Ising worm algorithm: derivations

Start with some hyperbolic identities

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\frac{d\sinh x}{dx} = \cosh x$$
$$\frac{d\cosh x}{dx} = \sinh x$$
$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = 1$$
$$\frac{d\tanh x}{dx} = 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x$$

Energy formula

$$Z = 2^N \cosh^{2N} \beta Z'$$
$$Z' = \sum_b \tanh^L \beta$$
$$L = \sum_i b_i = \text{number of links marked 1}$$
$$E = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$$
$$\frac{\partial Z}{\partial \beta} = 2^N \cdot 2N \cosh^{2N-1} \beta \frac{d\cosh\beta}{d\beta} Z' + 2^N \cosh^{2N} \beta \frac{dZ'}{d\beta} =$$
$$= 2^N \cdot 2N \cosh^{2N-1} \beta \sinh \beta Z' + 2^N \cosh^{2N} \beta \frac{dZ'}{d\beta} =$$
$$= 2^N \cosh^{2N} \beta \left[2N \tanh \beta Z' + \frac{dZ'}{d\beta} \right]$$
$$\frac{dZ'}{d\beta} = \frac{d\tanh\beta}{d\beta} \sum_b L \tanh^{L-1} \beta = \frac{1}{\tanh\beta \cosh^2 \beta} \sum_b L \tanh^L \beta$$
$$\boxed{E = -\tanh\beta \left[2N + \frac{\langle L \rangle}{\sinh^2 \beta} \right]}$$

Heat capacity formula

$$C = \frac{dE}{dT} = -\beta^2 \frac{dE}{d\beta}$$

$$E = -\tanh\beta \left[2N + \frac{1}{\sinh^2\beta} \frac{\sum L \tanh^L \beta}{\sum \tanh^L \beta} \right] = -\tanh\beta A$$

$$\frac{dE}{d\beta} = -\frac{d\tanh\beta}{d\beta} A - \tanh\beta \frac{dA}{d\beta} = -\frac{1}{\cosh^2\beta} A - \tanh\beta \frac{dA}{d\beta} = \frac{E\tanh\beta}{\sinh^2\beta} - \tanh\beta \frac{dA}{d\beta}$$

$$\frac{dA}{d\beta} = \langle L \rangle \frac{d}{d\beta} \frac{1}{\sinh^2\beta} + \frac{1}{\sinh^2\beta} \left(\frac{\sum L^2 \tanh^{L-1}\beta}{\sum \tanh^L \beta} - \frac{\left(\sum L \tanh^L \beta\right)\left(\sum L \tanh^{L-1}\beta\right)}{\left(\sum \tanh^L \beta\right)^2} \right) \frac{d\tanh\beta}{d\beta} =$$

$$= -2\langle L \rangle \frac{\cosh\beta}{\sinh^3\beta} + \frac{\langle L^2 \rangle - \langle L \rangle^2}{\sinh^2\beta \cosh^2\beta \tanh\beta} =$$

$$= \frac{1}{\sinh^2\beta \tanh\beta} \left(-2\langle L \rangle + \frac{\langle L^2 \rangle - \langle L \rangle^2}{\cosh^2\beta} \right)$$

$$\boxed{C = \frac{\beta^2}{\sinh^2\beta} \left(\frac{\langle L^2 \rangle - \langle L \rangle^2}{\cosh^2\beta} - E\tanh\beta - 2\langle L \rangle \right)}$$