UNIT – I

Probability & Random Variable

Short Answer questions

(1) Define probability, Conditional probability & Bayes Theorem

Defination of probability:

In a Random Experiment, Let there be an 'n' mutually exclusive and equally likely elementary events. Let E be an event of Experiment. The probability of E

$$P(E) = \frac{m}{n} = \frac{\text{Number of elementary events in } E}{\text{Total number of elementary events in the Random experiment}}$$

Probability – Axiomatic approach:-

Def: Let 'S' be a finite Sample Space. A real valued function P from the power set of 'S' into R is called a probability function on 'S' if the following three axioms are satisfied

i) Axiom of positivity: $P(E) \geq 0$ for every subset E of S

(ii) Axiom of Certainity:- $P(S) = 1$

iii) Axiom of Union:- If E_1 and E_2 are disjoint subset of 'S' Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Conditional probability:-

Def:- If E_1 and E_2 are two events in a sample space 'S' and $p(E_1) \neq 0$, Then the probability of E_2, after the event E_1 has occurred, is called the conditional probability of the event of E_2 given E_1 and is denoted by $p(E_2/E_1)$

$$p(E_2/E_1) = \frac{p(E_1 \cap E_2)}{p(E_1)}$$

Similarly we define $p(E_1/E_2) = \frac{p(E_1 \cap E_2)}{p(E_2)}$

Bayes Theorem

Statment:-

$E_1, E_2 -- E_n$ are n mutually exclusive and exhaustive events such that $p(E_i) > 0$ $(i = 1, 2 - n)$ in a sample space 'S' and A is any other event in S intersecting with every E_i (i.e A can only occur in combination with any one of the events $E_1, E_2 -- E_n$) such that $p(A) > 0$

if E_i is any of the events of $E_1, E_2 -- E_n$ where $p(E_1)\, p(E_2) --\, p(E_n)$ and $p(A/E_1)\, p(A/E_2) --- p(A/E_n)$ are known then

$$p(E_K/A) = \frac{p(E_K) \cdot p(A/E_K)}{p(E_1)p(A/E_1) + p(E_2)p(A/E_2) + --- + p(E_n)p(A/E_n)}$$

(2) Define Random variable & Types of Random variables

Random variable :-

Def:- A real variable X whose value is determined by the outcome of a random experiment is called a random variable

Types of Random variable

i) Discrete Random variable

A random variable X which can take only a finite number of discrete values in an interval of domain is called a discrete Random variable

(ii) Continuous Random variable:

A Random variable 'X' which can take values continuously i.e which takes all possible values in a given interval is called continuous random variable

(3) Define probability distribution function

Let 'x' be a random variable. Then the probability function associated with 'x' is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(s) \leq x$, $x \in R$

4 A Continous random variable has the p.d.f

$$f(x) = \begin{cases} K + \frac{x}{6} & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$ find 'k'

Sol:-

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^3 (K + x/6)\,dx = 1$$

$$\left[Kx + \frac{x^2}{12}\right]_0^3 = 1$$

$$3K + \frac{9}{12} = 1$$

$$3K = 1 - \frac{9}{12}$$

$$\cancel{3}K = \frac{\cancel{3}}{12} \Rightarrow K = \frac{1}{12}$$

5 Throwing two dice, find the probability of getting sum is even

Throwing two die $6^2 = 36$ out comes $= n$

faviorble out comes $m = 18$

$$\begin{bmatrix} (1,1) & (1,3) & (1,5) \\ (2,2) & (2,4) & (2,6) \\ (3,1) & (3,3) & (3,5) \\ (4,2) & (4,4) & (4,6) \\ (5,1) & (5,3) & (5,5) \\ (6,2) & (6,4) & (6,6) \end{bmatrix}$$

$$P(E) = \frac{18}{36} = \frac{1}{2}$$

(6) In a pack of Cards, find the probability of getting Ace (or) Spade

Sol:-

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$P(\text{Ace (or) Spade}) = P(\text{Ace}) + P(\text{Spade}) - P(\text{Ace} \cap \text{Spade})$$

$$= \frac{4c_1}{52c_1} + \frac{13c_1}{52c_1} - \frac{1c_1}{52c_1}$$

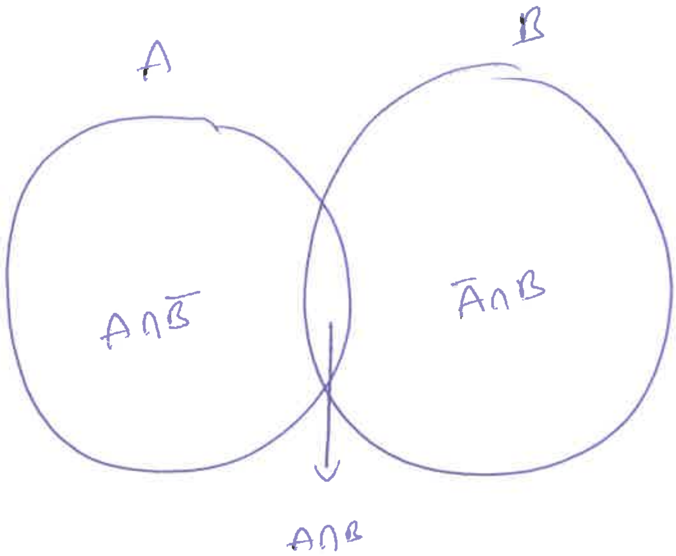
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$$

Long Answer questions

(1) State and prove addition theorem on probability

Statment:- if S is a Sample space and A and B are any Two events

$$P(A \text{ or } B) = P(A\cup B) = P(A) + P(B) - P(A\cap B)$$



using above diagram

$A\cup B = (A\cap\bar{B})\cup(A\cap B)\cup(\bar{A}\cap B)$

A and B disjoint sets by using Axiomatic def

$P(A\cup B) = P((A\cap\bar{B})\cup(A\cap B)\cup(\bar{A}\cap B))$

$P(A\cup B) = P(A\cap\bar{B}) + P(A\cap B) + P(\bar{A}\cap B) \rightarrow (1)$

$A = (A\cap\bar{B})\cup(A\cap B)$

$P(A) = P(A\cap\bar{B}) + P(A\cap B)$

$P(A\cap\bar{B}) = P(A) - P(A\cap B) \rightarrow (2)$

$B = (\bar{A}\cap B)\cup(A\cap B)$

$P(B) = P(\bar{A}\cap B) + P(A\cap B)$

$P(\bar{A}\cap B) = P(B) - P(A\cap B) \rightarrow (3)$

(2) & (3) sub in (1)

$P(A\cup B) = P(A) - P(A\cap B) + \cancel{P(A\cap B)} + P(B) - P(A\cap B)$

So

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

(2) Box 'A' Contain '5' red and '3' white marbles and box B contains '2' red and '6' white marbles. if marble is drawn from each box, what is the probability that they are both of same colour.

Sol:- Suppose E_1 = the event that the marble is drawn from box 'A' and is red

$$P(E_1) = \frac{1}{2} \cdot \frac{5}{8} = \frac{5}{16}$$

and E_2 = The event that the marble from box 'B' and Red

$$P(E_2) = \frac{1}{2} \cdot \frac{2}{8} = \frac{1}{8}$$

The probability that both the marbles are red is

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = \frac{5}{16} \cdot \frac{1}{8} = \frac{5}{128}$$

Let E_3 = The event that the marble from box A and is white

$$P(E_3) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$$

Let E_4 = The event of marble from box B is white

$$P(E_4) = \frac{1}{2} \cdot \frac{6}{8} = \frac{3}{8}$$

and $P(E_3 \cap E_4) = \frac{3}{16} \cdot \frac{3}{8} = \frac{9}{128}$

The probability that the marbles are of same colour

$= P(E_1 \cap E_2) + P(E_3 \cap E_4)$

$= \frac{5}{128} + \frac{9}{128} = \frac{14}{128} = \frac{7}{64} = 0.109$

(3) Two factories produce identical clocks. The production of The first factory Consists of 10,000 clocks of which 100 are defective. The second factory produces 20,000 clocks of which '300' are defective. What is the probability that a particular defective clock was produced in the first factory?

Sol: Two factories is denoted by A and B

Out put produced by first factory A = 10,000

Out put produced by second factory B = 20,000

probability that items produced by A are defective

$$P(D/A) = \frac{100}{10000} = 0.01$$

Similarly $P(D/B) = \frac{300}{20,000} = 0.015$

$$P(A) = \frac{10000}{30000} = \frac{1}{3} = 0.33$$

$$P(B) = \frac{20000}{30000} = \frac{2}{3} = 0.66$$

find

$P(A/D)$ = prob of defective clock was produced by first factory

$$P(A|D) = \frac{P(A) \times P(D/A)}{P(A) \times P(D/A) + P(B) P(D/B)}$$

$$= \frac{(0.33) \times (0.01)}{(0.33)(0.01) + (0.66) 0.015} =$$

4) $P(A) = \frac{2}{3}$ $\quad P(B) = \frac{1}{5}$

Then p.T $\frac{2}{15} \leq P(A \cap B) \leq \frac{1}{5}$

proof:

Using Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A) P(B)$$

$$P(A \cap B) = \frac{2}{3} \times \frac{1}{5}$$

$$P(A \cap B) = \frac{2}{15}$$

Since $\frac{2}{15} \leq \frac{1}{5}$ and $P(A \cap B) = \frac{2}{15}$

$$\Rightarrow \frac{2}{15} = P(A \cap B) \leq \frac{1}{5}$$

$$\frac{2}{15} \leq P(A \cap B) \leq \frac{1}{5}$$

5) From a lot of '10' items containing '3' defective. a Sample of 4 items is drawn at Random. Let the random variable X denote the number of defective items in the Sample. Find the probability distribution of 'X' when the Sample without Replacement

Sol:- obviously X takes the Values 0, 1, 2 or 3

Given total No of items = 10

No of good items = 7

No of defective items = 3

$$P(X=0) = P(\text{no defective}) = \frac{^7C_4}{^{10}C_4} = \frac{7!}{4!\,3!} \times \frac{4!\,6!}{10!} = \frac{1}{6}$$

$$P(X=1) = P(\text{one defective and 3 good items})$$

$$= \frac{^3C_1 \times {}^7C_3}{^{10}C_4} = \frac{3 \times 7!}{3!\,4!} = \frac{1}{2}$$

$$P(X=2) = P(\text{2 defective and 2 good items})$$

$$= \frac{^3C_2 \times {}^7C_2}{^{10}C_4} = \frac{3}{10}$$

$$P(X=3) = P(\text{3 defective and 1 Good item})$$

$$= \frac{^3C_3 \times {}^7C_1}{^{10}C_4} = \frac{7}{^{10}C_4} = \frac{1}{30}$$

The probability distribution of Random Variable 'X' as follow

$X = x_i$	0	1	2	3
$P(X=x_i)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(6) Let $f(x) = 3x^2$ when $0 \le x \le 1$ be the probability density function of a continuous random variable X. Determine 'a' and 'b' such That

i) $P(X \le a) = P(X > a)$ (ii) $P(X > b) = 0.05$

Sol:-

Given data $f(x) = 3x^2 \quad 0 \le x \le 1$

i) $P(X \le a) = P(X > a)$

$$\int_0^a f(x)\,dx = \int_a^1 f(x)\,dx$$

$$\int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx$$

$$\left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_0^a = \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_a^1$$

$$a^3 = [1 - a^3]$$

$$2a^3 = 1$$

$$a^3 = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

(11)

$P(X>b) = 0.05$

$$\int_b^1 3x^2\,dx = 0.5$$

$$\left[\frac{3x^3}{3}\right]_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$1 - 0.05 = b^3$$

$$b^3 = 0.05$$

$$b = (0.05)^{1/3}$$