Stat 982: Module 1, Homework 1

Due on September 6, 2022

Problem 1

Suppose that $X = (X_1, X_2, ..., X_n)$ has independent components, where each X_i is generated as follows. For independent random variables $W_i \sim N(\mu, 1)$ and $Z_i \sim Poisson(\mu)$, $X_i = W_i$ with probability p and $X_i = Z_i$ with probability $1 - p$. Suppose that $\mu \in [0, \infty)$. Use the factorization theorem and find low-dimensional sufficient statistics in the cases that:

(a) p is known to be $\frac{1}{2}$

(b) $p \in [0, 1]$ is unknown

Note: In the first case the parameter space is $\Theta = \left\{\frac{1}{2}\right\} \times [0, \infty)$, while in the second case it is $\Theta = [0, 1] \times [0, \infty)$.

Let $\lambda = \Updownarrow + \sqsubseteq$, where $\Updownarrow$ is the Lebesgue measure on $\mathbb{R}^+ = [0, \infty)$ and $\sqsubseteq$ is the counting measure on $\mathbb{N} = \{0, 1, 2, ...\}$. Note that $X_1 \overset{=}{d} R_1 W_1 + (1 - R_1) Z_1$

Problem 2

Suppose that X' is exponential with mean λ^{-1} (i.e. it has density $f_\lambda(x) = \lambda \exp(-\lambda x) I[x \geq 0]$ with respect to the Lebesgue measure on $\mathbb{R}^1$), but that one only observes $X = X' I[X' \geq 1]$. (There is interval censoring below $X' = 1$).

(a) Consider the measure μ on $\mathcal{X} = \{0\} \bigcup [1, \infty)$ consisting of a point mass of 1 at 0 plus the Lebesgue measure on $[1, \infty)$. Give a formula for the R-N derivative of P_λ^X with respect to μ on $\mathcal{X}$.

(b) Suppose that $X_1, ..., X_n$ are iid with the distribution P_λ^X. Find a two-dimensional sufficient statistic for this problem and argue that it is indeed sufficient.

(c) Argue carefully that your statisic from the previous part is minimal sufficient.

Problem 3

Let X be a sample from $P \in \mathcal{P}$ where $\mathcal{P}$ is a family of distributions on $(\mathbb{R}^k, \mathcal{B}^k)$. Show that if $T(X)$ is sufficient for $\mathcal{P}$ and $T = \psi(S)$, where ψ is measurable and $S(X)$ is another statistic, then $S(X)$ is sufficient for $\mathcal{P}$. (Hint: Consider first the special case that $\mathcal{P}$ is dominated by a σ-finite measure μ and then the general case).

Problem 4

Suppose that $X_1, X_2, ..., X_n$ are iid P_θ for $\theta = (\theta_1, \theta_2) \in (0, 1) \times \{1, 2\}$, where $P_{(\gamma,1)}$ is the Poisson(γ) distribution and $P(\gamma, 2)$ is the Bernoulli(γ) distribution. Find a two- dimensional minimal sufficient statistic for θ (and argue carefully for minimal sufficiency).

Problem 5

Suppose that $X_1, ..., X_n$ are iid $N(\gamma, \gamma^2)$ for $\gamma \in \mathbb{R}^1$. Find a minimal sufficient statistic and show that it is not complete.