Pairwise Ranking via Stable Committee Selection

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Abstract

In this project we will study the approximation algorithms for the problem of stable committe selection. We will mainly study the recent results by [17]. Given a set of m candidates with different weights each, n voters express their preferences over the candidates. The goal of the stable committee selection problem is to select a subset of candidates, whose weight is the sum of the weights of all the individual candidates in the subset, such that no coalition of voters strictly prefers another subset of proportional weight. The main contributions of [17] are a constant factor approximation to this problem, under some mild assumptions. Stability here can also be thought of as a group fairness notion, where a coalition of voters corresponds to a group of voters, whose preferences should not be neglected.

The problem of stable committee selection generalises several well-studied problems such as ranking, facility location, participatory budgeting, approval voting, network design, to name a few. In this project, we specifically study the problem of RANKING. *The way it is defined, this problem suffers pratical applicability. In this project, we define a variant of the ranking problem, which we call* PAIRWISE RANKING, *that overcomes the drawbacks of the ranking problem. If S and T are two committees, the preference of the voters are defined based on the pariwise winners between each of the candidates is S to those in T. This definition has practical applicability compared to the preference order based only on the top most ranked candidates in S and T. We show that the ordinal preferences defined this way satisfy all the conditions necessary to use the algorithms proposed in the paper, and thus, we obtain a 32-approximately stable committee.*

1 Introduction

Fair resource allocation has been a challenging problem in social choice theory. A famous example for this is the allocation of budget to social welfare projects. The memebers of the community each have ordinal preferences on the social welfare projects that can be carried out in the community. The goal of the governing body is then to choose a subset of projects that can be carried out within the budget limits while satisfying majority of the preferences of the community. We note that it may be impossible to satisfy all the memebers of the community at the same time due to potential transitive nature of the preferences of the voters. This is in some sense similar to the Arrow's impossibility theorem [2]. This impossibility result states the following (informally),

Theorem 1 (Arrow's Impossibility Theorem [2])**.** *When voters have three or more distinct alternatives (options), no ranked voting electoral system can convert the ranked preferences of individuals into a community-wide (complete and transitive) ranking while also meeting a specified set of criteria: unrestricted domain, non-dictatorship, Pareto efficiency, and independence of irrelevant alternatives (IIA).*

It is easy to see this theorem using cyclic preferences of the type in the famous game of *rock-paper-scissors*. Let v_1, v_2, and v_3 be the voters participating in the selection of a committee of size 1 from the set of committees $\{a, b, c\}$. The preferences of the voters are as follows,

1. v_1 has the preferences $\{a\} \succ \{b\} \succ \{c\}$.
2. v_2 has the preferences $\{b\} \succ \{c\} \succ \{a\}$.
3. v_3 has the preferences $\{c\} \succ \{a\} \succ \{b\}$.

Also, note that the preferences have to satisfy *transitivity*. Then, there will not be a committee of size 1 that is preferred by the majority of the voters. Therefore, majority of the voters are unhappy no matter what committee is chosen. More specifically, the IIA criteria is not satisfied in this case. This forms the motivation of the Arrow's impossibility theorem.

Fairness in preference aggregation. Arrow's impossibility theorem suggests that a preference aggregation algorithm may not be *fair* to certain individuals. The goal of fair aggregation of individual preferences, interests, or opinions, to reach a social welfare, has been to take into account the demographics of the individuals involved. To this end, several researcher have studied some relaxed variants of the Arrow's impossibility theorem by dropping some of the criteria such as IIA, non-dictatorship, etc.

The Voting/Condorcet paradox [9], Gibbard–Satterthwaite theorem [16], the median voter theorem [6], and May's theorem [18] are some of the well-known results in the social choice theory.

1.1 Stable Committees

In this section we introduce the problem of committee selection, and define the fairness notion called stability for the problem of committee selection. Committee selection is an abstract resource allocation model that generalizes many combinatorial problems such as ranking, network design, participatory budgeting, facility location to name a few. Formally, we are given a set of voters (agents) $\mathcal{N} = [n] = \{1, 2, \ldots, n\}$, and a set of candidates $\mathcal{C} = [m]$, where each candidate i is associated with a weight $s_i \geq 0$. A committee is a subset of candidates. For any such committee, $S \subseteq \mathcal{C}$, we define the weight of the committee as $w(S) = \sum_{i \in S} s_i$. Each voter $v \in \mathcal{N}$ expresses their ordinal preferences $\succeq_v$ over all possible committees. Let $S_1, S_2 \subseteq \mathcal{C}$ be any two committees. For any voter $v \in \mathcal{N}$, the voter is said to strictly prefer committee S_1 over S_2 iff $S_1 \succ_v S_2$. Similarly, weak preference of S_1 over S_2 is indicated by $S_1 \succeq_v S_2$. Note that for simplicity the authors show the results for weak preferences. The results easily extend to strong preferences as well. Following are the assumptions made about the preferences of any voter $v \in \mathcal{N}$ over the committees,

- **(Complete)** $\forall S_i, S_j \subseteq \mathcal{C}$, either $S_i \succeq_v S_j$ or $S_j \succeq_v S_i$.
- **(Transitive)** For any $S_1, S_2, S_3 \subseteq \mathcal{N}$, if $S_1 \succeq_v S_2$ and $S_2 \succeq_v S_3$, then $S_1 \succeq_v S_3$.
- **(Monotone)** If $S_1 \subseteq S_2$ then $S_2 \succeq_v S_1$.

Additionally we are also given a budget constraint $K \geq 0$. Then, the goal is to find a stable committee of size at most K. We use the following useful definition to formalize the definition of fairness in committee selection. Consider notion of pairwise score defined below.

Definition 1 (Pairwise Score). *Given two committees $S_1, S_2 \subseteq \mathcal{C}$, the pairwise score of S_2 over S_1 is the number of voters who strictly prefer S_2 to S_1 : $V(S_1, S_2) := |\{v \in N \mid S_2 \succeq_v S_1\}|$.*

Then the fairness in committee selection via stability is defined as follows,

Definition 2 (Stable Committees). *Given a committee $S \subseteq \mathcal{C}$ of weight at most K, the weight limit, we say that a committee $S' \subseteq \mathcal{C}$ of weight K' blocks S iff $V(S, S') \geq \frac{K'}{K} \cdot n$. A committee S is stable if there are no committees S' that block it.*

However, [8] show that a stable committee may not always exist. This is easy to show with the cyclic preferences similar to the ones shown in the previous example.

Therefore the authors in [17] find an approximate solution to the selection of stable committees. Formally, they consider the problem of selecting a c-approximately stable committee of weight at most K, defined as follows,

Definition 3 (c-Approximately Stable Committees). *Given a parameter $c \geq 1$, and a committee $S \subseteq \mathcal{C}$ of weight at most K, the weight limit, we say that a committee $S' \subseteq \mathcal{C}$ of weight K' c-blocks S iff $V(S, S') \geq c \cdot \frac{K'}{K} \cdot n$. A committee S is c-approximately stable if there are no committees S' that c-block it.*

1.2 Main Results

The main contribution of the paper is to show the existence of a 32-approximately stable committee of weight at most K.

Theorem 2 (32-approximately stable committee)**.** *For any monotone preference structure with n voters and m candidates, arbitrary weights and the cost-threshold $K \geq 0$, a 32-approximately stable committee of weight at most K always exists.*

Even though a lower bound of 2 on the approximation factor can be easily established, a constant factor upper bound has been elusive in the prior work, even in the special cases of APPROVAL SET and FACILITY LOCATION. Hence, this result is crucial for the study of stable committee selection.

1.3 Additional Results

Following is a compilation of all the corollaries as a result of Theorem 2. Additionally we also enlist two new preliminary results (Theorem 3 and Theorem 4) by the authors that improve the approximation factor in Theorem 2 for some special cases.

1. **Algorithm.** The authors give an algorithm to construct 32-approximately stable committee given n voters with preferneces over m candidates, arbitrary weights over the candidates, and a budget $K \geq 0$. However, running time of this algorithm is exponential in the size of the input. Hence, they also solve a variant of the c-approximately stable committee selection that is defined as follows,

   Definition 4 ((c, L)-Approximately Stable Committee)**.** *A committee $S \subseteq \mathcal{C}$ of weight at most an integer value K is (c, L)-approximately stable for $1 \leq L \leq K$ if there is no committee S' with at most L candidates such that $V(S,'') \geq c \cdot \frac{w(S')}{K} \cdot n$.*

2. **MWU.** The authors give a Multiplicative Weights Update (MWU) based polynomial time algorithm to compute approximate solution to stable committee selection where the committees can maximum be of size L as defined in Definition 4. Then we have the following corollary,

   Corollary 1. *For any $1 \leq L \leq K$, a $(32+\epsilon, L)$-approximately stable committee can be computed in time poly$\big(m^L, n, \frac{1}{\epsilon}\big)$.*

   Moreover, in the problem of RANKING, where the weights of the committees are additive, due to the observation that a committee is c-approximately stable iff it is $(c, 1)$-approximately stable, we have the following implication,

   Corollary 2. *For sufficiently small constant $\epsilon > 0$, a $(32 + \epsilon)$-approximately stable committee for Ranking and Facility Location preferences, even when candidates have arbitrary (additive) weights, can be computed in poly$\big(m, n, \frac{1}{\epsilon}\big)$ time.*

3. **Sub-additive weights.** The only property used in the proofs of the main results regarding the weights of the candidates is that for any two committees S and T, $w(S \cup T) \leq w(S) + w(T)$. Hence, we have the following corollary,

   Corollary 3. *There is a 32-approximately stable committee for any subadditive weight function $w(S)$ over committees, and any monotone preferences of the voters.*

4. **Multiple constraints.** The authors also show that their results extend to the case when there are multiple budget constraints. For example, consider construction projects taking place in a community. Then the weights of the candidates (library, auditorium, etc.) can vary for various resources (concrete, paints etc.) Then we have $\mathcal{Q}$ different sub-additive weight functions $w_1, \ldots, w_{\mathcal{Q}}$, and $\mathcal{Q}$ different budget constraints $K_1, \ldots, K_{\mathcal{Q}}$. The goal is to select a committee S such that all the $\mathcal{Q}$ constraints $w_j(S) \leq K_j$, for all $j \in [\mathcal{Q}]$ are satisfied. This problem can be formally defined as follows,

   Definition 5 (Stable Committees: Multiple Constraints)**.** *Given a committee $S \subseteq \mathcal{C}$ of weight at most $(K_1, K_2, \ldots, K_{\mathcal{Q}})$, a committee $S' \subseteq \mathcal{C}$ of weight $(K'_1, K'_2, \ldots, K'_{\mathcal{Q}})$ blocks S iff $V(S, S') \geq \frac{K'_j}{K_j} \cdot n$ for all $j \in [\mathcal{Q}]$. A committee S is stable if no committee S' blocks it.*

   For this variant also, the proof of the main result goes through with minor modificaitons, and thus we have the following corollary,

Corollary 4. *There is a* 32*-approximately stable committee in the setting with* $\mathcal{Q} \geq 1$ *resources.*

5. **Lower bound.** The authors show that for the special case of RANKING, the approximation factor can not be better than 2.

   Theorem 3. *In the unweighted Ranking setting, for any constant* $\epsilon > 0$*,* $(2 - \epsilon)$*-approximately stable deterministic committees of integral size* K *may not exist.*

6. **Exactly stable lotteries.** This is a stronger version of an intermediate result used in the proof of the main result, for a special case. Particularly, if the weights of the candidates are always 1, then for $K \in \{1, 2, 3\}$, the authors give a better construction than one used to prove the main result to obtain an exactly stable lottery.

   Theorem 4. *For unit-weight candidates and any number of voters with arbitrary monotone preferences, when* $K \in \{1, 2, 3\}$*, an exactly stable lottery always exists.*

2 Related work

The general idea of stable committee selection dates back to 1881 [10], and has been extensively studied since then. Related to the notion of stability is the concept of *core* in cooperative game theory. The term core was coined by Scarf [20] and was majorly studied in public-good settings. In the literature, researcher have often considered convex prefrences, where a voter has preferences over lotteries. Let there be additive utility functions u_v for each voter v over the lotteries Δ over the collection of the committees. Then we can define the stability with respect to the expected utlity of the voters for the lotteries. Formally, voters deviate to another lottery if their expected utility increases. Similar to the definitions in [17], we say that a lotter Δ' blocks Δ if there is a coalition of voters of size at least $n \cdot |S'|/K$ such that for all voters v in this coalition, we have,

$$\mathbf{E}_{S' \sim \Delta'} \left[u_v(S') \right] \geq \mathbf{E}_{S \sim \Delta} \left[u_v(S) \right],$$

with at least one strict inequality. This notion is an extension to the seminal work [15] in which Foley showed that a Lindah's market equillibrium always exists. One crucial difference between this notion of stability with that defined in [17] is that in the fromer the voters compare the expected utility fromt he lottery with the utility on deviation, while in the latter, a lottery is first realised, and then the voters who see higher utility deviate. These two notions are incomparable as existence of one notion of stability does not imply the exitence of the other. Lindahl's equillibrium gives a fractional stable committee. However, it is not known how to efficiently compute this fractional solution or to round the solution to give an approximately stable integer solution.

Stable committee selection is also similar to extensive work on voting rules [11]. Here, each voter gets a utility $u(S)$ if committee S is selected. From this, a score $\sigma(S)$ is constructed. Due to this score, any two committees are comparable and satisfy all the properties mentioned in Section 1. The goal is then to compute a committee that maximizes $\sum_v \sigma_v(S)$. For the preferences defined in APPROVAL SET, the Proportional Approval Voting (PAV) method which is about a century old [21], assigns scores $\sigma_v(S) \approx \log(1 + |A_v \cap S|)$, where A_v is the set of candidates approved by the voter v. Whereas, the Nash Welfare objective assigns a score $\sigma_v(S) = \log(u_v(S))$ [1, 4, 21, 12, 13, 14]. However, stability here means that there is no coalitioin of voters who when deviated get higher "utility". In general, the notion of stability based on utility requires that the utilities are given by an imputing cardinal utility functions. One can not use the more cannonical ordinal utilities, which is the case in the paper we study. Moreover, ordinal preferences enable us to study stability with respect to the deviation to a lower sized committees, unlike deviations in utilities. In some cases such notions of stability can be very strong a requirement. For example, in APPROVAL SET PAV method gives no better than $\Omega(\sqrt{K})$-approximation to a stable outcome. Hence, these two problems are not comparable.

In stable committee selection, the coalition of voters that deviate could be arbitrary. In cases where there is a certain cohesiveness among the voters such that only the subset of voters who are highly cohesive can deviate at a time (this is called *Justified representation* [3, 5]), much stronger guarantees can be shown. The algorithm called Proportional Approval Voting (PAV) [21] achieves near-optimal solutions. However, the PAV method can not approximate the core stability, which is the stability notion in the selection of stable committees. Another model is to approximate the utility of the deviating coalitions by a constant factor. A constant factor approximation has been obtained for clustering [7], and APPROVAL SET [19].

3 Main Result of [17]: 32-Approximately Stable Committees

First we note that an exactly stable committee is not always possible. This was proved by example in [8]. The authors construct a partially cyclic preference model on a set of 6 voters $\{1, 2, 3, 4, 5, 6\}$ and a set 6 candidates $\{a, b, c, d, e, f\}$ with unit weights. The preferences are given as follows,

Voters	Preferences
1	a $\succ$ b $\succ$ c $\succ$ d $\succeq$ e $\succeq$ f
2	b $\succ$ c $\succ$ a $\succ$ d $\succeq$ e $\succeq$ f
3	c $\succ$ a $\succ$ b $\succ$ d $\succeq$ e $\succeq$ f
4	d $\succ$ e $\succ$ f $\succ$ a $\succeq$ b $\succeq$ c
5	e $\succ$ f $\succ$ d $\succ$ a $\succeq$ b $\succeq$ c
6	f $\succ$ d $\succ$ e $\succ$ a $\succeq$ b $\succeq$ c

When $K = 3$, we can select at most three candidates since each candidate is of unit weight. Consider a partition of the candidates into the sets $\{a, b, c\}$ and $\{d, e, f\}$. It is easy to see that any committee can choose either at most one candidate from the first set and rest from the other, or at most one candidate from the second set and remaining from the other. W.l.o.g., assume that its the first case. Then we construct a committee S where the candidate a is chosen (or no one in $\{a, b, c\}$ is chosen). Then the voters $\{2, 3\}$ can form a coalition and deviate to choose a committee $S' = \{c\}$ instead. The committee S' blocks the committee S by Definition 2 since, $2 = V(S, S') = \frac{K'}{K} \cdot n = \frac{1}{3} \cdot 6 = 2$. Therefore, it is impossible to always find a stable committee. Hence, the authors use the notion of approximate stability as defined in Definition 3.

Key idea. In the paper, the authors first define a *lottery* as a distribution over all possible committees of $\mathcal{C}$ via which they capture the notion of "randomised" stability. This is the first such definition of randomised stability for committee selection. We will see that this is indeed one of the key ideas that enables a constant factor approximation factor for this problem. Informally, this definition enables the use of the Von Neumann's minimax theorem for the two-player zero sum games [22], and makes the analysis much simpler.

The proof of the main result proceeds in two steps, (1) constructing a lottery (randomization) over the set of committees of weight K that is 2-approximately stable and (2) iteratively rounding this solution to give a deterministic 32-approximately stable committee. Note that this is an existential result. Therefore, we ignore details of the exponential running of the algorithm to construct a 2-approximately lottery in step 1, and methods to improve it for special cases. Improving the running time for general case is beyond the scope of this project.

3.1 Step 1: Constructing 2-approximately stable lotteries.

The formal definition of a stability of lotteries is as follows,

Definition 6 (Stable lotteries)**.** *A distribution (or lottery) Δ over committees of weight at most K is c-approximately stable iff for all committees $S' \subseteq \mathcal{C}$ of weight K', we have*

$$\mathbf{E}_{S\sim\Delta}\left[V(S, S')\right] < c \cdot \frac{K'}{K} \cdot n.$$

In the paper the authors consider the dual formulation of selecting stable lotteries. In particular, they formulate it as a zero-sum game with two players – *attacker* and *defender*. Note that the pure strategies of both the defender and the attacker are the committees formed by the candidates. Since there m candidates, there are $t = 2^m$ possible committees. Therefore the strategy set is $2^{[m]}$, the power set of the candidate set,

$$2^{[m]} = \{\phi, \{1\}, \ldots, \{m\}, \{1, 2\}, \ldots, \{1, 2, \ldots, m\}\}.$$

Note that null set is also a possible committee if the budget $K < w_j, \forall j \in [m]$, i.e., the budget is not sufficient to include even one of the candidates. Now, mixed strategy is a distribution over the set of all the strategies, that is the committees. From Definition 6, a lottery is nothing but a mixed strategy of a player.

The defender plays mixed strategy Δ_d over the committes of weight at most K. This is the defender stable lottery. The attacker on the other hand plays a pure strategy S_a that c-blocks the defending lottery Δ_d. Then, the utilities are given as follows,

$$u_{\text{defender}} = \mathbf{E}_{S_d \sim \Delta_d}\left[V(S_d, S_a) - c \cdot \frac{w(S_a)}{K} \cdot n\right] = -u_{\text{attacker}}.$$

Clearly, defender gets a positive utility if the committee chosen by the attacker S_a does not c-block the committe S_d sampled from the lottery Δ_d, in expectation. Similarly, the attacker gets a positive utility, if the committee S_a c-blocks the committee S_d sampled from the lottery Δ_a, in expectation.

Then the existence of a c-approximately stable lottery is equivalent to the deciding feasiblity of the following minmax problem,

$$\min_{\Delta_d} \quad \max_{S_a} \quad \mathbf{E}_{S_d \sim \Delta_d}\left[V(S_d, S_a) - c \cdot \frac{w(S_a)}{K} \cdot n\right] < 0. \tag{1}$$

Due to the Von Neumann's minimax theorem of the two-palyer zero-sum games [22], the solution to the problem defined by (1) coincides with the solution of the following maxmin optimisation problem,

$$\max_{\Delta_a} \quad \min_{S_d : w(S_d) \leq K} \quad \mathbf{E}_{S_a \sim \Delta_a}\left[V(S_d, S_a) - c \cdot \frac{w(S_a)}{K} \cdot n\right] < 0, \tag{2}$$

where Δ_a is a lottery over committees of weight at most K chosen by the attacker. Notice that in this dual formulation, the attacker plays a mixed strategy whereas the defender plays a pure strategy. The advantage of this dual representation is that it is enough to show that given any mixed strategy (attacking lottery), we need to construct a deterministic defending committee S_d such that the committes in Δ_a do not c-block S_d in expectation. Therefore, this dual view provides a convenient tool for showing the existence of approximately stable lotteries.

We then have the following theorem about the existence of the approximately stable lotteries.

Theorem 5. *For any value K and all monotone preferenes, a 2-stable approximately stable lottery over committees of weight at most K always exists.*

We first show the next step in the algorithm for continuity and later give the proof-sketch of this theorem.

3.2 Step 2: Iteratively rounding stable lottery to a stable committee

From Theorem 5 we know that a 2-approximately stable lottery exists. In the paper, the authors only work with a brute-force algorithm to construct this 2-approximately stable lottery. This can be done by simply enumerating over all possible sets of committees S, S' and computing $V(S, S')$. This algorithm runs in time exponential in K. Since we only need to show existence of a stable committee, we proceed with this brute-force algorithm.

Let $\text{LOTTERY}(\mathcal{V}, K)$ be a sub-routine that gives a 2-approximately stable committee for the candidates $[m]$, voters given by the set $\mathcal{V}$ and the budget constraint $K \geq 0$. Using this as a subroutine, the authors provide Algorithm 1 that iteratively rounds a given lottery at each iteration to a constant fraction of candidates in the final stable committee that has an approximation factor of at most 32. Hence, this algorithm proves the existential result in Theorem 2.

The high-level idea of the proof is to fix a deviating committee S_a of weight $w(S_a)$ and come up with a committee T such that $V(T, S_a) < 32 \cdot \frac{w(S_a)}{K} \cdot n$. The proof proceeds by first defining a set of *good* committes $\mathcal{G}_v$ and *bad* committees $\mathcal{B}_v$ relative to a given lottery Δ for each voter v. A committee S is considered good (or bad) if the probability mass with repsect to Δ of all the committees that are better (not better) than S in the preference order of the voter v is not more than a constant. The idea is that the good committees appear sufficiently high up (with respect to Δ) in voter v's ranking, while the bad committees are lower down in the ranking.

Then the authors observe the following about the good and the bad committees,

1. Any committee S_a cannot lie in too many good sets relative to its weight.
2. There is some committee (with non-zero support in Δ) that does not lie in more than a constant fraction of the bad sets.

Algorithm 1 Iterated Rounding

$t \leftarrow 0$; $\mathcal{V}^{(0)} \leftarrow [n]$; $T^{(0)} \leftarrow \phi$; $K^{(0)} \leftarrow (1-\alpha)K$.
while $\mathcal{V}^{(t)} \neq \phi$ **do**
  $\Delta^{(t)} \leftarrow \textsc{Lottery}(\mathcal{V}^{(t)}, K^{(t)})$.
  Let $S^{(t)}$ be any committee such that $\left|\left\{v \in \mathcal{V}^{(t)} \;\middle|\; S^{(t)} \notin \mathcal{B}_v\left(\Delta^{(t)}\right)\right\}\right| \geq (1-\beta) \cdot |\mathcal{V}^{(t)}|$.
  $\mathcal{W}^{(t)} \leftarrow \left\{v \in \mathcal{V}^{(t)} \;\middle|\; S^{(t)} \notin \mathcal{B}_v\left(\Delta^{(t)}\right)\right\}$.
  $\mathcal{V}^{(t+1)} \leftarrow V^{(t)} \setminus \mathcal{W}^{(t)}$.
  $T^{(t+1)} \leftarrow T^{(t)} \cup S^{(t)}$.
  $K^{(t+1)} \leftarrow \alpha K^{(t)}$.
  $t \leftarrow t+1$.
end while
return $T^f \leftarrow T^{(t)}$.

3. For any committee S that is not a bad committee for voter v, if we have that S_a is preferred by the voter v over the committee S then S_a must be a good committee for v.
4. Then the crucial observation that follows immediately from all of the above is that,

   Any committee that does not lie in more than constant fraction of the bad sets is preferred by many voters over the attacking committee.

   Hence, it is straightforward to show that we need to add such a set in our final committee.

Because of the observation 4, we have the iterated rounding algorithm given in Algorithm 1.

Description of the algorithm. The full procedure is shown in Algorithm 1. The set $\mathcal{V}^{(t)}$ is the set of voters whose preferences have been largely satisfied by the end of tth iteration. Therefore, initially we set $\mathcal{V}^{(0)} := [n]$. The set $T^{(t)}$ is the set of candidates at iteration t. This keeps track of the candidates that are output. Hence, T^f is the final output. The budget $K^{(t)}$ is initially set to $(1-\alpha)K$ and reduced in each iteration. In each iteration t, the algorithm finds a 2-approximately stable lottery $\Delta^{(t)}$ given by the subroutine Lottery for the remaining set of voters and the remaining budget. If there is a committee $S^{(t)}$ such that for a constant fraction of the voters this committee is not bad, then by Observation 4, the attacking committee S_a will not be preferred by all these voters over $S^{(t)}$. Hence by including $S^{(t)}$, the algorithm essentially ensuring that in each itertion at least a constant fraction of the voters won't form coalition and deviate. These voters are removed from the set $\mathcal{V}^{(t+1)}$, $S^{(t)}$ is added (by union) to the final output, and the budget is decreased by a constant factor. When the voters are exhausted, the algorithm terminates.

3.3 Proof-Sketch of Theorem 5

In this section we show the proof-sketch of the result that a 2-approximately stable lottery always exists. Assume we are given a lottery Δ_a. We can assume Δ_a only has committees with weight at most $\frac{K}{2}$, because otherwise, any committee is preferred over a committees in Δ_a with weight more than $\frac{K}{2}$ Now the authors construct a defending committee S_d with weight at most K, such that the condition in Definition 6 is satisfied.

Suppose that the strategy Δ_a chooses S_1 with probability α_1, committee S_2 with probability $\alpha_2, \ldots,$ S_t with α_t, where $t = 2^{[m]}$. Let $\beta = \mathbf{E}_{S_a \sim \Delta_a}\left[\frac{w(S_a)}{K}\right] = \frac{\sum_{i=1}^{t} \alpha_i \cdot w(S_i)}{K}$ be the ratio between the expected total weight of the attacking strategy and K, the allowable weight for the defending strategy. We need to find an S_d of weight at most K so that:

$$\mathbf{E}_{S_a \sim \Delta_a}\left[V(S_d, S_a)\right] < 2\beta n. \tag{3}$$

We will construct a distribution Δ_d over committees S_d that satisfies a stronger property:

$$\Pr_{S_d \sim \Delta_d, S_a \sim \Delta_a}\left[S_a \succ_v S_d\right] < 2\beta \qquad \forall v \in [n]. \tag{4}$$

Let Z_i be an indicator random variable of the event $S_a \succeq_v S_d$. Then, due to the linearity of expectation, $\mathbf{E}[\sum_{i\in[n]} Z_i] = \sum_{i\in[n]} \mathbf{E}[Z_i]$. Therefore, we have that existence of a lottery Δ_d satisfying Equation (3) implies the existence of a deterministic committee S_d satisfying the same. Hence, the goal is now to construct a deterministic committee S_d, that will imply Theorem 5.

The trick of using duality of the zero-sum games. The authors use dependent rounding to construct S_d from the attacking lottery Δ_a. Hence, this is where the duality of the two-player zero-sum games does the trick. If this were not the case, we would have had to come up with a defending lottery, a distribution over the committees, which could be much harder.

The procedure of depending rounding comes up with random variables X_i for committee S_i given the attacking committee Δ_a such that the random variables satisfy (1) Almost-Integrality, (2) Correct Marginals, (3) Preserved Weight and (4) Negative Correlation properties. Then using the negative correlation and the monotonicity of the ordinal preferences of the voters, it is easy to show that Inequality 4 is satisfied for some $S_d \sim \Delta_d$. Hence, Theorem 5 follows.

3.4 Proof-Sketch of Theorem 2

To prove the Theorem 2, we need a formal definition of the subroutine LOTTERY, good and bad committees. These are as follows,

Definition 7. *Given candidate set $[m]$, voter set $\mathcal{V}'$, and committee size K', let* $\text{LOTTERY}(\mathcal{V}', K')$ *return a lottery Δ over committees of weight at most K' that is 2-approximately stable for the set of voters $\mathcal{V}'$. Similarly, let $V_{\mathcal{V}'}(S, S_a) = |\{v \in \mathcal{V}' \mid S_a \succ_v S\}|$.*

Let x_i be the probability that Δ includes committee S_i.

Definition 8. *Given a voter v, we define the set of* good *and* bad *committees relative to Δ, $\mathcal{G}_v(\Delta)$ and $\mathcal{B}_v(\Delta)$ respectively, as follows:*

$$\mathcal{G}_v(\Delta) = \left\{ S \subseteq \mathcal{C} \;\middle|\; \sum_{S_i \succeq_v S} x_i \leq 1 - \beta \right\} \qquad \text{and} \qquad \mathcal{B}_v(\Delta) = \left\{ S \subseteq \mathcal{C} \;\middle|\; \sum_{S_i \preceq_v S} x_i \leq \beta \right\}.$$

Then we have the following lemmas,

Lemma 1. *If $S \notin \mathcal{B}_v(\Delta)$, then $S_a \succ_v S$ only if $S_a \in \mathcal{G}_v(\Delta)$.*

Lemma 2. *Given $\Delta = \text{LOTTERY}(\mathcal{V}', K')$, we have the following upper and lower bounds:*

1. *For all committees S_a, we have $|\{v \in \mathcal{V}' \mid S_a \in \mathcal{G}_v(\Delta)\}| < \frac{2}{\beta} \cdot \frac{w(S_a)}{K'} \cdot |\mathcal{V}'|$.*
2. *There exists S with non-zero support in Δ such that $|\{v \in \mathcal{V}' \mid S \notin \mathcal{B}_v(\Delta)\}| \geq (1-\beta) \cdot |\mathcal{V}'|$.*

Both the lemmas follow from the definitions. For brievity we omit the proofs.

It is only remaining to show that the algorithm outputs a 32-approximately stable committee T^f of size at most K. We have that $w(T^f) \leq \sum_{t \geq 1} w(S^{(t)}) \leq \sum_t \alpha^{t-1}(1-\alpha)K \leq K$.

Finally, showing that the resulting set is approximately stable, completes the proof of Theorem 2.

Lemma 3. *When $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$, then T^f is a 32-approximately stable committee of weight at most K.*

Proof-Sketch. The proof crucially uses the monotonicity of the ordinal preferences to show that $V\left(T^f, S_a\right) \leq V_{\mathcal{W}^{(t)}}\left(S^{(t)}, S_a\right)$. Now due to Observation 4 of the good and the bad committees, we have that,

$$\left|\left\{ v \in \mathcal{V}^{(t)} \;\middle|\; S_a \in \mathcal{G}_v\left(\Delta^{(t)}\right) \right\}\right| < \frac{2}{\beta} \cdot \frac{w(S_a)}{K^{(t)}} \cdot |\mathcal{V}^{(t)}|.$$

Because β fraction of the voters are removed in every iteration of the algorithm, we have, $|\mathcal{V}^{(t+1)}| \leq \beta^t n$. Furthermore, $K^{(t)} = \alpha K^{(t-1)}$, so that $K^{(t+1)} = \alpha^t(1-\alpha)K$. Then it is easy to show that,

$$V\left(T^f, S_a\right) < \frac{2\alpha}{\beta(1-\alpha)(\alpha-\beta)} \cdot \frac{w(S_a)}{K} \cdot n.$$

This is minimized when $\beta = \frac{\alpha}{2}$. Setting $\alpha = \frac{1}{2}$, this is at most $32 \cdot \frac{w(S_a)}{K} \cdot n$, completing the proof. $\square$

4 My Ideas: PAIRWISE RANKING

In this section, we define a new preference model called PAIRWISE RANKING that overcomes the drawbacks of RANKING. We then show that, like RANKING, PAIRWISE RANKING is also a special case of stable committee selection. Hence, we finally show that this new preference model also has an approximate solution with approximation factor of 32.

Let there be n voters given by $\mathcal{V}$ and m candidates $\mathcal{C}$. For simplicity assume that $\mathcal{C} = [m]$. Then as defined earlier, a committee is a subset of the cadidates in $[m]$, that is, an element of the power set $2^{[m]}$. We assume that, in both the problems, each voter has a ranking over the candidates in $[m]$. Let us assume that the ranking of v over the candidates is given by the symbol $\sqsupset_v$. That is, if v ranks candidate i above the candidate j then we have $i \sqsupset_v j$. Then for a voter v, and any committe $S \in 2^{[m]}$, consider the following definition of top-k truncation of S.

Definition 9. *Given any committee $S \in 2^{[m]}$, we define S_k as the committee formed by choosing the top-k ranked candidates in S,*

$$S_k = \{i \in S : \sum_{j \in S} \mathbf{I}\left[j \sqsupset_v i\right] < k\},$$

where $\mathbf{I}[j \sqsupset_v i] = 1$ if j is ranked above i and 0 otherrwise.

This is nothing but the set of k highest ranked candidates among the candidates in S.

Then, the preference model in RANKING is defined as follows,

Definition 10 (RANKING[11])**.** *In* RANKING*, each candidate has unit weight. Each voter v has a ranking over candidates in $2^{[m]}$. In this case, $S \succ_v S'$ iff v's favorite candidate in S is ranked higher than her favorite candidate in S'.*

Here favorite candidate means the top most ranked candidate for voter v. Therefore in our notation the preference order in RANKING can be written as follows,

$$S \succeq_v S' \iff S_1 \sqsupset_v S'_1.$$

Note that by slight abuse of notation here we use $S_1 \sqsupset_v S'_1$ to mean that the candidate in the singleton set S_1 is ranked above the candidate in the singleton set S'_1. Before we define the PAIRWISE RANKING, consider the following useful definition,

Definition 11 (Pairwse Majority Score)**.** *Let each voter v have a ranking $\sqsupset_v$ over all the m candidates in the set $[m]$. Let $2^{[m]}$ be the collection of all the subsets of $[m]$. Let $S, S' \in 2^{[m]}$ be any two committees in $2^{[m]}$. Let $k = \min\{|S|, |S'|\}$. Then for voter v, the pairwise majority score function $h_v : 2^{[m]} \times 2^{[m]} \to \mathbf{Z}_{\geq 0}$ is given by the following,*

$$h_v(S, S') = \sum_{i \in S_k} \sum_{j \in S'_k} \mathbf{I}\left[i \sqsupset_v j\right].$$

That is, the pairwise majority score of S compared to S' is the number of pairs $(i, j) \in S_k \times S'_k$ such that candidate i is ranked above the candidate j in the ranking of voter v. Then we define PAIRWISE RANKING, which, to the best of our knowledge, has not been studied before,

Definition 12 (PAIRWISE RANKING)**.** *In* PAIRWISE RANKING*, each candidate has unit weight. Each voter v has a a ranking over candidates in $2^{[m]}$. In this case,*

$$S \succeq_v S' \iff h_v(S, S') \geq h_v(S', S).$$

We now show that this definition addresses several drawbacks of RANKING with respect to the preferences of the committees by a voter. Consider the following examples.

Example 1. *Let there be n voters and m candidates given the set $[m]$. W.l.o.g. let us assume that a voter v ranks candidates $i, j \in [m]$ such that $i \sqsupset_v j$ whenever $i < j$. That is v's ranking over $[m]$ is given by $1 \sqsupset_v 2 \sqsupset_v \cdots \sqsupset_v m$. Then, consider two committees $S = \{1, 2, 10\}$, and $S' = \{2, 3, 4\}$. In case of* RANKING*, $S \succ_v S'$ because $S_1 = \{1\}$, $S'_1 = \{2\}$, and $1 \sqsupset_v 2$. We have that $k = \min\{|S|, |S'|\} = 3$. Therefore, even in case of* PAIRWISE RANKING *we have $S \succ_v S'$. This is because $h_v(S, S') = 5 > 3 = h_v(S', S)$.*

We observe that the voter must be satisfied with this preference ordering because two of her top most ranked candidates are present in the preferred committee. However, consider another example,

Example 2. *Let $S = \{1, 19, 20\}$ and $S' = \{2, 3, 4\}$ be two committees. In this case,* RANKING *gives the preference order $S \succ_v S'$ because $S_1 = \{1\}$, $S'_1 = \{2\}$, and $1 \sqsupset_v 2$, as in case of the previous example. But* PAIRWISE RANKING *gives the preference order $S' \succ_v S$ because $h_v(S, S') = 3 < 6 = h_v(S', S)$.*

Clearly in Example 2, the voter might be left unsatisfied with RANKING model because apart from the first candidate, two other candidates are ranked very low in the preferred committee. In situations where each of these ranking positions carry a utility, the total utility of set S could be very small, hence undesirable. This is in fact a common occurence in many real world scenarios. Let us assume that in a city there are candidate plots for building hospitals. Depending on the location of their homes, let us assume each person has a ranking over the plots given by the distance between the person's home and the candidate plot. Now fix a person v. For v, let us assume the plot ranked at i is at a distance of i kilometers from her home. Then she would clearly prefer to have hospitals build at plots $\{2, 3, 4\}$ kilometers away rather than $\{1, 19, 20\}$, because in the former more hospitals are reachable faster than the those in the latter. This is exactly captured by our definition of PAIRWISE RANKING, and hence, is an important problem to study.

Note that it is not always the case that a utility function gives a ranking of the candidates or preference of the committees. For instance, it may be difficult to assign a value to rate movies, but it may be possible that a person has a ranking of the movies. But given a utility function we can always come up with the ranking or preferences. Therefore, in PAIRWISE RANKING we consider the more general ordinal preferences.

PAIRWISE RANKING is also a special case of stable committee selection. In what follows, we show that the preference model in PAIRWISE RANKING satisfies the necessary conditions to apply the result in [17], and hence show the existence of a 32-approximate solution.

4.1 A 32-approximate solution for PAIRWISE RANKING

Observe that Definition 12 gives ordinal preferences over the committees. Hence, this conditions is satisfied. Now, we have the following lemmas,

Lemma 4. *The preference model in* PAIRWISE RANKING *is complete.*

Proof. Fix a voter v. For any two committees $S, S' \in 2^{[m]}$ we have that either $h_v(S, S') \geq h_v(S', S)$ or $h_v(S', S) \geq h_v(S, S')$. Therefore, by Definition 12, either $S \succeq_v S'$ or $S' \succeq_v S$. Therefore, the preference model in PAIRWISE RANKING is complete. ☐

Lemma 5. *The preference model in* PAIRWISE RANKING *is monotone.*

Proof. Fix a voter v. Consider any two committees $S, T \in 2^{[m]}$ such that $S \subseteq T$. Let $S_0 = T_0 = \phi$. Then we have that $\forall k \in \{1, \ldots, |S|\}$, $i \in T_k \setminus T_{k-1}$, $j \in S_k \setminus S_{k-1}$,

$$i = j \qquad \text{or} \qquad i \sqsupset_v j.$$

It is easy to see that this is always true because otherwise, if $\exists k \in \{2, \ldots, |S|\}$ such that $i \in T_k \setminus T_{k-1}$, $j \in S_k \setminus S_{k-1}$, and $i \sqsubset_v j$, it implies that the candidate j in not in $T_k \setminus T_{k-1}$, therefore $S \not\subseteq T$, which is a contradiction. Therefore, the preference model in PAIRWISE RANKING is monotone. ☐

Some crucial observations from the proof of Theorem 2 are,

1. Monotonicity plays a major role in the proof of Theorem 2, especially because of the iterated rounding given in Algorithm 1. The algorithm outputs a committee T^f, which is iteratively constructed. That is, in iteration t, a committee $S^{(t)}$ that satisfies the preferences of at least a constant fraction of the voters is added to T^f by the set union operation. Therefore, $S^{(t)} \subseteq T^f$ after iteration t. Therefore, if monotonicity was not satisfied, it would not be possible to upper bound the fraction of voters that deviate from T^f by the sum of the fraction of voters that deviate from each of the $S^{(t)}$.

2. We also note that the use of top-k trucated sets S_k and S'_k in Definition 11 is essential for monotonicity. If we directly define the pairwise majority score on the sets S and S' monotonicity will not be satisfied.
3. Another crucial observation we make here is that nowhere in the proof of Theorem 2 did we require that the preference model is transitive. It is only sufficient that the preference model is complete over all possible committees and satisfies monotonicity.

Therefore, we have the following corollary,

Corollary 5 (32-approximate pairwise ranking)**.** *For the preference model given by* PAIRWISE RANKING *with n voters and m candidates, unit weights and the cost-threshold $K \geq 0$, a* 32-*approximately stable committee of weight at most K always exists.*

We also have the following corollary becasue PAIRWISE RANKING is defined for committees with unit-weights.

Corollary 6. *For unit-weight candidates and any number of voters with preference model given by* PAIRWISE RANKING*, when $K \in \{1, 2, 3\}$, an exactly stable lottery always exists.*

5 Conclusion and Open Questions

The preference model PAIRWISE RANKING proposed in this project is more practical than that of RANKING. We have shown that it is in fact a special case of the stable committee selection problem and hence there exists a constant factor approximate solution. The future work would include closing the gap between the lower bound of 2 and upper bound of 32 on this problem as well as the more general stable committee selection problem. We believe that a new special case such as PAIRWISE RANKING catalyses the research towards this goal while also being of high importance in real world applications. We formalise some of the open questions in stable committee selection,

1. **Deterministic exact committees in more specific settings.** UNWEIGHTED APPROVAL SET: In this setting, the candidates are unweighted. Among m candidates, each voter approves a subset of the candidates. Then the voter ranks (or prefers) committees based on the number of candidates in the committees the voter approves. Since the preference function here is monotone, transitive, and complete, we can use the algorithm proposed in [17] to find a 32-approximate stable committee. However, since this is a more specific setting, the question is, can we get an *exactly* stable committee in this case?
2. **Do exactly stable lotteries exist?** In [17], the authors have constructed a 2-approximately stable lotteries. The open question here is, does an *exactly* stable lottery exist? This has been resolved for RANKING and APPROVAL SET problems in [8]. However, for the generalised problem of stable committee selection this still remains an open problem.
3. **Closing the gap.** Currently the lower bound on the approximation factor for the stable committee selection is 2, whereas the upper bound is 32. It reamins as an open problem to close this gap. The authors conjecture that a 2-approximately stable committee always exists. In fact, they prove this for the specific problem of RANKING.
4. **Computing stable committees efficiently.** The bottleneck of the algorithm proposed in [17] is computing a 2-approximately stable lotteries, which takes time exponential in K. Efficient algorithms to compute approximately stable committees also is an open problem.

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