UNIT-V

Stochastic processes and Markov Chains

Short Answer questions

(1) Define Stochastic process, Markov Chain & Transition Matrix

A:- Stochastic process:- The family of all the random variables at particular time 't' is known as Stochastic process.

Ex:- A Queuing system, turblent fluid flow.

Markov chain:- A Stochastic process is said to be Markov process or chain if it satifies markov property. i.e if occurance future state is depends on present state.

i.e $P\{X_{n+1}=x_{n+1} \mid X_n = x_n\}$.

Transition matrix:- The probability of future state is depends on present state is known as Transition matrix.

i.e $P\{X_{n+1}=j \mid X_n=i\} = P_{ij}$

(2) If the transition probability matrix is $\begin{bmatrix} 0 & 0.2 & x \\ x & 0.1 & y \\ 0.1 & 0.2 & z \end{bmatrix}$ find x, y, z

A:- The matrix is said to be Transition probability

Matrix if it satifies following conditions

(a) It is a square matrix with non-negative elements.

(b) Sum of each row is equal to '1'

$\therefore \rightarrow 0+0.2+x=1 \Rightarrow x=0.8$

$\rightarrow x+0.1+y=1$

$0.8+0.1+y=1$

$y=0.1$

$\rightarrow 0.1+0.2+z=1$

$z=0.7$

$\therefore x=0.8,\ y=0.1,\ z=0.7$

(3) Define Regular Stochastic process matrix with Example.

Ans: A Matrix is said to be regular stochastic if some powers of 'p' becomes non-zero elements of matrix.

Ex:- $A=\begin{bmatrix} 0 & 1 \\ 1/4 & 3/4 \end{bmatrix}$, $A^2=\begin{bmatrix} \frac{1}{4} & 3/4 \\ 3/16 & 13/16 \end{bmatrix}$

$\therefore$ A is regular Matrix

(4) Find the equilibrium vector of $\begin{bmatrix} 1/4 & 3/4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Sol:- Given that $P = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$

Let X be the probability vector. We want to find X such that $XP = X$ & $\boxed{x_1 + x_2 = 1}$ —(3)

$$[x_1 \;\; x_2] \begin{bmatrix} \frac{1}{4} & 3/4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [x_1 \;\; x_2]$$

$$\frac{x_1}{4} + \frac{x_2}{2} = x_1 \Rightarrow -\frac{3}{4}x_1 + \frac{x_2}{2} = 0 \quad \text{—— (1)}$$

$$\frac{3x_1}{4} + \frac{x_2}{2} = x_2 \qquad \frac{3x_1}{4} - \frac{x_2}{2} = 0 \Rightarrow -\frac{3x_1}{4} + \frac{x_2}{2} = 0 \quad \text{—(2)}$$

eqn's (1) & (2) same.

$$\therefore \; -\frac{3x_1}{4} + \frac{x_2}{2} = 0 \Rightarrow -3x_1 + \cancel{2x_2} = 0$$
$$x_1 + x_2 = 1 \qquad 2x_1 + \cancel{2x_2} = 2$$

$$-5x_1 = -2$$
$$x_1 = 2/5$$

$$\therefore \; \frac{2}{5} + x_2 = 1$$
$$x_2 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$\therefore \; [x_1 \;\; x_2] = [2/5, 3/5]$$

(5)

(5) Recurrent State :- The state is said to be recurrent, if any time that we leave that state we will return to that state in the future with probability one.

Long Answer questions

(1) Define classification of states.

Ans:- **Classification of states:-**

① **Absorbing state:-** If $P_{ii}=1$ then 'i' is said to be absorbing state.

Ex:- $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

Here absorbing states are 1, 3, because $P_{11}=1$, $P_{33}=1$

② **Transient state:-** If $P_{ii}<1$ then 'i' is said to be transient state

Ex:- $P = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

∴ Here transient states are 1, 3. because $P_{11}<1$, $P_{33}<1$

③ **Return state:-** If $P_{ii}^{(n)}>0$ for some 'n' then 'i' is called return state.

∴ Ex:- $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

∴ Here Return states are 1, 2, 3

because $P_{11}>0$, $P_{22}>0$, $P_{33}>0$

Irreducible State:- If $P_{ij}^{(n)} > 0$ for some 'n' then it is irreducible State

$$P = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$P_{12}^{(1)} > 0$, $P_{21}^{(1)} > 0$, $P_{22}^{(1)} > 0$ But $P_{11} \neq 0$

$$P^2 = \begin{bmatrix} 1/3 & 2/3 \\ 2/9 & 7/9 \end{bmatrix}$$

$\therefore \quad P_{11} > 0$

$\therefore$ It is irreduciable State

Periodic State:- The Periodic of return state is defined as the GCD of all n Such that

$P_{ij}^{(n)} > 0$, $d_i = GCD\{n, P_{ii}^{(n)} > 0$

if $d_i > 1$ then State 'i' is Called Periodic State

Ex:- $P = \begin{bmatrix} 0 & 1 \\ 1/3 & 2/3 \end{bmatrix}$, $P^2 = \begin{bmatrix} 1/3 & 2/3 \\ 2/9 & 7/9 \end{bmatrix}$

$P_{11}^{(2)} > 0 \quad \therefore \; d_i = 1$

$P_{22}^{(1)} > 0$, $P_{22}^{(2)} > 0$

$\therefore \; d_i = GCD\{1, 2\} = 1$

$\therefore \; i = 1, 2$ are apeeriodic States.

(2) The transition probability matrix of a Markov Chain is given by $\begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$. Is this matrix irreducible?

Sol:-

Given that $P = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

Here $P_{11}^{(1)} > 0$, $P_{12}^{(1)} > 0$, $P_{21}^{(1)} > 0$, $P_{22}^{(1)} > 0$

$P_{23}^{(1)} > 0$, $P_{32}^{(1)} > 0$, $P_{33}^{(1)} > 0$

But $P_{13} = 0$, $P_{31} = 0$

Then $P^2 = \begin{bmatrix} 0.16 & 0.49 & 0.35 \\ 0.07 & 0.35 & 0.6 \\ 0.02 & 0.24 & 0.74 \end{bmatrix}$

Here $P_{13}^{(2)} > 0$, $P_{31}^{(2)} > 0$

. So it is irreducible

(3) A fair die tossed repeatedly. If X_n denotes the maximum of the number occuring in the first n tosses, find the transition probability matrix. Find also P^2.

Sol:-

State Space = $\{1, 2, 3, 4, 5, 6\}$

Let X_n = max of the number occuring in the first n trials = 3 (say)

Then $x_{n+1} = 3$, if $(n+1)^{th}$ trial results is 1, 2, or 3

$= 4$, if $(n+1)^{th}$ trial results is 4

$= 5$, if $(n+1)^{th}$ trial results is 5

$= 6$, if $(n+1)^{th}$ trial results is 6.

$\therefore \; P\{X_{n+1} = 3 / x_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 3/6$

$P\{X_{n+1} = i / x_n = 3) = 1/6$ when $i = 4, 5, 6$

$\therefore$ The tpm of Chain is

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 3/6 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 4/6 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

(4) If the transition probability matrix is $\begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$

and the initial probabilities are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ then find the probabilities after three periods. (b) Equilibrium Vector

Sol: G/T

$$P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

After one period $P_1 = P_0 . P$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.42 & 0.17 & 0.42 \end{bmatrix}$$

After two periods. $P_2 = P_1 . P$

$$= \begin{bmatrix} 0.42 & 0.17 & 0.45 \end{bmatrix} \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.21 & 0.4 \end{bmatrix}$$

After three Periods $P_3 = P_2 . P$

$$= \begin{bmatrix} 0.4 & 0.21 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.405 & 0.2 & 0.405 \end{bmatrix}$$

Equilibrium vector :-

let $X = [x_1 \; x_2 \; x_3]$ then $X = XP$ and

$x_1 + x_2 + x_3 = 1$

$$\therefore [x_1 \; x_2 \; x_3] = [x_1 \; x_2 \; x_3] \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$x_1 = 0.5x_1 + 0.5x_2 + 0.25x_3$$
$$x_2 = 0.25x_1 + 0.25x_3$$
$$x_3 = 0.25x_1 + 0.5x_2 + 0.5x_3$$

$$\Rightarrow \quad -0.5x_1 + 0.5x_2 + 0.25x_3 = 0 \quad \text{--- (1)}$$
$$0.25x_1 - x_2 + 0.25x_3 = 0 \quad \text{--- (2)}$$
$$0.25x_1 + 0.5x_2 - 0.5x_3 = 0 \quad \text{--- (3)}$$
$$\& \quad x_1 + x_2 + x_3 = 1 \quad \text{--- (4)}$$

Sub $x_1 = 1 - x_2 - x_3$ in (1) & (2)

$$\therefore -0.5(1 - x_2 - x_3) + 0.5x_2 + 0.25x_3 = 0$$

(1) $\Rightarrow$
$$-0.5 + x_2 + 0.75x_3 = 0 \Rightarrow x_2 + 0.75x_3 = 0.5 \quad \text{--- (5)}$$

(2) $\Rightarrow$
$$0.25(1 - x_2 - x_3) - x_2 + 0.25x_3 = 0$$

$$0.25 - 1.25x_2 \cancel{+0.25x_3} + \cancel{0.25x_3} = 0$$

$$\therefore \boxed{x_2 = \frac{0.25}{1.25} = \frac{1}{5}}$$

(5) $\Rightarrow$ $\frac{1}{5} + 0.75x_3 = 0.5 \Rightarrow \boxed{x_3 = 2/5}$

(1) $\Rightarrow$ $-0.5x_1 + \frac{0.5}{5} + 0.25(2/5) = 0$

$x_1 = 2/5$

(5) Suppose there are two market products of brands A and B respectually. let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given as follows

	A	B
A	0.9	0.1
B	0.5	0.5

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady state.

Sol:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

The steady state vector is

$X = XP$ where $x_1 + x_2 = 1$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

$x_1 = 0.9x_1 + 0.5x_2$

$x_2 = 0.1x_1 + 0.5x_2$

$\Rightarrow$ $\left.\begin{array}{l} -0.1x_1 + 0.5x_2 = 0 \\ 0.1x_1 - 0.5x_2 = 0 \end{array}\right\}$ These two eqn's are same

$$\therefore\ 0.1x_1 - 0.5x_2 = 0 \Rightarrow x_1 = \frac{0.5}{0.1}x_2$$

$$x_1 + x_2 = 1$$

$$\frac{0.5}{0.1}x_2 + x_2 = 1$$

$$6x_2 = 1$$

$$x_2 = 1/6$$

$$x_1 = 5/6$$

$$\therefore \text{ Vector is } [x_1 \ x_2] = \left[\frac{5}{6} \ \ 1/6\right].$$

(7) The tPm of a markov chain $\{x_n\}$, $n = 1,2,3 \ldots$ having 3 states 1, 2 & 3 $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ & the initial distribution is $P_0 = (0.7 \ 0.2 \ 0.1)$ Find

(1) $P(x_2 = 3)$ (2) $P\{x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2\}$.

Sol:- Given that $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.39 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

① $P(x_2=3) = \sum_{i=1}^{3} P\{x_2=3 / x_0=i\} \cdot P[x_0=i]$

$= P\{x_2=3 / x_0=1] \cdot P(x_0=1) + P\{x_2=3/x_0=2\}$

$\cdot P\{x_0=2\} + P\{x_2=3 / x_0=3\} \cdot P\{x_0=3\}$

$= P_{13}^{(2)} \cdot (0.7] + P_{23}^{(2)} (0.2) + P_{33}^{(2)} (0.1)$

$= (0.26)(0.7) + (0.34)(0.2) + 0.29(0.1)$

$= 0.279$

② $P\{x_1=3, x_0=2\} = P\{x_1=3 / x_0=2\} \cdot P\{x_0=2\}$

$= P_{23}^{(1)} \times 0.2$

$= 0.2 \times 0.2 = 0.04$

$P\{x_2=3, x_1=3, x_0=2\} = P\{x_2=3 / x_1=3, x_0=2\}$

$\times P\{x_1=3, x_0=2\}$

$= P\{x_2=3 / x_1=3\} \times P\{x_1=3, x_0=2\}$

$= P_{33}^{(1)} \cdot \times 0.04$

$= 0.3 \times 0.04$

$= 0.012$

$P\{x_3=2, x_2=3, x_1=3, x_0=2\}$

$= P\{x_3=2 / x_2=3, x_1=3, x_0=2\}$

$\times P\{x_2=3, x_1=3, x_0=2\}$

$= P\{x_3=2 / x_2=3\} \times 0.012$

$= P_{32}^{(1)} \times 0.012 = 0.4 \times 0.012$

$= 0.0048$