Comment on "Hausdorff Dimension of Critical Fluctuations in Abelian Gauge Theories"

In their Letter [1], Hove, Mo, and Sudbø derive a simple connection between the anomalous scaling dimension, η, of the U(1) universality class order parameter, $\phi(\mathbf{x})$, and the Hausdorff dimension, D_H, of critical loops:

$$\eta + D_H = 2 \, . \tag{1}$$

In the loop representation, the correlator $G(\mathbf{r}) = \langle \phi(\mathbf{r}) \phi^*(0) \rangle \propto r^{-(d-2+\eta)}$ describes the distribution of the end-points in open loops. For definiteness, one may think of the high-temperature-expansion loops for the lattice $|\phi|^4$-model.

The analysis of Ref. [1] might seem absolutely compelling, being just a translation of the hyperscaling hypothesis into the loop language: *At the critical point there should be about one loop of diameter r per volume element r^d* [2]. Nevertheless, given the result $\eta = 0.0380(4)$ of Ref. [3], the relation (1) is in strong contradiction with the value $D_H = 1.7655(20)$ which we obtained for the 3D $|\phi|^4$-model with suppressed leading corrections to scaling [3] (and also—with a bit less accuracy—for the standard bond-current model [4], and its special version with excluded loop overlaps and self-crossings). The simulations were done with the Worm algorithm [5].

The hidden flaw in the treatment of Ref. [1] is as follows. When introducing the self-similar expression

$$P(\mathbf{r}; N) \;\propto\; N^{-\rho} F(r/N^{\Delta}) \, , \quad \Delta = 1/D_H \tag{2}$$

for the probability to find the ends of an open loop of length N being distance $\mathbf{r}$ away from each other, which is then used to establish the connection between the open and closed loops, the authors take for granted that $F(0)$ is finite. While looking innocent, this is an *arbitrary* assumption, since the self-similar form (2) is valid only for $r \gg a$, where a is a microscopic cutoff (e.g., the lattice period). Strictly speaking, a closed loop of length N corresponds to $F(a/N^{\Delta})$ rather than to $F(0)$, and one has to work with the generic asymptotic form

$$F(x) \propto x^{\theta} \qquad \text{at} \qquad x \ll 1 \, , \tag{3}$$

with some exponent θ. With Eq. (3), the hyperscaling argument yields $\rho = (d-\theta)/D_H$, and from $G(r) \propto \int dN P(\mathbf{r}; N)$ one then obtains

$$\eta + D_H = 2 - \theta \, . \tag{4}$$

Using high-precision data for η and D_H mentioned above, we find $\theta = 0.1965(20)$.

It is instructive to explicitly verify Eq. (3) by simulating $P(r; N)$. In Fig. 1 we present results of such a simulation for the $|\phi|^4$-model. We plot the value of $P(r, N) N^{d\Delta}$ as a function of r for three different values of N. In view of the self-similarity of $P(r, N)$, the qualitative difference between the cases of $\theta \neq 0$ and $\theta = 0$ is readily seen. In the former case, curves for different values of N should merge for $r/N^{\Delta} \ll 1$—and they do in Fig. 1. In the latter case, as $r \to 0$ one should see a fan of curves with essentially different slopes and a common origin at $r = 0$.

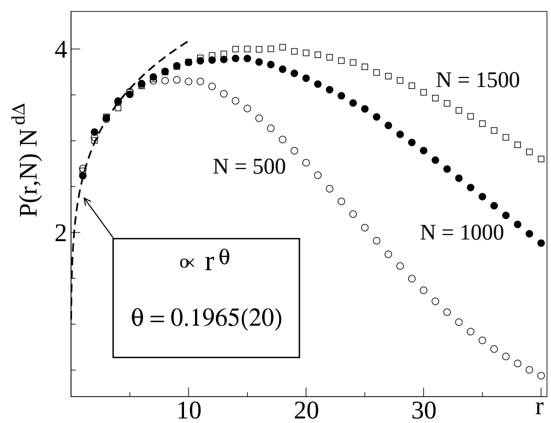

FIG. 1: Distribution of open loops over radii for three different values of N. The Worm algorithm simulation [5] was done for the loop representation (high-temperature expansion) of the 3D lattice $|\phi|^4$-model with $L = 192^3$ sites at the special critical point with suppressed leading corrections to scaling [3].

One important implication of Eq. (4) in the absence of additional relation between D_H, η and θ, is that the anomalous scaling dimension *can not* be deduced from simulations of closed loops which determine D_H only.

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