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Conformally Invariant Fractals and Potential Theory

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The multifractal (MF) distribution of the electrostatic potential near any conformally invariant fractal boundary, like a critical $O(N)$ loop or a Q-state Potts cluster, is solved in two dimensions. The dimension $\hat{f}(\theta)$ of the boundary set with local wedge angle θ is $\hat{f}(\theta) = \frac{\pi}{\theta} - \frac{25-c}{12}\frac{(\pi-\theta)^2}{\theta(2\pi-\theta)}$, with c the central charge of the model. As a corollary, the dimensions D_{EP} of the external perimeter and D_{H} of the hull of a Potts cluster obey the duality equation $(D_{\mathrm{EP}} - 1)(D_{\mathrm{H}} - 1) = \frac{1}{4}$. A related covariant MF spectrum is obtained for self-avoiding walks anchored at cluster boundaries.

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Fractals are by now a well-recognized and studied subject, yet they still pose great challenges in mathematical physics: to establish a complete theory of diffusion limited aggregates (DLA) or a rigorous theory of the tenuous fractal structures arising in critical phenomena, to name only two. Classical potential theory, i.e., that of the electrostatic or diffusion field near such random fractal boundaries, whose self-similarity is reflected in a *multifractal* (MF) behavior of the potential, is almost a *Terra Incognita*. In DLA, the potential, also called harmonic measure, actually determines the growth process, and its scaling properties are intimately related to those of the cluster itself [1]. In statistical fractals, the Laplacian field is created by the random boundary, and should be derivable, in a probabilistic sense, from the knowledge of the latter. A singular example was studied in Ref. [2], where the fractal boundary, the "absorber," was chosen to be a simple random walk (RW), or a self-avoiding walk (SAW), accessible to renormalization group methods near four dimensions.

In *two dimensions* (2D), conformal field theory (CFT) has lent strong support to the conjecture that statistical systems at their critical point produce *conformally invariant* (CI) fractal structures, examples of which are the continuum scaling limits of RW's, SAW's, critical Ising or Potts clusters, etc. Given the beautiful simplicity of the classical method of conformal transforms to solve 2D electrostatics of *Euclidean* domains, perhaps a *universal* solution is possible for the planar potential near a CI fractal.

A first exact example has been recently solved in 2D for the whole universality class of random or self-avoiding walks, and percolation clusters, which all possess the same harmonic MF spectrum [3] (see also [4]). In this Letter, I propose the general solution for the potential distribution near any conformal fractal in 2D. The exact multifractal spectra describing the singularities of the potential, or, equivalently, the distribution of wedge angles along the boundary, are obtained, and shown to depend only on the so-called *central charge c*, a parameter which labels the universality class of the underlying CFT. I devise conformal tools (linked to quantum gravity), which allow the mathematical description of random walks interacting with CI fractal structures, thereby yielding a complete, albeit probabilistic, description of the potential. The results are applied directly to well-recognized universal fractals, like $O(N)$ loops or Potts clusters. In particular, a subtle geometrical structure is observed in Potts clusters, where the *external perimeter* (EP), which bears the electrostatic charge, differs from the cluster's hull. Its fractal dimension D_{EP} is obtained exactly, generalizing the recently elucidated case of percolation [5].

Harmonic measure and potential.—Consider a single (conformally invariant) critical random cluster, generically called $\mathcal{C}$. Let $H(z)$ be the potential at exterior point $z \in \mathbb{C}$, with Dirichlet boundary conditions $H(w \in \partial C) = 0$ on the outer (simply connected) boundary ∂C of C, and $H(w) = 1$ on a circle "at ∞," i.e., of a large radius scaling like the average size R of C. From a well-known theorem due to Kakutani [6], $H(z)$ is identical to the *harmonic measure*, i.e., the probability that a random walker launched from z escapes to ∞ without having hit C. The multifractal formalism [7–10] characterizes subsets ∂C_α of boundary sites by a Hölder exponent α, and a Hausdorff dimension $f(\alpha) = \dim(\partial C_\alpha)$, such that their potential locally scales as

$$H(z \to w \in \partial C_\alpha) \approx (|z - w|/R)^\alpha, \tag{1}$$

in the scaling limit $a \ll r = |z - w| \ll R$, with a the underlying lattice constant. In 2D the *complex* potential $\varphi(z)$ [such that the electrostatic potential $H(z) = \mathrm{Im}\varphi(z)$ and field $|\mathbf{E}(z)| = |\varphi'(z)|$] reads for a *wedge* of angle θ, centered at w: $\varphi(z) = (z - w)^{\pi/\theta}$. By Eq. (1) a Hölder exponent α thus defines a local angle $\theta = \pi/\alpha$, and the (purely geometrical) MF dimension $\hat{f}(\theta)$ of the boundary subset with such θ is $\hat{f}(\theta) = f(\alpha = \pi/\theta)$.

Of special interest are the moments of H, averaged over all realizations of C

$$\mathcal{Z}_n = \left\langle \sum_{z \in \partial C(r)} H^n(z) \right\rangle, \tag{2}$$

where $\partial C(r)$ is shifted a distance r outwards from ∂C, and where n can be a real number. In the scaling limit, one expects these moments to scale as

$$\mathcal{Z}_n \approx (r/R)^{\tau(n)}, \tag{3}$$

where the multifractal scaling exponents $\tau(n)$ encode generalized dimensions, $D(n) = \tau(n)/(n - 1)$, which vary in a nonlinear way with n [7–10]; they obey the symmetric Legendre transform $\tau(n) + f(\alpha) = \alpha n$, with $n = f'(\alpha)$, $\alpha = \tau'(n)$. From Gauss' theorem [2] $\tau(1) = 0$. Because of the ensemble average (2), values of $f(\alpha)$ can become negative for some domains of α [2]. This Letter is organized as follows: I first present the main results in a universal way, then proceed with their derivation from conformal field theory, and finally specify them for the $O(N)$ and Potts models.

Exact multifractal dimensions and spectra.—Each conformally invariant random system is labeled by its *central charge* c, $c \leq 1$. The multifractal dimensions of a simply connected CI boundary then read explicitly

$$\tau(n) = \frac{1}{2}(n - 1) + \frac{25 - c}{24}\left(\sqrt{\frac{24n + 1 - c}{25 - c}} - 1\right),$$
$$D(n) = \frac{\tau(n)}{n - 1} = \frac{1}{2} + \left(\sqrt{\frac{24n + 1 - c}{25 - c}} + 1\right)^{-1}, \tag{4}$$
$$n \in \left[n^* = -\frac{1 - c}{24}, +\infty\right).$$

The Legendre transform reads

$$\alpha = \frac{d\tau}{dn}(n) = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{25 - c}{24n + 1 - c}}; \tag{5}$$

$$f(\alpha) - \alpha = \frac{25 - c}{24} \times \left[1 - \frac{1}{2}\left(2\alpha - 1 + \frac{1}{2\alpha - 1}\right)\right], \tag{6}$$
$$\alpha \in (\tfrac{1}{2}, +\infty).$$

Notice that the generalized dimensions $D(n)$ satisfy, for any c, $\tau'(n = 1) = D(n = 1) = 1$, or equivalently $f(\alpha = 1) = 1$, i.e., *Makarov's theorem* [11], valid for any simply connected boundary curve. From (4) and (5) we also note a fundamental relation, independent of c: $3 - 2D(n) = 1/\alpha = \theta/\pi$. We also have the *superuniversal* bounds: $\forall\, c, \forall\, n, \frac{1}{2} = D(\infty) \leq D(n) \leq D(n^*) = \frac{3}{2}$, hence $0 \leq \theta \leq 2\pi$. We arrive at the geometrical multifractal distribution of wedges θ along the boundary:

$$\hat{f}(\theta) = f\left(\frac{\pi}{\theta}\right) = \frac{\pi}{\theta} - \frac{25 - c}{12}\frac{(\pi - \theta)^2}{\theta(2\pi - \theta)}. \tag{7}$$

Remarkably enough, the second term also describes the contribution by a wedge to the density of electromagnetic modes in a cavity [12]. The maximum of $f(\alpha)$ corresponds to $n = 0$, and gives the dimension D_{EP} of the support of the measure, i.e., the *accessible* or *external perimeter* as $\sup_\alpha f(\alpha) = f(\alpha(0)) = D(0)$:

$$D_{\mathrm{EP}} = D(0) = \frac{3}{2} - \frac{1}{24}\sqrt{1 - c}\,(\sqrt{25 - c} - \sqrt{1 - c}). \tag{8}$$

This corresponds to a *typical wedge angle* $\hat{\theta} = \theta(0) = \pi/\alpha(0) = \pi(3 - 2D_{\mathrm{EP}})$.

The multifractal functions $f(\alpha) - \alpha = \hat{f}(\theta) - \frac{\pi}{\theta}$ are *invariant* when taken at primed variables such that $2\pi = \theta + \theta' = \frac{\pi}{\alpha} + \frac{\pi}{\alpha'}$, which corresponds to the complementary domain of the wedge θ. This condition reads also $D(n) + D(n') = 2$. This basic symmetry, first observed and studied in [13] for the $c = 0$ result of [3], is valid for *any* conformally invariant boundary.

In Fig. 1 are displayed the multifractal functions f, Eq. (6), corresponding to various values of $-2 \leq c \leq 1$, or, equivalently, to a number of components $N \in [0, 2]$, and $Q \in [0, 4]$ in the $O(N)$ or Potts models (see below). The generalized dimensions $D(n)$ (not shown) appear as quite similar for various values of c, and a numerical simulation would hardly distinguish the different universality classes, while the $f(\alpha)$ functions do distinguish these classes, especially for negative n. The singularity at $\alpha = \frac{1}{2}$, or $\theta = 2\pi$, in the multifractal functions f, or $\hat{f}$, corresponds to boundary points with a needle local geometry, and Beurling's theorem [14] indeed ensures the Hölder exponents α to be bounded below by $\frac{1}{2}$. This corresponds to large values of n, where, asymptotically, for *any* universality class, $\forall\, c$, $\lim_{n\to\infty} D(n) = \frac{1}{2}$.

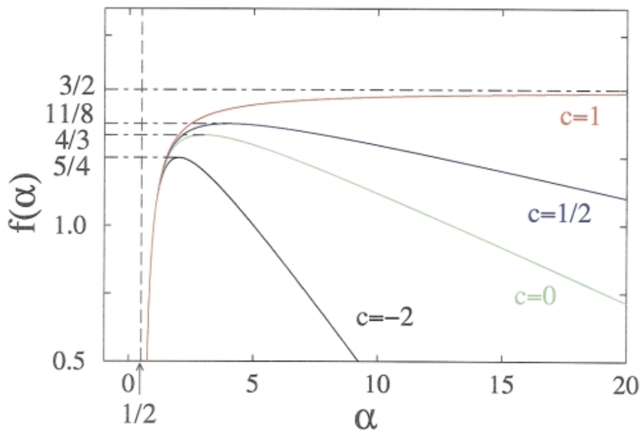

FIG. 1 (color). Universal harmonic multifractal spectra $f(\alpha)$. The curves are indexed by the central charge c, and correspond, respectively, to the following: 2D spanning trees ($c = -2$); self-avoiding or random walks, and percolation ($c = 0$); Ising clusters or $Q = 2$ Potts clusters ($c = \frac{1}{2}$); $N = 2$ loops, or $Q = 4$ Potts clusters ($c = 1$). The maximal dimensions are those of the *accessible* frontiers. The left branches of the various $f(\alpha)$ curves are largely indistinguishable, while their right branches split for large α, corresponding to negative values of n.

The right branch of $f(\alpha)$ has a linear asymptote $\lim_{\alpha\to\infty} f(\alpha)/\alpha = n^* = -(1-c)/24$. The *limit* multifractal spectrum is obtained for $c = 1$, which exhibits an *exact* example of a *left-sided* MF spectrum, with an asymptote $f(\alpha \to \infty, c = 1) \to \frac{3}{2}$ (Fig. 1). It corresponds to singular boundaries where $\hat{f}(\theta \to 0, c = 1) = \frac{3}{2} = D_{\mathrm{EP}}$, i.e., where the external perimeter is dominated by *fjords*, with typical angle $\hat{\theta} = 0$.

The $\alpha \to \infty$ behavior corresponds to moments of lowest order $n \to n^*$, where $D(n)$ reaches its maximal value: $\forall\, c, D(n^*) = \frac{3}{2}$, common to *all* simply connected, conformally invariant, boundaries. This describes almost inaccessible sites: define $\mathcal{N}(H)$ as the number of boundary sites having a given probability H to be hit by a RW starting at infinity; the MF formalism yields, for $H \to 0$, a power law behavior $\mathcal{N}(H)|_{H\to 0} \approx H^{-(1+n^*)}$ with an exponent $1 + n^* = \frac{23+c}{24} < 1$.

Conformal invariance and quantum gravity.—Let me now give the main lines of the derivation of exponents $\tau(n)$, hence $f(\alpha)$, by generalized *conformal invariance.* By definition of the H measure, n *independent* RW's, or Brownian paths $\mathcal{B}$ in the scaling limit, starting at the same point a distance r away from the cluster's hull $\partial\mathcal{C}$, and diffusing without hitting $\partial\mathcal{C}$, give a geometric representation of the nth moment, H^n, in Eq. (2) for n *integer.* Convexity yields analytic continuation for arbitrary n's. Let us introduce the notation $A \wedge B$ for two random sets conditioned to traverse, *without mutual intersection*, the annulus $\mathcal{D}(r,R)$ from the inner boundary circle of radius r to the outer one at distance R, and $A \vee B$ for two *independent*, thus possibly intersecting, sets [3]. With this notation, the probability that the Brownian paths and cluster are in a configuration $\partial\mathcal{C} \wedge (\vee\mathcal{B})^n \equiv \partial\mathcal{C} \wedge n$, is expected to scale for $R/r \to \infty$ as

$$\mathcal{P}_R(\partial\mathcal{C} \wedge n) \approx (r/R)^{x(n)}, \tag{9}$$

where the scaling exponent $x(n)$ depends on n. In terms of definition (9), the harmonic measure moments (2) simply scale as $\mathcal{Z}_n \approx R^2 \mathcal{P}_R(\partial\mathcal{C} \wedge n)$ [2,3], which, combined with Eq. (3), leads to $\tau(n) = x(n) - 2$.

To calculate these exponents, I use a fundamental mapping of the conformal field theory in the *plane* $\mathbb{R}^2$, describing a critical statistical system, to the CFT on a fluctuating abstract random Riemann surface, i.e., in the presence of *quantum gravity* [15,16]. Two universal functions U and V, depending only on the central charge c of the CFT, describe this map:

$$U(x) = x\,\frac{x-\gamma}{1-\gamma}, \qquad V(x) = \frac{1}{4}\,\frac{x^2-\gamma^2}{1-\gamma}, \tag{10}$$

with $V(x) \equiv U(\frac{1}{2}(x+\gamma))$ [17]. The parameter γ is the *string susceptibility exponent* of the random 2D surface (of genus zero), bearing the CFT of central charge c [15]; γ is the solution of $c = 1 - 6\gamma^2(1-\gamma)^{-1}, \gamma \leq 0$.

For two arbitrary random sets A, B, with boundary scaling exponents in the *half-plane* $\tilde{x}(A), \tilde{x}(B)$, the scaling exponent $x(A \wedge B)$, as in (9), has the universal structure [3,17]

$$x(A \wedge B) = 2V[U^{-1}(\tilde{x}(A)) + U^{-1}(\tilde{x}(B))], \tag{11}$$

where $U^{-1}(x)$ is the *positive* inverse function of U

$$U^{-1}(x) = \frac{1}{2}\,[\sqrt{4(1-\gamma)x + \gamma^2} + \gamma]. \tag{12}$$

$U^{-1}(\tilde{x})$ is, on the random Riemann surface, the boundary scaling dimension corresponding to $\tilde{x}$ in the half-plane $\mathbb{R} \times \mathbb{R}^+$, and the sum of U^{-1} functions in Eq. (11) is a *linear* representation of the product of two "boundary operators" on the random surface, as the condition $A \wedge B$ for two mutually avoiding sets is purely *topological* there. The sum is mapped back by the function V into the scaling dimensions in $\mathbb{R}^2$ [17].

For the harmonic exponents $x(n) \equiv x(\partial\mathcal{C} \wedge n)$ in (9), we use (11). The *external boundary* exponent $\tilde{x}(\partial\mathcal{C})$ obeys $U^{-1}(\tilde{x}) = 1 - \gamma$, which I derive either directly, or from Makarov's theorem: $\frac{dx}{dn}(n = 1) = 1$ [17]. The bunch of n independent Brownian paths have simply $\tilde{x}((\vee\mathcal{B})^n) = n$, since $\tilde{x}(\mathcal{B}) = 1$ [3]. Thus I obtain

$$x(n) = 2V[1 - \gamma + U^{-1}(n)]. \tag{13}$$

This finally gives from (10) and (12) $\tau(n) = x(n) - 2$,

$$\tau(n) = \frac{1}{2}\,(n-1) + \frac{1}{4}\,\frac{2-\gamma}{1-\gamma}\,[\sqrt{4(1-\gamma)n + \gamma^2} - (2-\gamma)],$$

from which Eq. (4) follows, Q.E.D.

This formalism immediately allows generalizations. For instance, in place of n random walks, one can consider a set of n *independent self-avoiding* walks $\mathcal{P}$, which avoid the cluster fractal boundary, except for their common anchoring point. The associated multifractal exponents $x(\partial C \wedge (\vee \mathcal{P})^n)$ are given by (13), with the argument n in U^{-1} simply replaced by $\tilde{x}((\vee \mathcal{P})^n) = n\tilde{x}(\mathcal{P}) = \frac{5}{8}n$ [3]. These exponents govern the universal multifractal behavior of the moments of the probability that a SAW escapes from C. One then gets a spectrum $\bar{f}$ such that $\bar{f}[\bar{\alpha} = \tilde{x}(\mathcal{P})\pi/\theta] = f(\alpha = \pi/\theta) = \hat{f}(\theta)$, thus unveiling the *same invariant* underlying wedge distribution as the harmonic measure, Q.E.D.

$O(N)$ and Potts cluster frontiers.—The $O(N)$ model partition function is that of a gas $\mathcal{G}$ of self- and mutually avoiding *loops* on a given lattice, e.g., the hexagonal lattice [18]: $Z_{O(N)} = \sum_{\mathcal{G}} K^{\mathcal{N}_B} N^{\mathcal{N}_P}$, with K and N two fugacities, associated, respectively, with the total number of occupied bonds $\mathcal{N}_B$, and with the total number $\mathcal{N}_P$ of loops, i.e., polygons drawn on the lattice. For $N \in [-2, 2]$, this model possesses a critical point (CP), K_c, while the whole *low-temperature* (low-T) phase, i.e., $K_c < K$, has critical universal properties, where the loops are *denser* that those at the critical point [18].

The partition function of the Q-state Potts model on, e.g., the square lattice, with a second order critical point for $Q \in [0, 4]$, has a Fortuin-Kasteleyn representation *at* the CP: $Z_{\mathrm{Potts}} = \sum_{\cup(C)} Q^{(1/2)\mathcal{N}_P}$, where the configurations $\cup(C)$ are those of reunions of clusters on the square lattice, with a total number $\mathcal{N}_P$ of polygons encircling all clusters, and filling the medial square lattice of the original lattice [18,19]. Thus the critical Potts model becomes a *dense* loop model, with a loop fugacity $N = Q^{1/2}$, while one can show that its *tricritical* point with site dilution corresponds to the $O(N)$ CP [20]. The $O(N)$ and Potts models thus possess the same "Coulomb gas" representations [18–20]: $N = \sqrt{Q} = -2\cos\pi g$, with $g \in [1, \frac{3}{2}]$ for the $O(N)$ CP, and $g \in [\frac{1}{2}, 1]$ for the low-T $O(N)$, or critical Potts, models; the coupling constant g of the Coulomb gas yields also the central charge: $c = 1 - 6(1-g)^2/g$. The above representation for $N = \sqrt{Q} \in [0, 2]$ gives a range of values $-2 \leq c \leq 1$; our results also apply for $c \in (-\infty, -2]$, corresponding, e.g., to the $O(N \in [-2, 0])$ branch, with a low-T phase for $g \in [0, \frac{1}{2}]$, and the CP for $g \in [\frac{3}{2}, 2]$.

The fractal dimension D_{EP} of the accessible perimeter, Eq. (8), is, like $c(g) = c(g^{-1})$, a symmetric function

$$D_{\mathrm{EP}} = 1 + \frac{1}{2}g^{-1}\vartheta(1 - g^{-1}) + \frac{1}{2}g\vartheta(1 - g), \tag{14}$$

where ϑ is the Heaviside distribution, thus given by two different analytic expressions on either side of the separatrix $g = 1$. The dimension of the *hull*, i.e., the complete set of outer boundary sites of a cluster, has been determined for $O(N)$ and Potts clusters [21], and reads $D_{\mathrm{H}} = 1 + \frac{1}{2}g^{-1}$, for the *entire* range of the coupling constant $g \in [\frac{1}{2}, 2]$. Comparing to Eq. (14), we therefore see that the accessible perimeter and hull dimensions *coincide* for $g \geq 1$, i.e., at the $O(N)$ CP (or for tricritical Potts clusters), whereas they *differ*, namely $D_{\mathrm{EP}} < D_{\mathrm{H}}$, for $g < 1$, i.e., in the $O(N)$ low-T phase, or for critical Potts clusters. This is the generalization to any Potts model of the effect originally found in percolation [22]. This can be directly understood in terms of the *singly connecting* sites (or bonds) where fjords close in the scaling limit. Their dimension is given by $D_{\mathrm{SC}} = 1 + \frac{1}{2}g^{-1} - \frac{3}{2}g$ [21]. Thus, for critical $O(N)$ loops, $g \in (1, 2]$ and $D_{\mathrm{SC}} < 0$, so there exist no closing fjords, which explains the identity: $D_{\mathrm{EP}} = D_{\mathrm{H}}$, whereas $D_{\mathrm{SC}} > 0$, $g \in [\frac{1}{2}, 1)$ for critical Potts clusters, or in the $O(N)$ low-T phase, where pinching points of positive dimension appear in the scaling limit, so that $D_{\mathrm{EP}} < D_{\mathrm{H}}$ (Table I). I then find from Eq. (14), with $g \leq 1$,

TABLE I. Dimensions for the critical Q-state Potts model; $Q = 0, 1, 2$ correspond, respectively, to spanning trees, percolation, and Ising clusters.

Q	0	1	2	3	4
c	-2	0	1/2	4/5	1
D_{EP}	5/4	4/3	11/8	17/12	3/2
D_{H}	2	7/4	5/3	8/5	3/2
D_{SC}	5/4	3/4	13/24	7/20	0

$$(D_{\mathrm{EP}} - 1)(D_{\mathrm{H}} - 1) = \frac{1}{4}. \tag{15}$$

The symmetry point $D_{\mathrm{EP}} = D_{\mathrm{H}} = \frac{3}{2}$ corresponds to $g = 1$, $N = 2$, or $Q = 4$, where, as expected, the dimension $D_{\mathrm{SC}} = 0$ of the pinching points vanishes.

For percolation, described either by $Q = 1$ or by the low-T $O(N = 1)$ model, with $g = \frac{2}{3}$, we recover the result $D_{\mathrm{EP}} = \frac{4}{3}$, recently derived in [5]. For the Ising model, described either by $Q = 2, g = \frac{3}{4}$, or by the $O(N = 1)$ CP, $g' = g^{-1} = \frac{4}{3}$, we observe that the EP dimension $D_{\mathrm{EP}} = \frac{11}{8}$ coincides, as expected, with that of the critical $O(N = 1)$ loops, which, in fact, appear as EP's. This is a particular case of a further duality relation between the critical Potts and CP $O(N)$ models: $D_{\mathrm{EP}}(Q(g)) = D_{\mathrm{H}}(O(N(g')))$, for $g' = g^{-1}, g \leq 1$.

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