Yuan–Ping Chen 1566360, Ex2&Latex
Hung–Ju Chen 1566699, Ex1&2

Homework 3

Professor Cormac Flanagan
CMPS 203 - Programming Language

February 4, 2017

Exercise 1. In the WHILE language, prove or disprove the equivalence of the two commands:

$$t := x;\ \ x := y;\ \ y := t \tag{1}$$

and

$$t := y;\ \ y := x;\ \ x := t \tag{2}$$

(where x, y, and t are distinct locations).

Proof. We first define (1) to S_1 and (2) to S_2. In the definition, S_1 and S_2 are semantically equivalent; this means that for all states s and s'

$$\langle S_1, s\rangle \rightarrow s' \ \ if\ and\ only\ if \ \ \langle S_2, s\rangle \rightarrow s'$$

So we assume the first state s to be Table 1. a, b and c can be any value. After the first command of S_1, the state s becomes s_1', which is Table 2. Then Table 3 and Table 4, which should be s'. However, after the first command of S_2, the state s becomes s_2', which is Table 5. Then Table 6 and Table 7, which should be exactly the same as Table 4, but it is not. As a result, we disprove the semantically equivalent of these two Statements. □

Exercise 2. In the WHILE language, prove that if

$$\langle while\ b\ do\ y := y - x,\ \ s\rangle \Downarrow s' \tag{3}$$

then there exists an integer k such that

$$s(y) = s'(y) + k \times s(x) \tag{4}$$

t	a
x	b
y	c

Table 1: s

t	b
x	b
y	c

Table 2: s_1'

t	b
x	c
y	c

Table 3: s_1''

t	b
x	c
y	b

Table 4: s'

t	c
x	b
y	c

Table 5: s'_2

t	c
x	b
y	b

Table 6: s''_2

t	c
x	c
y	b

Table 7: s'

Proof. The proof is by induction on the number k. We assume (3) always terminates and k equals the height of the derivation tree (3) minus 2.

For example, if the derivation tree is

$$\frac{\langle \text{ b, s}\rangle \Downarrow \text{true} \qquad \langle \text{y:=y-x, s } \rangle \Downarrow s' \qquad \dfrac{\langle \text{ b, } s''\rangle \Downarrow \text{false} \qquad \langle \text{ skip, } s''\rangle \Downarrow s'}{\langle \text{ while b do y:=y-x, } s'' \rangle \Downarrow s'}}{\langle \text{ while b do y:=y-x, s } \rangle \Downarrow s'}$$

the height of the tree is 3, and k=1. If k=0, the state doesnt't change because the derivation tree is

$$\frac{\langle \text{ b, s}\rangle \Downarrow \text{false} \qquad \langle \text{ skip, s}\rangle \Downarrow s'}{\langle \text{ while b do y:=y-x, s } \rangle \Downarrow s'}$$

which corresponds to (4) when k=0

$$s(y) = s'(y) + 0 \times s(x)$$

For the induction step, we assume the exercise statement holds for k $\leqslant k_0$, we shall prove it for k_0+1, so the derivation tree of (3) when k=k_0+1 is

$$\frac{\langle \text{ b, s}\rangle \Downarrow \text{true} \qquad \langle \text{y:=y-x, s } \rangle \Downarrow s_1 \qquad \dfrac{\langle \text{ b, } s_1\rangle \Downarrow \text{true} \qquad \left.\begin{matrix} \dfrac{\langle \text{ b, } s_k\rangle \Downarrow \text{true} \qquad \langle \text{y:=y-x, } s_k\rangle \Downarrow s_{k+1} \qquad \dfrac{\langle \text{ b, } s}{\langle}}{} \\ \vdots \end{matrix}\right\} k}{\langle \text{ while b do y:=y-x, } s_1 \rangle \Downarrow s_{k+1}}}{\langle \text{ while b do y:=y-x, s } \rangle \Downarrow s_{k+1}}$$

,note that s'=s_{k+1},

by induciton hypothesis, we know that

$$s(y) = s_k(y) + k \times s(x) \tag{5}$$

and in the height k+2 of the derivation tree, we have $\langle$ y:=y-x, s_k $\rangle \Downarrow s_{k+1}$, which equals s_k(y)=s_{k+1}(y)+x. Therefore, in (5), we get

$$s(y) = s_{k+1}(y) + x + k \times s(x)$$

$$= s_{k+1}(y) + (k+1) \times s(x)$$

so the exercise statement is proved.

□