Continuous probability Distribution &
Fundamental Sampling Distributions

Unit-3

Short Answer questions

(1) Define normal distribution

Ans:- A random Variable X is Said to have a normal distribution, if its density function is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} ; \quad -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma > 0$$

Where μ = mean, σ = S.D

(2) Define Gamma & Exponential distribution

Ans:- let X be a Continuous random variable, assuming only non-negative values, distributed, according to Gamma probability density function given by

$$f(x) = \begin{cases} \frac{\beta}{\Gamma(\alpha)} (\beta x)^{\alpha-1} e^{-\beta x}, & 0 < x < \infty, \; \alpha > 0, \beta > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Exponential:- A continuous random Variable X having the range $0 < x < \infty$ is Said to have exponential distribution if it has a probability density is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & , x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(3) A random Sample of size 80 is taken from a Population whose S.D is 15. Find the Standard error of means.

Sol:-

$n = 80, \quad \sigma = 15$

Standard error of mean $= \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{80}} = 1.68.$

(4) Find the Value of the finite Population Correction factor for n=10 & N=100

Sol:-

$N = 1000, \quad n = 10$

Correction factor $= \frac{N-n}{N-1} = \frac{1000-10}{1000-1} = 0.991$

(5) Define Simple, random, Purposive Sample.

Simple Sample:- Simple Sample is defind as every item in the population has eaual Chance.

Random Sample:- Random Sampling is Sample that it has an eaual probability of being Choosen.

purposive Sampling:- It is a non probability sampling Method, the elements selected for the sample are Choosen by the judgment of the researcher.

Long Answer questions

(1) If x is normally distributed with Mean 1 and S.D 0.6 obtain $P(x>0)$ & $P(-1.8 \le X \le 2.0)$

Sol:

Given that $\mu = 1$, $\sigma = 0.6$

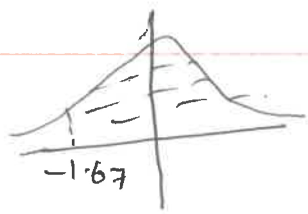
(1) When $x=0$, $z = \dfrac{x-\mu}{\sigma} = \dfrac{0-1}{0.6} = -1.67$

$\therefore P(x>0) = P(z>-1.67)$

$= 0.5 + A(1.67)$

$= 0.5 + 0.4525$

$= 0.9525$



(2) $P(-1.8 < X \le 2.0)$

When $x = -1.8$, $z = \dfrac{x-\mu}{\sigma} = \dfrac{-1.8-1}{0.6} = -4.67$

$x = 2.0$, $z = \dfrac{x-\mu}{\sigma} = \dfrac{2.0-1}{0.6} = 1.67$

$\therefore P(-1.8 < X \le 2) = P(-4.7 < z < 1.7)$

$= A(1.67) + A(-4.67)$

$= 0.4525 + 0.0691$

$= 0.5216$

(2) The marks obtained by 500 students is normally distributed with mean 65% & S.D 8%. Determine how many get more than 80%.

Sol:

$$\mu = 0.65, \quad \sigma = 0.08$$

$P(X > 0.8)$

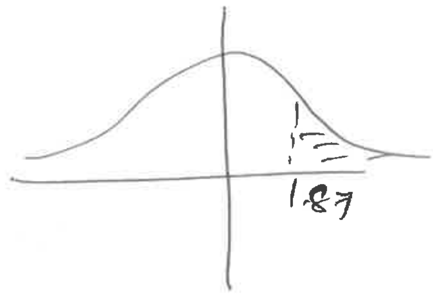
when $x = 0.8$, $z = \frac{x-\mu}{\sigma} = \frac{0.8-0.65}{0.08} = 1.87$

$\therefore P(X>0.8) = P(z>1.87)$

$= 0.5 - A(1.87)$

$= 0.5 - 0.4693$

$= 0.0307$



(3) Given that the mean heights of students in a class is 158 Cms with a S.D of 20cms. Find how many students height lie between 150 & 170 Cms.

Sol:

$$\mu = 158, \quad \sigma = 20$$

when $x = 150$, $z = \frac{x-\mu}{\sigma} = \frac{150-158}{20} = -0.4$

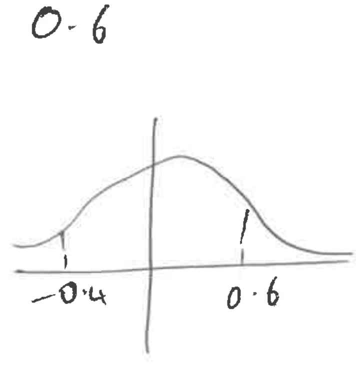
$x = 170$, $z = \frac{x-\mu}{\sigma} = \frac{170-158}{20} = 0.6$

$\therefore P(150 < X < 170) = P(-0.4 < z < 0.6)$

$= A(0.6) + A(-0.4)$

$= 0.2258 + 0.1554$

$= 0.3812$



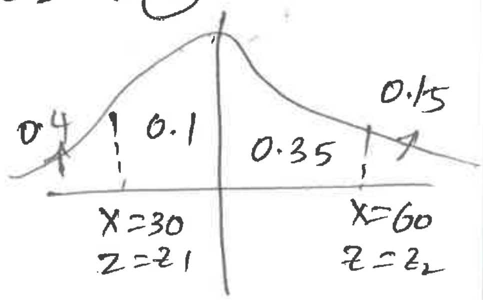
(3) The marks obtained in Statistics in a Certain exam found to be normally distributed. If 15% of the Students ≥ 60 marks, 40% < 30 marks, find mean & S.D

Sol:

$$P(x<30) = 0.4 \quad , \quad P(x \geq 60) = 0.15$$

when $x = 30$, $z = \frac{x-\mu}{\sigma} = \frac{30-\mu}{\sigma} = -z_1$ — (1)

when $x = 60$, $z = \frac{x-\mu}{\sigma} = \frac{60-\mu}{\sigma} = z_2$ — (2)



$\therefore \quad P(0<z<z_1) = P(-z_1<z<0)$

$= 0.5 - 0.4$

$= 0.1$

& $P(0<z<z_2) = 0.5 - 0.15 = 0.35$

from normal tables $z_1 = 0.25$ & $z_2 = 1.04$

$\therefore$ (1) $\Rightarrow$ $\frac{30-\mu}{\sigma} = -0.25$ & $\frac{60-\mu}{\sigma} = 1.04$

$30-\mu = -0.25\sigma \qquad 60-\mu = 1.04\sigma$

Solve above eqn's

$\therefore \quad \mu = 35.81, \quad \sigma = 23.26$

(5) The mean Voltage of a battery is '15' and standard deviation 0.2. Find the probability that four such batteries connected in series will have a combined voltage of 60.8 (or) more Volts

Sol: Given that

$\mu = 15 \qquad \sigma = 0.2$

~~To find~~ ~~$p(x \geq 60.8)$~~

~~When $x = 60.8$~~ ~~$z = \frac{60.8 - 15}{0.2}$~~

Let mean voltage of a batteries 1, 2, 3, 4 be $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$

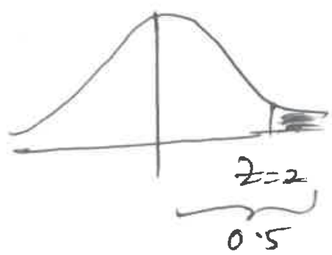
The mean of the series of the four batteries connected is

$\mu(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4) = \mu(\bar{x}_1) + \mu(\bar{x}_2) + \mu(\bar{x}_3) + \mu(\bar{x}_4) = 15 + 15 + 15 + 15 = 60$

$\sigma(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4) = \sqrt{\sigma^2(\bar{x}_1) + \sigma^2(\bar{x}_2) + \sigma^2(\bar{x}_3) + \sigma^2(\bar{x}_4)} = \sqrt{4(0.2)^2} = 0.4$

Let 'x' be the combined voltage of the series

When $x = 60.8 \qquad z = \frac{\bar{x} - \mu}{\sigma} = \frac{60.8 - 60}{0.4} = 2$



so $P(x \geq 60.8) = P(z \geq 2)$

$= 0.5 - A(2)$

$= 0.5 - 0.4772$

$= 0.0228.$

Mean of Normal distribution

Consider the Normal distribution with μ, σ as the parameters

then $f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $\quad -\infty < x < \infty$

$$\text{Mean} = \mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{x-\mu}{\sigma} = z$$

$=$

$$x = \mu + \sigma z$$

$$dx = 0 + \sigma dz$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z)\, e^{-\frac{1}{2}(z)^2}\, \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu\, e^{-\frac{z^2}{2}} dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma z\, e^{-\frac{z^2}{2}} dz$$

$e^{-\frac{z^2}{2}}$ is even function $\qquad z\, e^{-\frac{z^2}{2}}$ odd function

$$\left[\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx \quad f(x) \text{ is even} \atop \qquad = 0 \qquad\qquad f(x) \text{ is odd} \right]$$

$$= \frac{\mu}{\sqrt{2\pi}} \, 2 \int_0^\infty e^{-\frac{z^2}{2}} dz + 0$$

$$\frac{z^2}{2} = t$$

$$z^2 = 2t$$

$$z = \sqrt{2}\sqrt{t}$$

$$dz = \sqrt{2} \, \frac{1}{2} t^{\frac{1}{2}-1} dt$$

$$dz = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{t}} dt$$

$$= \frac{\mu}{\sqrt{2\pi}} \, 2 \int_0^\infty e^{-t} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{t}} dt$$

$$= \frac{2\mu}{\cancel{\sqrt{2}} \sqrt{\pi} \cancel{\sqrt{2}}} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \left(\Gamma\tfrac{1}{2} \right)$$

$$n - 1 = -\frac{1}{2}$$

$$n = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\left[\int_0^\infty e^{-x} x^{n-1} dx = \Gamma\tfrac{1}{2} \right)$$

$$= \frac{\mu}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$= \mu.$$

Variance of Normal distribution

$$Var(X) = E(X^2) - [E(X)]^2 \qquad (E(X) = \mu)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{x-\mu}{\sigma} = z$$

$$x = \mu + \sigma z$$

$$dx = \sigma dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z)^2 e^{-\frac{1}{2}(z^2)} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu^2 + \sigma^2 z^2 + 2\mu\sigma z) e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu^2 e^{-\frac{z^2}{2}} dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{z^2}{2}} dz + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\mu\sigma z e^{-\frac{z^2}{2}} dz$$

$e^{-\frac{z^2}{2}}$ is even function

$z^2 e^{-\frac{z^2}{2}}$ even function

$z\, e^{-z^2/2}$ odd function.

$$= \frac{\mu^2}{\sqrt{2\pi}} 2\int_0^\infty e^{-\frac{z^2}{2}} dz + \frac{\sigma^2}{\sqrt{2\pi}} 2\int_0^\infty z^2 e^{-\frac{z^2}{2}} dz + 0$$

Let $\frac{z^2}{2} = t$

$$z^2 = 2t$$

$$z = \sqrt{2}\sqrt{t}$$

$$dz = \sqrt{2}\, \frac{1}{2} t^{\frac{1}{2}-1} dt$$

$$dz = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{t}} dt$$

$$= \frac{\mu^2}{\sqrt{2\pi}} 2\int_0^\infty e^{-t} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{t}} dt + \frac{\sigma^2}{\sqrt{2\pi}} 2\int_0^\infty 2t\, e^{-t} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{t}} dt$$

$$= \frac{\cancel{2}\mu^2}{\cancel{\sqrt{2}}\sqrt{2}\sqrt{\pi}} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt + \frac{\cancel{4}\sigma^2}{\cancel{\sqrt{2}}\sqrt{2}\sqrt{\pi}} \int_0^\infty e^{-t} t^{\frac{1}{2}} dt$$

$n-1 = -\frac{1}{2}$ $\quad$ $n-1 = \frac{1}{2}$

$n = \frac{1}{2}$ $\quad$ $n = \frac{3}{2}$

$$= \frac{\mu^2}{\sqrt{\pi}} \left(\Gamma\tfrac{1}{2}\right) + \frac{2\sigma^2}{\sqrt{\pi}} \left(\Gamma\tfrac{3}{2}\right)$$

$$= \frac{\mu^2}{\cancel{\sqrt{\pi}}} (\cancel{\sqrt{\pi}}) + \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \frac{1}{\cancel{2}} \cancel{\sqrt{\pi}}$$

$$E(x^2) = \mu^2 + \sigma^2$$

$$\mathrm{var}(x) = (\mu^2 + \sigma^2) - \mu^2$$

$$\mathrm{Var}(x) = \sigma^2$$

→ A population consists of five numbers 2, 3, 6, 8 and 11. Consider all possible samples of size two which can be drawn with replacement from this population. Find

i) the mean of the population.

(ii) The standard deviation of the population

(iii) The mean of the sampling distribution of means

(iv) The standard deviation of the sampling distributions of means (i.e the standard error of means)

Sol:-

a) Mean of the population is given by

$$\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$$

b) Variance of the population (σ^2) is given by

$$\sigma^2 = \Sigma \frac{(x_i - \bar{x})^2}{n}$$

$$= \frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2}{5}$$

$$= \frac{16+9+0+4+25}{5} = 10.8$$

$$\sigma = \sqrt{10.8} =$$

$$= 3.29$$

c) Sampling with replacement (infinite population)

The total no. of samples with replacement is

$N^n = 5^2 = 25$ sample of size = 2

Here N = population size

n = sample size   The 25 samples

$$\begin{Bmatrix} (2,2) & (2,3) & (2,6) & (2,8) & (2,11) \\ (3,2) & (3,3) & (3,6) & (3,8) & (3,11) \\ (6,2) & (6,3) & (6,6) & (6,8) & (6,11) \\ (8,2) & (8,3) & (8,6) & (8,8) & (8,11) \\ (11,2) & (11,3) & (11,6) & (11,8) & (11,11) \end{Bmatrix}$$

Now mean of each of these 25 samples

The sample means are

$$\begin{Bmatrix} 2 & 2.5 & 4 & 5 & 6.5 \\ 2.5 & 3 & 4.5 & 5.5 & 7 \\ 4 & 4.5 & 6 & 7 & 8.5 \\ 5 & 5.5 & 7 & 8 & 9.5 \\ 6.5 & 7 & 8.5 & 9.5 & 11 \end{Bmatrix}$$

$$\mu_{\bar{x}} = \frac{\text{Sum of all sample means}}{25} = \frac{150}{25} = 6$$

So $\mu_{\bar{x}} = \mu$

d) $\sigma_{\bar{x}}^2 = \frac{(2-6)^2 + \cdots + (11-6)^2}{25} = \frac{135}{25} = 5.40$

and $\sigma_{\bar{x}} = \sqrt{5.40} = 2.32$

Clearly for finite population involving sampling with replacement (Infinite population)

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{(3.29)^2}{2} = 2.32$$

Solution of above problem with out Replacement (finite population)

Sol

i) $\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$

ii) $\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n}$

$$= \frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2}{5}$$

$\sigma^2 = 10.8$ so $\sigma = 3.29$

iii) Sampling with out replacement

The total no of samples with out replacement is $^N C_n = {}^5C_2 = 10$

The 10 samples are

$$\left\{ \begin{array}{llll} (2,3) & (2,6) & (2,8) & (2,11) \\ (3,6) & (3,8) & (3,11) & \\ (6,8) & (6,11) & & \\ (8,11) & & & \end{array} \right\}$$

The Corresponding Sample means are

$$\left\{\begin{matrix} 2.5 & 4 & 5 & 6.5 \\ 4.5 & 5.5 & 7 & \\ 7 & 8.5 & & \\ 9.5 & & & \end{matrix}\right\}$$

The mean of the sampling distribution of means

$$\mu_{\bar{x}} = \frac{2.5+4+5+6.5+4.5+5.5+7+7+8.5+9.5}{10} = 6$$

So $\mu_{\bar{x}} = \mu$

iv) The variance of sampling distributions of means

$$\sigma_{\bar{x}}^2 = \frac{(2.5-6)^2+(4-6)^2+ - - +(9.5-6)^2}{10} = 4.05$$

$$\sigma_{\bar{x}} = 2.01$$

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right) = \frac{10.8}{2}\left(\frac{5-2}{5-1}\right) = 4.05$$

for Sampling without Replacement