UNIT-5

Stochastic Processes and Markov Chains

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- Syllabus:

  Introduction to Stochastic Processes- Markov process. Transition Probability, Transition Probability Matrix, First order and Higher order Markov process, n – step transition probabilities, Markov Chain, Steady state condition, Markov analysis.

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- <u>Stochastic Process:</u>

  Families of random variables which are functions of say time t, are known as Stochastic processes or random processes or random functions.

  Mathematically, a stochastic process is a set of all random variables $\{X(t), t \in T\}$ depending on some real parameter like time 't'

The values assumed by random variables X(t) are called 'states' and the set of all possible values forms the state space of the process.

Example 1: Consider a Random event occurring in time, such as number of telephone calls received at a switch board. Suppose X(t) is the random variable which represents the number of incoming calls in an interval (0,t) of duration t units. The number of calls within a fixed interval of time, say one unit of time, is a random variable X(t) and the family $\{X(t), t \in T\}$ constitutes a stochastic process (T = [0,∞]).

Example 2: Consider a sample experiments like throwing a die. Suppose that X_n is the outcome of the n^{th} throw, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a family of random variable such that for a distinct value of n, one gets a distinct random X_n; So $\{X_n, n \geq 1\}$ constitutes a stochastic process known as Bernoulli process

<u>Example 3:</u> A patient's heart pulse during surgery.
Measured continuously during interval [0,T].
Stochastic variable X_t represents the occurrence of
a heartbeat at time t, $0 \le t \le T$. Hence,
X_t assumes only the values 0 (no heartbeat) and 1 (heartbeat)

A stochastic process is said to be discrete ifits state space is discrete. A discrete state process is also called chain. Also , if the index set t is discrete, then we have a discrete parameter process. Otherwise we have a continuous parameter process. A discrete parameter process is also called stochastic sequence $\{X_n\}, n \in T.$

Classification of stochastic processes:

1. Continuous stochastic process
2. Discrete stochastic process
3. Deterministic (random)stochastic proces
4. Non-deterministic stochastic process

Markov Process: If $\{X(t), t \in T\}$ is a stochastic process such that, given the value $X(s)$, the value of $X(t), t > s$, do not depend on the value of $X(u), u < s$,then the process is said to be a Markov process.

If for $t_1 < t_2 < .... < t_n < t$

$$P\{a \le X(t) \le b / X(t_1) = x_1, X(t_2) = x_{2,} ....X(t_n) = x_n\} = P\{a \le X(t) \le b / X(t_n) = x_n\}$$

Then the process $\{X(t), t \in T\}$ is a Markov Process

<u>Markov Chain</u>: Let $X(t)$ be a Markov process which takes only discrete values whether 't' is discrete or continuous is called as Markov Chain.

<u>Example</u>: Consider number of heads obtained in the first n tosses of a coin. The possible values are 0,1,2,.......n.

Let $S_{n+1} = \begin{Bmatrix} x+1 & if\ (n+1)^{th} & trial\ results\ in\ head \\ x & if\ (n+1)^{th} & trial\ results\ in\ trial \end{Bmatrix}$

$$S_n = X_0 + X_1 + X_2 + \cdots + X_n$$

i.e the conditional probability of S_{n+1} given S_n depends on S_n but not on the manner how the value of S_n is derived.

If we let $P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots \ldots \ldots X_1 = a_1]$

Such that $P[X_n = a_n / X_{n-1} = a_{n-1}]$ for every n.

A Morkov chain is a stochastic Model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

i.e Consider a system which can be in any one of a finite number of states $E_1, E_2, \dots \dots . E_n$. We also assume that the probability of the system being in a given state at the next trial depends only on its present state and not upon the states it may have been in earlier times. If any time, the system is in a state E_i , the probability of it being in the state E_j at the next trial is P_{ij} .

$$\text{i.e } P\{(E_i, E_j)\} = a_i P_{ij}$$

$$\text{Similarly } P\{(E_i, E_j, E_k)\} = a_i P_{ij} P_{jk}$$

Where a_i is the probability distribution of E_i

Here P_{ij} is called the **probability of a transition** from E_i

The transition probabilities P_{ij} will be arranged in a matrix form is called **matrix of transition probabilities** or **Transition Probability matrix**

Stochastic Matrix:

The transition probabilities P_{ij} will be arranged in a matrix of transition probabilities

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \cdots\cdots \\ P_{21} & P_{22} & P_{23} \cdots\cdots \\ P_{31} & P_{32} & P_{33} \cdots\cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Clearly P is a square matrix with non –negative elements and unit row sums. Such a matrix (finite or infinite) is called a **Stochastic matrix**

NOTE:

1. A stochastic matrix P is said to be **regular** if all the entries of some power P^m are positive.
2. A stochastic matrix P is not regular if '1'occures in the principal main diagonal.
3. If P and Q are stochastic matrices then product PQ is also a stochastic matrix.

Problems:

I. Which of the following matrices are stochastic

(i) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 0 & 1 \\ 1/3 & 1/4 \end{bmatrix}$

(iv) $D = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(v) $E = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

Soln:

(i) A is not a square matrix

$\therefore$ It is not stochastic

(ii) The matrix is a square matrix with non-negative entries and sum of the elements in each row is equal to 1

$\therefore$ B is stochastic

(iii) C is a square matrix but sum in each row is not equal to 1. So it is not stochastic

(iv) D is stochastic

(v) E is not stochastic , because it contains negative elements.

I. Which of the following matrices are regular.

(i) $$A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

(ii) $$B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

(iii) $$C = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution:

(i) A is not a regular matrix since 1 lies on the main diagonal.

(ii) $$B^2 = B.B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 3/8 & 3/8 & 1/4 \end{bmatrix}$$

$$B^3 = B^2.B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 7/16 & 7/16 & 1/8 \end{bmatrix}$$

Since some entries are zero, B is not Regular.

(i) $C^2 = C.C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$$C^3 = C^2.C = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$C^4 = C^3.C = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$C^5 = C^4.C = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 1/2 & 3/8 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Since all the entries of some power of C are positive, C is regular stochastic matrix.

<u>First order and higher order Markov process:</u>

The **Markov chain** of the **first order** is one for which each subsequent state depends only on the immediately preceding one. **Markov** chains of **second** or **higher orders** are the **processes** in which the next state depends on two or more preceding ones.

n – step transition probabilities:

The state transition probability matrix of a Markov chain gives the probabilities of transitioning from one state to another in a single time unit. Also, define an n - step transition probability matrix $P^{(n)}$ whose elements are the n -step transition probabilities.

A **state vector** is a vector (list) that records the probabilities that the system is in any given state at a particular step of the process.

For example, if we know for sure that it is raining today, then the state vector for today will be

(1, 0). But tomorrow is another day! We only know there's a 40% chance of rain and 60% chance of no rain, so tomorrow's state vector is (0.4, 0.6).

Classification of states:

A **state** is any particular situation that is possible in the system. For example, if we are studying rainy days, then there are two states:

1. It's raining today.
2. It's not raining today.

The system could have many more than two states also.

There are 3 states in Markov process

Recurrent State: A state i is said to be recurrent if

$$\sum P_{ii}{}^{(n)} = 1$$

Transient State: If there exist j and n such that

$\sum P_{ij}{}^{(n)} > 0 \; but \; \sum P_{ji}{}^{(m)} = 0$ for all m, then state 'i' is called transient state. i.e it process the property that it is possible with positive probability to pass from it to another state but it is not possible to return from that state to original state.

Irreducible State: If the two states i & j are such that, each is accessible from the other, then we say that the two states communicate. It is denoted by $i \leftrightarrow j$.Then ∃ an integer m & n such that

$$\sum P_{ij}{}^{(n)} > 0 \quad and \sum P_{ij}{}^{(m)} > 0$$

Problems:

Three Boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C. But C just as likely to throw the ball to B as to A. If C was the first person to throw the ball, find the probability that

(i) A has the ball
(ii) B has the ball
(iii) C has the ball after 3 throw

Solution:

$P_{11} \rightarrow$ Probability that the ball is thrown from A to A = 0

$P_{12} \rightarrow$ Probability that the ball is thrown from A to B = 1

$$P_{13} = 0, P_{21} = 0, P_{22} = 0, P_{23} = 1, P_{31} = \frac{1}{2}, P_{32} = \frac{1}{2}, P_{33} = 0$$

Then Transition Matrix $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

It is given 3 throws, so calculate P^3

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P^2 . P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Therefore the ball was first thrown by C, the initial probability distribution is given by row matrix

$$A^{(0)} = \left[P_1{}^{(0)} \quad P_2{}^{(0)} \quad P_3{}^{(0)}\right] = [0 \quad 0 \quad 1]$$

After 3 throws, the probability distribution is given by

$$\left[P_1{}^{(3)} \quad P_2{}^{(3)} \quad P_3{}^{(3)}\right] = A^{(0)}\, P^3 = [0 \quad 0 \quad 1] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

After 3 throws, the probability that the ball is with

(i) $A = \frac{1}{4}$

(ii) $B = \frac{1}{4}$

(iii) $C = \frac{1}{2}$

2. A person owning a scooter has the option to switch over to scooter, bike or a car next time with the probability of $\begin{bmatrix} 0.3 & 0.5 & 0.2 \end{bmatrix}$. If the transition probability matrix is $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$ what are the probability of vehicles elated to his fourth purchase.

Solution:

Given $P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$

We have to calculate for third purchase

$$P^2 = P.P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.295 & 0.345 & 0.36 \\ 0.255 & 0.385 & 0.36 \\ 0.275 & 0.325 & 0.4 \end{bmatrix}$$

$$P^3 = P^2.P = \begin{bmatrix} 0.295 & 0.345 & 0.36 \\ 0.255 & 0.385 & 0.36 \\ 0.275 & 0.325 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.277 & 0.351 & 0.372 \\ 0.269 & 0.359 & 0.372 \\ 0.275 & 0.345 & 0.380 \end{bmatrix}$$

Therefore probability of his fourth purchase is $\begin{bmatrix} 0.3 & 0.5 & 0.2 \end{bmatrix} P^3$

$$= \begin{bmatrix} 0.3 & 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 0.277 & 0.351 & 0.372 \\ 0.269 & 0.359 & 0.372 \\ 0.275 & 0.345 & 0.380 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2726 & 0.3538 & 0.3736 \end{bmatrix}$$

1. Consider a Markov Process with state space $S = \{0,1,2\}$ and transition matrix $P = \begin{bmatrix} p & q & 0 \\ {}^{1}/_{2} & 0 & {}^{1}/_{2} \\ p - {}^{1}/_{2} & {}^{7}/_{10} & {}^{1}/_{5} \end{bmatrix}$
   (i) What can you say about the values of p & q?
   (ii) Calculate the transition probability $P_{ij}^{(3)}$

Solution:

(i) From the third row if we sum, all the elements it should be equal to 1

i.e. $p - \frac{1}{2} + \frac{7}{10} + \frac{1}{5} = 1 \Rightarrow p = \frac{3}{5}$

put this in the first row $\Rightarrow p + q = 1 \Rightarrow q = \frac{2}{5}$

(ii) Transition probability

$$= {P_{ij}}^{(3)}$$

$$= \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/2 & 0 & 1/2 \\ 1/10 & 7/10 & 1/5 \end{bmatrix} \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/2 & 0 & 1/2 \\ 1/10 & 7/10 & 1/5 \end{bmatrix} \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/2 & 0 & 1/2 \\ 1/10 & 7/10 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.476 & 0.364 & 0.160 \\ 0.495 & 0.210 & 0.295 \\ 0.387 & 0.445 & 0.168 \end{bmatrix}$$

Classification of Chains:

Here is the following classifications

Absorbing Chain: A Markov chain is absorbing if

(i) It has at least one absorbing state. Here absorbing state means suppose j is absorbing iff $P_{jj} = 1$ & $P_{jk} = 0$, $k \neq j$

(ii) It is possible to go from every non absorbing state to at least one absorbing state.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$ Here first and last states are absorbing

Ergodic Chain: a chain in which every two states can communicate with each other and in which the system cannot be left is called ergodic chain.

A special chain of an ergodic chain is the regular chain in which for some n, P^n has no zero elements

Irreducible Chain: A Markov chain in which every state can be reached from every other state. Otherwise reducible.

All the states are reachable to one another so the matrix is Ergodic

1. The transition probability matrix of a Markov chain is given by

$$p = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$. Is this matrix irreducible?

Solution:

Consider the three states as 0,1,2.

In this chain we go from state 0 to state 1 with a probability of 0.7 and from state 1 to stat 2 with a probability 0.5

Thus it is possible to go from state 0 to state 2

$\therefore$ The chain is irreducible and all the states are recurrent.

THANK YOU