Convergence of Generative Adversarial Networks from the perspective of Game Theory

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Abstract

This paper explores the relationship between gradient descent and Nash equilibrium in game theory, focusing specifically on the intersection of these concepts in Generative Adversarial Networks (GANs). GANs have gained considerable attention in deep learning for their ability to generate realistic images. By viewing GANs as a classical game model, where two players maximize their own utility, research on Nash equilibrium in GANs has been extensively conducted.

Previous work on convergence in GANs has limitations in terms of scalability and strong assumptions. Recent studies have proposed alternative approaches, such as gradient descent on Hamilton functions, with Symplectic Gradient Adjustment (SGA) being a prominent example. This project aims to understand, analyze, and compare these approaches. Furthermore, the objective is to extend the research on equilibrium analysis to GAN variants, including Triple GAN, which involves three players: a generator, a discriminator, and a classifier. The findings from this research will contribute to advancing our understanding of equilibrium in GANs and its variants.

1 Introduction

The field of Game Theory has yielded numerous fascinating and compelling results pertaining to a wide range of games. Typically, the decision-making mechanisms of players in these games are often simple and analytically describable. However, when we consider the intersection of games with neural networks, which possess complex and difficult-to-explicitly-describe mechanisms, we are confronted with the challenge of dealing with the intricacies of gradient descent. In this project, we aim to explore the relationship between gradient descent and Nash equilibrium, emphasizing the perspective of convergence.

A notable instance where gradient descent and game theory intersect is in the realm of Generative Adversarial Networks (GANs) [1]. GANs have garnered significant attention in deep learning, operating with a generator and discriminator network trained to generate synthetic images that closely resemble real images. GANs inherently construct a classical game model, wherein two players strive to maximize their utility. Consequently, extensive research has been conducted on Nash equilibrium in the context of GANs. Notably, recent works have investigated convergence in GANs [2] [3]. Furthermore, researchers have explored variants of GANs that modify the model structure [4] [5].

While modifications to the structure of GANs have yielded promising results, it is worth noting that most gradient methods lack theoretical guarantees for these new structures, such as n-player games. To address this gap, a powerful gradient adjustment method for general differentiable games has been proposed [6]. Building upon the aforementioned works, this project seeks to comprehensively understand, analyze, and compare these approaches. The project adopts a Game Theory and Optimization Theory perspective to conduct a thorough examination, extending research on the equilibrium of GANs and their variants using the most powerful tools available. One specific objective is Triple GAN [4], which involves three players: a generator, a discriminator, and a classifier.

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The project includes experimental analysis, training curve analysis, and sample generation under various training configurations. Furthermore, philosophical reflections are drawn from the results, and comparisons are made with real-life examples.

2 Related Works

Generative adversarial Networks. In recent years, a novel framework has emerged for estimating generative models through an adversarial process [1]. This framework involves training two models simultaneously: a generative model (G) that captures the underlying data distribution, and a discriminative model (D) that estimates the probability of a sample originating from the training data rather than G. This training process can be conceptualized as a minimax two-player game. Within the space of arbitrary functions G and D, a unique solution exists for this framework. The optimal solution occurs when G accurately captures the training data distribution, while D outputs a probability of 1/2 for all samples. Notably, if G and D are defined as multilayer perceptrons, the entire system can be trained using backpropagation, eliminating the need for Markov chains or unrolled approximate inference networks during training or sample generation.

Numerical analysis of GANs. Several recent studies have examined the convergence properties of GANs. One such study proposes a two-time scale method for finding Nash equilibrium, although its scalability is limited as the number of players increases [2]. Additionally, we have come across another paper that investigates the lack of convergence in GANs [3]. The authors highlight that the eigenvalues of the Jacobian matrix corresponding to the GAN gradient vector field exhibit either zero real parts or excessively large imaginary parts. Based on theoretical analysis, the authors introduce a regularization method that proves beneficial for our research. However, these previous works do not readily apply to n-player games, including even simple three-player games.

Mechanics of n-Player Differentiable Games. Notably, we have found works that delve into n-player games and achieve performance on par with GANs. These studies identify the limitations of vanilla gradient descent and propose alternative approaches such as gradient descent on Hamilton functions. One notable approach is Symplectic Gradient Adjustment (SGA) [6]. Leveraging these ideas, the authors design a new, more stable algorithm for finding the Nash equilibrium of a smooth two-player game. This algorithm enables stable training of GANs across various architectures and divergence metrics.

Variants of GANs. Generative Adversarial Nets (GANs) have shown promise in image generation and semi-supervised learning (SSL). However, existing GANs in SSL face two challenges: (1) the generator and discriminator may not be optimal simultaneously, and (2) the generator lacks control over the semantics of generated samples. To tackle these problems, a variant called Triple-GAN has been proposed [4]. Triple-GAN introduces a third player, the classifier, alongside the generator and discriminator. The generator and classifier jointly model conditional distributions between images and labels, while the discriminator focuses on identifying fake image-label pairs.

3 Game structure of GANs

Discriminator D that distinguishes whether data x comes form the true distribution $p(x)$. Generator G that characterizes the conditional distribution in the other direction $p_g(x) \approx p(x)$.

The utility of D is

$$E_{x \sim p(x)}[\log(D(x)] + E_{z \sim p_z(z)}[\log(1 - D(G(z)))]$$

The utility of G is

$$-E_{x \sim p(x)}[\log(D(x)] - E_{z \sim p_z(z)}[\log(1 - D(G(z)))]$$

Thus the exact optimization objective is

$$\min_{G} \max_{D} E_{x \sim p(x)}[\log(D(x)] + E_{z \sim p_z(z)}[\log(1 - D(G(z)))]$$

Now we have two clear observations:

Proposition 1. GANs could be viewed as a two-player zero-sum game.

Proposition 2. $G \sim p$, and $D \equiv \frac{1}{2}$ is a Nash equilibrium. Suppose we have optimal D for x, notice that $L(D) = p_{\text{data}}(x) \cdot \log D + p_G(x) \cdot \log(1 - D)$, now we consider $\frac{\partial L}{\partial D} = 0$, thus we have $D^* = \frac{p_{\text{data}(x}}{p_{\text{data}(x)} + p_G(x)} = 0.5$ when we have a perfect generator.

4 Numerical analysis of GANs

The differentiable game defines a vector field

$$v(\phi, \theta) = \begin{pmatrix} \nabla_\phi f(\phi, \theta) \\ -\nabla_\theta f(\phi, \theta) \end{pmatrix}$$

Thus

$$v'(\phi, \theta) = \begin{pmatrix} \nabla^2_\phi f(\phi, \theta) & \nabla_{\phi,\theta} f(\phi, \theta) \\ -\nabla_{\phi,\theta} f(\phi, \theta) & -\nabla^2_\theta f(\phi, \theta) \end{pmatrix}$$

Now we analyse the Simultaneous Gradient Ascent method (SimGA):

while not converged do

$$\begin{aligned} v_\phi &\leftarrow \nabla_\phi f(\theta, \phi) \\ v_\theta &\leftarrow -\nabla_\theta f(\theta, \phi) \\ \phi &\leftarrow \phi + h \cdot v_\phi \\ \theta &\leftarrow \theta + h \cdot v_\theta \end{aligned}$$

And we have the definition of convergence of fixed point iterations: Let $F : \Omega \to \Omega$ be a continuously differential function on an open subset Ω of $\mathbb{R}^n$ and $\bar{x} \in \Omega$ be so that

1. $F(\bar{x}) = \bar{x}$
2. the absolute values of the eigenvalues of $F'(x)$ are all smaller than 1.

Then there is an open neighborhood U of $\bar{x}$ so that for all $x_0 \in U$, $F^{(k)}(x_0)$ converges to $\bar{x}$. And in practice, we would use a variant: $F(x) = x + hF(x)$, then $F'(x) = I + hG'(x)$.

Assume that $A \in \mathbb{R}^{n \times n}$ only has eigenvalues with negative real-part and let $h > 0$. Then the eigenvalues of the matrix $I + hA$ lie in the unit ball if.f

$$h < \min_{\lambda \text{ is an eigenvalue of } A} \frac{1}{|\text{Re}(\lambda)|} \frac{2}{1 + \left(\frac{\text{Im}(\lambda)}{\text{Re}(\lambda)}\right)^2}$$

Therefore, relate the SimGA process to this lemma, we can see that SimGA may not achieve convergence due to the fact that $G'(x)$ is not symmetric and can therefore have non-real eigenvalues, thus h should be bounded by small bounds. Now we define and analyse the Consensus optimization method:

Define L as $\frac{1}{2}||v||^2$, while not converged do

$$\begin{aligned} v_\phi &\leftarrow \nabla_\phi (f(\theta, \phi) - \gamma L(\theta, \phi)) \\ v_\theta &\leftarrow \nabla_\theta (-f(\theta, \phi) - \gamma L(\theta, \phi) \\ \phi &\leftarrow \phi + h \cdot v_\phi \\ \theta &\leftarrow \theta + h \cdot v_\theta \end{aligned}$$

With consensus optimization, we can rewrite $F(x)$ as

$$F(x) = x + h(I - \gamma v'(x)^T)v(x)$$

There is a useful lemma in Linear algebra[3], which is that for negative semi-definite matrix $A \in \mathbb{R}^{n \times n}$, we have

$$\max_{\lambda \text{ is an eigenvalue of } A} \frac{|\text{Im}(\lambda)|}{|\text{Re}(\lambda)|} \leq \frac{1}{c + 2\rho^2\gamma}$$

where $c = \min_{v \in \text{unit sphere in } \mathbb{C}^n} \frac{|\bar{v}^T(A + A^T)v|}{|\bar{v}^T(A - A^T)v|}$, and $|Av| \geq \phi|v|$. Therefore, we can see that under appropriate assumptions, the imaginary-to-real quotient can be made arbitrarily small. Thus this leads to a good convergence near a local Nash equilibrium.

5 Limitation of GANs and previous gradient ascent method

There are two main problems of vanilla GANs:

Problem 1. The generator and the discriminator may not be optimal at the same time. The vanilla game structure would lead to two dilemmas (1) When the discriminator is too strong, then gradient vanishing would happen, thus the generator cannot learn to generate well. (2) When the generator is too strong, then the discriminator may perform bad in prediction. Recall the Nash equilibrium we have stated before, when G has learned the exact distribution, then for D, none of strategy would make better-off than a random strategy.

Problem 2. The generator cannot control the semantics of the generated samples, especially in a semi-supervised context. The learning process of generator is based on the discriminator, but the discriminator can only judge real or fake, thus the generator cannot receive signals about the label information, then the generator cannot learn to control the semantics of the generated samples. Although there have been some techniques like Conditional-GANs, the discriminator solely takes the data as inputs, then in a semi-supervised context, it is still hard for generator to control semantic information as the ability of classification of discriminator is restricted as we state in **Problem 1**.

Limitation of previous gradient ascent methods are:

Limitation 1. Previous methods likely scale badly with the number of players, or may highly depend on gorgeous properties of zero-sum. There are also concerns that the property of zero-sum is useless when there are many players: any n-player game can be reformulated as a zero-sum $(n+1)$-player game where $\mathcal{L}_{n+1} = -\sum_{i=1}^{n} \mathcal{L}_i$.

Limitation 2. Consensus optimization, which works well in two players zero-sum games, fails in finding stable fixed points in general games. Here we can provide a example that Consensus optimization would converge to global maximum in general potential games: We consider a potential game with losses $\mathcal{L}_1(x,y) = \mathcal{L}_2(x,y) = -\frac{\mu}{2}(x^2+y^2)$ with μ large enough. Then we have

$$\xi = -\mu \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } H = -\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

Now we apply consensus optimization and get

$$\xi + \lambda \cdot H^T \xi = \mu(\lambda\mu - 1) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Descent on $\xi + \lambda \cdot H^T \xi$ would converge to the global maximum $(0,0)$ as $\lambda > \frac{1}{\mu}$.

6 More Game structure designs

Thus we redesign the game structure: Discriminator D that distinguishes whether a pair of data (x,y) comes form the true distribution $p(x,y)$. Classifier C that characterizes the conditional distribution $p_c(y|x) \approx p(y|x)$. Generator G that characterizes the conditional distribution in the other direction $p_g(x|y) \approx p(x|y)$. This structure is called "Triple GANs".

The utility of D is

$$E_{(x,y)\sim p(x,y)}[\log D(x,y)] + \alpha E_{(x,y)\sim p_c(x,y)}[\log(1-D(x,y)] + \beta E_{(x,y)\sim p_g(x,y)}[\log(1-D(G(y,z),y))]$$

The utility of C is

$$-\mathrm{KL}(p||p_c) - \gamma \mathrm{KL}(p_g||p_c)$$

The utility of G is

$$E_{(x,y)\sim p_g(x,y)}[\log D(G(y,z),y)] - \rho \mathrm{KL}(p_g||p_c)$$

where $\alpha, \beta \in (0,1)$ are constants controlling the relative importance of generation and classification from the perspective of discriminator. So as γ, ρ is.

For any fixed C and G, the best response of D is

$$D^*_{C,G}(x,y) = \frac{p(x,y)}{p(x,y) + \alpha p_c(x,y) + \beta p_g(x,y)}$$

Therefore, it turns out that Triple-GAN avoids the competition between C and G and the equilibrium of this game is achieved when $p(x,y) = p_g(x,y) = p_c(x,y)$.

7 Power of symplectic gradient adjustment

For more complex game structure, we introduce a more powerful gradient adjustment method: symplectic gradient adjustment. To illustrate this method, we first introduce the concepts of Hamilton games and Helmholtz decomposition. We decompose the second-order dynamics into two components:

1. The part of Potential games with symmetric Hessian matrix. (reducing to gradient descent on a implicit function.)
2. The part of Hamilton games with antisymmetric Hessian matrix. (akin to conservation laws.)

We have:

1. a set of players $[n]$
2. losses $\{l_i; R^d \to R\}_{i=1}^n$(twice continuously differentiable)
3. parameters $w = (w_1, ..., w_n) \in R^d$ and $w_i \in R^{d_i}$

And we define:

1. $\xi(w) = (\nabla_{w_1} l_1, ..., \nabla_{w_n} l_n)$
2. $H(w) = \nabla_w \xi(w)^T = \begin{pmatrix} \nabla^2_{w_1} l_1 & \nabla^2_{w_1,w_2} l_1 & \dots & \nabla^2_{w_1,w_n} l_1 \\ \dots & & & \\ \nabla^2_{w_n,w_1} l_n & \nabla^2_{w_n,w_2} l_n & \dots & \nabla^2_{w_n} l_n \end{pmatrix}$

Thus, the Hessian $H(w)$ of any vector field can be decomposed into two parts $H(w) = S(w) + A(w)$ where $S \equiv S^T$ and $A + A^T \equiv 0$. ($S(w)$ is the Potential part and $A(w)$ is the Hamilton part.) The construction is very easy: $S = \frac{1}{2}(H + H^T)$, and $A = \frac{1}{2}(H - H^T)$. Therefore, we can split any game into potential part and Hamilton part. For potential game, which is well-studied, gradient descent suffice to solve it. For Hamilton game, we also know that gradient descent on the Hamiltonian of the game $\mathcal{H} = \frac{1}{2}||\xi||_2^2$ can help approach a local Nash equilibrium as well. Thus, the only question is: how to combine them to solve the general game?

The exact solution to a general case is:

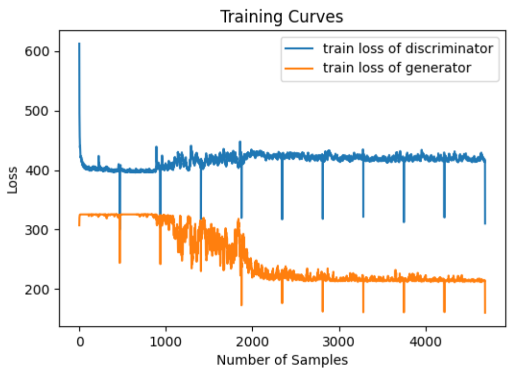
Algorithm 1 Symplectic Gradient Adjustment

Input: $losses\mathcal{L} = \{l_i\}_{i=1}^n, weights\mathcal{W} = \{w_i\}_{i=1}^n$
  $\epsilon \to [gradient(l_i, w_i) for (l_i, w_i) \in (\mathcal{L}, \mathcal{W})]$
  $A^T\xi \leftarrow get\_sym\_adj(\mathcal{L}, \mathcal{W})$
  if align **then**
    $\nabla\mathcal{H} \to [gradient(\frac{1}{2}||\xi||^2, w),$ for $w \in \mathcal{W}]$
    $\lambda \to sign(\frac{1}{d} < \xi, \nabla\mathcal{H} >< A^T\xi, \nabla\mathcal{H} > +\epsilon)$ // $\epsilon = 0.1$
  else
    $\lambda \to 1$
  end if
Output: $\xi + \lambda A^T\xi$

Now we explain why this process can find a Nash equilibrium. Firstly, we relate NE to stable fixed points in optimization theory. We state that a point w^* is a local Nash equilibrium if for $\forall i$, $\exists$ neighbor U_i of w_i^* s.t. $l_i(w_i, w_{-i}^*)$ for $w_i \in U_i$. If fixed point w^* is stable ($S(w) > 0$) then it is a local NE. This is due to the fact that if S is semi-positive definite then so are its $(d_i \times d_i)$ sub-matrices $S_i = \nabla^2_{w_i} l_i, \forall i$.

Let $\mathcal{H}(w) = \frac{1}{2}||\xi(w)||_2^2$. If the game is Hamiltonian then

1. $\nabla\mathcal{H} = A^T\xi$;
2. ξ preserves the level sets of ξ since $(\xi, \nabla\mathcal{H}) = 0$;
3. If the Hessian is invertible and $lim_{||w|| \to \infty}\mathcal{H}(w) = \infty$ then gradient descent on $\mathcal{H}$ converges to a local Nash equilibrium.



(a) curve of vanilla model


(b) samples of vanilla model

If $H(w)$ is invertible, then $\nabla\mathcal{H} = H^T\xi = 0$ if.f $\xi = 0$. Thus gradient descent on $\mathcal{H}$ converges to a fixed points of ξ. Recall that we have proposed *consensus optimization*, as $\xi + \lambda H^T\xi = \xi + \lambda\nabla\mathcal{H}$, and unfortunately it's not stable. To find stable fixed points, we state that an adjustment ξ_λ to the game dynamics should satisfy:

1. compatible with game dynamics: $< \xi_\lambda, \xi >= \alpha_1||\xi||^2$;
2. compatible with Potential dynamics: if the game is a Potential game then $< \xi_\lambda, \nabla\Phi >= \alpha_2||\nabla\Phi||^2$;
3. compatible with Hamiltonian dynamics: If the game is a Hamilton game then $< \xi_\lambda, \nabla\mathcal{H} >= \alpha_3||\nabla\mathcal{H}||^2$;
4. attracted to stable equilibria: in neighborhoods where S>0, require $\theta(\xi_\lambda, \nabla\mathcal{H}) \leq \theta(\xi, \nabla\mathcal{H})$;
5. repelled by unstable equilibria: in neighborhoods where S < 0, require $\theta(\xi_\lambda, \nabla\mathcal{H}) \geq \theta(\xi, \nabla\mathcal{H})$.

And we state that the SGA $\xi_\lambda = \xi + \lambda A^T\xi$ satisfies property 1-3 for $\lambda > 0$ with $\alpha_1 = 1 = \alpha_2$ and $\alpha_3 = \lambda$. The proof can be sketched as: First claim: $\lambda\xi^T A^T\xi = 0$ by skew-symmetry of A. Second claim: $A \equiv 0$ in a potential game $\xi_\lambda = \xi = \nabla\Phi$. Third claim: $< \xi_\lambda, \nabla\mathcal{H} >=< \xi_\lambda, H^T\xi >=< \xi_\lambda, A^T\xi >= \lambda$. And $\xi^T AA^T\xi = \lambda||\nabla\mathcal{H}||^2$ since $H = A$ by assumption and $\xi^T A^T\xi = 0$ by antisymmetry. Property 4-5 could be guaranteed by the exquisite design of the algorithm process. Finally, using the algorithm above, we select λ, s.t. it satisfies $\lambda \cdot \langle\xi, \nabla H\rangle \cdot \langle A^T\xi, \nabla H\rangle \geq 0$, in this way ξ_λ is bent towards stable and away from unstable fixed points.

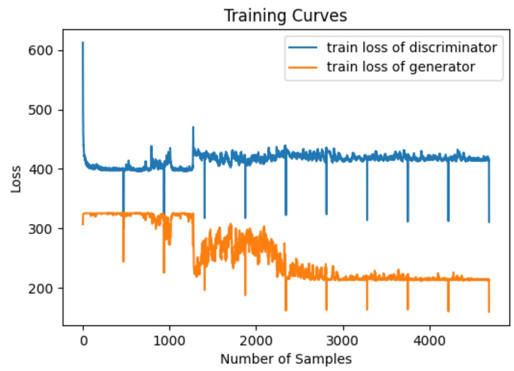
To conclude, by this algorithm which is similar to gradient descent, we optimize λ and get the result of $\xi + \lambda A^T\xi$, which can perform better in the process of Gradient Adjustment.

8 Experiments analysis

To evaluate the performance, we conducted experiments on the MNIST dataset and the distribution-mixture task using three optimization algorithms: SimGA, Consensus Optimization, and SGA, we also compared with Triple-GAN structure and Vanilla structure. The experimental code is openly available at `https://github.com/srzer/On-convergence-of-GAN`.

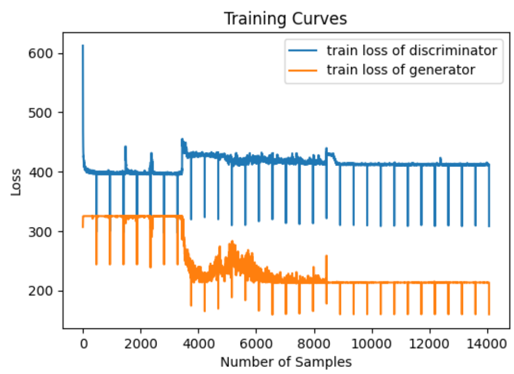
The training curves and generated samples are presented in figures (a), (b), (c), (d), (e), (f), (g), (h), (i), and (j). And for distribution mixture task, the training curves and visualized results are presented in figures (k), (l), (m), (n).

When comparing the vanilla model with consensus optimization, we observed that consensus optimization demonstrates the ability to stabilize the training process and enhance the quality of generated samples, as demonstrated in figure (f). However, the weight parameter of consensus, denoted as h, has a significant impact on the final results. If h is set too small, the performance does not significantly differ from simultaneous gradient ascent, as seen in figure (d). Conversely, if h is set too large,



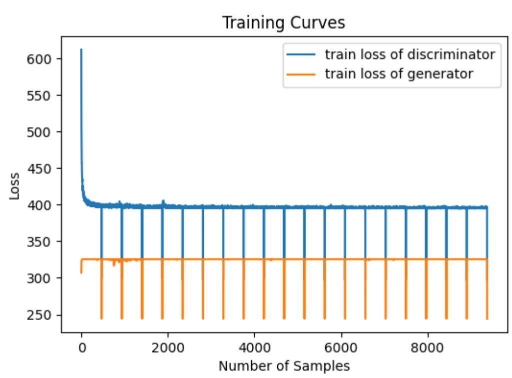
(c) curve of consensus optimization, $h = 10^{-4}$

(d) samples of consensus optimization, $h = 10^{-4}$

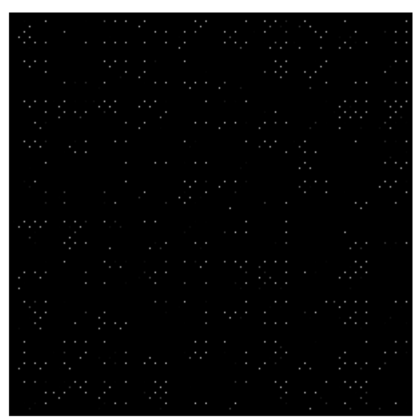


(e) curve of consensus optimization, $h = 5 \times 10^{-4}$

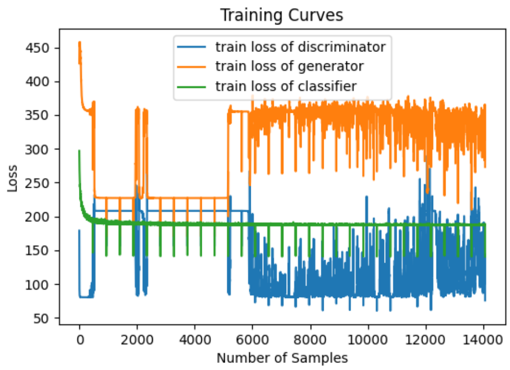
(f) samples of consensus optimization, $h = 5 \times 10^{-4}$



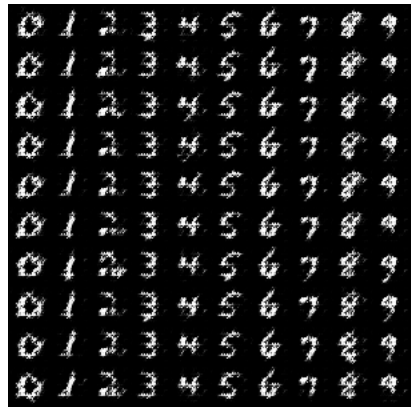
(g) failure curve of consensus optimization, $h = 10^{-3}$

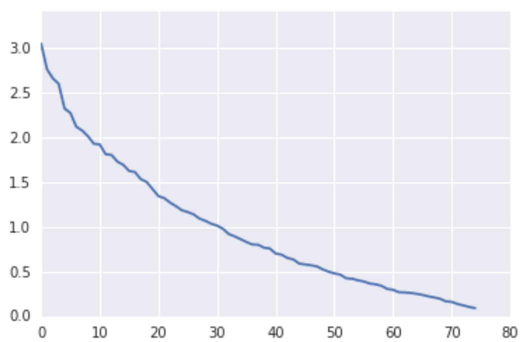


(h) failure sample of consensus optimization, $h = 10^{-3}$

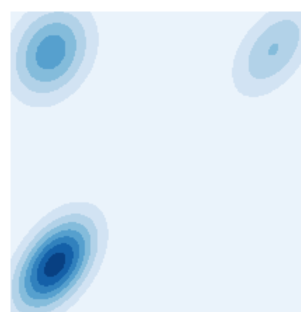


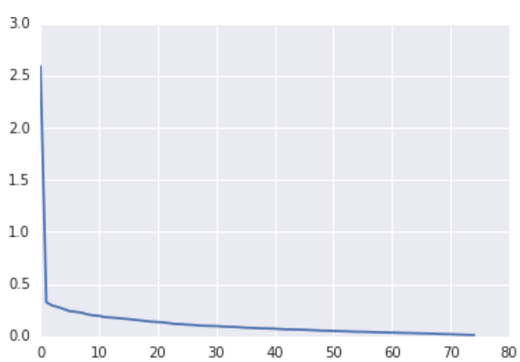
(i) curve of Triple GAN with SimGA, mode collapse


(j) sample of Triple GAN with SimGA, mode collapse

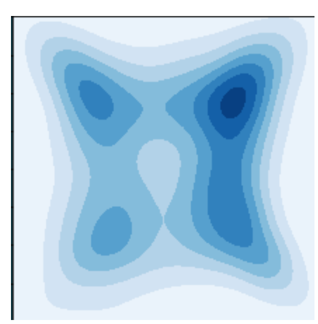


(k) loss of 3-mixed-gaussian


(l) 3-mixed-gaussian



(m) loss of 4-mixed-gaussian


(n) 4-mixed-gaussian

indicating a dominant influence of consensus in the utility, the performance degrades considerably in practice, as shown in figure (h).

Furthermore, after modifying the model structure to Triple GANs, we discovered that using SimGA leads to fast convergence but can result in mode collapse, as illustrated in figure (j). This implies that while SimGA may converge quickly, it may not generate diverse and high-quality samples.

To incorporate Symplectic Gradient Adjustment (SGA) into Generative Adversarial Networks (GANs), we can directly substitute the ascent process (such as SimGA or Consensus optimization) with SGA. In practical applications, when the training data is designed to explore the occurrence of mode collapse (where certain modes are skipped during sample generation), SGA demonstrates robustness comparable to Consensus optimization, surpassing the capabilities of SimGA. Additionally, SGA allows for the use of larger learning rates, facilitating faster convergence. Moreover, SGA can be effectively employed in multi-player structures like Triple-GAN without sacrificing its theoretical guarantees, unlike Consensus optimization.

When exploring SGA, we selected a well-known task: Gaussian mixtures. And we also simulate some composite distributions using SGA, and get the simulating graph. Its convergence performance is fast and accurate. Notably, we observed that SGA demonstrated exceptional adaptation to this task, without any instances of mode collapse, like (l) and (n). The convergence and performance of SGA were both remarkable.

9 Philosophical reflections

Although our primary objective in GANs is to learn the probability distribution, achieving this goal requires the design of complex game structures. To illustrate this, we conclude some philosophical reflections and compared with a couple of real-world examples:

Deliberately creating competitions can greatly benefit the final results. In vanilla GANs, we introduce a discriminator and a generator that engage in a competitive relationship. This competition drives the generator to improve its performance, resulting in higher-quality generated samples. It can be likened to a scenario where two fashion designers compete to create the most innovative and appealing designs, pushing each other to constantly raise the bar.

On the other hand, **reducing meaningless and malignant internal competition** can enhance efficiency and performance. In Triple GANs, we introduce a classifier alongside the generator and discriminator. This addition prevents unnecessary competition between the classifier's role of classifying and the discriminator's role of discriminating. By coordinating their efforts, they achieve a win-win outcome. One example of a situation where appropriate competition while avoiding malignant competition can enhance efficiency is in the context of a sales team within a company. In a sales team, it is crucial to have a competitive environment that drives individual performance and motivates sales representatives to achieve their targets. However, if the competition becomes too aggressive or cutthroat, it can lead to a negative work atmosphere, undermine teamwork, and ultimately harm the overall performance of the team. To strike the right balance, the company can implement a system where sales representatives compete for performance-based incentives, such as bonuses or recognition. This competition encourages individuals to strive for excellence and reach their sales targets, benefiting both the company and themselves.

In the context of GANs, the main objective is to find stable fixed points, without necessarily considering the exact losses of each player. This perspective gives rise to some interesting observations:

The **existence of consensus** can contribute to obtaining a better social welfare, which directly corresponds to the quality and convergence of the GAN. It is akin to a group of stakeholders in a decision-making process striving to reach a consensus that benefits everyone involved, resulting in a more effective and satisfactory outcome.

Sometimes, achieving the equilibrium requires a degree of **self-sacrifice**. In methods such as Symplectic Gradient Adjustment (SGA), the sign of a parameter (λ) may occasionally be flipped, causing the parameters to deviate from the strictly increasing direction. However, this sacrifice can help in finding stable fixed points and ultimately reaching a Nash equilibrium. This notion is reminiscent of situations where individuals make short-term sacrifices for the greater good, recognizing that it can lead to long-term stability and mutual benefits.

By examining these real-life examples and drawing parallels to GANs, we gain a deeper understanding of the complexities involved and the strategies employed to optimize their performance.

10 Workload Distribution

Collaboration.

1. We determined the project direction and conducted initial research.
2. We immersed ourselves in SGA papers, sharing insights and understanding of their main ideas and proofs.
3. We invested effort in preparing polished presentations and demos, contributing equally in the presentation.

Ruizhe Shi.

1. Analyzed and organized papers on GANs, discussing outcomes, advantages, and limitations.
2. Consolidated our findings and examined the mechanisms of Triple-GANs, conducted a small-scale simulation experiment for validation.
3. Refined our paper, adding details to demonstrate our comprehensive understanding.

Chenru Wen.

1. Guided the discussion of SGA part.
2. Reproduced an experiment of SGA.

References

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