Count data in gamlss package

A.10.1 Poisson distribution (PO)

Poisson distribution

The probability function of the Poisson distribution, denoted here as $\mathbf{PO}(\mu)$, is given by

$$p_Y(y|\mu) = P(Y = y|\mu) \quad = \quad \frac{e^{-\mu}\mu^y}{y!} \tag{A.47}$$

where $y = 0, 1, 2, \ldots$, where $\mu > 0$, with $E(Y) = \mu$ and $Var(Y) = \mu$. [See Johnson *et al.* (1993), p 151.] The moment ratios of the distribution are given by $\sqrt{\beta_1} = \mu^{-0.5}$ and $\beta_2 = 3 + \mu^{-1}$ respectively. Note that the Poisson distribution has the property that $E[Y] = Var[Y]$ and that $\beta_2 - \beta_1 - 3 = 0$. The coefficient of variation of the distribution is given by $\mu^{-0.5}$. The index of dispersion, that is, the ratio $Var[Y]/E[Y]$ is equal to one for the Poisson distribution. For $Var[Y] > E[Y]$ we have overdispersion and for $Var[Y] < E[Y]$ we have underdispersion or repulsion. The distribution is skew for small values of μ, but almost symmetric for large μ values.

A.10.2 Negative Binomial distribution (NBI, NBII)

First parameterization: Negative Binomial type I (NBI)

The probability function of the negative binomial distribution type I, denoted here as $\mathbf{NBI}(\mu,\sigma)$, is given by

$$p_Y(y|\mu,\sigma) = \frac{\Gamma(y+\frac{1}{\sigma})}{\Gamma(\frac{1}{\sigma})\Gamma(y+1)} \left(\frac{\sigma\mu}{1+\sigma\mu}\right)^y \left(\frac{1}{1+\sigma\mu}\right)^{1/\sigma}$$

for $y = 0, 1, 2, ...$, where $\mu > 0$, $\sigma > 0$ with $E(Y) = \mu$ and $Var(Y) = \mu + \sigma\mu^2$. [This parameterization is equivalent to that used by Anscombe (1950) except he used $\alpha = 1/\sigma$, as pointed out by Johnson *et al.* (1993), p 200, line 5.]

Second parameterization: Negative Binomial type II (NBII)

The probability function of the negative binomial distribution type II, denoted here as $\mathbf{NBII}(\mu,\sigma)$, is given by

$$p_Y(y|\mu,\sigma) = \frac{\Gamma(y+\mu/\sigma)\sigma^y}{\Gamma(\mu/\sigma)\Gamma(y+1)(1+\sigma)^{y+\mu/\sigma}}$$

for $y = 0, 1, 2, ...,$, where $\mu > 0$ and $\sigma > 0$. Note $E(Y) = \mu$ and $Var(Y) = (1+\sigma)\mu$, so σ is a dispersion parameter [This parameterization was used by Evans (1953) as pointed out by Johnson *et al* (1993) p 200 line 7.]

A.10.3 Poisson-inverse Gaussian distribution (PIG)

The probability function of the Poisson-inverse Gaussian distribution, denoted by **PIG**(μ,σ), is given by

$$p_Y(y|\mu,\sigma) = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \frac{\mu^y e^{1/\sigma} K_{y-\frac{1}{2}}(\alpha)}{(\alpha\sigma)^y y!}$$

where $\alpha^2 = \frac{1}{\sigma^2} + \frac{2\mu}{\sigma}$, for $y = 0, 1, 2, ..., \infty$ where $\mu > 0$ and $\sigma > 0$ and $K_\lambda(t) = \frac{1}{2}\int_0^\infty x^{\lambda-1}\exp\{-\frac{1}{2}t(x+x^{-1})\}dx$ is the modified Bessel function of the third kind. [Note that the above parameterization was used by Dean, Lawless and Willmot (1989). It is also a special case of the `gamlss.family` distribution SI(μ, σ, ν) when $\nu = -\frac{1}{2}$.]

A.10.4 Delaporte distribution (DEL)

The probability function of the Delaporte distribution, denoted by **DEL**(μ,σ,ν), is given by

$$p_Y(y|\mu,\sigma,\nu) = \frac{e^{-\mu\nu}}{\Gamma(1/\sigma)}\left[1 + \mu\sigma(1-\nu)\right]^{-1/\sigma} S \tag{A.48}$$

where

$$S = \sum_{j=0}^{y} \binom{y}{j} \frac{\mu^y \nu^{y-j}}{y!} \left[\mu + \frac{1}{\sigma(1-\nu)}\right]^{-j} \Gamma\left(\frac{1}{\sigma} + j\right)$$

for $y = 0, 1, 2, ..., \infty$ where $\mu > 0$, $\sigma > 0$ and $0 < \nu < 1$. This distribution is a reparameterization of the distribution given by Wimmer and Altmann (1999) p 515-516 where $\alpha = \mu\nu$, $k = 1/\sigma$ and $\rho = [1 + \mu\sigma(1-\nu)]^{-1}$. The mean of Y is given by $E(Y) = \mu$ and the variance by $Var(Y) = \mu + \mu^2\sigma\,(1-\nu)^2$.

A.10.5 Sichel distribution (SI, SICHEL)

First parameterization (SI)

The probability function of the first parameterization of the Sichel distribution, denoted by **SI**(μ,σ,ν), is given by

$$p_Y(y|\mu,\sigma,\nu) = \frac{\mu^y K_{y+\nu}(\alpha)}{(\alpha\sigma)^{y+\nu} y! K_\nu(\frac{1}{\sigma})} \tag{A.49}$$

where $\alpha^2 = \frac{1}{\sigma^2} + \frac{2\mu}{\sigma}$, for $y = 0, 1, 2, ..., \infty$ where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$ and $K_\lambda(t) = \frac{1}{2}\int_0^\infty x^{\lambda-1}\exp\{-\frac{1}{2}t(x+x^{-1})\}dx$ is the modified Bessel function of the third kind. Note that the above parameterization is different from Stein, Zucchini and Juritz (1988) who use the above probability function but treat μ, α and ν as the parameters. Note that $\sigma = [(\mu^2+\alpha^2)^{\frac{1}{2}} - \mu]^{-1}$.

Second parameterization (SICHEL)

The second parameterization of the Sichel distribution, Rigby, Stasinopoulos and Akantziliotou (2007), denoted by **SICHEL**(μ,σ,ν), is given by

$$p_Y(y|\mu,\sigma,\nu) = \frac{(\mu/c)^y K_{y+\nu}(\alpha)}{y!\,(\alpha\sigma)^{y+\nu}\,K_\nu\left(\frac{1}{\sigma}\right)} \tag{A.50}$$

for $y = 0, 1, 2, ..., \infty$, where $\alpha^2 = \sigma^{-2} + 2\mu(c\sigma)^{-1}$. The mean of Y is given by $E(Y) = \mu$ and the variance by $Var(Y) = \mu + \mu^2\left[2\sigma(\nu+1)/c + 1/c^2 - 1\right]$.

A.10.6 Zero inflated poisson (ZIP, ZIP2)

First parameterization (ZIP)

Let $Y = 0$ with probability σ and $Y \sim Po(\mu)$ with probability $(1-\sigma)$, then Y has a zero inflated Poisson distribution, denoted by **ZIP**(μ,σ,ν), given by

$$p_Y(y|\mu,\sigma) = \begin{cases} \sigma + (1-\sigma)e^{-\mu}, & \text{if} \quad y = 0 \\ (1-\sigma)\frac{\mu^y}{y!}e^{-\mu}, & \text{if} \quad y = 1,2,3,\ldots \end{cases} \tag{A.51}$$

See Johnson *et al* (1993), p 186, equation (4.100) for this parametrization. This parametrization was also used by Lambert (1992). The mean of Y in this parametrization is given by $E(Y) = (1-\sigma)\mu$ and its variance by $Var(Y) = \mu(1-\sigma)\left[1+\mu\sigma\right]$.

Second parameterization (ZIP2)

A different parameterization of the zero inflated poisson distribution, denoted by **ZIP2**(μ,σ,ν), is given by

$$p_Y(y|\mu,\sigma) = \begin{cases} \sigma + (1-\sigma)e^{-\left(\frac{\mu}{1-\sigma}\right)}, & \text{if} \quad y = 0 \\ (1-\sigma)\frac{\mu^y}{y!(1-\sigma)^y}e^{-\left(\frac{\mu}{1-\sigma}\right)}, & \text{if} \quad y = 1,2,3,\ldots \end{cases} \tag{A.52}$$

The mean of Y in (A.52) is given by $E(Y) = \mu$ and the variance by $Var(Y) = \mu + \mu^2\frac{\sigma}{(1-\sigma)}$.