BOUNDING COP NUMBER OF A GRAPH USING A NEW PARAMETER.

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What is Cops and Robbers?

The standard version of the game is played is played on an undirected reflexive graph.
The cops place themselves on vertices of the graph and then the robber places himself on a vertex. Each round, all cops make a move, followed by the robber. In a move, a player follows an edge of the graph to a new vertex.
The cops win if at any point, a cop and a robber are on the same vertex. If the robber can evade the cops indefinitely, then the robber wins.

Existing Results [1]

Cop number of a graph using an arbitrary subset $U \subseteq V$:

$$c(G) \leq \frac{|U|}{2} + l$$

Where l is the maximum cop number of a component of $G \setminus U$.

Cop number of a graph using vertex cover number vcn:

$$c(G) \leq \frac{vcn}{3} + 1$$

Our Result

We show that

$$c(G) \leq \frac{|U|}{3} + 9$$

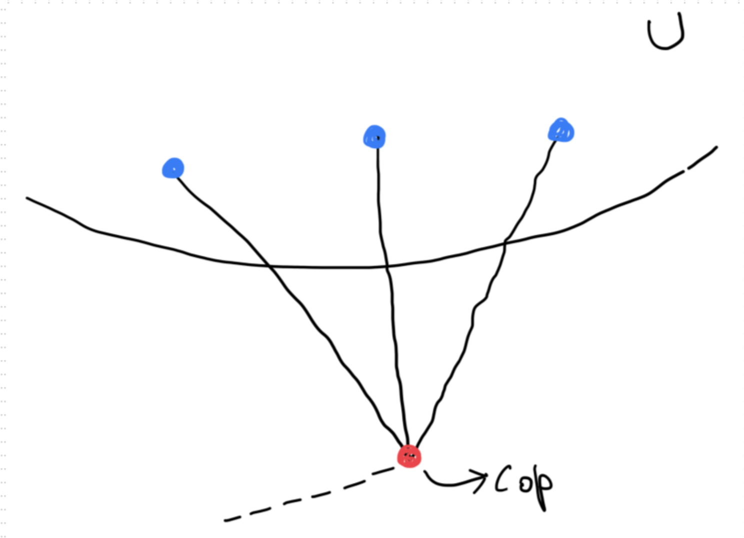
Where $U \subseteq V$ is a minimal subset s.t any component H of $G \setminus U$ is of size at most 2.

We think that this could be generalized to components of size at most k

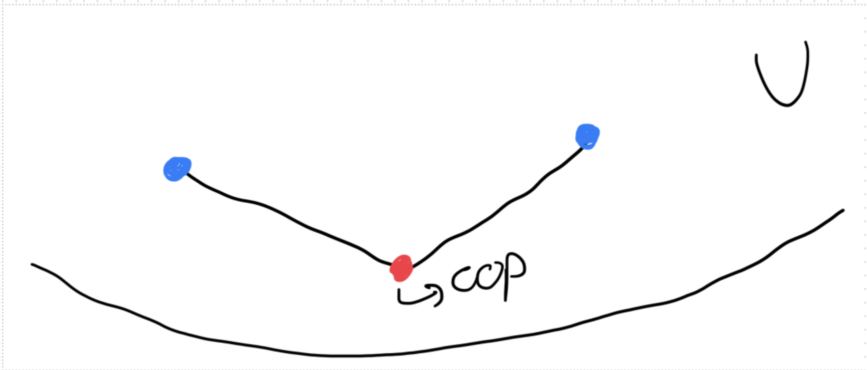
$$c(G) \leq \frac{|U|}{3} + f(k)$$

For some function f that grows with k

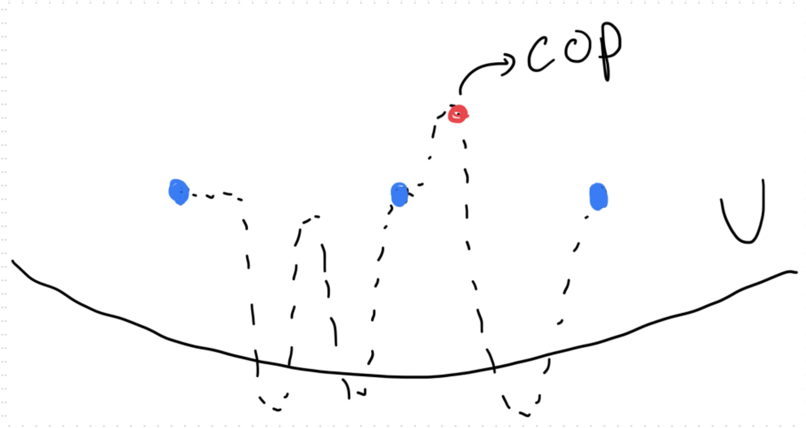
Reduction Rule 1: *If there is a vertex $v \in V(G) \setminus U$ such that $|N(v) \cap U| \geq 3$, then place a cop at v and delete $N[v]$.*



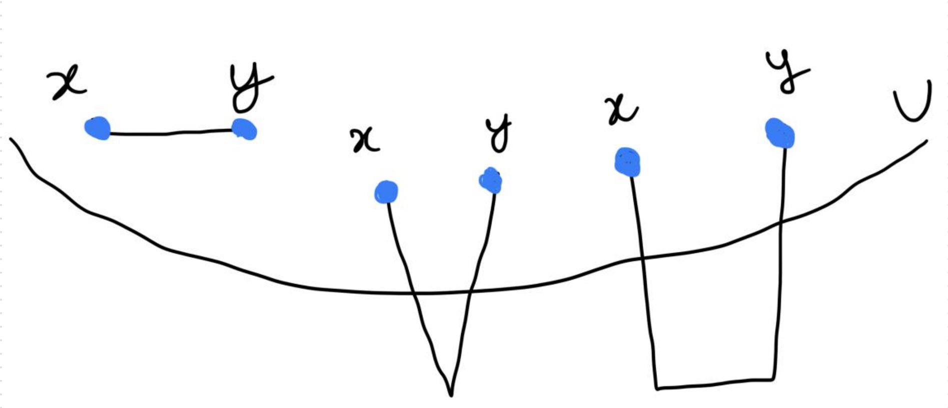
Reduction Rule 2: *If there is a vertex $v \in U$ such that $|N(v) \cap U| \geq 2$, then place a cop at v and delete $N[v]$.*



Reduction Rule 3: *If there exists an isometric(shortest) path with at least 3 vertices in U then place a cop on the path and delete it.*



Once RR1-3 cannot be applied anymore, any two remaining vertices $x, y \in U$ must have at least one path between them of the following form:

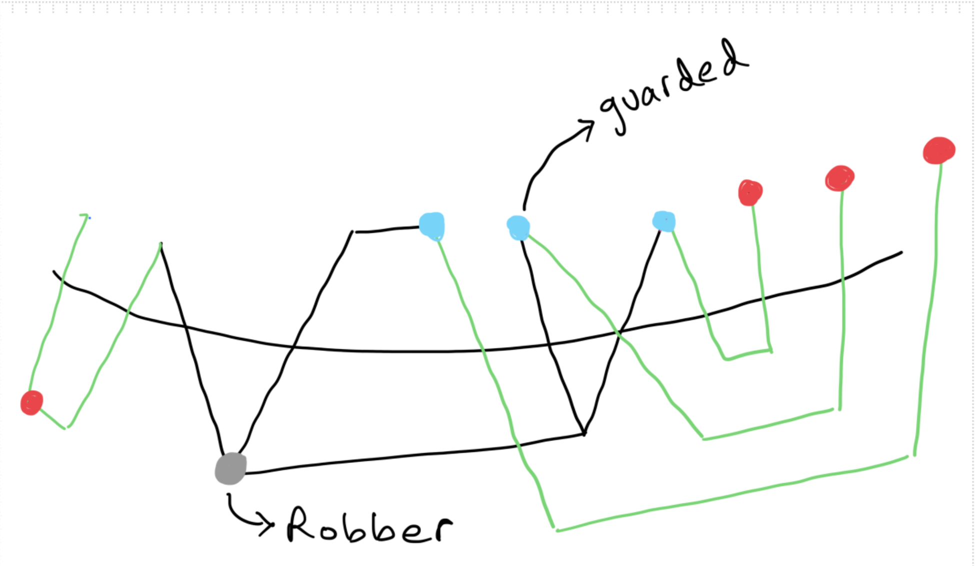


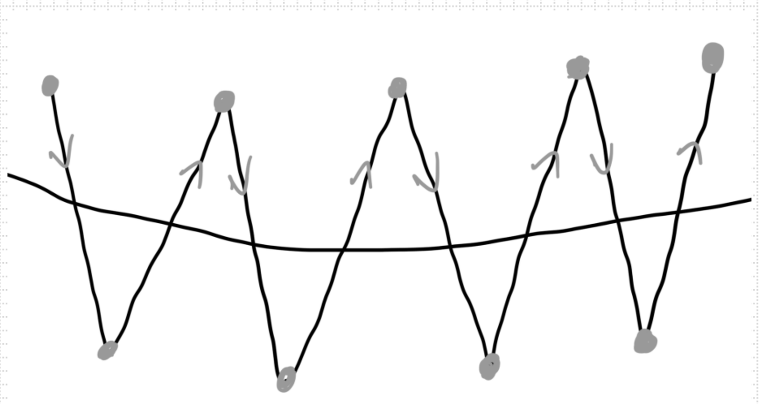
Each reduction removes at least 3 vertices from U, and adds exactly one cop. The remaining cover U' is such that

$$|U'| \leq |U| - 3r$$

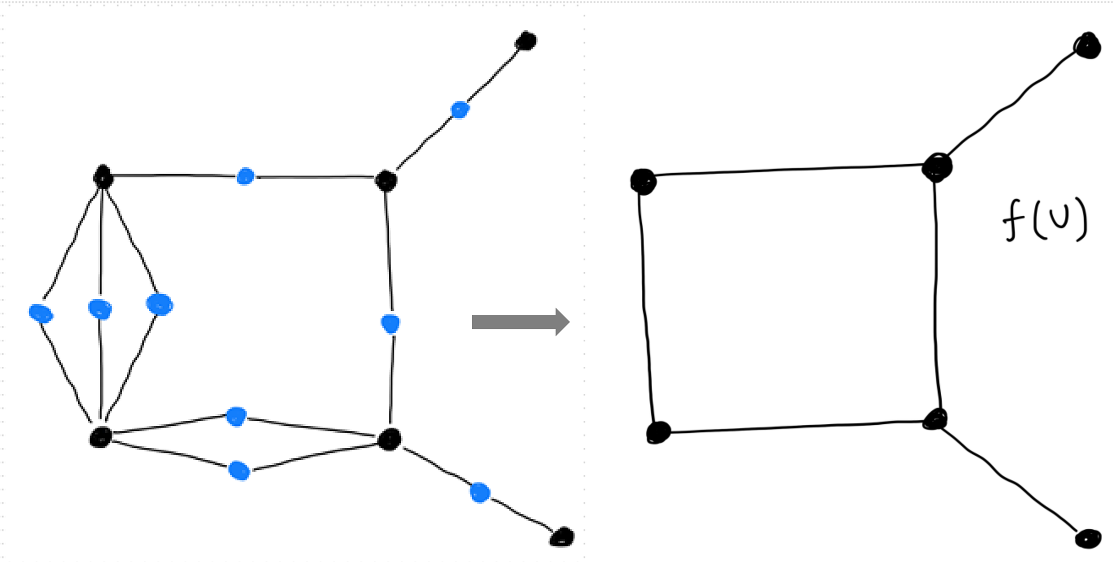
Where r is the number of times a reduction rule was applied.

In the reduced graph G', 7 cops can be added, to ensure that the robber must eventually move in a zigzag pattern, in and out of U'.





Now if we consider the subset of G' that is still traversable by the robber, after we have forced this zigzag motion on him, we get the following type of graph, with black vertices in U'.



We now define a reduction f, that retracts the blue vertices on to their black neighbors. Notice that $f(U')$ has vertices only from U'. We show that

$$c(G') \leq 1 + c(f(U'))$$

From the previous result we note that

$$c(f(U)) \leq \frac{vcn_{f(U)}}{3} + 1 \leq \frac{|U'|}{3} + 1$$

And so we have

$$c(G') \leq \frac{|U'|}{3} + 2$$

Combining the number of cops used in each step, we get our final result:

$$c(G) \leq r + 7 + c(G') \leq r + 7 + \frac{|U'|}{3} + 2 \leq \frac{|U|}{3} + 9$$

Other attempts we made:

- Before we attempted this problem, we explored the idea of parametrized algorithms for cop number, inspired by the FPT algorithm parametrized over vertex cover given in [1].
- There is an existing result, stating that a graph is 1-cop win iff it is *dismantlable*, where *dismantlability* is defined using *corners* of a graph. We attempted to extend this by defining a *k-corner* and a *k-dismantleable* graph but were unable to prove this result.
- While the above Reduction Rules are identical to those in [1], we attempted to extend them by adding further rules, to enforce more conditions on the reduced graph so that it would have a constant cop number, which is how the *vcn* bound given was found.
- When we were unable to make any progress using these approaches, we attempted to disprove any bound of the form $c(G) \leq \frac{|U|}{3} + k$, and instead pivoted to proving a new bound of

$$c(G) \leq \frac{|U|}{2 + \epsilon} + k$$

- We also tried to make more covers inside U to find the cop number of the residual graph.
- We tried to define a subcover U'' as a sort of vertex cover of the reduced graph U'. While this directly did not lead to a solution, it provided the final step to the bound proven above.

References

[1] Harmender Gahlawat and Meirav Zehavi (2023): Parameterized Analysis of the Cops and Robber Problem