
Licentiate Thesis

# Examining Hausdorff dimension and Scaling behaviour with Worm algorithm

Simon Rydell

Theoretical Particle Physics, Department of Theoretical Physics,
School of Engineering Sciences
Royal Institute of Technology, SE-106 91 Stockholm, Sweden

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# Abstract

2.3 The Metropolis Monte Carlo method The principle of importance sampling in statistical physics as suggested by Metropolis et al. [83] is as follows. In Fig. 2.1 we generated numbers $x$ and from a uniform distribution. In the evaluation of something like (2.1), we could instead imagine generating the configurations $\mu \equiv \{\theta_i\}_{i=1}^{N}$ not uniformly, but in such a way that the number of hits and misses are roughly equal. The estimated expectation value after $M$ generated configurations $\mu_n$ can be written

$$\langle O \rangle = \frac{\sum_{n=1}^{M} O_{\mu_n} e^{\beta H \mu_n}}{\sum_{n=1}^{M} e^{\beta H \mu_n}} \tag{1}$$

for states $\mu_n$ generated with uniform probability. If we instead were to draw configurations with probabilities according to their Boltzmann weights, the estimator is simply an unweighted arithmetic average

$$\langle O \rangle \approx \sum_{n=1}^{M} O_{\mu_n} \tag{2}$$

for states $\mu_n$ generated with probability $e^{-\beta \mu_n}$

The remaining question is then how one generates configurations with probability according to their Boltzmann weights. In practice, the importance sampling discussed above is obtained with a Markov chain through the Metropolis method [83]. Metropolis Monte Carlo works by proposing a new configuration $j$ from an old configuration $i$ with a proposal distribution $g(i \to j)$, which is accepted with a specified probability $A(i \to j) = \min(1, e^{-\beta \Delta E})$ where $\Delta E = E_j - E_i$. Consider the time evolution of the probability of being in state $i$,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \sum_j [p_j P(j \to i) - p_i P(i \to j)] \tag{3}$$

where $P(i \to j)$ is the probability of transitioning from $i$ to $j$. The first and second terms on the right-hand side correspond to the rate of transitioning into and out of the state i respectively. In equilibrium the time derivative is zero and (2.5) is fulfilled if (but not only if)

$$p_j P(j \to i) = p_i P(i \to j) \tag{4}$$

The condition of (2.6) is called detailed balance and is a sufficient but not necessary condition for Metropolis Monte Carlo given that states are generated ergodically, see [84, 88]. Ergodicity means that each state is reachable from every other state in a finite time, however a non-ergodic calculation may still yield useful information within an ergodic class [84]. In a simulation, the probability of transitioning from i to j is the product of the proposal and acceptance probabilities, that

is $P(i \to j) = g(i \to j)A(i \to j)$ which after insertion into the balance equation (2.6) gives

$$\frac{p_i}{p_j}\frac{g(j \to i)\min(1, e^{-\beta(E_i - E_j)})}{g(i \to j)\min(1, e^{-\beta(E_j - E_i)})} = \frac{e^{-\beta E_i}}{e^{-\beta E_j}} \tag{5}$$

assuming that the proposal distribution is symmetric. The Metropolis Monte Carlo method thus generates a chain of configurations where each configuration appears with a probability proportional to its Boltzmann weight. The estimation of thermal averages can thus be done via arithmetic averaging of the form (2.4) by starting with some initial configuration $\mu_0$ and generating a chain $\mu_1, \ldots, \mu_M$ via the Metropolis scheme. 2.5 Phase transitions: definition and classification

The thermodynamical bulk free-energy density $f = \beta - 1\ln(Z)/V$ depends on the coupling parameters of the Hamiltonian, and thermodynamic functions are calculated via differentiation of the free energy. Such thermo-ynamic functions may exhibit discontinuities at certain sets of coupling parameters, for which the free energy is not analytic. Such sets can be taken as definitions of phase boundaries which separate phases (i.e. regions of analyticity). The crossing of a phase boundary is then understood as a phase transition. It should be noted that the definition in terms of analytic free-energy is not completely general, considering the BKT-transition. For further details of mathematical aspects of phase transitions, we refer to [94] and for a more physical discussion to [95].

By the Ehrenfest classification [96] phase transitions are classified by the order of the derivative of the free energy which has a discontinuity at the critical point. If a first order derivative (e.g. entropy, internal energy) is discontinuous, then the phase transition is of first order. If a second order derivative (e.g. heat capacity) is discontinuous, the phase transition is of second order, and so on.

The Ehrenfest classification is however discouraged [95, 98], for example there may be divergences rather than dis- continuities in thermodynamic functions (as for the 2D Ising model without external field, where the heat capacity diverges logarithmically at $T_c$). The modern classification of phase transitions is binary where a phase transition is either 'first-order' (there is a latent heat for a temperature-driven transition) or 'continuous' (no latent heat) [98, 95].

5. Partition function A problem of statistical mechanics is to calculate the partition function, that is calculating the trace of $e^{-\beta H}$, a task most straightforwardly written as

$$Z = \sum \langle \alpha_n | e^{-\beta H} | \alpha_n \rangle \tag{6}$$

$$= \sum e^{-\beta E_i} \tag{7}$$

where the $|\alpha_n\rangle$ is a complete orthonormal set and $E_i$ the energy eigenvalues of $H$. In this form, it is necessary to determine the eigenvalues $E_i$ of the Hamiltonian operator, and then carry out the sum over all corresponding weights $e^{-\beta E_i}$, as

Feynman put it [132] a 'hopelessly difficult' calculation. Instead we reformulate the problem in terms of imaginary time path integrals [132, 69, 133].

Imaginary-time path integral reformulation of the partition function First, we outline the reformulation of (5.5) in terms of path integrals, with the Bose-Hubbard model (5.2) in mind. Following the derivation in [69], first denote the statistical operator It then follows that

$$\rho(\beta) = e^{-\beta H} \tag{8}$$
$$\partial\rho = -H\rho \tag{9}$$

## 0.1 Derivation for Ising 2D Worm

The Ising model energy

$$E = -J \sum_{\langle ij \rangle} S_i S_j \tag{10}$$

Let $K = \beta J$ where $\beta = 1/k_{\mathrm{B}}T$.

$$\beta E = -K \sum_{\langle ij \rangle} S_i S_j \tag{11}$$

The partition function $Z$.

$$Z = \sum_{\text{all states}} e^{-\beta E} = \sum_{\text{all states}} e^{K \sum_{\langle ij \rangle} S_i S_j} = \sum_{\text{all states}} \Pi_{\langle ij \rangle} e^{K S_i S_j} \tag{12}$$

Since $S_i S_j = \pm 1$ in the Ising model, Euler identities can be used to expand the exponential in (12).

$$\begin{aligned} e^{K S_i S_j} &= \frac{e^K + e^{-K}}{2} + S_i S_j \frac{e^K - e^{-K}}{2} \\ &= \cosh(K) + S_i S_j \sinh(K) \\ &= \{T = \tanh(K)\} \\ &= (1 + T S_i S_j) \cosh(K) \end{aligned}$$

For $N$ spins there are 2N bonds, therefore the partition function is

$$\begin{aligned} Z &= \sum_{\text{all states}} \Pi_{\langle ij \rangle} (1 + T S_i S_j) \cosh(K) \\ &= \cosh^{2N}(K) \cdot 2^N \left( 2^{-N} \sum_{\text{all states}} \Pi_{\langle ij \rangle} (1 + T S_i S_j) \right) \\ &= \cosh^{2N}(K) \cdot 2^N Z' \end{aligned}$$

And

$$\begin{aligned} Z' &= 2^{-N} \sum_{\text{all states}} \Pi_{\langle ij \rangle} (1 + T S_i S_j) \\ &= 2^{-N} \sum_{S_1 = \pm 1} \sum_{S_2 = \pm 1} \cdots \sum_{S_N = \pm 1} \left( 1 + T \sum_{l=1} S_i S_j + T^2 \sum_{l=2} (S_i S_j)(S_{i'} S_{j'}) + \ldots \right) \end{aligned}$$

Where the sums $\sum_{l=L}$ should be interpreted as the sum over all sets where the link length is $L$. Link length is the coupling between $S$-terms as

$$\begin{aligned} (S_1 S_2) &\ - \text{ Open with length 1} \\ (S_1 S_2)(S_2 S_3) &\ - \text{ Open with length 2} \\ (S_1 S_2)(S_2 S_3)(S_3 S_4)(S_4 S_1) &\ - \text{ Closed with length 4} \end{aligned}$$

Since $\sum_{S_i = \pm 1} S_i = 0$, only terms with an even number of $S_i$ are contributing to $Z'$. We will call these terms closed, indicating that they represent a closed loop if we were to draw the link lengths between the sites. The sum over all contributing terms gives a factor of $2^N$, canceling the $2^{-N}$.

We can now rewrite $Z'$ in terms of loop lengths.

$$Z' = \sum_L g(L) T^L \tag{13}$$

Where $g(L)$ is the number of loops with length $L$. Finally we can write the expression for the partition function.

$$Z = 2^N \cosh^{2N}(K) \sum_L g(L) T^L \tag{14}$$

### 0.1.1 Energy calculation

$$E = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{J}{Z} \frac{\partial Z}{\partial K} \tag{15}$$

Therefore

$$\begin{aligned}\frac{\partial Z}{\partial K} &= 2^N 2N\cosh^{2N-1}(K)\cdot\frac{\partial\cosh(K)}{\partial K}Z' + 2^N\cosh^{2N}(K)\cdot\frac{\partial Z'}{\partial K}\\ &= 2^N\cosh^{2N}(K)\left(2N\tanh(K)Z' + \frac{\partial Z'}{\partial K}\right)\\ &= 2^N\cosh^{2N}(K)\tanh(K)\left(2NZ' + \frac{1}{\tanh(K)}\frac{\partial Z'}{\partial K}\right)\end{aligned}$$

Examine $\tanh^{-1}(K)\frac{\partial Z'}{\partial K}$.

$$\begin{aligned}\tanh^{-1}(K)\frac{\partial Z'}{\partial K} &= \tanh^{-1}(K)\frac{\tanh(K)}{\partial K}\sum_L g(L)L\tanh^{L-1}(K)\\ &= \frac{1}{\tanh^2(K)\cosh^2(K)}\sum_L g(L)L\tanh^L(K)\\ &= \frac{Z'}{\sinh^2(K)}\frac{\sum_L g(L)L\tanh^L(K)}{\sum_L g(L)\tanh^L(K)}\\ &= \frac{Z'}{\sinh^2(K)}\langle L\rangle\end{aligned}$$

And finally

$$E = -J\tanh(K)\left(2N + \frac{\langle L\rangle}{\sinh^2(K)}\right) \tag{16}$$

where

$$\langle L\rangle = \frac{\sum_L g(L)L\tanh^L(K)}{\sum_L g(L)\tanh^L(K)} \tag{17}$$

### 0.1.2 Heat capacity

$$C = \frac{\partial E}{\partial T} = -\beta^2\frac{\partial E}{\partial\beta} = -K\beta\frac{\partial E}{\partial K} \tag{18}$$

Let $A = 2N + \frac{1}{\sinh^2(K)}\langle L\rangle$. Then

$$\frac{E}{J} = -\tanh(K)A \tag{19}$$

and

$$\frac{1}{J}\frac{\partial E}{\partial K} = -\frac{\partial\tanh(K)}{\partial K}A - \tanh(K)\frac{\partial A}{\partial K} \tag{20}$$

where

$$\begin{aligned}\frac{\partial A}{\partial K} &= \langle L\rangle \frac{\partial \sinh^{-2}}{\partial K} + \tanh^{-1}(K)\sinh^{-2}(K)\left(\langle L^2\rangle - \langle L\rangle^2\right)\frac{\partial \tanh(K)}{\partial K} \\ &= \frac{1}{\sinh^2(K)\tanh(K)}\left(-2\langle L\rangle + \frac{\langle L^2\rangle - \langle L\rangle^2}{\cosh^2(K)}\right)\end{aligned}$$

and finally

$$C = \frac{K^2}{\sinh^2(K)}\left(\frac{\langle L^2\rangle - \langle L\rangle^2}{\cosh^2(K)} - E\tanh(K) - 2\langle L\rangle\right) \tag{21}$$

# Preface

# Contents

# Part I

# Introduction and background material

# Chapter 1

# Introduction

# Chapter 2

# Background

## 2.1 *n*-vector model

The $n$-vector model describes a classical system of $n$-dimensional classical spins $s_i$ of unit length interacting on a lattice. It is a generalization of the Ising model where each spin can have a continuous set of values. The Hamiltonian is then

$$H = -J \sum_{\langle ij \rangle} s_i \cdot s_j \tag{2.1}$$

where $J$ is the bond strength and $\langle ij \rangle$ refers to a nearest neighbour interaction.

## 2.2 XY model

A special case of the $n$-vector model is the XY model when $n = 2$. Here the spins are two dimensional rotors as $s_i = (\cos\theta_i,\ \sin\theta_i)$. This yields the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \tag{2.2}$$

The partition function is therefore

$$Z = \prod_i \int \frac{\mathrm{d}\theta_i}{2\pi} e^{-\beta H} = \prod_i \int \frac{\mathrm{d}\theta_i}{2\pi} e^{K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \tag{2.3}$$

where $K = J\beta$.

### 2.2.1 Loop expansion

Since equation (2.3) is invariant under the transformation $\theta_i - \theta_j \to \theta_i - \theta_j + 2\pi n$, $n \in \mathbb{Z}$, it can be expanded using the identity

$$e^{\alpha \cos\beta} = \sum_{\gamma=-\infty}^{\infty} I_\gamma(\alpha) e^{i\gamma\beta} \tag{2.4}$$

where $I_\gamma(\alpha)$ is the modified Bessel function. Using that $e^{\sum_i x_i} = \prod_i e^{x_i}$

$$Z = \prod_i \int \frac{\mathrm{d}\theta_i}{2\pi} \sum_{J_{\langle ij\rangle}=-\infty}^{\infty} \prod_{b=\langle ij\rangle} I_{J_{\langle ij\rangle}}(K) e^{iJ_{\langle ij\rangle}(\theta_i-\theta_j)} \tag{2.5}$$

$$= \prod_i \sum_{J_b} \left( \int \frac{\mathrm{d}\theta_i}{2\pi} e^{iN_i(\theta_i-\theta_j)} \right) \left( \prod_b I_{J_b} \right) \tag{2.6}$$

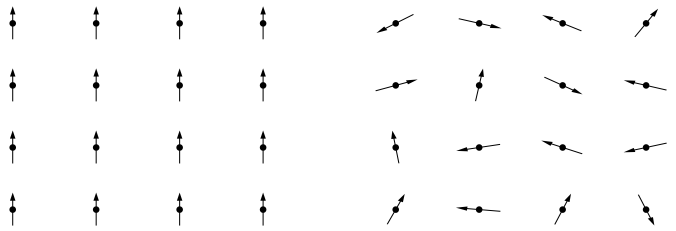
Where in the last step, $\prod_{\langle ij\rangle} e^{iJ_{\langle ij\rangle}(\theta_i-\theta_j)} = e^{iN_i(\theta_i-\theta_j)}$. $N_i$ is therefore the sum of $J$ for the nearest neighbours of site $i$. Noting that

$$\int \frac{\mathrm{d}\theta_i}{2\pi} e^{iN_i(\theta_i-\theta_j)} = C\delta_{N_i 0} \tag{2.7}$$

leads to the conclusion that the sum of incoming and outgoing flux $J$ into a site $i$ must be zero, in other words, the configurations are divergence free. This in turn means that a configuration of the system must contain closed loops of flux $J$.

### 2.2.2 Winding number

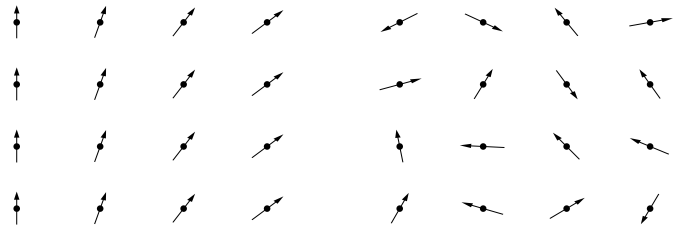
In the ground state all the spins are aligned, while at higher energy states, the spins are pointed in random directions as can be seen in figure (2.1).



(a) Ground state (b) Higher energy state

Figure 2.1: Energy states for XY model

Therefore, making a constant phase shift $\Phi_\mu = \frac{A}{L}$ of $\theta_i - \theta_j$ in the $\mu$ direction would change the energy drastically for the ground state while, on a statistical average, not change the higher states energy at all (see figure (2.2)).



(a) Ground state (b) Higher energy state

Figure 2.2: A phase shift for $\mu = x$

The free energy change $\Delta F$ for such a shift is

$$\Delta F = L^d \cdot \frac{1}{2}\rho_s \left(\frac{A}{L}\right)^2 \Rightarrow \rho_s = \lim_{A\to 0} L^{2-d}\frac{\partial^2 \Delta F}{\partial A^2} \tag{2.8}$$

where $d$ is the dimension and $\rho_s$ is the superfluid density which is zero for a high energy state. The free energy is

$$F = -T\ln(Z) \Rightarrow F'' = T\left(\left(\frac{Z'}{Z}\right) - \left(\frac{Z''}{Z}\right)^2\right) \tag{2.9}$$

where $F' = \partial F/\partial A$. Examining $Z$ from (2.6) with the added shift

$$Z = \prod_i \int \frac{\mathrm{d}\theta_i}{2\pi} \sum_{J_{\langle ij\rangle}=-\infty}^{\infty} \prod_{b=\langle ij\rangle} I_{J_{\langle ij\rangle}}(K) e^{iJ_{\langle ij\rangle}(\theta_i-\theta_j+\Phi_\mu)} \tag{2.10}$$

$$= \prod_i \sum_{J_b} \left(\int \frac{\mathrm{d}\theta_i}{2\pi} e^{iN_i(\theta_i-\theta_j)}\right)\left(\prod_b I_{J_b}\right) \cdot e^{iA\frac{1}{L}\sum_i J_{i,i+\mu}} \tag{2.11}$$

$$\tag{2.12}$$

where in the last step

$$\prod_i \left( \prod_{\langle ij \rangle} e^{iJ_{\langle ij \rangle}\Phi_\mu} \right) = \tag{2.13}$$

$$\{\Phi_\mu \neq 0 \text{ only for neighbours in the } \mu \text{ direction}\} = \tag{2.14}$$

$$\prod_i \left( e^{iJ_{i,i+\mu}\Phi_\mu} \right) = \tag{2.15}$$

$$e^{iA\frac{1}{L}\sum_i J_{i,i+\mu}} \tag{2.16}$$

Introduce the winding number in the $\mu$ direction as

$$W_\mu = \frac{1}{L} \sum_i J_{i,i+\mu} \tag{2.17}$$

Intuitively, this describes the net flux in the $\mu$-direction. Given a loop within the bounds of the lattice, the winding number is always zero. This is since an equal amount of flux in $+\mu$ as in $-\mu$ is needed to form a loop. However, this is not the case for a percolating cluster going, for example, from $-\mu$ to $+\mu$ connecting with periodic boundary conditions. For such a 'winding' cluster, the winding number will be $+1$. An example can be seen in figure (2.3).

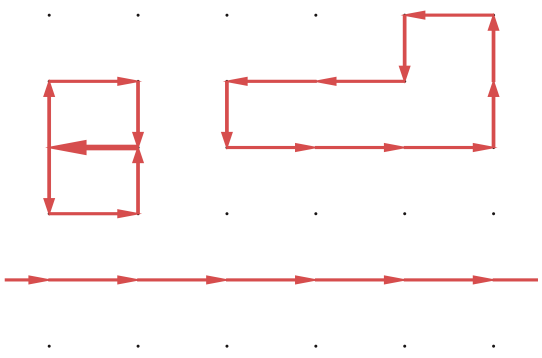


Figure 2.3: Three flux clusters on a square lattice. One percolating cluster with $W_x = +1$. The size of arrow corresponds to the number of flux quanta between two sites.

Using the definition (2.17) for the winding number in the partition function in equation (2.12) yields

$$Z = \sum_{J_b} \left( \prod_b I_{J_b} \right) \prod_i \left( \int \frac{\mathrm{d}\theta_i}{2\pi} e^{iN_i(\theta_i - \theta_j)} \right) \cdot e^{iAW_\mu} \tag{2.18}$$

$$= \sum_{J_b,\ N_i=0} Z_0 \cdot e^{iAW_\mu} \tag{2.19}$$

$$= \sum_{W_\mu} Z_0 \cdot e^{iAW_\mu} \tag{2.20}$$

Using this result in equation (2.9)

$$\frac{\partial^2 F}{\partial A^2} = T \left( \left( \frac{\sum_{W_\mu} (iW_\mu) Z_0 e^{iAW_\mu}}{\sum_{W_\mu} Z_0 e^{iAW_\mu}} \right)^2 - \frac{\sum_{W_\mu} (-W_\mu^2) Z_0 e^{iAW_\mu}}{\sum_{W_\mu} Z_0 e^{iAW_\mu}} \right) \tag{2.21}$$

$$= T \left( -\langle W_\mu \rangle^2 + \langle W_\mu^2 \rangle \right) \tag{2.22}$$

$$= T \langle W_\mu^2 \rangle \tag{2.23}$$

Where $\langle W_\mu \rangle = 0$ since there is an equal chance of percolating from $-\mu$ to $\mu$ as the other way around.

The superfluid density can finally be determined as

$$\rho_s = L^{2-d} T \langle W_\mu^2 \rangle \tag{2.24}$$

# Chapter 3

# Method

# Chapter 4

# Graph Division and Hausdorff Dimension

## 4.1 Worm Algorithm

## 4.2 Graph Labeling

### 4.2.1 Hoshen Kopelman

To find the clusters a modified version of the Hoshen Kopelman algorithm was used. A raster scan is used to label disjoint sets into groups with some canonical label[1]. It is a variant on the union-find algorithm and is most easily described through the associated functions. Intuitively, applying the find function on a site $i$ returns the canonical, often implemented as the smallest, label in the cluster that $i$ belongs to. Union uses find to ensure that two sites $i$ and $j$ are connected by setting the canonical label of $i$ to that of $j$ (or vice versa).

An example implementation would be to have a 2D graph without periodic boundary conditions of zeros and ones, where a site is occupied if it has a one associated with it, and unoccupied otherwise. A disjoint set here is a number of occupied sites neighbouring each other with unoccupied sites surrounding them. For simplicity the scan can start in the lower left corner, moving right and up, while search for neighbours left and down, ensuring that if a neighbouring site is occupied, it has been labeled before.

Start by setting each site to a unique label, putting all sites in individual clusters. Go through the lattice until an occupied site $i$ is found. Search the neighbours below and to the left. If none of these neighbours are occupied, label $i$ have a unique label and move to the next site. If $i$ has one occupied neighbour it must have been labeled before, so $i$ inherits the neighbours label. Finally if both neighbours are occupied, site $i$ must be connecting a cluster and a union is performed on the neighbours to

join their labels. A final pass through the lattice using the find function ensures that all sites have their canonical label.

In this paper an occupied site corresponds to a site with connections to the neighbouring sites (in the 2D example above, each site could have four such connections). In the original paper by Hoshen and Kopelman the labels for the sites who did not originally carry the canonical label, were set to a negative integer, symbolizing that they were aliases. A positive value was used at the canonical label, showing the number of sites in that cluster. This was not used in this project since the number of links in a cluster is not necessarily equal to the number of sites.

## 4.3 Fractals

Everyone agrees that the dimension of a point is zero, and that of a smooth line is one, but what about a set of points? A definition could be to say that the dimension is the minimum number of coordinates needed to describe every point in the set. Effectively, a point would describe itself, and a curve could be parametrized to the distance of some point on the same curve.

The situation is more complex when examining fractals. Take for example the Koch curve, it starts out as a line segment of length $L_0$, and successively adds a 'bump', making the total length $L_1 = 4/3 \cdot L_0$. Iterating $n$ times gives a line length of $L_n = (4/3)^n \cdot L_0$, and so the final fractal length is infinite.

Any two point on the final curve has a distance of infinity between them, so parametrization is impossible. But the area is still finite, so the dimension should intuitively be somewhere between one and two.

A useful concept here is the similarity dimension, defined by the scaling of each iteration. If $m$ is the number of similar elements after an iteration and $r$ is the scaling factor, the dimension is defined by $m = r^d$, or equivalently

$$d = \frac{\ln m}{\ln r} \tag{4.1}$$

So for the Koch curve, each segment is divided into fourths with each having one third the length from the previous iteration, giving it a dimension of $\ln 4 / \ln 3 \approx 1.26$.

## 4.4 Box dimension

## 4.5 Graph Dividing Algorithm

In order to calculate the box dimension the lattice needs to be divided into boxes of decreasing size. A step by step instruction of a graph dividing algorithm is provided below, and an implementation in pseudocode is available in the Appendix at Section 8.1.

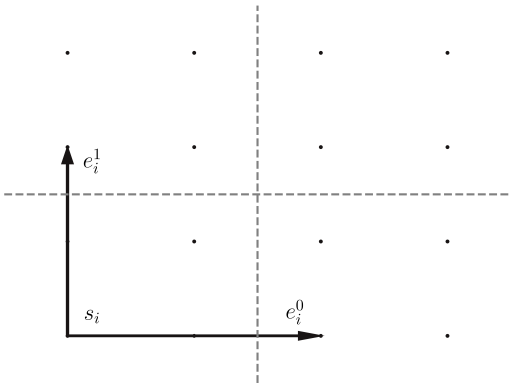



Figure 4.1: One step in the graph dividing algorithm where $l_i = 4$. $e_i^0$ and $e_i^1$ are drawn from site $s_i$ and summed permutations of these will give the starts for the next boxes. The next iteration of boxes are shown via the dividing dotted lines.

For brevity some abbreviations are introduced.

| | |
|---|---|
| $d$ = dimension | $l_i$ = side length of the current box |
| $l_0$ = side length of the smallest box allowed | $e_i^j$ = vector of length $l_i/2$ in the $j$'th direction |
| perm($v$) = All permutations of $v$ | $s_i$ = starting site of the current box |

1. If $l_i \geq l_0$, go to 2, else stop.
2. Save all sites in the current box, starting for $s_i$ going $l_i$ in $d$ directions.
3. Find all starting points for new boxes.
   (i) Form the matrix $E = (e_i^0, e_i^1, \ldots, e_i^d)^T$
   (ii) For all vectors $v_k$ in perm$(0, 0, \ldots, 0)$, perm$(1, 0, \ldots, 0)$, $\ldots$, perm$(1, 1, \ldots, 1)$, create the new start $s_k$ as
   $$s_k = v_k E$$
4. For each start $s_k$:
   (i) $s_i = s_k$, $l_i = l_i/2$
   (ii) Go to 1.

# Chapter 5

# Connection between Hausdorff Dimension and Scaling Behaviour

# Chapter 6

# Results

## 6.1 Error Estimation

In this thesis a number of error estimation techniques were used to know how much data was needed for each measurement. In this section the techniques will be described intuitively.

### 6.1.1 Bootstrap

Bootstrap is a resampling method to examine a probability distribution. In this thesis it was used to estimate the error propagation of parameters in curve fitting.

Given a set $\boldsymbol{x}$ of $N$ measurements from an unknown distribution $\hat{\phi}$, some statistical calculation of interest can be done as $\theta = s(\boldsymbol{x})$. A resampling $\boldsymbol{x}_0$ of $\boldsymbol{x}$ comprised of $N$ random measurements from $\boldsymbol{x}$ (where one measurement can be included several times), can then be used to calculate $\theta_0^* = s(\boldsymbol{x}_0)$. Repeating this $N_B$ times gives an estimate $\theta^* = (\theta_0^*, \theta_1^*, ..., \theta_{N_B}^*)$ of the distribution $\hat{\theta}$. Assuming $N_B$ is large then, by the central limit theorem, $\hat{\theta}$ is a normal distribution with some standard deviation $\sigma_\theta$ that can be used as an error estimation for $\theta$.

# Chapter 7

# Summary and conclusions

# Chapter 8

# Appendix

## 8.1 Pseudo Code for Box Division Algorithm

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