 + 15x^2 - 11x + 85\right]$

Graph :-





$\frac{1}{10}\left(x^2 + 7x + 10\right)$
$1 + \frac{17}{20}x + \frac{1}{60}x^3$
$\frac{1}{40}\left(-x^3 + 15x^2 - 11x + 85\right)$
$(5, 7)$
$(3, 4)$
$(0, 1)$
15
10
5
-5
-15
-10
-5
0
5
10
15