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TECHNICAL REPORT
ON
SAFE- AND ECO- DRIVING CONTROL FOR CONNECTED AND AUTOMATED ELECTRIC VEHICLES USING ANALYTICAL STATE-CONSTRAINED OPTIMAL SOLUTION

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Motivation

The motivation behind this work is our keen interest to work on the Autonomous and Intelligent systems. Now-a-days, Speed advisory Systems and Autonomous Cruise Control are grabbing the attention of the leading Automobile manufacturers. Although, there are many proposed systems on these topics, still there is a significant gap between proposals and practical implementation when safety and energy efficiency is concerned. Hence, we wanted to work on an autonomous driving system that is very energy efficient while guaranteeing safety of the user.

Abstract

Speed advisory systems have been proposed for connected vehicles to minimize energy consumption over a planned route. However, for their practical diffusion, these systems must adequately consider the presence of preceding vehicles. In this report, we have discussed and highlighted the details of the paper Jihun Han, Antonio Sciarretta, Luis Leon Ojeda, Giovanni De Nunzio and Laurent Thibault, *"safe- and eco-driving control for connected and automated Electrical Vehicles Using Analytical and State-Constrained Optimal Solution"*, June 2018. In this paper, a Safe- and Eco-driving control system is proposed for connected and automated vehicles to accelerate or decelerate optimally while guaranteeing vehicle safety constraints. Their work mainly defines the minimum intervehicle distance and maximum road speed limit as the state constraints and formulates an optimal control problem to minimize the energy consumption. An analytical state-constrained solution is also derived for real-time use. This technical report details on all the ideas that forms the base for the paper and eases the effort of the reader in understanding their work. The key mathematical formulations are also emphasized and derived wherever necessary. The report finally discusses the feasible range of terminal conditions proposed in the paper that are adjusted to guarantee the existence of the analytical solution for various driving scenarios.

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Chapter 1

Introduction

1.1 Connected and automated vehicles

Development of communication, automation technologies and global navigation systems lead to Connected and automated vehicles (CAVs) which improve traffic flow stability and throughput. CAV's are vehicles that use any of the different communication technologies to communicate with the driver, other cars on the road (vehicle-to-vehicle [V2V]), roadside infrastructure (vehicle-to-infrastructure [V2I]), and the "Cloud" [V2C]. This technology can be used to not only improve vehicle safety, but also to improve vehicle efficiency and commute times

1.2 Adaptive Cruise Control

Adaptive cruise control (ACC) is an automated function to track a desired speed by maintaining a prescribed inter-vehicle distance. These systems promise more vehicle safety and provide a great scope to innovate more autonomous and safe driving practice.

1.3 Objective of the work

The main objective of the work presented in the paper is to save energy and maximize energy efficiency by adding optimization based Eco-driving functions to ACC in CAV's. In this work, speed/distance tracking and energy efficiency are considered in single cost function, which is to be minimized in model predictive control (MPC) framework by using Pontryagin's minimum principle (PMP). Eco-driving techniques can also be employed for speed advisory systems (SAS) to minimize energy consumption cost function.

1.4 Key Ideas

The presence of preceding vehicle as state constraint is considered as an extension for existing Adaptive cruise Control (vehicle safety) and speed advisory systems (energy efficiency). Here, a closed-form state-constrained optimal control solution is investigated and a computationally efficient MPC methodology is used to optimize the energy consumption of a CAV driving in a traffic stream. The solution is formulated such that it ensures the feasibility of an analytical solution while avoiding collision and respecting the speed limitations.

Chapter 2

Safe- and Eco- Driving Control Problem

2.1 System Model

Here the car of mass 'm' is moving on road with inclination of $\alpha(s)$ with the ground. Total 5 forces act upon the car. One of them is traction force which drives the car and remaining forces retard the motion of car.

System Model is given by

$$\begin{aligned} \dot{s} &= v, & (2.1) \\ m\dot{v} &= F_t - (F_a + F_r + F_q) - F_b, \\ &= F_t - (\rho_a c_d A_f v^2/2 - c_r mg - mg\sin(\alpha(s))) - F_b & (2.2) \end{aligned}$$

Following are the forces where F_t, F_a, F_r, F_g,and Fb are the traction force at the wheels, aerodynamic drag resistance, rolling resistance, hill climbing resistance, mechanical brake force, respectively;
s vehicle's position and v is the speed;
m is the vehicle mass;
ρ_a is the external air density;
A_f is the vehicle frontal area;
c_d is the aerodynamic drag coefficient;
c_r is the rolling resistance coefficient;
g is the gravity acceleration;
α is the road slope as a function of the position;

Since the car used here is electric vehicle, the electric motor connected to transmission drives the car. It is assumed that there is no slip at the wheels.The traction force is given as follows [4]

$$F_t = \left(T_m \eta_t^{sign(T_m)} R_t\right)/r, \tag{2.3}$$

where T_m is the motor torque;
R_t is the transmission ratio;
η_t is the transmission efficiency;
r is the wheel radius

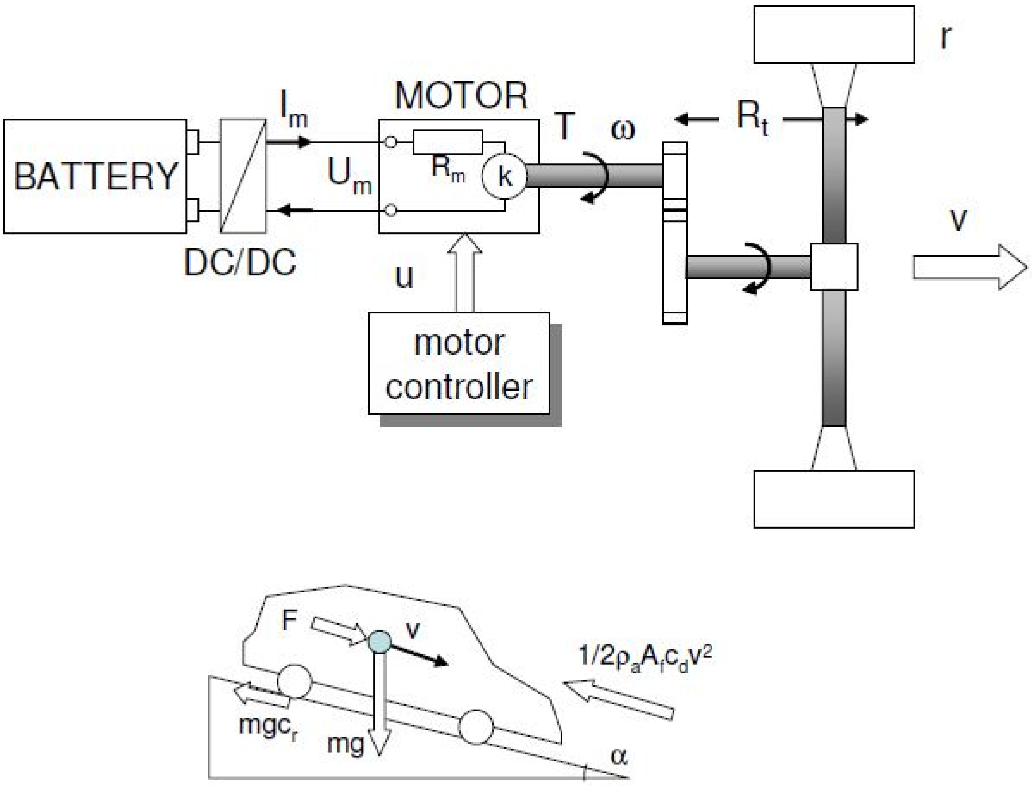


Figure 2.1: System Model

$$
\begin{aligned}
V_a &= i_a R_a + k\omega_m, && (2.4)\\
i_a &= T_m/k, && (2.5)\\
Then, \quad P_m &= V_a i_a, &&\\
&= \omega_m T_m + (R_a/k^2)T_m^2, &&\\
&= b_1 v T_m + b_2 T_m^2 && (2.6)
\end{aligned}
$$

where $b_1 = R_t/r, b_2 = R_a/k^2, V_a, i_a,$ and R_a are some effective voltage, current and resistance respectively. k is the motor torque constant. The rotational motor speed is $\omega_m = R_t v/r$. If the electro-chemical power drained from or supplied to the battery system is Pb , then the energy consumption of an electric vehicle is computed by

$$E_f = \int_0^{t_f} P_b dt \tag{2.7}$$

In eco-driving studies for electric vehicles, the electro-chemical conversion efficiency in the battery can be simplified to a constant value or neglected. Here, P_b is set to P_m.

2.2 Problem Statement

Here the main objective of problem is to minimize the energy consumption of the host electric vehicle ensuring safety of vehicle. Hence two state-inequality constraints are formulated in the aspects of speed and position respectively as follows:

$$
\begin{aligned}
h_1(t) &= v(t) - v_{max} \leq 0, && (2.8)\\
h_2(t) &= s(t) - (s_p(t) - \delta_s) \leq 0, && (2.9)
\end{aligned}
$$

where s_p is the position of the preceding vehicle.

In the first constraint, speed cannot exceed the maximum speed limit (v_{max}).On the other hand, the second constraint is set by the requirement that the inter-vehicle distance is always larger than a minimum safe gap (δ_s),

Our Optimal Control problem has pure state inequality constrains. These are the constrains which do not directly affect the control variable. If the constraint is of p_{th} order, we will have to differentiate p times until the control variable explicitly appears in the constraint as $h_i^{(p)}(x,u,t)$. This makes it tough to handle with the Pontryagin's minimum principle. These are of the form

$$h_i(x,t) < 0, \quad i = 1,2,...r \tag{2.10}$$

If the p^{th} order derivative is directly dependent on the control variable as below

$$[h_i^{(p)}]_u \neq 0, \quad [h_i^{(k)}]_u = 0, \quad k = 1,2,...,p-1$$

where $[h_i^{(p)}]_u$ is the p^{th} order derivative that is directly dependent on the control variable. This represents a mixed state inequality constrain.

The control inputs T_m and F_b are bounded as,

$$
\begin{aligned}
T_{m,min} &\leq T_m(t) \leq T_{m,max}, && (2.11)\\
F_{b,min} &\leq F_b(t) \leq 0, && (2.12)
\end{aligned}
$$

2.3 Problem Formulation

A model predictive control (MPC) approach is used to solve the safe- and eco-driving control problem in real-time. At every time step, the MPC computes an optimal control trajectory over a finite prediction horizon (t_p), and this process with feedback of current vehicle information is repeated as the prediction horizon recedes. If the control inputs are defined by $u = T_m$ and$W = F_b/m$,

The MPC Problem is written as

$$
\begin{aligned}
\text{minimize } J &= \int_{t_0}^{t_0+t_p} \left(b_1 v u + b_2 u^2\right) dt, && (2.13)\\
\text{subject to } \dot{s} &= v, && (2.14)\\
\dot{v} &= c_1 \eta_t^{sign(u)} u - (c_2 v^2 + c_0) - W, && (2.15)\\
u_{min} &\leq u(t) \leq u_{max}, && (2.16)\\
W_{min} &\leq W(t) \leq 0, && (2.17)\\
h_1(t) &= v(t) - v_{max} \leq 0, && (2.18)\\
h_2(t) &= s(t) - (s_p(t) - \delta_s) \leq 0, && (2.19)
\end{aligned}
$$

where $c_1 = R_t/(rm)$, $c_2 = \rho_0 A_f c_d/(2m)$, and $c_0 = g(c_r + \sin(\alpha(s)))$,
while t_0 is the current time. Initial and terminal state constraints are

$$s(t_0) = s_0, v(t_0) = v_0$$

$$s(t_0 + t_p) = S, v(t_0 + t_p) = V,$$

where S is a desired terminal position, s_0 is the current position, v_0 is the current speed, and V is a desired terminal speed of the receding horizon. state-contrained optimal solutions are analytically derived under some assumptions discussed in the next chapters.

Chapter 3

Analytical State-Constrained Solution

3.1 Assumptions

To find the analytical solution for this control problem we make the following assumptions. [5]

- There is no transmission loss (i.e., $\eta = 1$)
- There is no mechanical braking force (i.e., $W = 0$)
- There are no constraints on control input ($u_{max} = -u_{min} \to \infty$)
- Constant acceleration of the preceding vehicles. ($a_p(t) = a_{p.0}, t \in [0.t_p]$
- There is no aerodynamic drag, (for simplifying the higher order computations in future) ($c_2 = 0$)
- The road on which the vehicles is moviing is flat, i.e., $\alpha(s) = 0$

3.2 Solution for Unconstrained Case

From 2.13 we can write

$$L = b_1 v u + b_2 u^2,$$

and the Hamiltonian is given by

$$\begin{aligned} H &= L + \lambda^T \dot{x}, \\ &= b_1 v u + b_2 u^2 \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} \end{aligned}$$

After making the necessary assumptions,the Hamiltonian can be written as

$$H = b_1 v u + b_2 u^2 + \lambda_1 v + \lambda_2 (c_1 u - c_0) \tag{3.1}$$

The necessary optimality conditions from the Pontryagin's minimum principle are given by [2]

$$\frac{\partial H}{\partial u} = b_1 v + 2 b_2 u + \lambda_2 c_1 = 0$$

hence,

$$\begin{aligned} u &= -\frac{b_1 v + c_1\lambda_2}{2b_2} && (3.2)\\ \dot{\lambda_1} &= -\frac{\partial H}{\partial s} = \lambda_2 \frac{d\gamma(s)}{ds}, && (3.3)\\ \dot{\lambda_2} &= -\frac{\partial H}{\partial v} = -[b_1 u + \lambda_1 - 2\lambda_2 c_2 v] && (3.4) \end{aligned}$$

where $\gamma(s) = g(c_r + \sin(\alpha(s)))$

3.2.1 Two-point boundary value problem

We compute the two point boundary value problem (TPBVP) based on the above equations as follows

$$\begin{aligned} \dot{v} &= c_1 u - c_0, && (3.5)\\ \dot{s} &= v, && (3.6)\\ \dot{\lambda_1} &= \lambda_2 g cos(\alpha(s)), && (3.7)\\ \dot{\lambda_2} &= -b_1\frac{(b_1 v + \lambda_2 c_1)}{2b_2} + \lambda_1 - 2\lambda_2 c_2 v, && (3.8)\\ s(0) &= 0, s(t_p) = S && (3.9)\\ v(0) &= 0, v(t_p) = V && (3.10) \end{aligned}$$

in the above equations the s, v, u are the primal variables, whereas the λ_1, λ_2 are the dual variables. These variables can be rewritten using the derivatives of s considering the stationarity conditions [6] $\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = 0$

$$v = \dot{s}$$

u can be rewritten as

$$u = \frac{\ddot{s} + c_2\dot{s}^2 + \gamma(s)}{c_1}$$

here c_0 is written as $\gamma(s)$

from stationarity conditions $\frac{\partial H}{\partial u} = 0$. we can write

$$\begin{aligned} \lambda_2 &= \frac{1}{c_1}(-b_1 v - 2b_2 u)\\ &= \frac{1}{c_1}\left(-b_1\dot{s} - \frac{2b_2}{c_1}\left(\ddot{s} + c_2\dot{s}^2 + \gamma(s)\right)\right) && (3.11) \end{aligned}$$

performing $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0$ we get,

$$\begin{aligned} 0 &= b_1\dot{v} + 2b_2\dot{u} + c_1\dot{\lambda_2}\\ 0 &= b_1\ddot{s} + 2b_2\frac{d}{dt}\left[\frac{1}{c_1}\left(\ddot{s} + c_2\dot{s}^2 + \gamma(s)\right)\right] - c_1[b_1 u + \lambda_1 - 2\lambda_2 c_2 v]\\ \lambda_1 &= \frac{b_1}{c_1}\ddot{s} + \frac{2b_2}{c_1^2}\left[\dddot{s} + 2c_2\dot{s}\ddot{s} + \dot{\gamma(s)}\right] + 2\lambda_2 c_2\dot{s} - b_1\left[\frac{\dot{s}}{c_1} + c_2\dot{s}^2 + \gamma(s)\right] \end{aligned}$$

by performing $\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = 0$ we get the equations in the fourth order of s terms. It can be rewritten in the factorized form as

$$As^{[4]} + B(\dot{s})\, s^{[3]} + C(s, \dot{s}, \ddot{s})\, s^{[2]} + D\left(s, \dot{s}, \ddot{s}, s^{[3]}\right) \dot{s} - c_1 \lambda_2 (s, \dot{s}, \ddot{s}) \gamma(\dot{s}) = 0 \tag{3.12}$$

The end point conditions associated to the single variable fourth-order differential equations are

$$s(0) = 0$$

By assuming no aerodynamic drag force and a flat road, we take $c_2 = o$ and $\alpha(s) = 0$, then the fourth-order differential equation will be

$$s^{[4]} = 0$$

Which mean the $s(t)$ is the interpolating polynomial of order three. The adjoining variables are

$$\lambda_1 = -\frac{b_1}{c_1}\dot{s} + \frac{b_1}{c_1}\ddot{s} + \frac{2b_2}{c_1^2}\dddot{s} \tag{3.13}$$

$$\lambda_2 = \frac{1}{c_1}\left(-b_1 \dot{s} - \frac{2b_2}{c_1}\ddot{s}\right) \tag{3.14}$$

The optimal control input for an unconstrained case can be expressed as a linear function of time as below,

$$u^*(t) = k_1 t + k_2,$$

at $t = 0$,

$$u^*(0) = -\frac{b_1 v_0 + c_1 \lambda_{2.0}}{2b_2} = k_2$$

at $t = t_p$,

$$\begin{aligned} k_1 = \frac{du}{dt} &= -\frac{1}{2b_2}\left[b_1 \dot{v} + c_1 \dot{\lambda}_2\right] \\ &= \frac{-1}{2b_2}\left[b_1 c_1 u - b_1 c_0 - c_1 b_1 u - c_1 \lambda_{1.0}\right] \\ &= \frac{1}{2b_2}\left[b_1 c_0 + c_1 \lambda_{1.0}\right] \end{aligned}$$

the final equation can be written as

$$u^*(t) = \frac{t}{2b_2}\left[b_1 c_0 + c_1 \lambda_{1.0}\right] - \frac{b_1 v_0 + c_1 \lambda_{2.0}}{2b_2}$$

where $\lambda_1^*(0) = \lambda_{1.0}$ and $\lambda_2^*(0) = \lambda_{2.0}$

Now, by enforcing the terminal conditions $s^*(t_p) = S$ and $v^*(t_p) = V$, we get a system of two linear equations in two unknowns $\lambda_{1.0}$ and $\lambda_{2.0}$,

$$\begin{aligned} S &= \frac{c_1^2 t_p^3 \lambda_{1.0}}{12b_2} - \frac{3c_1^2 t_p^2 \lambda_{2.0}}{12b_2} + \frac{b_1 c_0 c_1 t_p^3}{12b_2} - \frac{(2b_2 c_0 + b_1 c_1 v_0)\, 3t_p^2}{12b_2} \\ &+ b_2\left(v_0 t_p + s_0\right) \end{aligned} \tag{3.15}$$

$$V = \frac{3c_1^2 t_p^2 \lambda_{1.0}}{12b_2} - \frac{6c_1^2 t_p \lambda_{2.0}}{12b_2} + \frac{3b_1 c_0 c_1 t_p^2}{12b_2} - \frac{(2b_2 c_0 + b_1 c_1 v_0)\, 6t_p^2}{12b_2} + v_0 \tag{3.16}$$

the above equations can e written as

$$\begin{bmatrix} S \\ V \end{bmatrix} = \begin{bmatrix} s^*(t_p) \\ v^*(t_p) \end{bmatrix} = A \begin{bmatrix} \lambda_{1.0} \\ \lambda_{2.0} \end{bmatrix} + B, \tag{3.17}$$

where,

$$A = \frac{c_1^2 t_p}{12b_2} \begin{bmatrix} t_p^2 & -3t_p \\ 3t_p^2 & -6 \end{bmatrix}, B = \frac{1}{12b_2} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{3.18}$$

while $B_1 = b_1 c_0 c_1 t_p^3 - 3(2b_2 c_0 + b_1 c_1 v_0)\, t_p^2 + 12b_2(v_0 t_p + s_0)$ and
$B_2 = 3b_1 c_0 c_1 t_p^2 - 6(2b_2 c_0 + b_1 c_1 v_0)\, t_p + 12b_2 v_0$.
solving the above equation we find the optimal control trajectory of the unconstrained case.

3.3 Solution for state-constrained Case

As we have mentioned in 2.13 we have pure state inequality constraints, hence this paper follows the Indirect adjoining method [7].

First the Lagrange is formed in the following way,

$$L(x,u,t) = H(x,u,t) + \mu h^{(p)}(x,u,t),$$

where $\mu h(x,t) = 0, \mu \geq 0$.
Imposing only $h(x,u,t)^{(p)} \leq 0$ (i.e.,) whenever $h(x,t) = 0$ does not prevent the trajectory from violating $h(x,t) \leq 0$ because it cannot guarantee that $h(x,t)^{(q)} \leq 0$ for $q = 1, ..., p-1$. From this fact, tangency conditions,

$$\psi = \begin{bmatrix} h^(0) \\ h^(1) \\ . \\ . \\ . \\ h^(p-1) \end{bmatrix} = 0, \tag{3.19}$$

must be added at the entry time [1]. As these tangency conditions form the interior point constraints, the necessary optimality conditions for this are now given by the following

$$\begin{aligned} u^*(t) &= arg \min_{u \in \Omega(x^*,t)} H\left(u^*, x^*, \lambda^*, t\right), &(3.20) \\ \dot{x}^*(t) &= L_\lambda^*(u^*, x^*, \lambda^*, \mu^*, t), &(3.21) \\ \dot{\lambda}^*(t) &= L_x^*(u^*, x^*, \lambda^*, \mu^*, t), &(3.22) \end{aligned}$$

where $\mu^* h^{(p)} = 0$, $\mu^* \geq 0$, and $\Omega = \{h^{(p)} \leq 0 \text{ if } h = 0\}$,

Due to below jump conditions the costate variables may be discontinuous at the entry time to satisfy the tangency conditions.

$$\lambda^*(\tau^-) = \lambda^*(\tau^+) + \sum_{j=0}^{p-1} \pi_j h_{x^*}^{(j)}(x^*, \tau), \tag{3.23}$$

$$H^*(\tau^-) = H^*(\tau^+) - \sum_{j=0}^{p-1} \pi_j h_t^{(j)}(x^*, \tau), \tag{3.24}$$

The speed and position constraint equations can be written as below assuming the constant acceleration as,

$$h_1(t) = v(t) - v_{max}, \tag{3.25}$$
$$h_2(t) = s(t) - (s_{p.0} + v_{p.0} t + a_{p.0} t^2/2) \tag{3.26}$$

The tangency conditions are obtained by differentiating (3.25) and (3.26) the constraints until the control variable explicitly appears. The speed constraint is a first order constraint obtained as

$$\begin{aligned} h_1^1(\tau) &= \dot{v(\tau)}, \\ &= c_1 u - c_0 \end{aligned} \tag{3.27}$$

The position constraint is a second order constraint equation obtained as,
first derivative,

$$\begin{aligned} h_2^1(\tau) &= \dot{s(\tau)} - \left[\dot{s_{p.0}} + \dot{v_{p.0}}\tau + v_{p.0} + a_{p.0}\tau + \dot{a_{p.0}}\frac{\tau^2}{2} \right] \\ h_2^1(\tau) &= v(\tau) - \left[v_{p.0} + a_{p.0}\tau \right] \end{aligned} \tag{3.28}$$

second derivative,

$$\begin{aligned} h_2^2(\tau) &= \dot{v(\tau)} - \left[\dot{v_{p.0}} + \dot{a_{p.0}}\tau + a_{p.0} \right] \\ h_2^2(\tau) &= \dot{v(\tau)} - a_{p.0} \\ &= c_1 u - c_0 - a_{p.0} \end{aligned} \tag{3.29}$$

The tangency conditions are the derivatives of our constraints before the control variable explicitly appears, from (3.25)(3.26) and (3.28) we can write the tangency conditions as,

$$\Psi_1 = v(\tau) - v_{max} = 0, \tag{3.30}$$

$$\Psi_2 = \begin{bmatrix} s(\tau) - \left(s_{p.0} + v_{p.0}\tau + a_{p.0}\frac{\tau^2}{2} \right) \\ v(\tau) - \left(a_{p.0}\tau + v_{p.0} \right) \end{bmatrix} = 0, \tag{3.31}$$

where $\tau \in [0, t_p]$ represents entry time among the junction times.
Now the Lagrangian (3.19) can be formed from (3.27) and (3.29) as

$$\begin{aligned} L &= H + \mu_1 h_1^1 + \mu_2 h_2^2, \\ L &= H + \mu_1(c_1 u - c_0) + \mu_2(c_1 u - c_0 - a_{p.0}) \end{aligned} \tag{3.32}$$

we can find the following as

$$
\begin{aligned}
\dot{s} &= \frac{\partial L}{\partial \lambda_1} = v^*(t), && (3.33)\\
\dot{v} &= \frac{\partial L}{\partial \lambda_2} = (c_1 u^* - c_0) = v^{\dot{*}}(t), && (3.34)\\
\dot{\lambda_1^*} &= \frac{\partial L}{\partial s} = \left(-\lambda_2^* \mu_1^* + \mu_2^*\right) g \cos\alpha(s), && (3.35)\\
\dot{\lambda_2^*} &= \frac{\partial L}{\partial v} = b_1 u^* + \lambda_1^* && (3.36)
\end{aligned}
$$

The optimal control trajectory of a state-constrained case is a quadratic function. The optimal speed trajectory is a second order polynomial of the form [8]

$$
\begin{aligned}
u^*(t) &= \frac{1}{c_1}\left(c_0 - \frac{4v_0}{t_p} - \frac{2V}{t_p} + \frac{6S}{t_p^2} + \frac{6v_0 t}{t_p^2} + \frac{6Vt}{t_p^2} - \frac{12St}{t_p^3}\right), && (3.37)\\
v^*(t) &= v_0 - \frac{4v_0 t}{t_p} - \frac{2Vt}{t_p} - \frac{6St^2}{t_p^3} + \frac{6St}{t_p^2} + \frac{3v_0 t^2}{t_p^2} - \frac{3Vt^2}{t_p^2} && (3.38)
\end{aligned}
$$

We have three cases mentioned below to find out the optimal control $u^*(t)$ based on which state-constraint is active

- Active Speed only constraint case
- Active Position only constraint case
- Both active position and speed constrain case

3.3.1 Speed only constraint case

Jump conditions for the speed only constraint case are obtained as below,

$$
\begin{aligned}
\lambda(\tau^+)^T &= \lambda(\tau^-)^T - \begin{bmatrix}\pi_{0.h_1} & \pi_{1.h_1}\end{bmatrix} \begin{bmatrix} \frac{\partial h_1^0(\tau)}{\partial s} & \frac{\partial h_1^0(\tau)}{\partial v} \\ \frac{\partial (0)}{\partial s} & \frac{\partial (0)}{\partial v} \end{bmatrix} && (3.39)\\
&= \lambda(\tau^-)^T - \begin{bmatrix}\pi_{0.h_1} & \pi_{1.h_1}\end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
\lambda_1^*(\tau^+) &= \lambda_1^*(\tau^-), && (3.40)\\
\lambda_2^*(\tau^+) &= \lambda_2^*(\tau^-) - \pi_{0.h_1} && (3.41)\\
H(\tau^+) &= H(\tau^-) + \begin{bmatrix}\pi_{0.h_1} & \pi_{1.h_1}\end{bmatrix} \begin{bmatrix} \frac{\partial h_1^0(\tau)}{\partial \tau} \\ \frac{\partial (0)}{\partial \tau} \end{bmatrix} && (3.42)\\
&= H(\tau^-) + \begin{bmatrix}\pi_{0.h_1} & \pi_{1.h_1}\end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
H(\tau^+) &= H(\tau^-) && (3.43)
\end{aligned}
$$

From above jump conditions and continuous control input, the jump parameter of speed co-state variables, $\pi_{0.h1}$, is zero, thereby resulting in the continuity of both co-state variables without any jumps

If the speed of the unconstrained solution exceeds the maximum road speed limit, a speed-constrained optimal solution must be computed. This optimal control is defined by three phases,

$$u^*(t) = \begin{cases} k_1 t + k_2 & [0, t_1) \\ u_{c.v} & [t_1, t_2) \\ k_1(t - t_2) + u_{c.v} & [t_2, t_p] \end{cases}, \tag{3.44}$$

where $k_1 = (b_1 c_0 + c_1 \lambda_{1.0}) / (2b_2)$ and $k_2 = -(b_1 v_0 + c_1 \lambda_{2.0}) / (2b_2)$

The boundary control input is obtained by making $h_1^1 = 0$

$$\begin{aligned} h_1^1 &= 0, \\ c_1 u_{c.v} - c_0 &= 0, \\ u_{c.v} &= \frac{c_0}{c_1} \end{aligned} \tag{3.45}$$

Since there are two interior-point conditions (i.e., $u^*(t_1) = u_{c.v}, v^*(t_1) = v_{max}$) and two boundary conditions, four unknown parameters ($\lambda_{1.0}, \lambda_{2.0}, t_1$, and t_2) . The solution for this system is obtained from the (3.18) as,

$$A_{1.1} t_1^2 + A_{1.2} t_1 + A_{1.3} = 0, \tag{3.46}$$

$$t_2 = (B_{1.1} t_1 + B_{1.2}) / B_{1.3}, \tag{3.47}$$

for $0 < t_1 < t_2 < t_p$, with

$$\lambda_{1.0} = C_{1.1} + C_{1.2} / t_1^2, \tag{3.48}$$

$$\lambda_{2.0} = D_{1.1} + D_{1.2} / t, \tag{3.49}$$

where coefficients

$$\begin{aligned} A_{1.1} &= v_0(3v_{max}^2 - 3v_0 v_{max} + v_0^2), \\ A_{1.2} &= -6(Sv_0^2) + Sv_{max}^2 - t_p v_{max}^2 + 2t_p v_0 v_{max}^2 - t_p^2 v_0^2 v_{max} - 2Sv_0 v_{max}), \\ A_{1.3} &= -9(t_p^2 v_{max}^2 - S^2 v_0 + S^2 v_{max} - 2St_p v_{max}^2 - t_p^2 v_0 v_{max}^2 + 2Sv_0 v_{max}), \\ B_{1.1} &= v_0 - v_{max}, B_{1.2} = 3(s_0 - S) + t_p V + 2t_p v_{max}, B_{1.3} = V - v_{max}, \\ C_{1.1} &= -(b_1 c_0) / c_1, C_{1.2} = 4b_2(v_0 - v_{max}) / c_1^2, \\ D_{1.1} &= -(2b_2 c_0 + b_1 c_1 v_0) / c_1^2, D_{1.2} = 4b_2(v_0 - v_{max}) / c_1^2 \end{aligned}$$

3.3.2 Position only constraint case

Jump conditions for position only constraint case are obtained as below,

$$
\begin{aligned}
\lambda(\tau^+)^T &= \lambda(\tau^-)^T - \begin{bmatrix}\pi_{0.h_2} & \pi_{1.h_2}\end{bmatrix} \begin{bmatrix} \frac{\partial h_2^0(\tau)}{\partial s} & \frac{\partial h_2^0(\tau)}{\partial v} \\ \\ \frac{\partial h_2^1(\tau)}{\partial s} & \frac{\partial h_2^1(\tau)}{\partial v} \end{bmatrix} \\
= \lambda(\tau^-)^T &- \pi_{j,h2}^T \begin{bmatrix} \frac{\partial}{\partial s}\left[s(\tau) - \left(s_{p.0} + v_{p.0}\tau + a_{p.0}\frac{\tau^2}{2}\right)\right] & \frac{\partial}{\partial v}\left[s(\tau) - \left(s_{p.0} + v_{p.0}\tau + a_{p.0}\frac{\tau^2}{2}\right)\right] \\ \\ \frac{\partial}{\partial s}\left[v(\tau) - \left(a_{p.0}\tau + v_{p.0}\right)\right] & \frac{\partial}{\partial v}\left[v(\tau) - \left(a_{p.0}\tau + v_{p.0}\right)\right] \end{bmatrix} \\
&= \lambda(\tau^-)^T - \begin{bmatrix}\pi_{0.h_2} & \pi_{1.h_2}\end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\lambda_1^*(\tau^+) = \lambda_1^*(\tau^-) - \pi_{0.h_2} \tag{3.50}
$$

$$
\lambda_2^*(\tau^+) = \lambda_2^*(\tau^-) - \pi_{1.h_2} \tag{3.51}
$$

$$
\begin{aligned}
H(\tau^+) &= H(\tau^-) + \begin{bmatrix}\pi_{0.h_2} & \pi_{1.h_2}\end{bmatrix} \begin{bmatrix} \frac{\partial h_2^0(\tau)}{\partial \tau} \\ \\ \frac{\partial h_2^1(\tau)}{\partial \tau} \end{bmatrix} \\
&= H(\tau^-) + \begin{bmatrix}\pi_{0.h_2} & \pi_{1.h_2}\end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \tau}\left[s(\tau) - \left(s_{p.0} + v_{p.0}\tau + a_{p.0}\frac{\tau^2}{2}\right)\right] \\ \\ \frac{\partial}{\partial \tau}\left[v(\tau) - \left(a_{p.0}\tau + v_{p.0}\right)\right] \end{bmatrix} \\
&= H(\tau^-) + \begin{bmatrix}\pi_{0.h_2} & \pi_{1.h_2}\end{bmatrix} \begin{bmatrix} 0 - \left(0 + v_{p.0} + a_{p.0}\tau\right) \\ 0 - a_{p.0} \end{bmatrix}
\end{aligned}
$$

$$
H(\tau^+) = H(\tau^-) - \pi_{0.h_2}\left(a_{p.0}\tau + v_{p.0}\right) - \pi_{1.h_2} a_{p.0} \tag{3.52}
$$

The position co-state, $\lambda_1^*(t)$, is discontinuous at the entry time because of non-zero jump parameter, $\pi_{0.h_2}$. On the other hand, another jump parameter, $\pi_{1.h_2}$, should be zero, i.e., the speed co-state, $\lambda_2^*(t)$, is continuous. Position only case the constant control input is given by making $h_2^2(t) = 0$,

$$
\begin{aligned}
h_2^2 &= 0, \\
c_1 u_{c.s} - c_0 - a_{p.0} &= 0, \\
c_0 + a_{p.0} &= c_1 u_{c.s}, \\
u_{c.s} &= \frac{c_0 + a_{p.0}}{c_1}
\end{aligned} \tag{3.53}
$$

Optimal control of the boundary interval case is expressed in the similar form as the speed only constrained case, as follows:

$$
u^*(t) = \begin{cases} k_1 t + k_2 & [0, t_1) \\ u_{c.s} & [t_1, t_2)\ , \\ k_3(t - t_2) + u_{c.s} & [t_2, t_p] \end{cases} \tag{3.54}
$$

where $k_1 = (b_1 c_0 + c_1 \lambda_{1.0}) / (2b_2), k_2 = -(b_1 v_0 + c_1 \lambda_{2.0}) / (2b_2)$ and $k_3 = k_1 + c_1 \pi 0.h_2 / (2b_2)$
A system of five nonlinear equations and five unknowns ($\lambda_{1.0}, \lambda_{2.0}, t_1, t_2$,and$\pi_{0.h_2}$)are obtained. The solution is as follows,

$$
\begin{aligned}
t_1 &= A_{2.b.1} / A_{2.b.2}, &(3.55)\\
t_2 &= B_{2.b.1} / B_{2.b.2}, &(3.56)
\end{aligned}
$$

for $0 < t_1 < t_2 < t_p$, with

$$
\begin{aligned}
\lambda_{1.0} &= C_{2.b.1} + C_{2.b.2} / t_1^2, &(3.57)\\
\lambda_{2.0} &= D_{2.b.1} + D_{2.b.2} / t_1, &(3.58)\\
\pi_{0.h_2} &= \frac{E_{2.b.1} + \left(E_{2.b.2} t_2^2 + E_{2.b.3} t_2 + E_{2.b.4}\right) / t_1^2}{(t_2 - t_p)^2}, &(3.59)
\end{aligned}
$$

where coefficients

$$
\begin{aligned}
A_{2.b.1} &= -3(s_0 - s_{p.0}), A_{2.b.2} = v_0 - v_{p.o},\\
B_{2.b.1} &= -(3s_{p.o} - 3S + t_p V + 2t_p v_{p.o} + (a_{p.o} t_p^2)/2), B_{2.b.2} = v_{p.o} - V + a_{p.o} t_p,\\
C_{2.b.1} &= -(b_1 c_0)/c_1, C_{2.b.2} = 4b_2(v_0 - v_{p.o})/c_1^2,\\
D_{2.b.1} &= -(2a_{p.o} b_2 + 2b_2 c_0 + b_1 c_1 v_0)/c_1^2, D_{2.b.2} = 4b_2(v_0 - v_{p.0})/c_1^2,\\
E_{2.b.1} &= 4b_2(v_{p.0} - V + a_{p.0} t_p)/c_1^2, E_{2.b.2} = 4b_2(v_0 - v_{p.0})/c_1^2,\\
E_{2.b.3} &= -8b_2 t_p (v_0 - v_{p.0})/c_1^2, E_{2.b.4} = 4b_2 t_p^2 (v_0 - v_{p.0})/c_1^2.
\end{aligned}
$$

As the activation level of the position constraint becomes looser, the boundary interval (i.e, when the constraint equals to zero) vanishes and becomes a contact point.

$$
u^*(t) = \begin{cases} k_1 t + k_2 & [0, t_1) \\ k_3(t - t_2) + u^*(t_1) & [t_1, t_p] \end{cases}, \tag{3.60}
$$

where $k_1 = (b_1 c_0 + c_1 \lambda_{1.0}) / (2b_2), k_2 = -(b_1 v_0 + c_1 \lambda_{2.0}) / (2b_2)$ and $k_3 = k_1 + c_1 \pi 0.h_2 / (2b_2)$

The solution of this system is obtained as:

$$
A_{2.c.1} t_1^3 + A_{2.c.2} t_1^2 + A_{2.c.3} t_1 + A_{2.c.4} = 0, \tag{3.61}
$$

for $0 < t_1 < t_P$, with

$$
\begin{aligned}
\lambda_{1.0} &= C_{2.c.1} + C_{2.c.2} / t_1^2 + C_{2.c.3} / t_1^3, &(3.62)\\
\lambda_{2.0} &= D_{2.c.1} + D_{2.c.2} / t_1 + D_{2.c.3} / t_1^2, &(3.63)\\
\pi_{0.h_2} &= \frac{E_{2.c.1} + E_{2.c.2} / t_1 + E_{2.c.3} / t_1^2 + E_{2.c.4} / t_1^3}{(t_1 - t_p)^2}, &(3.64)
\end{aligned}
$$

$$
\begin{aligned}
A_{2.c.1} &= v_0 - V + a_{p.0} t_p, \\
A_{2.c.2} &= 3s_0 - 3S - 2t_p v_0 + t_p V + 4t_p v_{p.0} + (a_{p.0} t_p^2)/2, \\
A_{2.c.3} &= -6t_p(s_0 - s_{p.0)} + t_p^2(v_0 - v_{p.0}), A_{2.c.4} = 3t_p^2(s_0 - s_{p.0}), \\
C_{2.c.1} &= -(b_1 c_0)/c_1, C_{2.c.2} = 12b_2(v_0 - v_{p.0})/c_1^2, \\
C_{2.c.3} &= 24b_2(s_0 - s_{p.0})/c_1^2, \\
D_{2.c.1} &= -(2a_{p.0}b_2 + 2b_2 c_0 + b_1 c_1 v_0)/c_1^2, D_{2.c.2} = 8b_2(v_0 - v_{p.0})/c_1^2, \\
D_{2.c.3} &= 12b_2(s_0 - s_{p.0})/c_1^2, \\
E_{2.c.1} &= 4b_2(v_0 - V + a_{p.0} t_p)/c_1^2, E_{2.c.2} = 16b_2 t_p(v_0 - v_{p.0})/c_1^2, \\
E_{2.c.3} &= 12b_2 t_p(2(s_0 - s_{p.0}) - t_p(v_0 - v_{p.0}))/c_1^2, E_{2.c.4} = 24b_2 t_p^2(s_0 - s_{p.0})/c_1^2.
\end{aligned}
$$

3.3.3 Both Speed and Position constraint case

For the both constraints active case, there can be several sequences how the constraints become active. The below optimal control is formed by considering the speed constraint to be active first and then the position constraint.

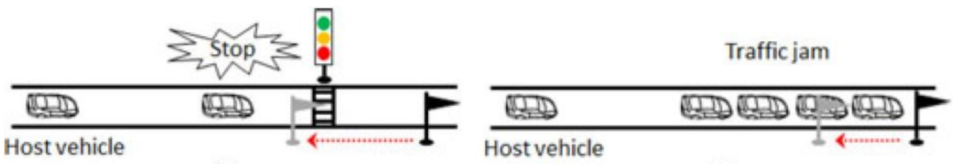
$$
u^*(t) = \begin{cases} k_1 t + k_2 & [0, t_1) \\ u_{c.v} & [t_{1.1}, t_{1.2}) \\ k_1(t - t_{1.2}) + u_{c.v} & [t_{1.2}, t_{2.1}) \\ u_{c.s} & [t_{2.1}, t_{2.2}) \\ k_3(t - t_{2.2}) + u_{c.s} & [t_{2.2}, t_p] \end{cases}, \tag{3.65}
$$

where $k_1 = (b_1 c_0 + c_1 \lambda_{1.0})/(2b_2)$, $k_2 = -(b_1 v_0 + c_1 \lambda_{2.0})/(2b_2)$ and $k_3 = k_1 + c_1 \pi 0.h_2/(2b_2)$ For this optimal control we solve seven nonlinear equations in seven unknowns $(\lambda_{1.0}, \lambda_{2.0}, t_{1.1}, t_{1.2}, t_{2.1}, t_{2.2} \text{and} \pi_{0.h_2})$.

Chapter 4

Conditions for Feasibility of Analytical solution

The existence of analytical state constrained solution is determined by terminal conditions at final time. Terminal speed must be lower than the maximum speed limit. Preceding vehicle's driving and maximum speed limit affects feasible terminal position. Consider two consecutive vehicles driving on the same lane; the host vehicle does not overtake the preceding vehicle or change the direction of movement on the planned route. terminal position condition is infeasible in 2 scenarios - In the first scenario, the preceding vehicle will stop before arriving at S, whereas in the second scenario, it will drive too slow to arrive at S within t_p. Thus, there is need to define feasible range of terminal position and to guarantee the existence of the state-constrained solution.



(a) (b)

Figure 4.1: situations where terminal position condition could be in-feasible

4.1 Maximum Terminal Position

When only the speed constraint is active, the active speed constraints generates the boundary interval and penalizes the terminal position of host vehicle, When this boundary interval (t_1, t_2) expands to the terminal position (i.e, $t_2 = t_p$) then $v(t) = v_{max}$ Then the maximum terminal position can be

$$\begin{aligned} s(t) &= s_0 + v(t)t, \\ s(t_p) &= s_0 + v_{max}t_p, \end{aligned}$$

where s_0 is the initial position of the vehicle.

So the condition for maximum terminal position is

$$
\begin{aligned}
S_{max.1} &\leq s(t_p), && (4.1)\\
S_{max.1} &\leq s(t_p) = s_0 + v_{max} t_p && (4.2)
\end{aligned}
$$

Similarly ,for the active position constraint case , when the boundary interval expands to terminal position, i.e., $t_2 = t_p$, then

$$
s(t) = s_{p.0} + v_{p.0} t_p + (a_{p.0} t^2)/2,
$$
$$
s(t_p) = s_{p.0} + v_{p.0} t_p + (a_{p.0} t_p^2)/2;
$$

$$
\begin{aligned}
S_{max.2} &\leq s(t_p); && (4.3)\\
S_{max.2} &\leq s(t_p) = s_{p.0} + v_{p.0} t_p + (a_{p.0} t_p^2)/2 && (4.4)
\end{aligned}
$$

When both the constraints are active in a sequence such that the speed constraint precedes the position constraint then the above equation still works. If the position constraint precedes the speed constraint, the terminal position is more penalized than above case.

During the exception case when the exit time of position constraint $(t_{2.2})$ equals the entry time of the speed constraint$(t_{1.1})$, the exit time of speed constraint should equal the prediction horizon in order to have a safe terminal position. If the entry time $(t_{1.1})$ of speed constraint is smaller than the t_p,

$$
s(t_p) = s_0(t_{1.1}) + v_{max}(t_p - t_{1.1}, \tag{4.5}
$$

We know

$$
v(t_{1.1}) = v_{p.0} + a_{p.0} t_{1.1} \tag{4.6}
$$

We have maximum speed at $t_{1.1}$ in this case can be v_{max}

$$
\begin{aligned}
v_{max} &= v_{p.0} + a_{p.0} t_{1.1}, && (4.7)\\
t_{1.1} &= (v_{max} - v_o)/a_{p.0}, && (4.8)
\end{aligned}
$$

substituting in 4.5 we get

$$
S \leq S_{max.3}(t_p) = s_{p.0} - \frac{(v_{max} - v_{p.0})^2}{2a_{p.0}} + v_{max} t_p \tag{4.9}
$$

Given all these conditions the maximum terminal position will be the minimum of smax1 smax2 smax3

$$
S_{max}(t_p) = min(S_{max.1}, S_{max.2}, Smax.3),
$$

If $S_{max.1} = S_{max.2}$,

$$
\begin{aligned}
s_0 + v_{max} t_p &= s_{p.0} + v_{p.0} t_p + a_{p.0} t_p^2/2, && (4.10)\\
\frac{a_{p.o} t_p^2}{2} + (v_{p.0} - v_{max}) t_p + (s_{p.0} - s_0) &= 0, && (4.11)\\
F_1 + F_2 t_{p.th.1} + F_3 t_{p.th.1}^2 &= 0 && (4.12)
\end{aligned}
$$

For second threshold,

$$t_{p.th.2} = t_{1.1}$$

If there exist $t_{p.th.1}$ and $t_{p.th.2}$ satisfying

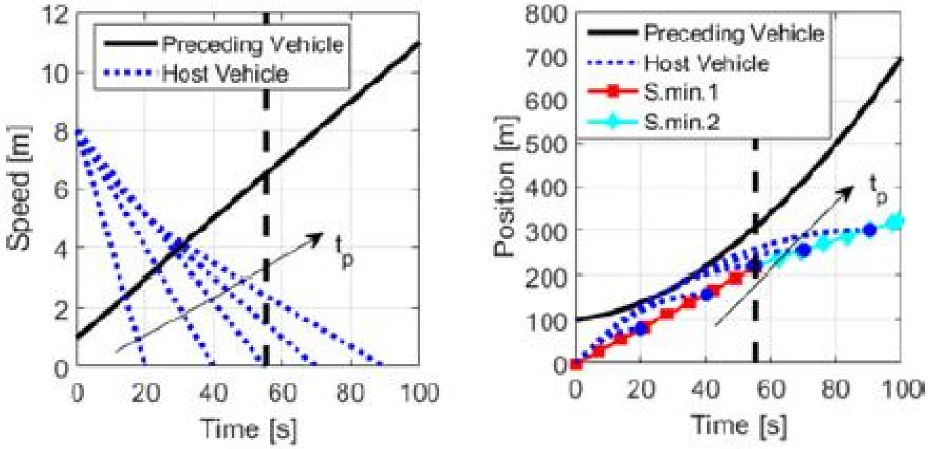
$$0 < t_{p.th.1} < t_{p.th.2}$$

Hence the maximum terminal position can be

$$S_{max}(t_p) = \begin{Bmatrix} S_{max.1} & t_p \in [0, t_{p.th.1}) \\ S_{max.2} & [t_{p.th.1}, t_{p.th.2}) \\ S_{max.3} & [t_{p.th.2}, \infty) \end{Bmatrix}. \tag{4.13}$$

4.2 Minimum Terminal Position

Since the optimal speed profile is quadratic function of time, host vehicle drives backward near the end of the prediction horizon. This function also has zero terminal speed, it changes from concave to convex as terminal position decreases. Hence the feasible minimum terminal position is when the speed profile becomes linear. i.e., the slope becomes zero ($k_1 = 0$) in 3.2.1. Hence $v^l(t) = v_0 - v_0 t/t_p$ and $s^l(t_p) = s_0 + v_0 t - v_0 t^2/(2t_p)$. Depending on the which constraint is active, we find two minimum terminal positions.



(a) **(b)**

Figure 4.2: Trajectories of the host vehicle generating the minimum terminal position for five different values of (a) tp : speed and (b) position. Bold black line is the trajectory of the preceding vehicle, and dashed black line is $t_{p.th.3}$.

$$S_{min.1}(t_p) = s^l(t_p) = s_0 + v_0 t_p/2, \tag{4.14}$$

In the second case, the contact time, t_c , such that $v^*(t_c) = v_p(t_c)$ and $s^*(t_c) = s_p(t_c)$ exists in order to enforce the position constraint. This position-constrained solution having linear speed profile after t_c results in the minimum terminal position, as follows:

$$S_{min.2}(t_p) = s_p(t_c) + v_p(t_c)(t_p - t_c),$$
$$= s_{p.0} + \frac{v_{p.0}t_c}{2} + \frac{v_{p.0} + a_{p.0}t_c}{2} t_p, \tag{4.15}$$

and t_c is computed imposing $k_3 = (b_1c_0 + c_1(\lambda_{1.0} + \pi 1.p))/(2b_2) = 0$

$$F_5 t_c^2 + F_6 t_c + F_7 = 0$$

for

$$0 < t_c < t_{p.s},$$

where $F_5 = 2v_0 - 3v_{p.0} - a_{p.0}t_{p.s}, F_6 = 6(s_0 - s_{p.0}) + 2t_{p.s}(v_{p.0} - v_0), F_7 = 6t_{p.s}(s_{p.0} - s_0)$.

The threshold that activates the position constraint, $t_{p.th.3}$, exists only if there exists the touch point such that $s_l(t_{un.c}) = s_p(t_{un.c}) = 0$. Using the discriminant of the condition for $t_{un.c}, t_{p.th.3}$ is written as

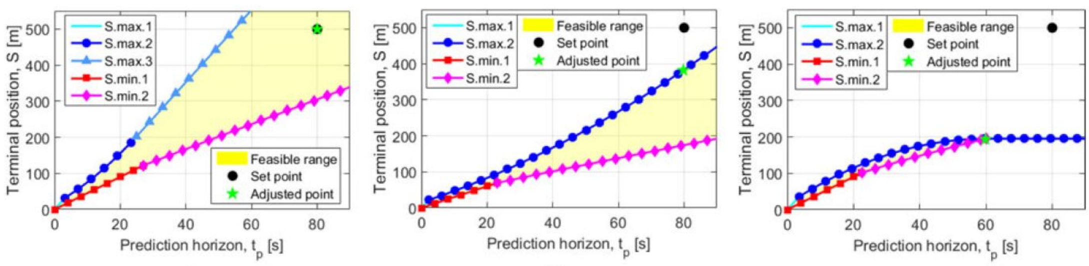
$$t_{p.th.3} = \frac{2(s_{p.0} - s_0)v_0}{(v_{p.0} - v_0)^2 - 2a_{p.0}(s_{p.0} - s_0)} \tag{4.16}$$

with $t_{p.th.3}$, the mnimum terminal position can be written as

$$S_{min(t_p)} = \begin{Bmatrix} S_{min.1} & t_p \in [0, t_{p.th.3}) \\ S_{min.2} & t_p \in [t_{p.th.3}, \infty] \end{Bmatrix}. \tag{4.17}$$

Figure 4.2 shows that the speed trajectory becomes a linear function over the whole horizon or the sub-horizon if the position constraint is active. Moreover, the corresponding terminal position points build up the minimum terminal position curve ($S_{min}(t_p)$).

4.3 Feasible range



(a) (b) (c)

Figure 4.3: Feasible range of terminal position with $(t_{p.s}, S_s) = (80, 500)$ for three scenarios: (*a*) normal, (*b*) abnormal non-stop, and (*c*) abnormal stop.

The feasible range is defined as the area in the plane (t_p, S) between the maximum and minimum terminal position curves as shown in figure 4.3. Suppose that the point $(t_{p,S})$ was set to $(t_{p.s}, S_s)$ ("Set point"). In the normal scenario, the set point is a feasible terminal condition, whereas in the abnormal scenarios, it becomes an infeasible terminal condition and thus must be corrected to be in the feasible range as $(t_{p.a}, S_a)$ ("Adjusted point"). In the non-stop scenario, only the terminal position must be adjusted, whereas in the stop scenario, the prediction horizon must be also shrunk. In other terms, when the preceding vehicle is braking suddenly and sharply (stop scenario), the host vehicle must take an action to stop itself optimally considering the stopping distance and time of the preceding vehicle. The adjusted terminal position must be as far as possible to avoid large torque values afterwards that cause unnecessary energy losses. This adaptation of the terminal position condition guarantees the existence of the analytical solution, and thus improves robustness of the MPC with respect to uncertain driving of the preceding vehicle.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

This report presents about a novel work on safe- and eco- driving control system based on analytical solution of minimal-energy torque input for Electrical Connected and Autonomous Vehicles. This report emphasizes on the key objectives that are achieved in the paper and derives the mathematical formulations on the same. Furthermore, the feasible range of terminal conditions that guarantee the existence of the analytical solution are also discussed.

5.2 Future scope

The paper which is discussed in this report aims to investigate CAV's influence on mixed traffic with different penetration rates. In future, it can be further extended to multi-lane driving scenarios also.

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