Hausdorff dimension of clusters at T_c generated by the Worm algorithm

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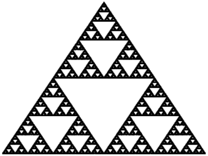
Table of contents

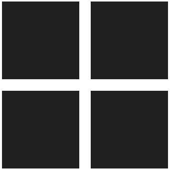
1. Fractals
2. Algorithms for Working with Graph Patterns
3. Ising Model
4. XY Model

Fractals

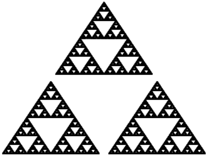
Scaling Mass



Scaling Mass



$$L \to \frac{1}{2}L$$



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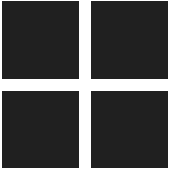
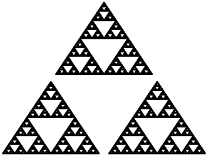
Scaling Mass



$$L \to \frac{1}{2}L$$
$$\left(\frac{1}{2}\right)^d M = \frac{1}{4}M$$



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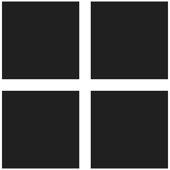
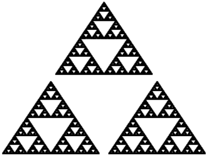
Scaling Mass





$$L \to \frac{1}{2} L$$

$$\left(\frac{1}{2}\right)^d M = \frac{1}{4} M$$

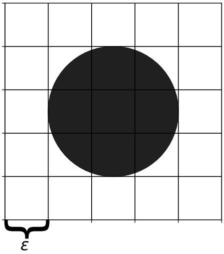
$$d = 2$$

$$L \to \frac{1}{2} L$$

$$\left(\frac{1}{2}\right)^d M = \frac{1}{3} M$$

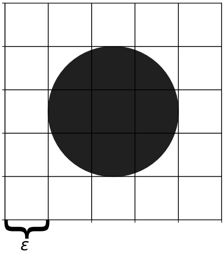
$$\begin{aligned} d &= \log_2(3) \\ &\approx 1.585 \end{aligned}$$

Box Counting Method



$$N \sim \frac{1}{\epsilon^d}$$

Box Counting Method



$$N \sim \frac{1}{\epsilon^d}$$

$$d = \lim_{\epsilon \to 0^+} \frac{\ln N(\epsilon)}{\ln 1/\epsilon}$$

Hausdorff Dimension

$$\mathcal{H}_d(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{B \in \mathcal{B}} (\mathrm{diam}(B))^d \right\}$$

Hausdorff Dimension

$$\mathcal{H}_d(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{B \in \mathcal{B}} (\mathrm{diam}(B))^d \right\}$$

$$\dim_H(A) = \inf\{d > 0 : \mathcal{H}_d(A) = 0\}$$

Algorithms for Working with Graph Patterns

Worm Algorithm

- Graph configurations
- Metropolis Steps

Labeling and Box Dimension

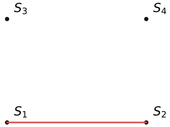
- Hoshen Kopelman Algorithm
- Graph Dividing

Ising Model

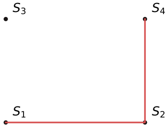
Ising Loop Expansion

$$Z \propto \sum_{\{S\}} \left(1 + \tanh(K) \sum_{l=1} S_i S_j + \tanh^2(K) \sum_{l=2} (S_i S_j)(S_{i'} S_{j'}) + \dots \right)$$

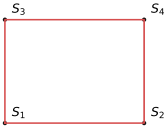
Ising Loop Expansion



a: $(S_1 S_2),\ L = 1$

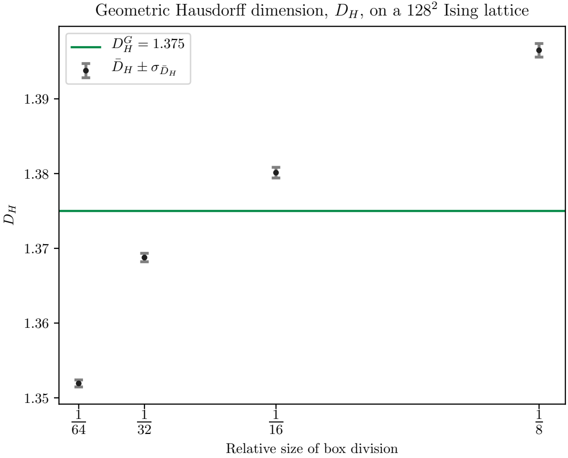


b: $(S_1 S_2)(S_2 S_4),\ L = 2$

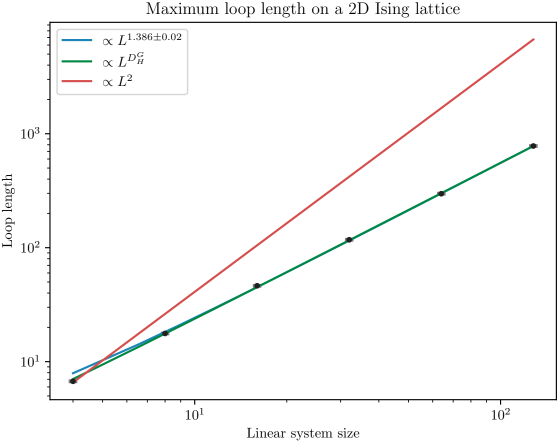


c: $(S_1 S_2)(S_2 S_4)(S_4 S_3)(S_3 S_1),\ L = 4$

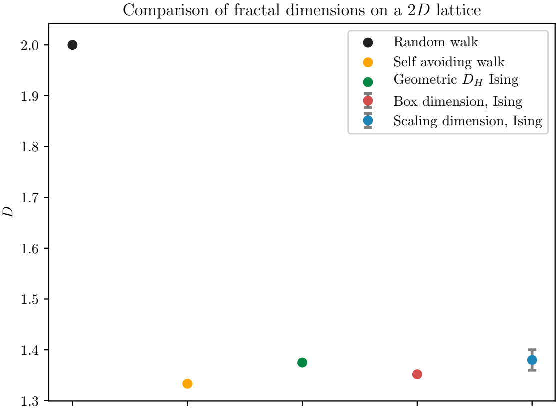
Box Dimension



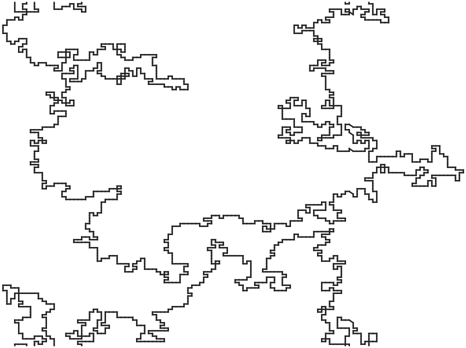
Scaling Dimension



Comparison of Dimensions $2D$ Ising



Largest Ising Loop on a 128^2 Lattice



$2D$ Ising Animation

XY Model

XY Loop Expansion

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$Z = \prod_i \int \frac{\mathrm{d}\theta_i}{2\pi} \prod_{\langle ij \rangle} e^{K \cos(\theta_i - \theta_j)}$$

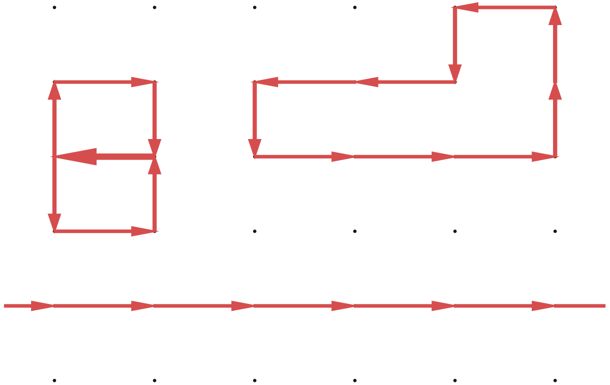
XY Loop Expansion

$$Z \sim \int \frac{\mathrm{d}\theta_i}{2\pi} e^{i \sum_{\langle ij \rangle} j_{\langle ij \rangle}(\theta_i - \theta_j)}$$

XY Loop Expansion

$$
\begin{aligned}
Z &\sim \int \frac{\mathrm{d}\theta_i}{2\pi} e^{i \sum_{\langle ij \rangle} j_{\langle ij \rangle} (\theta_i - \theta_j)} \\
&\sim \delta_{0, \sum_{\langle ij \rangle} j_{\langle ij \rangle}}
\end{aligned}
$$

XY Loop expansion

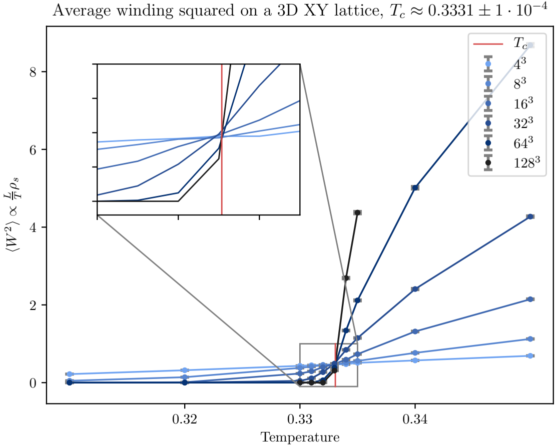


Villain Approximation

$$E = \frac{1}{2}\sum_i j_i^2$$

Winding Number

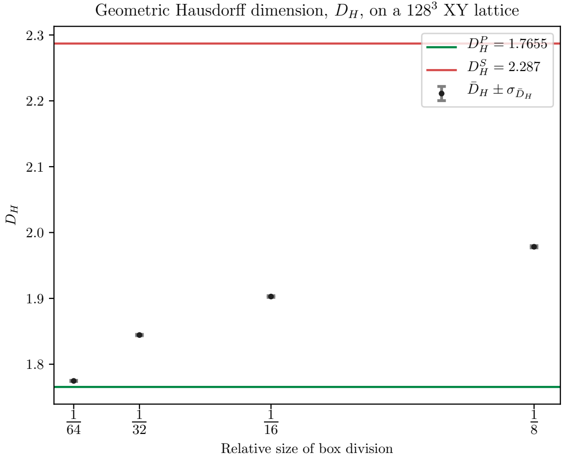
$$\rho_s = L^{2-d}\, T \langle W_\mu^2 \rangle$$


Average winding squared on a 3D XY lattice, $T_c \approx 0.3331 \pm 1 \cdot 10^{-4}$
T_c
4^3
8^3
16^3
32^3
64^3
128^3
$\langle W^2 \rangle \propto \frac{L}{T}\rho_s$
0
2
4
6
8
0.32
0.33
0.34
Temperature

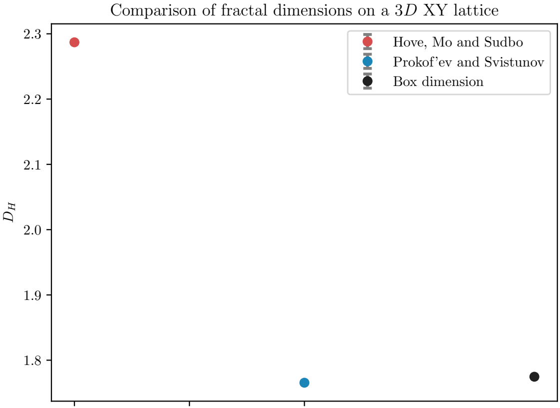
$3D$ **XY Model Hausdorff Dimension**

- Hove, Mo and Sudbo (2000): $D_H = 2.287 \pm 4 \cdot 10^{-3}$
- Prokof'ev and Svistunov Comment (2005): $D_H = 1.765 \pm 2 \cdot 10^{-3}$

Box Counting Method $3D$ XY



Comparison of Dimensions $3D$ XY



$3D$ **XY Animation** - 4^3 **system**

$3D$ XY Animation - Largest cluster

Summary

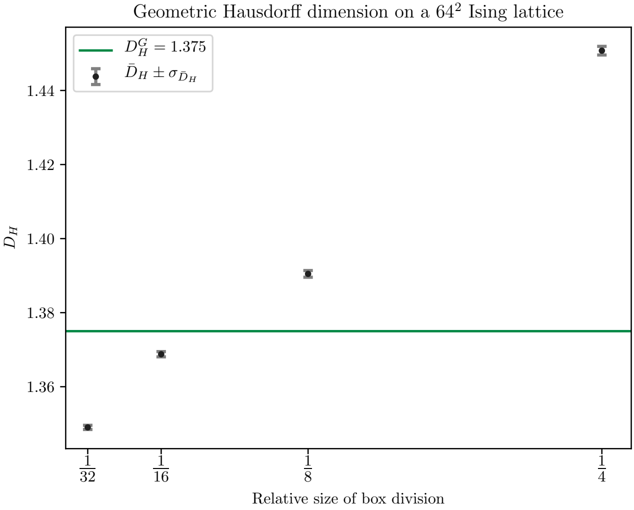
	D_H
Box	1.35193(5)
Scaling	1.38(2)
D_H^G	1.375
SAW	1.33
Random Walk	2

Table 1: $2D$ Ising

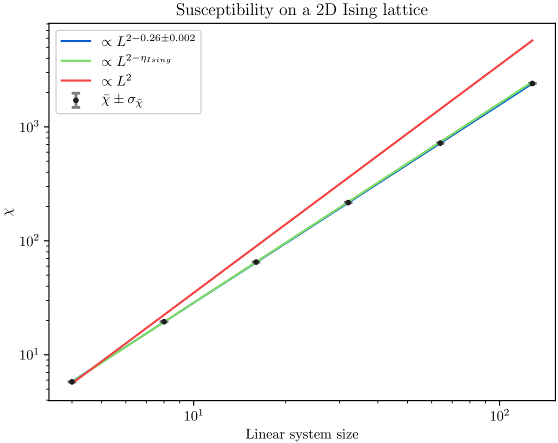
	D_H
Box	1.77468(4)
Prokof'ev	1.765(2)
Sudbo	2.287(2)

Table 2: $3D$ XY

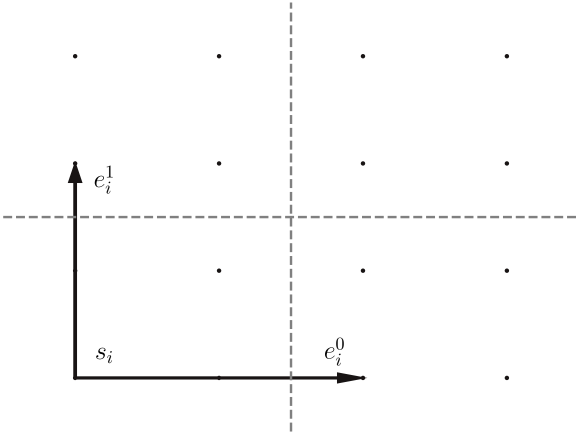
Extra slides: Box Dimension 64^2 Ising



Extra slides: Susceptibility $2D$ Ising



Extra slides: Graph Dividing Algorithm



Extra slides: Graph Dividing Algorithm