Ch. 5: Comparing Two Proportions

Navigation

By Date

- March 10: start - end
- March 12: start - end
- March 17: start - end

By Section

- 5.1: start - end
- 5.2: start - end
- 5.3: start - end

5.1: Comparing Two Groups

Summarizing Two Categorical Variables

Quentin Tarantino's films are world-famous. A dedicated 538.com reporter watched all of the films, recording the incidence of curse words and deaths in each movie (along with the time of the occurrence, and the word) (data).

You would like to know if the frequency of curse words relative to deaths is the same between *Kill Bill: Volume 1* and *Kill Bill: Volume 2*.

	Movie	Type	Time
1	Kill Bill: Vol. 2	word	1.18
2	Kill Bill: Vol. 2	word	4.83
3	Kill Bill: Vol. 1	word	7.12
4	Kill Bill: Vol. 1	word	8.10
5	Kill Bill: Vol. 1	word	8.12
6	Kill Bill: Vol. 2	word	9.02
7	Kill Bill: Vol. 2	word	9.62
8	Kill Bill: Vol. 1	word	10.45
9	Kill Bill: Vol. 1	word	10.50

Summarizing Two Categorical Variables

- What is the observational unit?
- Which columns do we need?
- How can we summarize this data?

Summarizing Two Categorical Variables

- What is the observational unit?
  Each instance of an event (curse word or death) in a *Kill Bill* movie
- Which columns do we need?
  Movie and Type
- How can we summarize this data?
  For each movie, add up the total number of deaths and curse words

A **two-way table** (or *contingency table*) of counts can be used when we have 2 categorical variables

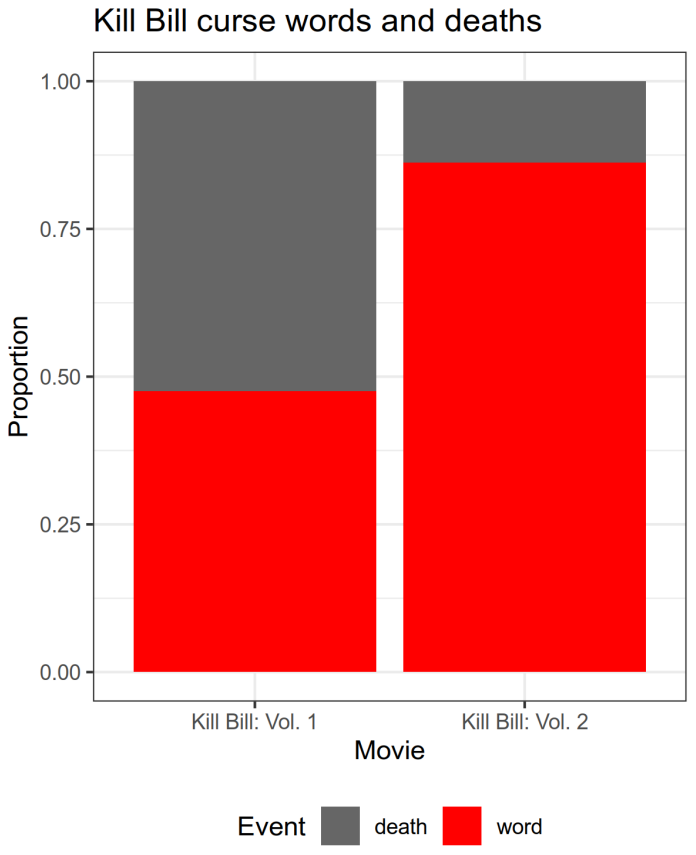
Movie	Death	Word	Total
Kill Bill: Vol. 1	63	57	120
Kill Bill: Vol. 2	11	69	80
Total	74	126	200

Summarizing Two Categorical Variables

Movie	Death	Word	Total
Kill Bill: Vol. 1	63	57	120
Kill Bill: Vol. 2	11	69	80
Total	74	126	200

- Do the counts differ? Does *Kill Bill: Vol 1* have a higher death:curse word ratio than Kill Bill: Vol 2?
- Group sizes differ - 120 events in *Kill Bill: Vol 1*, 80 in *Kill Bill: Vol 2*

Visualizing Categorical Variables



segmented bar graphs show the relative proportions of observational units in each category.

These relative proportions are called **conditional proportions**

- Of the events in *Kill Bill 1*, what proportion are deaths?
- Of the events in *Kill Bill 2*, what proportion are curse words?

The mathematical notation for conditional proportions looks like this:

P(death | Kill Bill 1) = 0.525

Read it as "The probability of an event being a death given that the movie is *Kill Bill 1* is 0.525"

Numerical Summaries of Contingency Tables

The **relative risk** is the ratio of conditional proportions between two categories.

Movie	Death	Word	Total
Kill Bill: Vol. 1	63	57	120
Kill Bill: Vol. 2	11	69	80
Total	74	126	200

In this case, we can calculate the relative risk of a death event in *Kill Bill 1* **relative to** the risk of a death event in *Kill Bill 2*

$$RR = \frac{\hat{p}_{KB1}}{\hat{p}_{KB2}} = \frac{63/120}{11/80} = 3.8182$$

Of observed events in the Kill Bill movies, we are 3.8182 times more likely to see a death in *Kill Bill 1* than we are in *Kill Bill 2*

Practice: Categorical variables

The Titanic sank in 1912 after striking an iceberg, and did not have enough lifeboats on board for all passengers. The mantra of "Women and children first" is well known, but does it hold up under examination? Were women more likely to survive the Titanic disaster?

Sex	No	Yes	Total
Female	126	344	470
Male	1364	367	1731
Total	1490	711	2201

- What are the observational units?
- What are the two variables?
- What variable is explanatory and what is the response?
- Calculate the relative risk of dying on the titanic for men vs. women

Exploration 5.1: Murderous Nurse?

For several years in the 1990s, Kristen Gilbert worked as a nurse in the intensive care unit (ICU) of the Veterans Administration Hospital in Northampton, Massachusetts. Over the course of her time there, other nurses came to suspect that she was killing patients by inject-ing them with the heart stimulant epinephrine. Gilbert was eventually arrested and charged with these murders.

Part of the evidence presented against Gilbert at her murder trial was a statistical analysis of 1,641 eight-hour shifts during the time Gilbert worked in the ICU. For each of these shifts, researchers recorded two variables: whether or not Gilbert worked on the shift and whether or not at least one patient died during the shift.

- What are the observational units? Shifts
- Classify each variable as categorical or quantitative:
  - Whether or not Gilbert worked on the shift: categorical
  - Whether or not at least one patient died during the shift: categorical
- Which variable is the:
  - Explanatory variable: Whether or not Gilbert worked on the shift
  - Response variable: Whether or not a patient died during the shfit

Exploration 5.1: Murderous Nurse?

	working	not working	Total
Death	40	34	74
No Death	217	1350	1567
Total	257	1384	1641

Using this data (from question 7), work through Exploration 5.1 in your groups

5.2: Comparing Two Proportions (Simulation-Based Approach)

Example 5.2 - Swimming with Dolphins

Is swimming with dolphins therapeutic for patients suffering from clinical depression?

- 30 subjects (18-65) with mild to moderate depression
- Randomly assigned to one of two treatment groups:
  - 1 hour of swimming and snorkeling each day
  - 1 hour of swimming and snorkeling each day with dolphins
- Evaluate depression at the end of two weeks:
  (yes/no) did subjects experience a significant decline in depression

---

- Observational unit?
- Experiment or Observational study?

- Variables:
  - Explanatory:
  - Response:

Example 5.2 - Swimming with Dolphins

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---

- Observational unit? the patient
- Experiment or Observational study?
  Experiment - random assignment to treatment groups
- Variables:
  - Explanatory: treatment groups (dolphins/no dolphins)
  - Response: depression status (substantial improvement/no substantial improvement)

Example 5.2 - Swimming with Dolphins

- Null hypothesis (in words)
- Alt hypothesis (in words)
- In symbols:
- Parameter of interest (in words and symbols):

Example 5.2 - Swimming with Dolphins

- Null hypothesis (in words)
  Whether or not someone swims with dolphins has no association with whether or not someone shows substantial improvement
- Alt hypothesis (in words)
  Swimming with dolphins increases the probability of substantial improvement in depression symptoms (there is an association)
- In symbols:

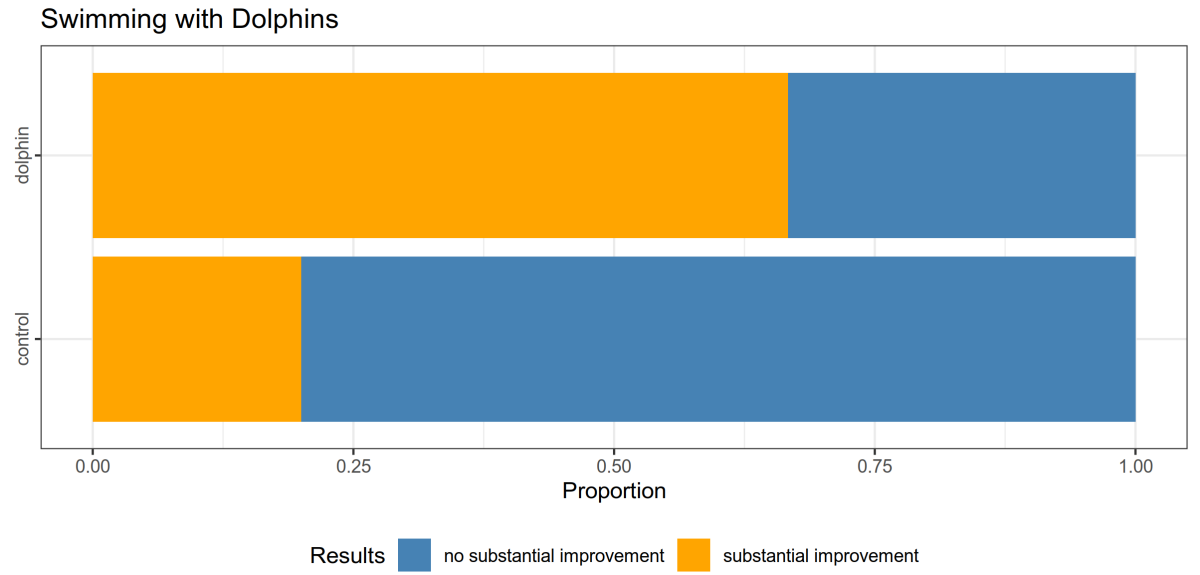
$$H_0 : \pi_{dolphins} - \pi_{control} = 0$$
$$H_A : \pi_{dolphins} - \pi_{control} > 0$$

- Parameter of interest (in words and symbols):

  The difference in the two probabilities
  $\pi_{dolphins} - \pi_{control}$

Example 5.2 - Swimming with Dolphins

	Dolphin therapy	Control group	Total
Substantial Improvement	10	3	13
No Substantial Improvement	5	12	17
Total	15	15	30



Example 5.2 - Swimming with Dolphins

$$\hat{p}_{dolphins} = 10/15 = 0.6667$$
$$\hat{p}_{control} = 3/15 = 0.2$$

- Statistic: $\hat{p}_{dolphins} - \hat{p}_{control} =$
- Explanations for the difference?
  -
  -

Example 5.2 - Swimming with Dolphins

$$\hat{p}_{dolphins} = 10/15 = 0.6667$$
$$\hat{p}_{control} = 3/15 = 0.2$$

- Statistic: $\hat{p}_{dolphins} - \hat{p}_{control} = 0.4667$
- Explanations for the difference?
  - Random chance alone (corresponds to H_0)
  - Swimming with dolphins really helps depression
- Simulation:
  - If H_0 is true, dolphin therapy is no more effective than the control, and we would still have 13 improvers and 17 nonimprovers
  - If we randomly assign outcome labels to explanatory variable groups and compute a simulation statistic $p^*_{dolphins} - p^*_{control}$ ...

Example 5.2 - Swimming with Dolphins Simulation

- Physical method:
  1. label index cards with "improve" (13 cards) and "not improve" (17 cards)
  2. Shuffle
  3. Count out 15 cards for the "dolphin therapy" group and 15 cards for the control group
  4. Calculate proportions and subtract
  5. Repeat N times
  6. Count how many N are greater than or equal to $\hat{p}_{dolphins} - \hat{p}_{control}$

Example 5.2 - Swimming with Dolphins Simulation

- App:
  1. Other applets -> Dolphin Study
  2. Enter real data into the 2x2 table
  3. Check "Show Shuffle Options"
  4. Enter sample statistic under "Count Samples"
  5. Click the "Count" button

Example 5.2 - Swimming with Dolphins

Categorical Response	Quantitative Response		Other Ap	
One Proportion	Descriptive Statistics	Matched Pairs	Theory-Based Inference	Sam
Two Proportions	One Mean	Corr/Regression	Power Simulation	Simu
Multiple Proportions	Multiple Means		Dolphin Study	Ranc

Sample Data (2x2: ☑)

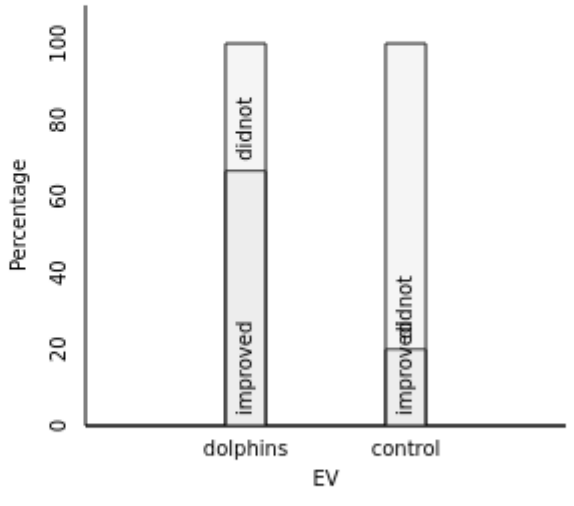
	dolphins	control	Totals
improved	10	3	13
didnot	5	12	17
Totals	15	15	30

Use Table

Clear

Sample Data

Bar graph



Success: improved

(dolphins - control)

Example 5.2 - Swimming with Dolphins

Show Shuffle Options

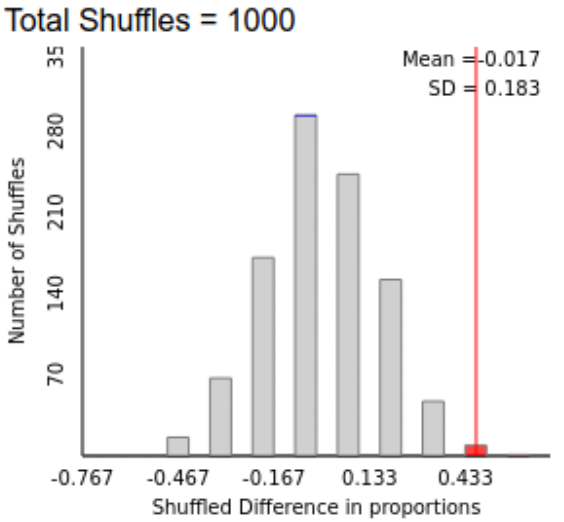
Number of Shuffles 1000

Shuffle | Cards | Data | Plot

Most Recent Shuffle

Group A
Success
9
Failure
6

Group B
Success
4
Failure
11



Show previous

Count Samples Greater than ≥ .467

Count

Count= 10/1000 (0.0100)

Overlay normal distribution

Example 5.2 - Swimming with Dolphins

To recap:

- Null Hypothesis:
- One repetition:
- Statistic:
- Strength of Evidence:

Example 5.2 - Swimming with Dolphins

To recap:

- Null Hypothesis:
  No difference in probability of substantial improvement
- One repetition:

  Random assignment of response outcomes to the two groups
  Note: We always assign the **response outcome** to the **explanatory variable**
- Statistic:

  Difference in conditional proportions, $p^*_{dolphin} - p^*_{control}$
- Strength of Evidence:

  Proportion of simulation samples with $p^*_{dolphin} - p^*_{control} \geq 0.4667$
  $= 10/1000 = 0.0100$

Example 5.2 - Swimming with Dolphins

- Interpretation:

We have strong evidence against the null hypothesis that the difference in the proportion of individuals whose depression improved between the treatment and control groups is due to random chance alone.

We reject H_0 (p = 0.0010) and conclude that there is evidence that swimming with dolphins increases the likelihood that individuals will have substantial improvement in their clinical depression.

Interpreting Tests for Difference of Proportions

Generic version:

We (reject/fail to reject) the null hypothesis that

the difference in the proportion of (response variable)

between the (two levels of the explanatory variable)

(is equal to/is greater than/is less than) (null hypothesis value) because (p = ...).

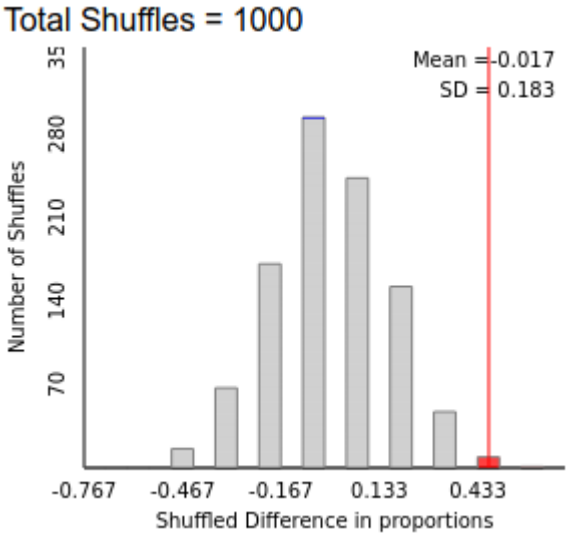
We conclude that (conclusion in the context of the problem).

Example 5.2 - Swimming with Dolphins - Confidence Intervals

We can use the 2SD method to get confidence intervals for the difference in proportions

$$\hat{p}_{dolphin} - \hat{p}_{control} \pm 2\sigma$$
$$= 0.467 \pm 2 * 0.183$$
$$= (0.101, 0.833)$$

We are 95% confident that the difference in the percent of individuals whose depression improved after swimming with dolphins is between 10.1% and 83.3% higher than the percent of individuals whose depression improved and did not swim with dolphins.



You can work this in proportions and multiply by 100 to get a percent if that is easier for you to think about. The simulation SD will be for the proportion, so wait until the end of the calculation to multiply by 100.

Example 5.2 - Swimming with Dolphins - Confidence Intervals

We can use the 2SD method to get confidence intervals for the difference in proportions

$$\begin{aligned}&\hat{p}_{dolphin} - \hat{p}_{control} \pm 2\sigma \\ &= 0.467 \pm 2 * 0.183 \\ &= (0.101, 0.833)\end{aligned}$$

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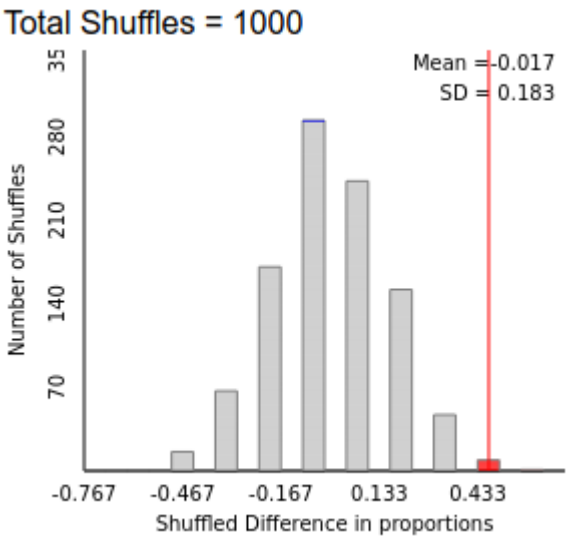


Confidence Intervals

$$\hat{p}_1 - \hat{p}_2 \pm 2 * SD$$

- Does the interval contain zero?
  - If yes, there is no significant difference between the two groups
  - If no,
    - Does the interval lie entirely above 0 (if H_A is greater than 0)?
    - Does the interval lie entirely below 0 (if H_A is less than 0)?
- If the interval does not contain your H_0 value, then a hypothesis test would be significant at the $\alpha = 0.05$ level.

Relative Risk

We can also conduct significance tests for the relative risk:

- Hypotheses:
  - Null: The relative risk of improvement in depression symptoms between those who swim with dolphins and those who do not is 1

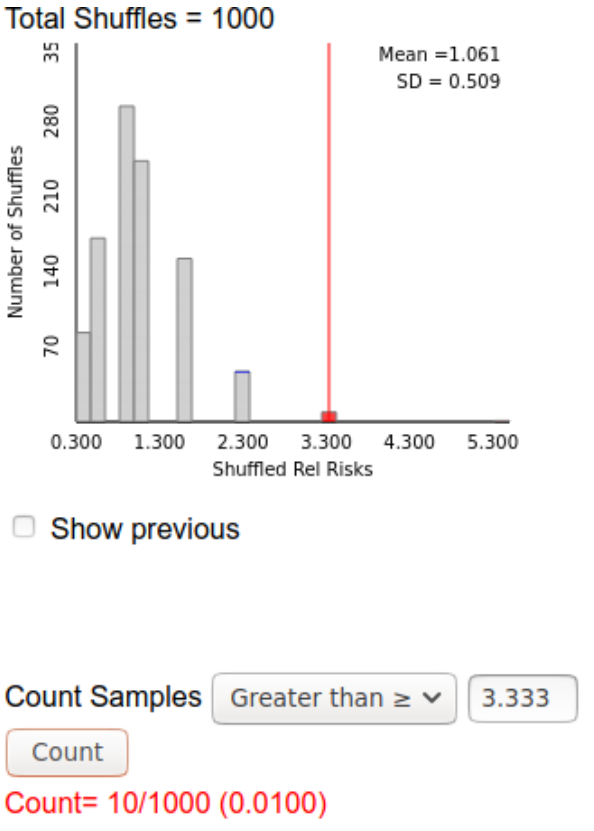
$$\left\{\hat{p}_{dolphins} = \hat{p}_{control}\right\} \text{ implies } \left\{\frac{\hat{p}_{dolphins}}{\hat{p}_{control}} = 1\right\}$$

  - Alt: The relative risk of improvement in depression symptoms between those who swim with dolphins and those who do not is greater than 1

$$\left\{\hat{p}_{dolphins} > \hat{p}_{control}\right\} \text{ implies } \left\{\frac{\hat{p}_{dolphins}}{\hat{p}_{control}} > 1\right\}$$

Example 5.2: Relative Risk

- Statistic: Observed relative risk = $\frac{0.6667}{0.2} = 3.333$
- Simulate: Same strategy, but now compute $\frac{p^*_{dolphin}}{p^*_{control}}$
- Strength of Evidence: p = 0.0010
  Note that the p-value is the same - this will always happen (with the same simulation).
- Conclusion:
  We reject H_0 and conclude that there is strong evidence that the relative risk of improvement in depression symptoms between those who swam with dolphins and those who do not is greater than 1.



Factors affecting Strength of Evidence

- The larger the difference between $\hat{p}_1$ and $\hat{p}_2$, the _____ the evidence that the population proportions differ
- The larger the sample size, the _____ the evidence that the population proportions differ
- The difference between the proportions is the _____ of the confidence interval
- Larger sample sizes produce _____ confidence intervals

Factors affecting Strength of Evidence

- The larger the difference between $\hat{p}_1$ and $\hat{p}_2$, the stronger the evidence that the population proportions differ
- The larger the sample size, the stronger the evidence that the population proportions differ
- The difference between the proportions is the midpoint of the confidence interval
- Larger sample sizes produce narrower confidence intervals

5.3: Comparing Two Proportions

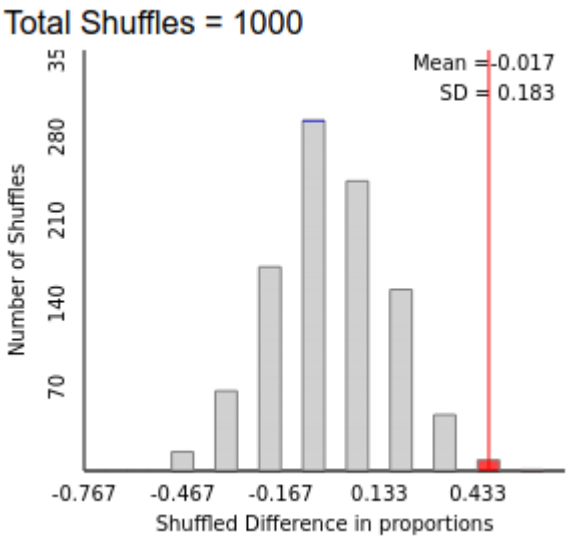
Theory-Based Approach

Normal Distributions

As in Chapter 1, we often see symmetric, bell-shaped distributions for simulated differences between proportions

For Theory-Based Inference, we need:

- Validity conditions (when can we use it)
- Formula for the standard error of the statistic (under H_0)



Z test for Difference Between Two Proportions

$$z = \frac{\text{observed statistic} - \text{hypothesized value}}{\text{standard error of statistic under } H_0}$$

$$= \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\text{standard error of } (\hat{p}_1 - \hat{p}_2) \text{ under } H_0}$$

H_0 is that $\pi_1 = \pi_2$, so let

$$\hat{p} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$$

be the **combined proportion of success** (aka pooled proportion)

Z test for Difference Between Two Proportions

What about the standard error?

In the one-proportion case, our standard error was $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

With two proportions, we can pool our groups to get $\hat{p} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$

... but we also need a pooled sample size: $\frac{1}{\frac{1}{n_1} + \frac{1}{n_2}}$

This is a formula that takes into account the difference in sample sizes between groups (smaller group = larger variance in $\hat{p}$ for the group).

$$\text{Standard error of } \hat{p} = \sqrt{\frac{\hat{p}(1-\hat{p})}{\frac{1}{\frac{1}{n_1} + \frac{1}{n_2}}}} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Z test for Difference Between Two Proportions

CAUTION!

The pooled $\hat{p}$ is only used for the standard error calculation! Don't get it confused with $\hat{p}_1 - \hat{p}_2$!

Z test for Difference Between Two Proportions

$$z = \frac{\text{observed statistic} - \text{hypothesized value}}{\text{standard error of statistic under } H_0}$$

$$= \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\text{standard error of } (\hat{p}_1 - \hat{p}_2) \text{ under } H_0}$$

$$= \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $\hat{p} = \dfrac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$

Confidence Interval: Difference Between Two Proportions

- With a z-test, we are working as if H_0 is true
  - $\hat{p}_1 = \hat{p}_2$ means that we can pool proportions
  - $\hat{p}$ simplifies the standard error calculation
- With a confidence interval:
  - we're estimating the value
  - We don't have a null hypothesis

For a confidence interval for the difference of two proportions:

$$SE = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Confidence Interval: Difference Between Two Proportions

The general formula for a confidence interval is

$$\text{sample statistic} \pm \text{multiplier} \times SE$$

So for a difference of two proportions, the CI formula is:

$$(\hat{p}_1 - \hat{p}_2) \pm \text{multiplier} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

We will use a multiplier of 2 in class - this corresponds to approximately a 95% confidence interval.

Validity Conditions for theory-based approach

- 10 observations in each of the four cells of the 2x2 table
- Required for a theory-based test or for the interval for the difference in two proportions.