UNIT-II

Mathematical Expectation &

Discrete probability Distributions

Short Answer questions

(1) Define Expectation & variance of random variable

Expectation for Discrete Variable

A Random variable 'x' assumes the values $x_1, x_2, x_3 \ldots x_n$ with respectives probabilites $p_1, p_2 \ldots p_n$ Then $E(X) = x_1p_1 + x_2p_2 + \ldots + x_np_n$

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Continuous Case:

'x' is Continuous Random variable and $f(x)$ is probability density function Then

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

(2) P.T $E(aX+b) = aE(X)+b$ and

$Var(X+k) = Var(X)$

Sol:-

$$E(aX+b) = \sum_{i=1}^{n} (ax_i+b)\, p(X=x_i)$$

$$= a\sum_{i=1}^{n} x_i\, p(X=x_i) + b\sum_{i=1}^{n} p(X=x_i)$$

$$= a\, E(X) + b(1)$$

$$= a\, E(X) + b$$

$$Var(X+k) = \int_{-\infty}^{\infty} (x+k)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} (x+k) f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} (x^2+2kx+k^2) f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x) + k \int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + 2k \int_{-\infty}^{\infty} x f(x)\,dx + k^2 - \left[\int_{-\infty}^{\infty} x f(x)\,dx + k\right]^2$$

$$= E(X^2) + 2k\,E(X) + k^2 - [E(X)+k]^2$$

$$= E(X^2) + \cancel{2k\,E(X)} + \cancel{k^2} - [E(X)]^2 - \cancel{k^2} - \cancel{2k\,E(X)}$$

$$= E(X^2) - E(X)^2$$

$$Var(X+k) = Var(X)$$

(3) Define Binomial distribution & Poisson distribution

Def: A Random Variable X has a Binomial distribution if it assumes only non-negative values and its probability density function is given by

$$P(X=r) = P(r) = \begin{cases} {}^{n}C_r\, p^r q^{n-r} & r = 0,1,2,\dots n;\ q = 1-p \\ 0 & \text{otherwise} \end{cases}$$

Poisson distribution:

A random variable 'X' is said to follow a poisson distribution if it assumes only non-negative values and its probability density function is given by

$$p(x,\lambda) = p(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & x=0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Here $\lambda>0$ is called the parameter of the distribution.

(4) The mean of Binomial distribution is '3' & var $\frac{9}{4}$ find 'n'

mean $np=3$ var $npq=\frac{9}{4}$

$$\frac{\cancel{n}\cancel{p}q}{\cancel{n}\cancel{p}} = \frac{\frac{\cancel{9}^{3}}{4}}{\cancel{3}} = \frac{3}{4} \qquad q=\frac{3}{4} \quad p=\frac{1}{4}$$

$np=3$

$n\frac{1}{4}=3 \qquad n=12$

(5) The prob of poisson variate 'X' taking the value 1 & 2 are equal Find i) M (ii) $P(X\geq 1$

(iii) $P(1<X<4)$

Sol:

$$P(X=1) = P(X=2)$$

$$\frac{\cancel{e^{-\lambda}} \cancel{\lambda^1}}{1!} = \frac{\cancel{e^{-\lambda}} \lambda^{\cancel{2}}}{2!} \Rightarrow \lambda = 2$$

$$\text{mean} = \mu = \lambda = 2$$

(ii) $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - \frac{e^{-2} 2^0}{0!} = 1 - \frac{1}{e^2}$$

(iii) $P(1 < X < 4) = P(X=2) + P(X=3)$

$$= \frac{e^{-2} 2^{\cancel{2}}}{\cancel{2}!} + \frac{e^{-2} 2^3}{3!}$$

$$= \frac{2}{e^2} + \frac{e^{-2}\, 2 \times \cancel{2} \times 2}{3 \times \cancel{2}} = \frac{2}{e^2} + \frac{4}{3e^2}$$

(6) Derive mean of Geometric distribution

Geometric distribution

$$P(X=x) = q^{x-1} p \quad \text{where } q = 1-p$$

where p is success of out come
q is failure of out come
x is number of trails required to get a first success

Mean:

$$\mu = E(X) = \sum_{x=1}^{\infty} x\, P(X=x)$$

$$= \sum_{x=1}^{\infty} x q^{x-1} p$$

$$= p[1 \cdot q^{1-1} + 2q^{2-1} + 3q^{3-1} + \cdots]$$

$$= p[1 + 2q + 3q^2 + 4q^3 + \cdots]$$

$$= p[(1-q)^{-2}]$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$\text{mean} = \mu = \frac{1}{p}$$

long Answer questions

(1) A Random variable 'x' has following probability function

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Determine k (ii) Evaluate $P(X<6)$, $P(X\geq 6)$, $P(0<X<5)$

(iii) if $P(X\leq \cancel{k}4)>\frac{1}{2}$ find the minimum value of 'k'

iv) Determine the distribution function of 'x'

v) mean vi) Variance

Sol:-

i) $$\sum_{x=1}^{7} p(x)=1$$

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$10k^2+9k=1$$

$$10k^2+9k-1=0$$

$$(10k-1)(k+1)=0$$

$$k=\frac{1}{10}=0.1$$ (since $p(x)\geq 0$

so $k\neq -1$)

So $k=0.1=\frac{1}{10}$ sub in above distribution

table

x	0	1	2	3	4	5	6	7
$p(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(ii) $P(X<6) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$
$+ P(X=4) + P(X=5)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = 0.81$

$P(X \geq 6) = 1 - P(X<6) = 1 - 0.81 = 0.19$

$P(0<X<5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} = 0.8$

(iii) The required minimum value of 'k'

$P(X \leq 1) = P(X=0) + P(X=1) = 0 + \frac{1}{10} = 0.1$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = 0.1 + \frac{2}{10} = 0.3$

$P(X \leq 3) = P(X \leq 2) + P(X=3) = 0.3 + \frac{2}{10} = 0.5$

$P(X \leq 4) = P(X \leq 3) + P(X=4) = 0.5 + \frac{3}{10} = 0.8 > 0.5 = \frac{1}{2}$

$P(X \leq 4) = 0.8 > 0.5 = \frac{1}{2}$

minimum value of $k = 4$

iv) The distribution function of X is given by following table

X	$F(x) = P(X \le x)$
0	0
1	$0 + \frac{1}{10} = \frac{1}{10}$
2	$\frac{1}{10} + \frac{2}{10} = \frac{3}{10}$
3	$\frac{3}{10} + \frac{2}{10} = \frac{5}{10}$
4	$\frac{5}{10} + \frac{3}{10} = \frac{8}{10}$
5	$\frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$\frac{81}{100} + \frac{2}{100} = \frac{83}{100}$
7	$\frac{83}{100} + \frac{17}{100} = \frac{100}{100} = 1$

v) $\text{Mean} = \sum_{i=0}^{7} x_i p_i$

$= 0 \times 0 + 1 \times \frac{1}{10} + 2 \times \frac{2}{10} + 3 \times \frac{2}{10} + 4 \times \frac{3}{10} + 5 \times \frac{1}{100} + 6 \times \frac{2}{100} + 7 \times \frac{17}{100}$

$= 3.66$

vi) $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum_{i=0}^{7} x_i^2 p_i$

$= 0^2 \times 0 + 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{2}{10} + 4^2 \times \frac{3}{10} + 5^2 \times \frac{1}{100} + 6^2 \times \frac{2}{100} + 7^2 \times \frac{17}{100} = 16.8$

$Var(X) = 16.8 - (3.66)^2$

$= 16.8 - 13.3956 = 3.4044$

(2) Derive mean and variance of Binomial distribution

Sol:- The Binomial probability distribution is given by

$$p(r) = n_{C_r} p^r q^{n-r} \quad r=0,1,2 \dots n \quad q=1-p$$

$$\text{Mean of } X = \mu = E(X) = \sum_{r=0}^{n} r\, p(X=r)$$

$$= \sum_{r=0}^{n} r\, n_{C_r} p^r q^{n-r}$$

$$= 0 \times q^n + 1 \times n_{C_1} p q^{n-1} + 2 n_{C_2} p^2 q^{n-2} + \dots + n p^n$$

$$= npq^{n-1} + 2\frac{n(n-1)}{2!} p^2 q^{n-2} + \dots + np^n$$

$$= np[q^{n-1} + (n-1)pq^{n-2} + \dots + p^{n-1}]$$

$$= np(q+p)^{n-1} \qquad [q+p=1]$$

$$= np$$

Variance of Binomial distribution

$$V(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \sum_{r=0}^{n} r^2\, p(X=r)$$

$$= \sum_{r=0}^{n} r^2\, n_{C_r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} [r(r-1)+r]\, n_{C_r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1)\, {}^{n}C_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\, {}^{n}C_r\, p^r q^{n-r}$$

$$= \Big[0(0-1)\, {}^{n}C_0\, p^0 q^{n-0} + 1(1-1)\, {}^{n}C_1\, p^1 q^{n-1} + 2(2-1)\, {}^{n}C_2\, p^2 q^{n-2} + \cdots$$

$$\cdots + n(n-1)\, {}^{n}C_n\, p^n q^{n-n}\Big] + np$$

$$= 2\, \frac{n(n-1)}{\cancel{2!}} p^2 q^{n-2} + \cdots + n(n-1) p^n\Big] + np$$

$$= n(n-1)p^2 \big[q^{n-2} + \cdots + p^{n-2}\big] + np$$

$$= n(n-1)p^2 [q+p]^{n-2} + np$$

$$E(x^2) = n(n-1)p^2 + np$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$var(x) = npq$$

(3) Ten Coins are tossed. Find the probability of getting greater than (or) equal to 6 heads

Sol:-

Binomial distribution

$$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

prob of getting head $p = \frac{1}{2}$ $\quad q = 1-p = \frac{1}{2}$

$n = 10$

find

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_{6}\left(\tfrac{1}{2}\right)^{6}\left(\tfrac{1}{2}\right)^{10-6} + {}^{10}C_{7}\left(\tfrac{1}{2}\right)^{7}\left(\tfrac{1}{2}\right)^{10-7}$$

$$+ {}^{10}C_{8}\left(\tfrac{1}{2}\right)^{8}\left(\tfrac{1}{2}\right)^{10-8} + {}^{10}C_{9}\left(\tfrac{1}{2}\right)^{9}\left(\tfrac{1}{2}\right)^{10-9}$$

$$+ {}^{10}C_{10}\left(\tfrac{1}{2}\right)^{10}\left(\tfrac{1}{2}\right)^{10-10}$$

$$= \cancel{210}\,(210 + 120 + 45 + 10 + 1)\left(\tfrac{1}{2}\right)^{10}$$

$$= 0.37$$

4) Derive mean and variance of poisson distribution.

Sol:-

The poisson probability distribution is given by

$$P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad r = 0,1,2 \ldots$$

$$\text{Mean} = E(X) = \mu = \sum_{r=0}^{\infty} r \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} \cancel{r} \frac{e^{-\lambda}\lambda^r}{\cancel{r}(r-1)!}$$

$$= e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^r}{(r-1)!}$$

$$= e^{-\lambda}\left[\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \ldots\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\text{variance}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{r=0}^{\infty} r^2 P(X=r)$$

$$= \sum_{r=0}^{\infty} (r(r-1) + r)\, P(X=r)$$

$$= \sum_{r=0}^{\infty} (r(r-1))\, P(X=r) + \sum_{r=0}^{\infty} r\, P(X=r)$$

$$= \sum_{r=0}^{\infty} r(r-1) \frac{e^{-\lambda}\lambda^r}{r!} + \lambda$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} r\cancel{(r-1)} \frac{\lambda^r}{r\cancel{(r-1)}(r-2)!} + \lambda$$

$$= e^{-\lambda} \sum_{r=2}^{\infty} \frac{\lambda^r}{(r-2)!} + \lambda$$

$$= e^{-\lambda} \left[\frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots \right] + \lambda$$

$$= e^{-\lambda} \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right] + \lambda$$

$$= \cancel{e^{-\lambda}} \lambda^2 \cancel{e^{\lambda}} + \lambda = \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \cancel{\lambda^2} + \lambda - [\cancel{\lambda}]^2$$

$$Var(X) = \lambda$$

5) P.T poisson distribution is the limiting case of Binomial distribution

Proof: The poisson distribution can be derived as a limiting case of the Binomial distribution under the following conditions

i) p, the probability of the occurrence of the is very small

ii) n is very very large, where 'n' is no. of trials i.e $n \to \infty$

iii) np is a finite quantity say $np = \lambda$ then λ is called the parameter of the poisson distribution.

Binomial distribution

$$p(X=r) = nC_r p^r q^{n-r}$$

$$= nC_r p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} p^r \frac{(1-p)^n}{(1-p)^r} \rightarrow (1)$$

put $np = \lambda$ then $n = \frac{\lambda}{p}$

$$p(r) = \frac{\frac{\lambda}{p}\left(\frac{\lambda}{p}-1\right)\left(\frac{\lambda}{p}-2\right)\cdots\left(\frac{\lambda}{p}-r+1\right)}{r!} \cdot p^r \cdot \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{\lambda(\lambda-p)(\lambda-2p)\cdots[\lambda-(r-1)p]}{r!\,p^r} p^r \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{\lambda(\lambda-p)(\lambda-2p)\cdots[\lambda-(r-1)p]}{r!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{(1-p)^r}$$

As $n \to \infty$, $p \to 0$ so that $np = \lambda$ we have

$$p(r) = \frac{\lambda \cdot \lambda \cdots r \text{ factors}}{r!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{p\to 0} \frac{1}{(1-p)^r}$$

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \quad \left[\because \lim_{p\to 0} (1-p)^r = 1 \text{ for a given } r\right]$$

$$= \frac{\lambda^r}{r!} e^{-\lambda} \quad \left[\because \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = \lim_{n\to\infty} \left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda} = e^{-\lambda}\right]$$

$p(r)$ = probability of 'r' successes $= \dfrac{e^{-\lambda}\lambda^r}{r!}$

This is known as poisson distribution

where $r = 0, 1, 2 \ldots$

(6) Average number of accidents on any day on a national highway is 1.6. Determine the probability that the no of accidents is

i) At least one (ii) at most one

Sol:-

Given data avg = mean = $\lambda = 1.6$

poisson distribution $P(X=r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$

ii) p(at ~~least~~ most one) $= P(X \le 1) = P(X=0) + P(X=1)$

$$= \frac{e^{-1.6}(1.6)^0}{0!} + \frac{e^{-1.6}(1.6)^1}{1!}$$

$$= \frac{1}{e^{1.6}}(1+1.6) = \frac{2.6}{e^{1.6}} = 0.52$$

(bi) $P(\text{at least one}) = P(X \geq 1)$

$$= 1 - P(X=0)$$

$$= 1 - \frac{e^{-1.6}(1.6)^0}{0!}$$

$$= 1 - \frac{1}{e^{1.6}} = 0.80$$

7). Six Cards are drawn from a pack of 52 Cards Find the probability that

i) at least three are diamonds

(ii) 4 are diamonds

Sol:

$P(\text{getting a diamond card}) = \frac{13C_1}{52C_1} = \frac{13}{52} = \frac{1}{4}$

$$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$$

i) $P(\text{at least three are diamond}) = P(X \geq 3)$

$n = 6 \quad p = \frac{1}{4} \quad q = \frac{3}{4}$

$$= 1 - P(X<3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[6C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^6 + 6C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^5 + 6C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4\right]$$

$$= 1 - [0.177 + 0.355 + 0.2966]$$

$$= 0.1714$$

(ii) $P(\text{4 are diamonds}) = P(X=4)$

$$= {}^{6}C_{4}\left(\frac{1}{4}\right)^{4}\left(\frac{3}{4}\right)^{2}$$

$$= 15 \times 0.0039 \times 0.56$$

$$= 0.0329$$

8) A sample of '5' items is selected at random from a box containing 15 items of which '8' are defective find i) mean (ii) variance of defective items.

$$P(\text{defective item}) = \frac{8}{15}$$

$$q = 1 - \frac{8}{15} = \frac{7}{15}$$

Sample size $n = 5$

i) $\text{mean} = np = \cancel{5} \times \frac{8}{\cancel{15}_{3}} = \frac{8}{3} = 2.66$

(ii) $\text{variance} = npq = \cancel{5} \times \frac{8}{\cancel{15}_{3}} \times \frac{7}{15} = \frac{56}{45}$

9) Chebyshev's theorem:

Chebyshev's theorem states that the proportion (or) percentage of any data set that lies within 'k' standard deviation of the mean where 'k' is any positive integer >1 is atleast $1 - \frac{1}{k^2}$