Unit-II

Mathematical Expectation

Expectation of a Discrete Random Variable

Suppose a random variable X assumes the values $x_1, x_2, \cdots x_n$ with respective probabilities $p_1, p_2, \cdots p_n$. Then the mathematical Expectation (or) mean (or) Expected value of x, denoted by $E(X)$ (or) μ, is defined as the sum of products of different values of 'x' and the corresponding probabilities

$$\therefore \mu = E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

$$\Rightarrow \boxed{\mu = E(X) = \sum_{i=1}^{n} p_i x_i}$$

In General, The expected value of any function $g(x)$ of a random variable X is defined as

$$\boxed{E(g(x)) = \sum_{i=1}^{n} p_i \, g(x_i)}$$

Properties: Let X & Y are two random variables and 'a' and 'b' are constants then

(a) $E(a) = a$

(b) $E(aX) = a E(X)$

(c) $E(aX \pm b) = a E(X) \pm b$

(d) $E(aX \pm bY) = a E(X) \pm b E(Y)$

(e) If X, Y, Z are independent Random variables then $E(XYZ) = E(X) E(Y) E(Z)$

Note: $E(\frac{1}{x})$ and $\frac{1}{E(x)}$ are not same

Variance: If X is a random variable, then the mathematical Expectation of $(X-\mu)^2$ is defined to be the variance of the random variable X, and is denoted by $Var(X)$ (or) $V(X)$ (or) σ^2. Then

$$\sigma^2 = Var(X) = \boxed{V(X) = E(X-\mu)^2}$$

$$\boxed{\sigma^2 = V(X) = E(X^2) - \mu^2}$$

(or) $$\boxed{\sigma^2 = V(X) = E(X^2) - [E(X)]^2}$$

Standard Deviation: It is the +ve square root of the variance. $\therefore S.D = \sqrt{\sigma^2} = \sigma$

$$\boxed{\sigma = \sqrt{E(X^2) - [E(X)]^2}}$$

Properties:

(i) Variance of Constant is zero i.e. $V(K) = 0$

(ii) If K is a Constant, then $V(KX) = K^2 V(X)$

(iii) If X is a discrete random variable & 'a', 'b' are Constants then $V(aX+b) = a^2 V(X)$

(iv) If X & Y are Two independent Random variables Then $V(X \pm Y) = V(X) \pm V(Y)$

(*) ~~Prove that~~ ~~Proof of~~ Prove that $E(aX+b) = aE(X)+b$ where a, b are constant and X is Random variable.

Proof: By definition of Expection we have

$$E(ax+b) = \sum_{i=1}^{n} (ax_i+b)p_i = \sum_{i=1}^{n} (ax_ip_i + bp_i)$$

$$= \sum_{i=1}^{n} ax_ip_i + \sum_{i=1}^{n} bp_i$$

$$= a\sum_{i=1}^{n} x_ip_i + b\sum_{i=1}^{n} p_i$$

$$\boxed{E(ax+b) = aE(x)+b} \quad \left(\because \sum_{i=1}^{n} p_i = 1\right)$$

(*) prove that $V(ax+b) = a^2V(x)$ where $V(x)$ is variance of Random variable & a, b are Constants.

Proof:- Let $Y = ax+b$ —— ①

$\Rightarrow E(Y) = E(ax+b) = aE(x)+b$ —— ②

Now ① - ② $\Rightarrow$ $Y - E(Y) = (ax+b) - (aE(x)+b)$

$Y - E(Y) = a[x - E(x)]$

Squaring and taking Expectation on both sides

$$E[Y-E(Y)]^2 = E[a(x-E(x))]^2$$

$$= E(a^2(x-E(x))^2)$$

$$E[Y-E(Y)]^2 = a^2 E[x-E(x)]^2$$

$$V(Y) = a^2V(x)$$

$$\Rightarrow \boxed{V(ax+b) = a^2V(x)} \quad (\because ①)$$

(*) A random variable X has the following distribution

x	1	2	3	4	5	6
p(x)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

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then find (i) the Mean (ii) Variance

Sol^n (i) Mean $\mu = \sum_{i=1}^{6} p_i x_i = p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 + p_5 x_5 + p_6 x_6$

$$= \frac{1}{36}\cdot 1 + \frac{3}{36}\cdot 2 + \frac{5}{36}\cdot 3 + \frac{7}{36}\cdot 4 + \frac{9}{36}\cdot 5 + \frac{11}{36}\cdot 6$$

$$= \frac{1}{36}(1+6+15+28+45+66)$$

$$= \frac{161}{36}$$

$$\boxed{\mu = 4.47}$$

(ii) Variance $(\sigma^2) = \sum_{i=1}^{6} p_i x_i^2 - \mu^2$

$$= \left[\frac{1}{36}(1)^2 + \frac{3}{36}(2)^2 + \frac{5}{36}(3)^2 + \frac{7}{36}(4)^2 + \frac{9}{36}(5)^2 + \frac{11}{36}(6)^2\right] - (4.47)^2$$

$$= \frac{1}{36}[1+12+45+112+225+396] - (4.47)^2$$

$$= \frac{791}{36} - 19.9808 = 21.97 - 19.981$$

$$\sigma^2 = 1.9912$$

(*) A random variable X has the following probability function:

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Determine k (ii) Mean (iii) Variance.

Sol^n (i) Since $\sum_{x=0}^{7} P(x) = 1$

$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 10k^2 + 9k - 1 = 0$

$(10k-1)(k+1) = 0$

$k = -1$ (or) $k = 1/10 = 0.1$

Since $k \neq -1$ $(\because P(1) = k > 0)$

$\therefore \boxed{k = \frac{1}{10}}$

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(ii) Mean $(\mu) = \sum_{i=0}^{7} P_i x_i$

$= 0(0) + 1(K) + 2(2K) + 3(2K) + 4(3K) + 5(K^2) + 6(2K^2) + 7(7K^2+K)$

$= 66K^2 + 30K$

$= 66(0.1)^2 + 30(0.1)$

$= 0.66 + 3$

$\mu = \underline{3.66}$

(iii) Variance $(\sigma^2) = \sum_{i=0}^{7} P_i x_i^2 - \mu^2$

$= [0(0)^2 + K(1)^2 + 2K(2)^2 + 2K(3)^2 + 3K(4)^2 + K^2(5)^2 + (2K^2)(6)^2 + (7K^2+K)7^2] - (3.66)^2$

$= [K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K] - 13.3956$

$= 440K^2 + 124K - 13.3956$

$= 440(0.1)^2 + 124(0.1) - 13.3956$

$= 4.4 + 12.4 - 13.3956$

$\sigma^2 = 3.4044$

⊛ A random sample with replacement of size '2' is taken from $S = \{1, 2, 3\}$. Let the random variable X denote the sum of the two ~~no. of~~ numbers taken.

(i) Write the probability distribution of X

(ii) Find the Mean (iii) Find the variance

Sol) Sample Space $S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Now $X(s) = \{2, 3, 4, 5, 6\}$

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$P(2) = P(X=2) = \frac{1}{9}$ $\quad$ $P(3) = P(X=3) = \frac{2}{9}$

$P(4) = P(X=4) = \frac{3}{9}$ $\quad$ $P(5) = P(X=5) = \frac{2}{9}$

$P(6) = P(X=6) = \frac{1}{9}$

(i) The probability distribution of X is

X	2	3	4	5	6
$P(X)$	$\frac{1}{9}$	$2/9$	$3/9$	$2/9$	$1/9$

(ii) Mean $(\mu) = \sum p_i x_i$

$= 2(\frac{1}{9}) + 3(\frac{2}{9}) + 4(3/9) + 5(2/9) + 6(1/9)$

$= \frac{1}{9}(2+6+12+10+6)$

$= \frac{36}{9}$

$\boxed{\mu = 4}$

(iii) Variance $(\sigma^2) = \sum p_i x_i^2 - \mu^2$

$= \left[\frac{1}{9}(2)^2 + \frac{2}{9}(3)^2 + \frac{3}{9}(4)^2 + \frac{2}{9}(5)^2 + \frac{1}{9}(6)^2\right] - 4^2$

$= \frac{1}{9}[4 + 18 + 48 + 50 + 36] - 16$

$= \frac{156}{9} - 16$

$\boxed{\sigma^2 = 1.33}$

(*) A random variable X has the following probability function

$X=x$	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	2K	0.3	K

Ans: (i) $K = 0.1$

(ii) $\mu = 0.8$

(iii) $\sigma^2 = 2.16$

Find (i) K (ii) Mean (iii) Variance

(8) A fair die is tossed. Let the random variable X denote the twice the number appearing on the die:

(i) write the Probability distribution of X

(ii) the Mean (iii) the Variance

Ans: $\mu = 7$

$\sigma^2 = 11.67$

Measures of Central Tendency for Continuous probability Distribution: Let $f(x)$ be the probability density function of a continuous random variable X. then.

(i) **Mean:** Mean of a distribution is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

In general, mean (or) Expectation of any function $\phi(x)$ is given by $E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x)\,dx$.

(ii) **Median:** Median is the point which divides the entire distribution into two equal parts. In case of continuous distribution, median is the point which divides the total area into two equal parts. Thus if X is defined from a to b and M is the median, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

Solving for M, we get the median

(iii) **Mode:** mode is the value of x for which $f(x)$ is maximum. Mode is thus given by $f'(x) = 0$ and $f''(x) < 0$ for $a < x < b$

(iv) Variance: Variance of a distribution is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

(or)

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) - \mu^2$$

(v) Mean deviation:- Mean deviation about the mean (μ) is given by $\int_{-\infty}^{\infty} |x-\mu|\, f(x)\, dx$

(*) A Continuous random variable has the ~~probt~~ probability density function $f(x) = \begin{cases} kx e^{-\lambda x}, & \text{for } x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

Determine (i) k (ii) Mean (iii) Variance

Sol: (i) Since the total probability is unity we have

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \Rightarrow \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow 0 + k \int_0^{\infty} x e^{-\lambda x}\, dx = 1$$

$$\Rightarrow k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$$

$$\Rightarrow \frac{k}{\lambda^2} = 1 \Rightarrow \boxed{k = \lambda^2}$$

(ii) Mean $(\mu) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-\infty}^{0} x f(x) dx + \int_{0}^{\infty} x f(x) dx$$

$$= 0 + K \int_{0}^{\infty} x \cdot x e^{-\lambda x} dx = \lambda^2 \int_{0}^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda^2 \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right]_0^{\infty}$$

$$= \lambda^2 \left[(0 - 0 + 0) - \left(0 - 0 - \frac{2}{\lambda^3} \right) \right]$$

$$= \lambda^2 \left(\frac{2}{\lambda^3} \right)$$

$$\boxed{\mu = \frac{2}{\lambda}}$$

(iii) Variance $(\sigma^2) = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

$$= \left[\int_{-\infty}^{0} x^2 f(x) dx + \int_{0}^{\infty} x^2 f(x) dx \right] - \mu^2$$

$$= \left[0 + \int_{0}^{\infty} x^2 \cancel{K} x e^{-\lambda x} dx \right] - \mu^2$$

$$= \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x} dx - \left(\frac{2}{\lambda} \right)^2$$

$$= \lambda^2 \left[x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right]_0^{\infty} - \left(\frac{2}{\lambda} \right)^2$$

$$= \lambda^2 \left[(0 - 0 + 0 - 0) - \left(0 - 0 + 0 - \frac{6}{\lambda^4} \right) \right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$\boxed{\sigma^2 = \frac{2}{\lambda^2}}$$

(*) Probability density function of a random variable X

is $f(x) = \begin{cases} \frac{1}{2}\sin x & , \text{for } 0 \le x \le \pi \\ 0 & , \text{elsewhere} \end{cases}$

Find the mean, mode, and median of the distribution and also find the probability between 0 and $\pi/2$.

Sol^n (i) Mean of the distribution $(\mu) = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_{-\infty}^{0} x f(x) dx + \int_{0}^{\pi} x f(x) dx + \int_{\pi}^{\infty} x f(x) dx$$

$$= 0 + \int_{0}^{\pi} x \left(\tfrac{1}{2}\sin x\right) dx + 0$$

$$= \frac{1}{2}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\pi(-\cos\pi) + \sin\pi + 0 - 0\right]$$

$$(\mu) = \frac{\pi}{2}$$

(ii) Mode is the value of x for which $f(x)$ is maximum

Now $f'(x) = \frac{1}{2}\cos x$

For $f(x)$ to be maximum, $f'(x) = 0 \Rightarrow \frac{1}{2}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}$

$f''(x) = -\frac{1}{2}\sin x$

at $x = \frac{\pi}{2}$ $\quad f''(x) = -\frac{1}{2} < 0$

Hence $f(x)$ is maximum at $\boxed{x = \frac{\pi}{2}}$

$\therefore$ Mode of the distribution is given by $x = \frac{\pi}{2}$

(iii) If M is the median of the distribution, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^M \frac{1}{2}\sin x\,dx = \int_M^{\pi} \frac{1}{2}\sin x\,dx = \frac{1}{2}$$

Solving $\int_0^M \frac{1}{2}\sin x\,dx = \frac{1}{2} \Rightarrow \int_0^M \sin x\,dx = 1$

$$\Rightarrow (-\cos x)_0^M = 1 \Rightarrow (-\cos M + 1) = 1$$

$$\Rightarrow \cos M = 0 \Rightarrow M = \frac{\pi}{2}$$

$\therefore$ Median of the distribution $= \frac{\pi}{2}$

Thus Mean $=$ mode $=$ median $= \frac{\pi}{2}$

(iv) $P(0 < x < \frac{\pi}{2}) = \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{1}{2}\sin x\,dx$

$$= \frac{1}{2}(-\cos x)_0^{\pi/2} = -\frac{1}{2}(0-1) = \frac{1}{2}$$

If X is the continuous random variable whose density function is $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$

Find $E(25X^2 + 30X - 5)$

Sol^n Given $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 \le x < 2 \\ 0 & \text{elsewhere.} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x f(x)\,dx + \int_0^1 x f(x)\,dx + \int_1^2 x f(x)\,dx + \int_2^{\infty} x f(x)\,dx$$

$$= 0 + \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx + 0$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(2\frac{x^2}{2} - \frac{x^3}{3}\right)_1^2 = \left(\frac{1}{3} - 0\right) + \left(4 - \frac{8}{3} - 1 + \frac{1}{3}\right)$$

$$E(x) = 1$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{0} x^2 f(x)\,dx + \int_0^1 x^2 f(x)\,dx + \int_1^2 x^2 f(x)\,dx + \int_2^{\infty} x^2 f(x)\,dx$$

$$= 0 + \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx + 0$$

$$= \int_0^1 x^3\,dx + \int_1^2 (2x^2 - x^3)\,dx$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left(2\frac{x^3}{3} - \frac{x^4}{4}\right)_1^2$$

$$= \frac{1}{4} - 0 + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{14}{3} + \frac{1}{2} - 4 = \frac{28 + 3 - 24}{6}$$

$$E(x^2) = \frac{7}{6}$$

$$\therefore E(25x^2 + 30x - 5) = 25\,E(x^2) + 30\,E(x) - E(5)$$

$$= 25\left(\frac{7}{6}\right) + 30(1) - 5 \qquad (\because E(a) = a)$$

$$= 25\left(\frac{7}{6} + 1\right)$$

$$= 25\left(\frac{13}{6}\right)$$

$$= \frac{325}{6}$$

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(*) The cumulative distribution function for a continuous random variable X is $F(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Find (i) the density function $f(x)$

(ii) Mean and (iii) variance of the density function

Ans: $\boxed{\mu = \frac{1}{8}}$ $\boxed{\sigma^2 = -\frac{9}{64}}$

(*) For the continuous probability function $f(x) = kx^2 e^{-x}$ when $x \geq 0$ find (i) K (ii) Mean (iii) Variance.

Ans: $\boxed{k = \frac{1}{2}}$ $\boxed{\mu = 3}$ $\boxed{\sigma^2 = 3}$

(*) The probability density $f(x)$ of a continuous random variable is given by $f(x) = ce^{-|x|}$, $-\infty < x < \infty$. Show that $c = \frac{1}{2}$ and find the mean & variance of the distribution. Also find the probability that the variate lies between 0 and 4.

Ans: $\mu = 0$	$\sigma^2 = 2$	$P(0 \leq x \leq 4) = 0.49$

Mean (or) Expectation of Multiple Random Variables

(*) Let X, Y be descrete ~~two~~ two dimensional random variable with joint probability distribution $P(X=x, Y=y)$ then Mean (or) Expectation of random variable function $g(x, y)$ is

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) \, P(X=x, Y=y)$$

(*) If X, Y is continuous two dimensional random variables with joint density function $f(x,y)$ then.

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dx \, dy$$

(*) (a) If $g(x,y) = x$

$$\mu_x = E(X) = \begin{cases} \sum_x \sum_y x \, p(x,y) = \sum_x x \, P_X(x) & \text{for discrete R.V} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \, f(x,y)\, dx\,dy = \int_{-\infty}^{\infty} x \, f_X(x)\, dx & \text{for Continuous R.V.} \end{cases}$$

where $P_X(x)$ = marginal distribution of X

$f_X(x)$ = Marginal density ~~distribution~~ of X

If $g(x,y) = Y$

$$\mu_y = E(Y) = \begin{cases} \sum_x \sum_y y \cdot p(x,y) = \sum_y y \cdot P_Y(y) & \text{for discrete R.V} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \, f(x,y)\, dx\,dy = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy & \text{for Continuous R.V} \end{cases}$$

where $P_Y(Y)$ = Marginal distribution of X

$f_Y(Y)$ = " " density " "

(*) Conditional Expectation of Joint 2-D R.V

$$E\left(\frac{Y}{X}\right) = \begin{cases} \sum_x \sum_y \frac{y}{x} \, P(x,y) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y}{x} \, f(x,y)\, dx\,dy \end{cases}$$

$$E\left(\frac{X}{Y}\right) = \begin{cases} \sum_x \sum_y \frac{x}{y} \, P(x,y) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x}{y} \, f(x,y)\, dx\,dy \end{cases}$$

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~~Variance and~~ Co-Variance of Multiple R.V.s

~~Variance~~ ⇒ Let X, Y be 2D Random variables with joint probability distribution $P(x,y)$ & joint density function $f(x,y)$, then Co-variance of X and Y is defined as

$$\sigma_{XY} = Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

$$= \begin{cases} \sum_x \sum_y (x-\mu_x)(y-\mu_y) P(x,y) & \text{for discrete R.V} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\,dx\,dy & \text{for continuous R.V.} \end{cases}$$

$$\Rightarrow \boxed{Cov(X,Y) = E(XY) - E(X)\,E(Y)}$$

$$\boxed{\sigma_{XY} = Cov(X,Y) = E(XY) - \mu_x \mu_y}$$

⊛ Let X, Y be two random variables each taking values −1, 0, and 1 and it has following joint probability distribution.

X \ Y	−1	0	1
−1	0	0.2	0
0	0.1	0.2	0.1
1	0.1	0.2	0.1

(i) Show that X & Y have different expectations.

(ii) Find Covariance (X, Y)

(iii) Find Var(X) & Var(Y)

(iv) Find Conditional probability distribution of X given that Y=0.

Sol) Given that

X \ Y	-1	0	1	Total
-1	0	0.2	0	0.2
0	0.1	0.2	0.1	0.4
1	0.1	0.2	0.1	0.4
Total	0.2	0.6	0.2	

Marginal distribution of X

Marginal distribution of Y

(i) $E(X) = \sum x \, p_X(x)$

$= -1 \cdot (0.2) + 0 \cdot (0.4) + 1(0.4)$

$= -0.2 + 0.4$

$E(X) = 0.2$

$E(Y) = \sum y \, p_Y(y)$

$= -1 \cdot (0.2) + 0 (0.6) + 1 \cdot (0.2)$

$= -0.2 + 0.2$

$E(Y) = 0$

$\therefore E(X) \neq E(Y)$

(ii) $\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)}$

$E(XY) = \sum_x \sum_y xy \, p(x,y)$

$= (-1)(-1)(0) + (-1)(0)(0.2) + (1)(-1)(0) + (0)(-1)(0.1)$
$+ (0)(0)(0.2) + (0)(1)(0.1) + (1)(-1)(0.1) + (1)(0)(0.2)$
$+ (1)(1)(0.1)$

$= 0 + 0 + 0 + 0 + 0 + 0 - 0.1 + 0 + 0.1$

$E(XY) = 0$

$\therefore$ $Cov(X,Y) = E(XY) - E(X)E(Y)$

$= 0 - (0)(0)$

$Cov(X,Y) = 0$

(iii) $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P_X(x) = (-1)^2(0.2) + (0)^2(0.4) + (1)^2(0.4)$

$= 0.2 + 0.4$

$= 0.6$

$E(Y^2) = \sum y^2 P_Y(y) = (-1)^2(0.2) + (0)^2(0.6) + (1)^2(0.2)$

$= 0.2 + 0.2$

$= 0.4$

$Var(X) = E(X^2) - [E(X)]^2 = 0.6 - (0.2)^2$

$= 0.6 - 0.04$

$Var(X) = 0.56$

$Var(Y) = E(Y^2) - [E(Y)]^2 = 0.4 - (0)^2 = 0.4$

(iv) Conditional probability distribution of X given that Y,

is $P(X=x / Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$

If $Y=0$ conditional probability distribution of x given that $Y=0$ is

~~$P_Y(y=0) = P(X=1, Y=y)$~~

~~If Y=0~~ $P(X=x / Y=0) = \frac{P(X=x, Y=0)}{P(Y=0)}$

$P(Y=0) = P(X=-1, Y=0) + P(X=0, Y=0) + P(X=1, Y=0)$

$= 0.2 + 0.2 + 0.2$

$= 0.6$

If $x = (-1)$ $\quad P(X=-1/Y=0) = \frac{P(X=-1, Y=0)}{P(Y=0)} = \frac{0.2}{0.6} = \frac{1}{3}$

If $x = 0$ $\quad P(X=0/Y=0) = \frac{P(X=0, Y=0)}{P(Y=0)} = \frac{0.2}{0.6} = 1/3$

If $x = 1$ $\quad P(X=1/Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{0.2}{0.6} = 1/3$

(*) The Random variables X and Y have joint probability distribution given below then find Covariance of X, Y.

X \ Y	0	1	2
0	3/28	3/14	1/28
1	9/28	3/14	0
2	3/28	0	0

(*) The Random Variables X and Y have following joint probability density function $f(x,y) = \begin{cases} 2-x-y & 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$

Find (i) marginal density functions of X & Y

(ii) conditional density function of X & Y

(iii) Var(X) & Var(Y)

(iv) cov (X, Y)

Sol (i) Marginal density functions of X & Y is

$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$

$\Rightarrow f_X(x) = \int_{y=0}^{1} (2-x-y)\,dy$

$$= \left[(2-x)y - \frac{y^2}{2}\right]_{y=0}^{1} = (2-x)(1-0) - \frac{(1-0)}{2}$$

$$= 2 - x - \frac{1}{2}$$

$$f_X(x) = \frac{3}{2} - x \quad , \quad 0 \le x \le 1$$

marginal distribution of Y is

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx = \int_{x=0}^{1} f(x,y)\,dy$$

$$= \int_{x=0}^{1} (2-x-y)\,dy = \left[(2-y)x - \frac{x^2}{2}\right]_{x=0}^{1}$$

$$= \left[(2-y)(1-0) - \left(\frac{1-0}{2}\right)\right] = \frac{3}{2} - y \;,\; 0 \le y \le 1$$

(ii) Conditional density function of X given $Y=y$ is

$$f(X/y) = \frac{f(x,y)}{f_Y(y)} = \frac{2-x-y}{\frac{3}{2}-y} \quad , \quad 0 \le x \le 1, 0 \le y \le 1$$

Conditional density function of Y given $X=x$ is

$$f\left(\frac{Y}{X}\right) = \frac{f(x,y)}{f_X(x)} = \frac{2-x-y}{\frac{3}{2}-x} \quad , \quad 0 \le x \le 1, 0 \le y \le 1$$

(iii)

$$E(X) = \int_{x=-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{x=0}^{1} x\left(\frac{3}{2}-x\right)dx$$

$$= \left[\frac{3}{2}\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{1} = \frac{3}{2}\frac{(1)}{2} - \frac{1}{3}$$

$$= \frac{9-4}{12}$$

$$E(X) = 5/12$$

$$E(X^2) = \int_{x=-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{x=0}^{1} x^2\left(\frac{3}{2}-x\right)dx = \left[\frac{3}{2}\frac{x^3}{3} - \frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(Y) = \int_{y=-\infty}^{\infty} y\, f_Y(y)\, dy = \int_{y=0}^{1} y\left(\frac{3}{2}-y\right) dy$$

$$= \left[\frac{3}{2}\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{1} = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

$$E(Y^2) = \int_{y=-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_{y=0}^{1} y^2\left(\frac{3}{2}-y\right) dy = \left[\frac{3}{2}\frac{y^3}{3} - \frac{y^4}{4}\right]_{y=0}^{1}$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$Var(Y) = E(Y^2) - [E(X)]^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

(iv)

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} xy\,(2-x-y)\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2xy - x^2y - xy^2)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[2x\frac{y^2}{2} - x^2\frac{y^2}{2} - x\frac{y^3}{3}\right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} \left(x - \frac{x^2}{2} - \frac{x}{3}\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{6} - \frac{x^2}{6}\right]_{x=0}^{1}$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{6}$$

$$E(XY) = \frac{1}{6}$$

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Now $Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

$$= \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12}$$

$$= \frac{-1}{144}$$

(*) The joint probability density function of continuous random variables x, y is given below.

$$f(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

find $Cov(x,y)$

Sol) Marginal density function of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{y=0}^{x} 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_{y=0}^{x} = 4x(x^2 - 0)$$

$f_X(x) = 4x^3 \quad 0 \le x \le 1$

Marginal density function of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_{x=y}^{1} 8xy\,dx = 8y\left(\frac{x^2}{2}\right)_{x=y}^{1} = 4y(1-y^2)$$

$f_Y(y) = 4y - 4y^3 \quad 0 \le y \le 1$

$$E(X) = \int_{x=-\infty}^{\infty} x f_X(x)\,dx = \int_{x=0}^{1} x(4x^3)\,dx = 4\int_{x=0}^{1} x^4\,dx$$

$$= 4\left(\frac{x^5}{5}\right)_{x=0}^{1} = \frac{4}{5}(1-0) = \frac{4}{5}$$

$$E(Y) = \int_{y=-\infty}^{\infty} y f_Y(y)\,dy = \int_{y=0}^{1} y(4y - 4y^3)\,dy$$

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$$= \int_{y=0}^{1} (4y^2 - 4y^4)\,dy = \left(4\frac{y^3}{3} - 4\frac{y^5}{5}\right)_{y=0}^{1}$$

$$= \frac{4}{3} - \frac{4}{5} = 4\left(\frac{5-3}{15}\right) = \frac{8}{15}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} xy\,(8xy)\,dy\,dx = 8\int_{x=0}^{1}\int_{y=0}^{x} x^2y^2\,dy\,dx$$

$$= 8\int_{x=0}^{1} x^2\left(\frac{y^3}{3}\right)_{y=0}^{x} dx = \frac{8}{3}\int_{x=0}^{1} x^2(x^3-0)\,dx$$

$$= \frac{8}{3}\left(\frac{x^6}{6}\right)_{x=0}^{1} = \frac{\cancel{8}^{4}}{3}\left(\frac{1-0}{\cancel{6}_{3}}\right)$$

$$= \frac{4}{9}$$

Now $\quad Cov(X,Y) = E(XY) - E(X)E(Y)$

$$= \frac{4}{9} - \frac{4}{5}\cdot\frac{8}{15}$$

$$= \frac{4}{225}$$

Correlation Coefficient :- $r_{xy} = \dfrac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$

where, σ_{xy} = Covariance of x, y

σ_x = Standard deviation of x

σ_y = Standard deviation of y

Mean & Variance of linear Combination of multiple R.V

If X, Y be joint 2D Random variables, then

(i) $E(X \pm Y) = E(X) \pm E(Y)$

(ii) $E[aX \pm bY] = aE(X) \pm bE(Y)$

(iii) $E[g(x,y) \pm h(x,y)] = E[g(x,y)] \pm E[h(x,y)]$

(iv) $E[g(x) \pm h(y)] = E[g(x)] \pm E[h(y)]$

(v) If X, Y are independent R.V. then $E(XY) = E(X) \cdot E(Y)$

and $\mathrm{Cov}(X,Y) = 0$

(vi) If $\mathrm{Cov}(X,Y) = \sigma_{XY} = 0$ the X, Y are need not be independent R.V.

(vii) $\boxed{\mathrm{Var}(aX+bY+c) = \sigma^2_{aX+bY+c} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}}$

$\boxed{V(aX+bY+c) = a^2V(X) + b^2V(Y) + 2ab\,\mathrm{Cov}(X,Y)}$

Let X & Y be Random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$ then find variance of Random variable $Z = 3X - 4Y + 8$

Sol

we have $\sigma^2_{aX+bY+c} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}$

Given $Z = 3X - 4Y + 8$ Here $a = 3$ $b = -4$ & $c = 8$

$\sigma_X^2 = 2$, $\sigma_Y^2 = 4$; & $\sigma_{XY} = -2$

$\therefore \sigma_Z^2 = \sigma^2_{3x-4y+8} = 3^2(2) + (-4)^2 4 + 2(3)(-4)(-2)$

$= 18 + 64 + 48$

$= 130$

(*) Let x & y be R.V with variance $\sigma_x^2 = 2, \sigma_y^2 = 3$ find variance of R.V $z = 3x - 2y + 5$. If x, y are independent.

$\boxed{\text{Ans: } 30}$ $\boxed{\text{Hint: } \sigma_{xy} = 0}$

Chebychev's Inequality:⇒ If X is a random variable with mean μ and variance σ^2, then for any +ve number k, we have $P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

$$P\{|x-\mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$$

(*) A R.V. X has mean $\mu = 8$ and variance $\sigma^2 = 9$ for unknown probability distribution then find

(i) $P(-4 < x < 20)$ (ii) $P(|x - 8| \geq 6)$

Sol) Given $\mu = 8$, $\sigma^2 = 9$, i.e. $\sigma = 3$

(i) we have $P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

$\Rightarrow P(\mu - k\sigma < x < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$

$\Rightarrow P(8 - 3k < x < 8 + 3k) \geq 1 - \frac{1}{k^2}$

If $k = 4$

$P(8 - 12 < x < 8 + 12) \geq 1 - \frac{1}{4^2}$ ($\because$ $8 - 3k = -4$, $8 + 4 = 3k$, $k = 4$)

$\Rightarrow P(-4 < x < 20) \geq \frac{15}{16}$

(ii) we have $P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

$P(|x-8| \geq 3k) \leq \frac{1}{k^2}$

If $k=2 \Rightarrow P(|x-8| \geq 6) \leq \frac{1}{2^2} = \frac{1}{4}$

$\Rightarrow P(|x-8| \geq 6) \leq \frac{1}{4}$

(*) A. R.v. X has mean $\mu = 10$, $\sigma^2 = 4$ then find

(i) $P(5 < X < 15)$ (ii) $P(|x-10| < 3)$, (iii) $P(|x-10| \geq 3)$

(*) For geometric distribution $p(x) = 2^{-x}$, $x = 1, 2, 3 \ldots$

Prove that chebychev's inequality gives $P(|x-2| \leq 2) \geq \frac{1}{2}$

while the actual probability is $\frac{7}{8}$.

Sol) Given $p(x) = 2^{-x} = \frac{1}{2^x}$ $x = 1, 2, 3 \ldots$

$$\text{Mean } (\mu) = \sum_{x=1}^{\infty} x\, p(x) = \sum_{x=1}^{\infty} x \cdot \frac{1}{2^x}$$

$$= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots$$

$$= \frac{1}{2}\left(1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2^2}\right) + 4\left(\frac{1}{2^3}\right) + \ldots\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} \qquad (\because (1-a)^{-2} = 1 + 2a + 3a^2 + 4a^3 + \ldots)$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = \frac{2^2}{2}$$

$$\boxed{\mu = 2}$$

$$E(x^2) = \sum_{x=1}^{\infty} x^2 \cdot P(x) = \sum_{x=1}^{\infty} x^2 \left(\frac{1}{2^x}\right)$$

$$= 1\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2^2}\right) + 3^2\left(\frac{1}{2^3}\right)$$

$$= \frac{1}{2}\left(1 + 4\left(\frac{1}{2}\right) + 9\left(\frac{1}{2^2}\right) + \cdots\right)$$

$$= \frac{1}{2}\left(1 + 4A + 9A^2 + \cdots\right) \qquad (\text{put } A = \tfrac{1}{2})$$

$$= \frac{1}{2}(1+A)(1-A)^{-3}$$

$$= \frac{1}{2}\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)^{-3} \qquad (\because A = \tfrac{1}{2})$$

$$= \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^{-3}$$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot 2^3$$

$$E(x^2) = 6$$

$$V(x) = \sigma^2 = E(x^2) - \mu^2 = 6 - (2)^2 = 2$$

$$\therefore \boxed{\sigma = \sqrt{2}}$$

By Chebychev's inequality, we have

$$P(|x-\mu| \le K\sigma) \ge 1 - \frac{1}{K^2}$$

$$\Rightarrow P(|x-2| \le K\sqrt{2}) \ge 1 - \frac{1}{K^2}$$

If $K = \sqrt{2}$ $\quad P(|x-2| \le \sqrt{2}\cdot\sqrt{2}) \ge 1 - \frac{1}{(\sqrt{2})^2}$

$$P(|x-2| \le 2) \ge 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow P(|x-2| < 2) \ge \frac{1}{2}$$

~~Aunt~~ Actual probability is given by

$$P\{|x-2| < 2\} = P(0 < x < 4) = P(x=1) + P(x=2) + P(x=3)$$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{4+2+1}{8}$$

$$= \frac{7}{8}$$

(*) If x is the number appearing on a die when it is thrown, show that the chebychev's inequality give $P(|x-\mu| \geq 2.5) < 0.47$. while the actual probability is zero

---

(*) x is a random variable such that $E(x)=3$ and $E(x^2)=13$. Determine the lower bound for $p(-2<x<8)$. Using chebychev's inequality. $\boxed{\text{Ans:- } \frac{21}{25}}$

Probability Distributions

there are two types of Distributions

(1) Discrete probability Distributions

ⓐ Binomial distribution

ⓑ Poission distribution

ⓒ Geometric distribution

(2) Continuous probability Distributions.

ⓐ Normal Distribution

ⓑ student's t- Distribution

ⓒ F-distribution

ⓓ Chi-square Distributions

Bernoulli's Distribution: A random variable X which takes two values 0 and 1 with probability q and p respectively i.e. $P(X=0)=q$ and $P(X=1)=p$, $q=1-p$ is called a Bernoulli's discrete random variable and is said to have a Bernoulli's distribution. The probability function of Bernoulli's distribution can be written as

$$P(x) = p^x q^{1-x} = p^x (1-q)^{1-x}, \quad x=0,1$$

Binomial Distribution: A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x) = p(x) = \begin{cases} {}^nC_x\, p^x q^{n-x}, & x=0,1,2,\dots n, \quad q=1-p. \\ 0 & \text{otherwise} \end{cases}$$

where n = no. of trials, x = no. of success, p = prob. of getting success, q = prob. of getting failure

* The Binomial distribution function is given by

$$F_X(x) = P(X \le x) = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r}$$

Conditions of Binomial Distribution

1. Trials are repeated under identical conditions for a fixed number of times, say 'n' times.
2. There are only two possible outcomes. i.e. success or failure for each trial.
3. The probability of success in each trial remains

constant and does not change from trial to trial

(4) The trials are independent, i.e. the probability of an events in any trial is not affected by the results of any other trial.

Constants of Binomial Distribution

(1). **Mean:** $\mu = E(X) = \sum_{r=0}^{n} r \cdot p(r)$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r\, p^r q^{n-r}$$

$$= 0 \times q^0 + 1 \times {}^nC_1\, p q^{n-1} + 2 \times {}^nC_2\, p^2 q^{n-2} + \dots + n\, {}^nC_n\, p^n q^{n-n}$$

$$= np\, q^{n-1} + 2 \cdot \frac{n(n-1)}{2!} p^2 q^{n-2} + 3 \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + \dots + (n) p^n q^0$$

$$= np\left[q^{n-1} + (n-1) p\, q^{n-2} + \frac{(n-1)(n-2)}{2!} p^2 q^{n-3} + \dots + p^{n-1}\right]$$

$$= np\,[q+p]^{n-1} \quad \text{(Using Binomial theorem)}$$

$$= np\,[1]^{n-1} \quad (\because p+q=1)$$

$$\boxed{\mu = E(X) = np}$$

(2) **Variance:** $\sigma^2 = V(X) = E(X^2) - [E(X)]^2$

$$= \sum_{r=0}^{n} r^2 p(r) - \mu^2$$

$$= \sum_{r=0}^{n} (r^2 + r - r)\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} [r(r-1) + r]\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} r(r-1)\, p(r) + \sum_{r=0}^{n} r\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + \mu - \mu^2$$

$$= 2\cdot {}^nC_2\, p^2 q^{n-2} + 3\cdot 2\, {}^nC_3\, p^3 q^{n-3} + \dots + n(n-1)\, {}^nC_n\, p^n q^{n-n} + \mu - \mu^2$$

$$= 2\cdot \frac{n(n-1)}{2!} p^2 q^{n-2} + 6\, \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + \dots + n(n-1)\, p^n + \mu - \mu^2$$

$$= n(n-1)p^2 \left[q^{n-2} + (n-2)\, p\, q^{n-3} + \frac{(n-2)(n-3)}{2!} p^2 q^{n-4} + \dots + p^{n-2} \right] + \mu - \mu^2$$

$$= n(n-1)p^2 (q+p)^{n-2} + \mu - \mu^2$$ (using Binomial Expansion)

$$= n(n-1)p^2 + \mu - \mu^2$$ ($\because p+q=1$)

$$= n(n-1)p^2 + np - (np)^2$$ ($\because \mu = np$)

$$= np\,[(n-1)p + 1 - np]$$

$$= np\,[\cancel{np} - p + 1 - \cancel{np}]$$

$$= np(1-p)$$ ($\because p+q=1 \Rightarrow q = 1-p$)

$$\boxed{\sigma^2 = npq}$$

(3) Standard Deviation : $\boxed{\sigma = \sqrt{npq}}$

(4) Mode: Mode = $\begin{cases} \text{integral part of } (n+1)p, & \text{if } (n+1)p \text{ is not an integer} \\ (n+1)p \text{ and } (n+1)p - 1 & \text{if } (n+1)p \text{ is an integer} \end{cases}$

Recurrence Relation for the Binomial Distribution

we know that $P(r) = {}^nC_r\, p^r q^{n-r}$ —— (1)

$\therefore P(r+1) = {}^{n}C_{r+1}\, p^{r+1} q^{n-(r+1)}$ — ②

$$\frac{②}{①} \Rightarrow \frac{P(r+1)}{P(r)} = \frac{{}^{n}C_{r+1}\, p^{r+1} q^{n-r-1}}{{}^{n}C_{r}\, p^{r} q^{n-r}} = \frac{\frac{n!}{(n-(r+1))!\,(r+1)!}\, p^{r+1} q^{n-r-1}}{\frac{n!}{(n-r)!\,r!}\, p^{r} q^{n-r}}$$

$$= \frac{(n-r)(n-(r+1))!\; \cancel{r!}}{(n-(r+1))!\,(r+1)\cdot \cancel{r!}} \cdot \frac{p}{q}$$

$$\boxed{\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1}\cdot\frac{p}{q}}$$

(or)

$$\boxed{P(r+1) = \frac{(n-r)p}{(r+1)q}\cdot P(r)}$$

Binomial Frequency Distribution: If 'n' independent trials constitute one experiment and this experiment is repeated N times, then the frequency of r successes is $N\cdot {}^{n}C_{r}\, p^{r} q^{n-r}$

Since the probabilities of $0, 1, 2, \ldots n$ successes in 'n' trials are given by the terms of the binomial expansion of $(q+p)^n$, therefore in N sets of n trials the theoretical frequencies of $0, 1, 2 \ldots r, \ldots n$ successes will be given by the terms of expansion of $N(q+p)^n$.

The possible no. of successes and their frequencies is called a Binomial Frequency Distribution.

(*) Ten Coins are thrown simultaneously. Find the probability of getting atleast (i) seven heads (ii) Six heads. (iii) one head.

Sol^n P = probability of getting a head $= \frac{1}{2}$

q = " " " " tail i.e. not getting a head $= \frac{1}{2}$

The probability of getting 'r' heads in a throw of 10 Coins is $P(X=r) = P(r) = {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r}$, $r = 0, 1, 2 \ldots 10$

(i) Probability of getting at least seven heads is given by $P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}$$

$$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9}{2 \times 1} \left(\frac{1}{2}\right)^{10} + 10 \left(\frac{1}{2}\right)^{10} + 1 \left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}} [120 + 45 + 10 + 1]$$

$$= \frac{176}{1024}$$

$$= 0.1719$$

(ii) Probability of getting atleast six heads is given by $P(X \geq 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{10-6}$

$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \left(\frac{1}{2^{10}}\right)$

$= \frac{210}{1024} = 0.20507$

(ii) P (at least 6 heads) = $P(X \geq 6)$

$= P(X=6) + P(X \geq 7)$

$= {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{10-6} + \frac{176}{1024}$

$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \cdot \frac{1}{2^{10}} + \frac{176}{1024}$

$= \frac{210}{1024} + \frac{176}{1024}$

$= \frac{386}{1024}$

$= 0.37695$

(iii) P(atleast one head) $= P(X \geq 1)$

$= 1 - P(X=0)$

$= 1 - {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10-0}$

$= 1 - 1\left(\frac{1}{2^{10}}\right)$

$= 1 - \frac{1}{1024}$

$= \frac{1023}{1024}$

* Two dice are thrown five times. Find the probability of getting 7 as sum (i) atleast once (ii) exactly two times (iii) $P(1 < X \leq 5)$

Sol) p = the probability of getting a sum 7 in a single throw of a pair of dice

$$= \frac{6}{36} = \frac{1}{6}$$

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Number of trials, $n = 5$

(i) $P(\text{at least once}) = P(X \geq 1) = 1 - P(X=0)$

$$= 1 - {}^5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{5-0}$$

$$= 1 - 1 \cdot \frac{5^5}{6^5}$$

$$= 1 - \left(\frac{5}{6}\right)^5$$

(ii) $P(\text{exactly two times}) = P(X=2)$

$$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{5-2}$$

$$= \frac{5 \times \cancel{4}^2}{\cancel{2} \times 1} \cdot \frac{5^3}{6^5} = \frac{2}{6} \cdot \frac{5^4}{6^4}$$

$$= \frac{1}{3}\left(\frac{5}{6}\right)^4$$

(iii) $P(1 < X < 5) = P(X=2) + P(X=3) + P(X=4)$

$$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 + {}^5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + {}^5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$$

$$= \frac{5 \times \cancel{4}^2}{\cancel{2} \times 1} \cdot \frac{5^3}{6^5} + \frac{5 \times \cancel{4}^2}{\cancel{2} \times 1} \cdot \frac{5^2}{6^5} + 5 \cdot \frac{5}{6^5}$$

$$= \frac{1}{6^5}\left(5^4 \times 2 + 5^3 \times 2 + 5^2\right)$$

$$= \frac{5^2}{6^5}(50+10+1)$$

$$= 61 \cdot \frac{5^2}{6^5}$$

* Determine the Binomial distribution for which the mean is 4 and variance 3.

Soln Given mean $\mu = np = 4$ — (1)

variance $\sigma^2 = npq = 3$ — (2)

$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$

$\therefore p = 1-q = 1-\frac{3}{4} = \frac{1}{4}$

$\therefore (1) \Rightarrow n \cdot (1/4) = 4 \Rightarrow n = 16$

Hence the Binomial distribution is

$$P(X=x) = p(x) = \begin{cases} 16_{C_x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x} & x = 0,1,2 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

* Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls (c) either 2 or 3 boys (d) atleast one boy. Assume equal probabilities for boys and girls.

Hint: let the no. of boys in each family $= x$

$p = q = \frac{1}{2}$ $n = 5$

(a) $P(X=3) = \frac{5}{16}$ (250 families)

(b) $P(X=0) = \frac{1}{32}$ (25 families)

(c) $P(X=2) + P(X=3) = \frac{5}{8}$ (500 families)

(d) $P(X \geq 1) = \frac{31}{32}$ (775 families)

(*) The Mean & Variance of a binomial distribution are 2 and $8/5$. Find 'n'. Ans: (n=10)

(*) Fit a binomial distribution to the following data

x	0	1	2	3	4
f	30	144	342	287	164

Hint: n = no of trial = 4 $N = \sum f_i$

$\mu = np = \frac{2345}{975}$ $p = 0.6$

x
P(x)
Expected frequency ($N \times P(x)$)

(*) The mean of Binomial distribution is 3 and the variance is $\frac{9}{4}$. Find (i) The value of n (ii). $P(X \geq 7)$ (iii) $P(1 \leq X \leq 6)$

Ans: (i) n=12, (ii) 0.0142. (iii) 0.82

Geometric Distribution:

A random Variable X is said to follow Geometric distribution if it assumes only non-negative values and its probability distribution is given by

$$P(X=x) = p(x) = q^{x-1} p, \quad x=1,2,3,\ldots$$
$$q = 1-p$$

where p = Probability of success of an outcome

$q = 1-p$

x = No. of trials required to get (first) success

(OR) No of failures preceding the first success

(*) Mean of the Geometric distribution $= \frac{1}{p}$

<u>proof</u> $E(X) = \sum_{x=1}^{\infty} x\, p(X=x)$

$$= \sum_{x=1}^{\infty} x q^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x q^{x-1}$$

$$= p(1+2q+3q^2+4q^3+\cdots)$$

$$= p(1-q)^{-2} = \frac{p}{(1-q)^2} = \frac{p}{p^2}.$$

$$\boxed{\mu = E(X) = \frac{1}{p}}$$

(*) <u>Variance</u>: $V(X) = E(X-\bar{X})^2 = E(X^2) - \mu^2$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 p(X=x)$$

$$= \sum_{x=1}^{\infty} x^2 q^{x-1} p \qquad \boxed{\cancel{p \sum_{x=1}^{\infty} x^2 q^{x-1}}}$$

$$= \sum_{x=1}^{\infty} (x^2 - x + x) q^{x-1} p$$

$$= \sum_{x=1}^{\infty} [x(x-1) + x] q^{x-1} p$$

$$= \sum_{x=1}^{\infty} x(x-1) q^{x-1} p + \sum_{x=1}^{\infty} x q^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x(x-1) q^{x-1} + \frac{1}{p} \qquad \left(\because \text{mean } E(X) = \sum_{x=1}^{\infty} x q^{x-1} p = \frac{1}{p}\right)$$

$$= p(2q + 6q^2 + 12q^3 + \cdots) + \frac{1}{p}$$

$$= 2pq(1 + 3q + 6q^2 + \cdots) + \frac{1}{p}$$

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$$E(x^2) = 2pq(1-q)^{-3} + \frac{1}{p}$$

$$= 2pq(p)^{-3} + \frac{1}{p}$$

$$E(x^2) = \frac{2q}{p^2} + \frac{1}{p}$$

Now $V(x) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2}$

$$= \frac{q+q+p-1}{p^2} = \frac{q+1-1}{p^2} \quad (\because p+q=1)$$

$$\boxed{\sigma^2 = V(x) = \frac{q}{p^2}}$$

$\therefore$ Standard Deviation $\boxed{(\sigma) = \frac{\sqrt{q}}{p}}$

(*) Let x be the number of days until the closing price of a certain stock shows a gain over the previous day's closing price. Assume that x is a geometric random variable with $p=0.5$, the probability of a gain in the price from one day to the next day.

(i) Find the mean and standard deviation of x.

(ii) what is the probability that more than 2 days pass before a gain in price from one day to the next is observed?

Sol: Given $p = 0.5$; $q = 1 - p = 0.5$

we have $p(x) = q^{x-1} p$

(i) Mean $= \frac{1}{p} = \frac{1}{0.5} = 2$

Variance $= \frac{q}{p^2} = \frac{0.5}{(0.5)^2} = \frac{1}{0.5} = 2$

Standard deviation $= \sqrt{\text{variance}} = \sqrt{2} = 1.4142$

(ii) Required Probability $= p(x > 2)$

$= 1 - p(x \leq 2)$

$= 1 - (p(1) + p(2))$

$= 1 - (q^{1-1} p + q^{2-1} p)$

$= 1 - (p + qp)$

$= 1 - (0.5 + (0.5)(0.5))$

$= 1 - 0.75$

$= \underline{0.25}$

---

* ~~A die~~ A die is tossed until 6 appears. Find the probability that it must be cast more than '5' times.

Sol: $p =$ probability of getting $6 = \frac{1}{6}$

$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

If X is the no. of tosses required for the first success, then $P(X = x) = q^{x-1} p$ for $x = 1, 2, 3, \ldots$

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$\therefore$ Required Probability $= P(X>5)$

$= 1 - P(X \le 5)$

$= 1 - (P(1) + P(2) + P(3) + P(4) + P(5))$

$= 1 - [q^{1-1}p + q^{2-1}p + q^{3-1}p + q^{4-1}p + q^{5-1}p]$

$= 1 - p(1 + q + q^2 + q^3 + q^4)$

$= 1 - \frac{1}{6}\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4\right)$

$= 1 - 0.59$

$= \underline{0.41}$

(*) Find the mean and variance of the geometric distribution given by $P(x) = 2^{-x}$ (or) $\left(\frac{1}{2}\right)^x$ $x = 1, 2, 3, \ldots$

Ans:- $\boxed{\mu = 2}$, $\boxed{\sigma^2 = 2}$

(*) If probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6^{th} attempt

Ans:- $\boxed{P(X=6) = \frac{1}{64}}$

Poisson Distribution: Poisson distribution is can be derived as a limiting case of binomial distribution under the conditions.

(i) the probability of occurrence of the event (p) is very small i.e. $p \to 0$

(ii) the no. of independent trials is very very large i.e. $n \to \infty$

(iii) Mean np is finite quantity and it is called Parameters of poisson distribution.

Definition: A Discrete Random variable 'x' is said to follows poisson distribution, if it assumes non-ve values

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda} \cdot \lambda^x}{x!} & x = 0,1,2,3 \ldots \\ 0 & \text{otherwise.} \end{cases}$$

ex: No. of printing mistakes per page in a large text.

Mean:

$$\mu = E(X) = \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \frac{\lambda}{0!} + e^{-\lambda} \frac{\lambda^2}{1!} + e^{-\lambda} \frac{\lambda^3}{2!} + e^{-\lambda} \frac{\lambda^4}{3!} + \ldots$$

$$= e^{-\lambda} \lambda \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$\boxed{\mu = E(x) = \lambda}$$

Variance: $\sigma^2 = V(x) = E(x^2) - \mu^2$

$$= \sum_{x=0}^{\infty} x^2 p(x) - \mu^2$$

$$= \sum_{x=0}^{\infty} (x^2 - x + x)\, p(x) - \mu^2$$

$$= \sum_{x=0}^{\infty} x(x-1)\, p(x) + \sum_{x=0}^{\infty} x\, p(x) - \mu^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + \mu - \mu^2$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} + \lambda - \lambda^2$$

$$= \left[e^{-\lambda} \frac{\lambda^2}{0!} + e^{-\lambda} \frac{\lambda^3}{1!} + e^{-\lambda} \frac{\lambda^4}{2!} + e^{-\lambda} \frac{\lambda^5}{3!} + \cdots \right] + \lambda - \lambda^2$$

$$= e^{-\lambda} \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right] + \lambda - \lambda^2$$

$$= e^{-\lambda} \lambda^2 \cdot e^{\lambda} + \lambda - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\sigma^2 = V(x) = \lambda}$$

$\therefore$ Standard deviation $\boxed{\sigma = \sqrt{\lambda}}$

Mode of Poisson distribution

$$\text{Mode} = \begin{cases} \text{Integral Part of } \lambda & \text{if } \lambda \text{ is not an integer} \\ \lambda \ \& \ \lambda - 1 & \text{if } \lambda \text{ is integer} \end{cases}$$

Recurrence Relation :-

$$P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \qquad P(x+1) = \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!}$$

$$\frac{P(x+1)}{P(x)} = \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!} \cdot \frac{x!}{e^{-\lambda} \lambda^{x}}$$

$$\Rightarrow \frac{P(x+1)}{P(x)} = \frac{\lambda}{x+1}$$

$$\Rightarrow \boxed{P(x+1) = \frac{\lambda}{(x+1)} \cdot P(x)}$$

(*) If the Probability that an individual Suffers a bad reaction from a certain injection is 0.001 find The Probability that out of 2000 individuals

(i) exactly 3 (ii) more than '2' individuals

(iii) none (iv) more than 1 individuals Suffer a bad reaction.

Soln Given

$p = 0.001$

$n = 2000$

∴ mean $(\mu) = \lambda = np = 2000(0.001)$

$= 2$

let x = No. of persons Suffer bad reaction

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!} = \frac{e^{-2} 2^{x}}{x!} \quad \text{where } x = 0, 1, 2 \ldots$$

(i) $P(X=3) = \frac{e^{-2}2^3}{3!} = \frac{1}{e^2}\cdot\frac{8}{6} = \frac{4}{3e^2}$

(ii) $P(X>2) = 1-P(X\le 2)$

$= 1-[P(X=0)+P(X=1)+P(X=2)]$

$= 1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}\right]$

$= 1-\frac{1}{e^2}[1+2+2]$

$P(X>2) = 1-\frac{5}{e^2}$

(iii) $P(X=0) = \frac{e^{-2}\cdot 2^0}{0!} = \frac{1}{e^2}$

(iv) $P(X>1) = 1-P(X\le 1) = 1-[P(X=0)+P(X=1)]$

$= 1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}\right]$

$= 1-e^{-2}[1+2]$

$= 1-\frac{3}{e^2}$

Average no. of accidents on any day on a national high way is 1.8. Determine the probability that the no. of accidents are (i) at least one (ii) at most one.

Sol) Given $\lambda = 1.8$

(i) $P(X\ge 1) = 1-P(X=0) = 1-\frac{\lambda^0 e^{-\lambda}}{0!} = 1-e^{-1.8}$

(ii) $P(X \leq 1) = P(X=0) + P(X=1) = \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!}$

$= e^{-\lambda} + e^{-\lambda}(1.8)$

$= e^{-\lambda}(1+1.8)$

$= e^{-1.8}(2.8)$

(*) If the Random Variable has Poisson distribution such that $P(1) = P(2)$ Find (i) Mean (ii) $P(4)$ (iii) $P(X \geq 1)$ (iv) $P(1 < X < 4)$

Sol) we have $P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$

Given $P(1) = P(2)$

$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \Rightarrow \lambda = \frac{\lambda^2}{2} \Rightarrow \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda(\lambda - 2) = 0$

$\Rightarrow \lambda = 0, 2$

(i) Mean $\lambda = 2$

(ii) $P(X=4) = \frac{e^{-\lambda}\lambda^4}{4!} = \frac{e^{-2}2^4}{4!} = \frac{16}{4\times3\times2} \times \frac{1}{e^2}$

$P(X=4) = \frac{2}{3e^2}$

(iii) $P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - \frac{e^{-2}2^0}{0!}$

$= 1 - e^{-2}$

$= 1 - \frac{1}{e^2}$

(iv) $P(1 < X < 4) = P(X=2) + P(X=3)$

$= \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!}$

$= e^{-2}\left[2 + \frac{4}{3}\right]$

$= \frac{10}{3e^2}$

(4) If a Poisson distribution is such that ~~P(X=1)~~ $P(X=1)\cdot\frac{3}{2} = P(X=3)$. Find (i) $P(X \geq 1)$ (ii) $P(X \leq 3)$

Sol^n) We have $P(X) = \frac{\lambda^x e^{-\lambda}}{x!}$

Given $P(X=1)\cdot\frac{3}{2} = P(X=3)$

$$\frac{\lambda e^{-\lambda}}{1!}\cdot\frac{3}{2} = \frac{\lambda^3 e^{-\lambda}}{3!}$$

$$\Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3$$

λ cannot be negative. Hence $\lambda = 3$

(i) $P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - \frac{3^0 e^{-3}}{0!}$

$= 1 - e^{-3}$

$= 0.9502$

(ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!}$

$= e^{-3}\left[1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!}\right]$

$= e^{-3}(13)$

$= 0.6472$

(*) The average no. of phone calls/minute coming into a switch board between 2 p.m and 4 p.m is 2.5. Determine the probability that during one particular minute there will be (i) 4 or fewer (ii) more than 6 calls.

Sol (***)

Mean $(\mu) = \lambda = 2.5$

For the Poisson distribution $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$

(i) Probability that the 4 or fewer phone calls/minute is

$$P(x \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \frac{\lambda^4 e^{-\lambda}}{4!}$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!}\right]$$

$$\cancel{= e^{-0.25}\left[1 + 0.2 + \frac{(0.25)^2}{2!} + \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!}\right]}$$

$$\cancel{= e^{+0.25}}$$

$$= e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$$

$$= e^{-2.5}(10.856)$$

$$= \underline{0.8911}$$

(ii) Probability that more than 6 phone calls/minute is $P(X > 6) = 1 - P(X \le 6)$

$$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6)]$$

$$= 1 - \left[\frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \frac{\lambda^4 e^{-\lambda}}{4!} + \frac{\lambda^5 e^{-\lambda}}{5!} + \frac{\lambda^6 e^{-\lambda}}{6!}\right]$$

$$= 1 - e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \frac{\lambda^5}{5!} + \frac{\lambda^6}{6!}\right]$$

$$= 1 - e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!} + \frac{(2.5)^5}{5!} + \frac{(2.5)^6}{6!}\right]$$

$$= 1 - e^{-2.5}\left[12.0097\right]$$

$$= 1 - 0.9858$$

$$= 0.0141$$

(*) If X is poisson variate such that $P(X=0) = P(X=1)$, find $P(0)$ and using recurrence relation, find the probability at $x = 1, 2, 3, 4$ and 5.

Soln Given $P(X=0) = P(X=1)$

$$\frac{\lambda^0 e^{-\lambda}}{0!} = \frac{\lambda^1 e^{-\lambda}}{1!} \Rightarrow \lambda = 1$$

Now $P(0) = P(X=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = e^{-1} = 0.368$

$P(1) = P(X=1) = \frac{\lambda^1 e^{-\lambda}}{1!} = \lambda e^{-\lambda} = 1 \cdot e^{-1} = 0.368$

$P(2) = P(X=2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{1^2 \cdot e^{-1}}{2!} = \frac{0.368}{2} = 0.184$

$P(3) = P(X=3) = \frac{\lambda^3 e^{-\lambda}}{3!} = \frac{1^3 e^{-1}}{3!} = \frac{0.368}{6} = 0.0613$

$P(4) = P(X=4) = \frac{\lambda^4 e^{-\lambda}}{4!} = \frac{1^4 e^{-1}}{4!} = \frac{0.368}{24} = 0.0153$

$P(5) = P(X=5) = \frac{\lambda^5 e^{-\lambda}}{5!} = \frac{1^5 e^{-1}}{5!} = \frac{0.368}{120} = 0.00306$

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(*) Suppose 2% of the people on the average are left handed. Find (i) the probability of finding 3 or more left handed (ii) the probability of finding ≤ 1 left handed.

Ans: given $\lambda = 2$
$P(X \geq 3) = 0.3233$
$P(X \leq 1) = 3e^{-2}$

(*) If a bank receives on an average 6 bad cheques per day what are the probability that it will receive

(i) 4 bad cheques on any given day

(ii) 10 bad cheques on any two consecutive days

Ans: (i) $\lambda = 6$ $P(X=4) = 0.1339$
(ii) consecutive day $\lambda = 2 \times 6 = 12$
$P(X=10) = 0.105$

(*) Fit a Poisson distribution calculates theoretical frequencies for the following data.

x	0	1	2	3	4
f	109	65	22	3	1

Soln To fit a Poisson distribution find the parameter λ of the distribution from the given data

λ = Arithmatic mean = $\frac{\sum f_i x_i}{\sum f_i} = \frac{122}{200} = 0.61$

Total frequency $N = \sum f_i = 200$

Thus the poisson distribution that fit to the given data $P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$\therefore \; f(x) = N \cdot P(x)$

$$f(0) = N \cdot P(0) = 200 \times \frac{(0.61)^0 e^{-0.61}}{0!} = 200 \times 0.543 = 108.6$$

$$f(1) = N \cdot P(1) = 200 \times \frac{(0.61)^1 e^{-0.61}}{1!} = 200 \times 0.331 = 66.3$$

$$f(2) = N \cdot P(2) = 200 \times \frac{(0.61)^2 e^{-0.61}}{2!} = 200 \times 0.1 = 20.2$$

$$f(3) = N \cdot P(3) = 200 \times \frac{(0.61)^3 e^{-0.61}}{3!} = 200 \times 0.02 = 4.1$$

$$f(4) = N \cdot P(4) = 200 \times \frac{(0.61)^4 e^{-0.61}}{4!} = 200 \times 0.0022 = 0.64$$

Hence theoretical frequencies are

108.6, 66.3, 20.2, 4.1, 0.64